PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 4, August 1981

A WEAKLY INFINITE-DIMENSIONAL COMPACTUM WHICH IS NOT COUNTABLE-DIMENSIONAL

ROMAN POL[1]

ABSTRACT. A compact metric space is constructed which is neither a countable union of zero-dimensional sets nor has an essential map onto the Hilbert cube.

We consider only separable metrizable spaces and a compactum means a compact space.

A space is *countable-dimensional* if $X = \bigcup_{i=1}^{\infty} X_i$ with X_i zero-dimensional; a space X is *weakly infinite-dimensional* if for each countable family $\{(A_i, B_i)\colon i = 0, 1, \dots\}$ of pairs of disjoint closed sets in X there are partitions S_i between A_i and B_i (i.e., closed sets separating A_i and B_i in X) with $\bigcap_{i=0}^{\infty} S_i = \varnothing$ [**A-P**, Chapter 10, §47], [**N**].

Countable-dimensional spaces are weakly infinite-dimensional[2] and an old open question of P. S. Aleksandrov [**A1**, §4, Hypothesis] (cf. also [**A2**], [**A-P**, Chapter 10], [**S**], [**N**, Problem 13-7]) asked whether the converse is true for compacta.[3] In this note we present a counterexample, i.e., we describe a compactum X with the properties indicated in the title.

The existence of such an X is an easy consequence of the following lemma.

LEMMA. *There exists a topologically complete space Y which is totally disconnected but not countable-dimensional* (*not even weakly infinite-dimensional*).

The existence of such a space Y follows immediately from a construction in [**R-S-W**] (see also Comment A). More specifically, if one performs the construction in Example 4.5 of [**R-S-W**] using, as indicated in Remark 4.4, the Hilbert cube instead of the $n + 1$-dimensional cube, then one obtains a compactum M and a continuous map $p\colon M \to \Delta$ onto the Cantor set Δ such that each subset of M which maps onto Δ is not weakly infinite-dimensional (see Proposition 3.4 and Remark 4.1 of [**R-S-W**]). It is, however, well known that in this situation there exists a G_δ-set $Y \subset M$ which intersects each fiber $p^{-1}(t)$ in exactly one point [**B**, p. 144, Exercise 9a], [**Ku2**, Chapters IV, IX], and this is the space Y we need.

Received by the editors April 21, 1980.
1980 *Mathematics Subject Classification*. Primary 54F45.

[1]This paper was written while the author was visiting the University of Washington.

[2]This follows from the fact [**H-W**, Chapter II, §2, F] that, given two closed disjoint sets $A, B \subset X$ and a zero-dimensional set $E \subset X$, there is a partition in X between A and B disjoint from E; cf. also [**H-W**, Chapter IV, §6, A].

[3]For nonmetrizable compact spaces, a counterexample was recently constructed by Fedorčuk [**F**].

Now, let Y be as in the lemma and let X be a compactification of Y with countable-dimensional remainder $X \setminus Y$ (the existence of X, which requires only the completeness of Y, is a well-known fact following easily from a theorem of Kuratowski [**Ku1**, Théorème 2]; cf. also [**E**, 4.15]). It is enough to check that X is weakly infinite-dimensional. Let $\{(A_i, B_i): i = 0, 1, \dots\}$ be as above and let $X \setminus Y = \bigcup_{i=1}^{\infty} X_i$ with X_i zero-dimensional. Let S_i be a partition in X between A_i and B_i disjoint from X_i, $i = 1, 2, \dots$ (see footnote 2), and let $S = \bigcap_{i=1}^{\infty} S_i$. The set S, being a compact subset of the totally disconnected space Y, is zero-dimensional, and thus (see footnote 2) there is a partition S_0 in X between A_0 and B_0 disjoint from S. Since $\bigcap_{i=0}^{\infty} S_i = \varnothing$, we are done.

Comments. A. The construction of Rubin, Schori and Walsh [**R-S-W**] which we have used (in fact, a simpler variant of this construction is enough for our purpose) is closely related to a construction of Lelek [**L**, Example, p. 81] which follows an old idea going back at least to Knaster [**Kn**]. It is our feeling that the space Y did not appear in the literature much earlier only because it seemed that there was no reason for such a construction (even the authors of [**R-S-W**] noted Y only as a by-product of a certain much more powerful technique they developed). Probably, the old construction of Mazurkiewicz [**Ma**] of totally disconnected topologically complete spaces M_n with $\dim M_n = n$ can also be adapted to obtain Y; it also seems quite probable that $Y = M_1 \times M_2 \times \dots$ has the desired property.

B. The existence of X, together with some results obtained in [**P1**], yields the following two statements:

(a) There is a weakly infinite-dimensional compactum S containing compact subspaces of arbitrarily large transfinite dimension (see [**H-W**, Chapter IV, §6, B] or [**E**] for the definition).

(b) The second question formulated by Henderson in [**H**, p. 168] has a positive answer, while the first question has a negative answer even for compacta which are countable-disjoint unions of finite polytopes.

The special construction of Y which we have applied also allows one to choose an X which maps continuously onto the Cantor set by a map with countable-dimensional fibers.

C. The space X shows that weak infinite-dimensionality is not a hereditary property. An idea of Michael [**Mi**] can be also used to define two (noncomplete) subspaces A, B of X which are weakly infinite-dimensional, but their product $A \times B$ is not weakly infinite-dimensional [**P2**].

References

[**A1**] P. S. Aleksandrov, *The present status of the theory of dimension*, Uspehi Mat. Nauk **6** (1951), 43–68. (Russian)

[**A2**] ______, *On some results in the theory of topological spaces obtained within the last twenty-five years*, Uspehi Mat. Nauk **15** (1960), 25–95. (Russian)

[**A-P**] P. S. Aleksandrov and B. A. Pasynkov, *Introduction to dimension theory*, "Nauka", Moscow, 1973. (Russian)

[**B**] N. Bourbaki, *Topologie générale*, 2nd ed., Chap. 9, Hermann, Paris, 1958.

[**E**] R. Engelking, *Transfinite dimension*, Surveys in General Topology, Ed., G. M. Reed, 1980.

[**F**] V. V. Fedorčuk, *Infinite-dimensional compact Hausdorff spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978); English transl. in Math. USSR Izv. **13** (1979), 445–460.

[H] D. W. Henderson, *A lower bound for transfinite dimension*, Fund. Math. **63** (1968), 167–173.

[H-W] W. Hurewicz and H. Wallman, *Dimension theory*, Van Nostrand, Princeton, N. J., 1948.

[Kn] B. Knaster, *Sur les coupures biconnexes des espaces euclidiens de dimension* $n > 1$ *arbitraire*, Mat. Sb. **19** (1946), 9–18. (Russian)

[Ku1] K. Kuratowski, *Sur le prolongement des fonctions continues et les transformations en polytopes*, Fund. Math. **24** (1935), 258–268.

[Ku2] ______, *Topology*. Vol. II, PWN, Warsaw, 1968.

[L]. A. Lelek, *Dimension inequalities for unions and mappings of separable metric spaces*, Colloq. Math. **23** (1971), 69–91.

[Ma] S. Mazurkiewicz, *Sur les problèmes* κ *et* λ *de Urysohn*, Fund. Math. **10** (1927), 311–319.

[Mi] E. Michael, *The product of a normal space and a metric space need not be normal*, Bull. Amer. Math. Soc. **69** (1963), 375–376.

[N] K. Nagami, *Dimension theory*, Academic Press, New York, 1970.

[P1] R. Pol, *On classification of weakly infinite-dimensional compacta*, Fund. Math. (to appear).

[P2] ______, *A remark on A-weakly infinite-dimensional spaces*, General Topology Appl. (to appear).

[R-S-W] L. R. Rubin, R. M. Schori and J. J. Walsh, *New dimension-theory techniques for constructing infinite-dimensional examples*, General Topology Appl. **10** (1979), 93–102.

[S] Ju. M. Smirnov, *On dimensional properties of infinite-dimensional spaces*, Proc. Sympos. Prague 1961, Czechoslovak Acad. Sci., Prague, 1962.

Wydział Matematyki, Uniwersytet Warszawski, Pałac Kultury i Nauki IX p., 00-901 Warszawa, Poland