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A WEIGHTED NORM INEQUALITY FOR FOURIER SERIES

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Let $Mf(x)=\sup|S_nf(x)|$, where S_nf denotes the nth partial sum of the Fourier series of f. We will show

$$(1)\qquad w \in A_p,\ p > 1, \quad \text{implies} \quad \int [Mf]^p w \leqq C \int |f|^p w.$$

Recall that a nonnegative weight function $w \in A_p$, $p>1$, if there is a constant K such that

$$\left(\int_I w\right)\left(\int_I w^{-1/(p-1)}\right)^{p-1} \leqq K\,|I|^p$$

for all intervals I. The A_p condition, $p>1$, characterizes all weights w for which the mapping of f into the Hardy-Littlewood maximal function of f is bounded on the weighted L^p space $L^p(w)$. (See Muckenhoupt **[6]**.) This fundamental fact leads to boundedness on $L^p(w)$ for other operators which can be associated with the Hardy-Littlewood maximal function. For example, the conjugate function and more general singular integrals are of this type. Also, $\int |S_nf-f|^p w \to 0$ $(n\to\infty)$ if and only if $w \in A_p$, $p>1$. (See Hunt, Muckenhoupt and Wheeden **[5]** and Coifman **[3]**.) It follows that the inequality in (1) holds only if $w \in A_p$, $p>1$.

Our proof of (1) follows closely the proof in Coifman **[3]**. We will prove a Burkholder-Gundy type distribution function inequality which relates the weighted distribution functions of modified versions of Mf and the Hardy-Littlewood maximal function of f. (See Burkholder and Gundy **[1]**.) To do this we will use the boundedness of M on L^r, $r>1$, and an extremely useful consequence of the A_p condition which relates the w-weighted measure and the Lebesgue measure of certain types of sets. This useful property is closely related to the development of Muckenhoupt **[6]** and was first explicitly used in connection with a distribution

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function inequality by Fefferman in an unpublished paper. The distribution function inequality implies the $L^p(w)$ norm of Mf is majorized by a constant multiple of the $L^p(w)$ norm of the modified Hardy-Littlewood maximal function of f. (1) then follows from Muckenhoupt's result on the $L^p(w)$ boundedness of the Hardy-Littlewood maximal function.

Let

$$H_r f(x) = \sup_{h>0} \left(\frac{1}{2h} \int_{|x-t|<h} |f(t)|^r \, dt \right)^{1/r}, \qquad r \geqq 1.$$

Note that $H_1 f$ is the usual Hardy-Littlewood maximal function and so $\int [H_1 f]^s w \leqq C \int |f|^s w$ if $w \in A_s$, $s>1$. (See Muckenhoupt [6].) Since $H_r f = (H_1(|f|^r))^{1/r}$, it follows that

$$\text{(2)} \qquad w \in A_{p/r},\ r < p, \quad \text{implies} \quad \int [H_r f]^p w \leqq C \int |f|^p \, w.$$

We will need to use (2) for some $r>1$. This is possible because of the following fundamental result of Muckenhoupt [6]:

$$\text{(3)} \qquad w \in A_p,\ p > 1, \quad \text{implies} \quad w \in A_{p/r} \quad \text{for some } 1 < r < p.$$

Following Carleson [2], we replace Mf by

$$M^* f = \sup_n \left| \int_{|x-t|<\pi} e^{-int} f(t)/(x-t) \, dt \right|.$$

In fact, we will use

$$M^{**} f = \sup_n \sup_{\varepsilon>0} \left| \int_{\varepsilon<|x-t|<\pi} e^{-int} f(t)/(x-t) \, dt \right|.$$

Standard arguments imply

$$\text{(4)} \quad Mf \leqq C(H_1 f + M^* f) \leqq C(H_1 f + M^{**} f) \leqq C(H_1 f + H_1(M^* f)).$$

From (4) and (2) with $r=1$ we see that we may replace Mf by $M^{**}f$ in (1). Also, since $\int [M^* f]^r \leqq C \int |f|^r$, $r>1$, (see Hunt [4]) we have

$$\text{(5)} \qquad r > 1 \quad \text{implies} \quad \int [M^{**} f]^r \leqq C \int |f|^r.$$

Given $w \in A_p$, $p>1$, choose r as in (3). We will prove

$$\text{(6)} \qquad m_w(M^{**} f > 3\lambda,\ H_r f \leqq \gamma\lambda) \leqq C(\gamma) m_w(M^{**} f > \lambda),$$

where $C(\gamma) \to 0$ $(\gamma \to 0)$. $(m_w(E) = \int_E w.)$

Given this weighted distribution function inequality it is easy to complete the proof of (1). From (6) we obtain

$$m_w(M^{**} f > 3\lambda) \leqq m_w(H_r f > \lambda\gamma) + C(\gamma) m_w(M^{**} f > \lambda).$$

Hence,

$$p\int_0^\infty \lambda^{p-1} m_w(M^{**}f > 3\lambda)\, d\lambda$$

$$\leqq p\int_0^\infty \lambda^{p-1} m_w(H_r f > \gamma\lambda)\, d\lambda + C(\gamma)p\int_0^\infty \lambda^{p-1} m_w(M^{**}f > \lambda)\, d\lambda$$

and so $\int [M^{**}f]^p w \leqq [\gamma^{-p}/(3^{-p}-C(\gamma))] \int [H_r f]^p w$. (2) then implies (1).

To prove (6) note that the set $(M^{**}f > \lambda)$ is open, so $(M^{**}f > \lambda) = \bigcup I_j$, where the intervals $I_j = (\alpha_j, \alpha_j + \delta_j)$ are disjoint and $M^{**}f(\alpha_j) \leqq \lambda$. It is then sufficient to prove

$$\text{(7)} \qquad m_w(x \in I_j : M^{**}f > 3\lambda,\ H_r f \leqq \gamma\lambda) \leqq C(\gamma) m_w(I_j).$$

We may clearly assume there is a point $z_j \in I_j$ with $H_r f(z_j) \leqq \gamma\lambda$.

Let $\bar{I}_j = (\alpha_j - 2\delta_j, \alpha_j + 2\delta_j)$,

$$\begin{aligned} f_1(x) &= f(x), & x &\in \bar{I}_j, \\ &= 0, & x &\notin \bar{I}_j, \end{aligned} \quad \text{and } f_2 = f - f_1.$$

m will denote Lebesgue measure.

Using (5) we have

$$m(M^{**}f_1 > \lambda) \leqq \lambda^{-r}\int [M^{**}f_1]^r \leqq C\lambda^{-r}\int |f_1|^r$$

$$\leqq C\lambda^{-r}[H_r f(z_j)]^r m(I_j) \leqq C\gamma^r m(I_j).$$

For any $x \in I_j$, n and $\varepsilon > 0$,

$$\left| \int_{\varepsilon < |x-t| < \pi} e^{-int} f_2(t)/(x-t)\, dt - \int_{\varepsilon < |\alpha_j - t < \pi} e^{-int} f_2(t)/(x-t)\, dt \right|$$

is majorized by $C_0 H_1 f(z_j) \leqq C_0 H_r f(z_j) \leqq C_0 \gamma\lambda$. It follows that $x \in I_j$ implies

$$M^{**}f_2(x) \leqq M^{**}f(\alpha_j) + C_0\gamma\lambda \leqq (1 + C_0\gamma)\lambda,$$

and so

$$M^{**}f(x) \leqq M^{**}f_1(x) + M^{**}f_2(x) \leqq M^{**}f_1(x) + (1 + C_0\gamma)\lambda.$$

Hence, $M^{**}f(x) > 3\lambda$, $x \in I_j$, implies $M^{**}f_1(x) > \lambda$ if $1 + C_0\gamma < 2$. Collecting results we obtain

$$\text{(8)} \qquad m(x \in I_j : M^{**}f(x) > 3\lambda,\ H_1 f \leqq \gamma\lambda) \leqq C\gamma^r m(I_j).$$

(7) follows immediately from (8) and the following consequence of the A_p condition:

(9) If $w \in A_p$, any p, then there are positive constants C and δ such that

for any interval I and measurable set E, $m(E\cap I)\leqq \varepsilon m(I)$ implies $m_w(E\cap I)\leqq C\varepsilon^\delta m_w(I)$.

To prove (9) we use the fact that $w\in A_p$, any p, implies there is $s>1$ and a constant C such that

$$\left(\int_I w^s\right)^{1/s} \leqq C\,|I|^{(1/s)-1}\int_I w \tag{10}$$

for all intervals I. (See Muckenhoupt [6].) If $(1/s)+(1/s')=1$, Hölder's inequality and (10) imply

$$\int_{E\cap I} w \leqq (m(E\cap I))^{1/s'}\left(\int_I w^s\right)^{1/s} \leqq C(m(E\cap I)/m(I))^{1/s'}\int_I w.$$

This gives (9) and completes our proof.

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