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A WEIGHTED NORM INEQUALITY FOR ROUGH SINGULAR INTEGRALS

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Abstract. We prove a weighted norm inequality for homogeneous singular integrals when only an H^1-size condition is assumed on the restriction of the kernel to the unit sphere. We also give several applications of this inequality.

1. Introduction. Let $\boldsymbol{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and S^{n-1} the unit sphere in $\boldsymbol{R}^n$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let $\Omega(x)|x|^{-n}$ be a homogeneous function of degree $-n$ with $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.1}$$

where $x' = x/|x|$ for any $x \neq 0$.

Let $b(t)$ be a measurable function on $(0, \infty)$ and γ a real number larger than one. We say that $b \in \Delta_\gamma$ if $\|b\|_{\Delta_\gamma} = \sup_{R>0} R^{-1} \int_0^R |b(t)|^\gamma dt < \infty$. We also define $\Delta_\infty = L^\infty(\boldsymbol{R}_+)$. Clearly, $\Delta_\infty \subseteq \Delta_\gamma \subseteq \Delta_\lambda$ for any $1 < \lambda \leq \gamma$. Suppose that $\Gamma(t)$ is a strictly monotonic C^1 function on the interval $(0, \infty)$. We define the singular integral operator $T_{\Gamma,b}(f)$ by

$$T_{\Gamma,b}(f)(x) = \text{p.v.} \int_{\boldsymbol{R}^n} K(y) f(x - \Gamma(|y|)y')dy, \tag{1.2}$$

where $y' = y/|y| \in S^{n-1}$, $K(y) = b(|y|)\Omega(y')|y|^{-n}$ and $f \in \mathcal{S}(\boldsymbol{R}^n)$, the space of Schwartz functions.

For the sake of simplicity, we denote $T_{\Gamma,b} = T_b$ if $\Gamma(t) = t$ and denote $T_{\Gamma,b} = T$ if $\Gamma(t) = t$ and $b(t) \equiv 1$.

The investigation of the operators T_b began with Calderón-Zygmund's pioneering study of the operator T (see [CZ1], [CZ2]). The operator T_b, whose kernel has the additional roughness in the radial direction due to the presence of b, was first studied by R. Fefferman ([Fe]) and subsequently by many other authors ([C], [Ch], [CR], [DR], [Fa], [FP1], [FP2], [H1], [JL], [KS], [Na], [Wa1]). The best result concerning the size of Ω, so far, is the following theorem.

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THEOREM A (see [FP1]). *If $\Omega \in H^1(S^{n-1})$, $\Gamma(t)$ satisfies the conditions either in the following Theorem 1 or Theorem 2 (in particular $\Gamma(t) = t$) and $b \in \Delta_2$, then $T_{\Gamma,b}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

The proof of Theorem A is exactly the same as that of [FP1], in which the function Γ is slightly different. In [FP1], we proved the theorem for $b \in L^\infty(\mathbf{R}_+)$ and pointed out that the theorem still remains true if $b \in \Delta_2$ by an easy treatment (see [FP2, Theorem 7.5]).

The space $H^1(S^{n-1})$ in Theorem A represents the Hardy space on the unit sphere, whose definition will be reviewed in section 2. In order to make comparisons among the conditions imposed on Ω in other known results, we point out that on S^{n-1}, for any $q > 1$, $L^q(S^{n-1}) \subseteq L\log^+ L(S^{n-1}) \subseteq H^1(S^{n-1})$ and all inclusions are proper.

On the other hand, the weighted L^p boundedness of T_b has also been studied by a number of authors ([ABKP], [Du], [H2], [KW], [MW], [Wa2]). However, all papers mentioned above require a stronger condition than $\Omega \in H^1(S^{n-1})$ even in the simplest case $b = 1$.

In this paper, we are interested in studying the weighted L^p boundedness of $T_{\Gamma,b}$ for certain radial weights, which were introduced in [Du]. Our result, in which the $L\log^+ L$ size on Ω in [Du] is weakened by the H^1 size, is an outgrowth of a recent result in [Du] by Duoandikoetxea. We will also give several applications of our result, including the power weights $|x|^\alpha$; L^p boundedness of certain commutators; and the boundedness of $T_{\Gamma,b}$ on the Herz spaces and on the Morrey spaces. It is well-known that both commutators and Morrey spaces play an important role in studying the solvability and the regularity of solutions to partial differential equations with discontinuous coefficients (see [FR] and [Hu]).

In order to state our main theorems, we first give the definitions of certain weights.

DEFINITION 1.1. Suppose that $\omega(t) \geq 0$ and $\omega \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$. For $1 < p < \infty$, we say that $\omega \in A_p(\mathbf{R}_+)$ if there is a constant $C > 0$ such that for any interval $I \subset \mathbf{R}_+$,

$$\left(|I|^{-1}\int_I \omega(r)dr\right)\left(|I|^{-1}\int_I \omega(r)^{-1/(p-1)}dr\right)^{p-1} \leq C < \infty. \tag{1.3}$$

If there is a constant $C > 0$ such that

$$\omega^*(r) \leq C\omega(r) \quad \text{for a.e.} \quad r \in \mathbf{R}_+, \tag{1.4}$$

where ω^* denotes the standard Hardy-Littlewood maximal function of ω on $\mathbf{R}_+$, then we say $\omega \in A_1(\mathbf{R}_+)$.

DEFINITION 1.2. If $\omega(x) = \nu_1(|x|)\nu_2(|x|)^{1-p}$, where either $\nu_i \in A_1(\mathbf{R}_+)$ is decreasing or $\nu_i^2 \in A_1(\mathbf{R}_+)$, $i = 1, 2$, then we say $\omega \in \tilde{A}_p(\mathbf{R}_+)$.

DEFINITION 1.3. For $1 < p < \infty$, we denote

$$\bar{A}_p(\mathbf{R}_+) = \{\omega(x) = \omega(|x|) : \omega(t) > 0, \omega(t) \in L^1_{\mathrm{loc}}(\mathbf{R}_+) \text{ and } \omega^2(t) \in A_p(\mathbf{R}_+)\}.$$

Let $A^I_p(\mathbf{R}^n)$ be the weight class defined by using all n-dimensional intervals with sides parallel to coordinate axes. In what follows, for $p \in (1, \infty)$, any measurable function f and any weight ω, we define

$$\|f\|_{L^p(\omega)} \equiv \left(\int_{R^n} |f(x)|^p \omega(x)dx\right)^{1/p}.$$

Thus the weighted L^p spaces associate to the weight ω is defined by

$$L^p(R^n, \omega(x)dx) = \{f : \|f\|_{L^p(\omega)} < \infty\}.$$

By [DL], we know that $\bar{A}_p(R_+) \subseteq \tilde{A}_p(R_+)$. Also, if $\omega(t) \in \tilde{A}_p(R_+)$, then we know from [Du] that the Hardy-Littlewood maximal function Mf is bounded on $L^p(R^n, \omega(|x|)dx)$. Thus, if $\omega(t) \in \tilde{A}_p(R_+)$, then $\omega(|x|) \in A_p(R^n)$, where $A_p(R^n)$ is the Muckenhoupt weight (see [GR] for the definition). Let $\tilde{A}_p^I = \tilde{A}_p \cap A_p^I$ and $\bar{A}_p = \bar{A}_p \cap A_p^I$.

In [Du], Duoandikoetxea proved the following theorem.

THEOREM B (see [Du, Theorem 7]). *If $\omega \in \tilde{A}_p(R_+)$ for $1 < p < \infty$, then T is bounded on $L^p(\omega)$ provided $\Omega \in L\log^+ L(S^{n-1})$.*

Now we are in a position to state our results.

THEOREM 1. *Let $b \in \Delta_\gamma$ for $\gamma \geq 2$, $1 < p < \infty$. Let Γ be a nonnegative C^1 function on $(0, \infty)$ satisfying:*

(a) *Γ is strictly increasing and $\Gamma(2t) \geq \lambda\Gamma(t)$ for all t and some $\lambda > 1$,*

(b) *Γ satisfies a doubling condition, $\Gamma(2t) \leq c\Gamma(t)$ for all t and some $c > 0$,*

(c) *$\Gamma'(t) \geq C_1\Gamma(t)/t$ for all t and some $C_1 > 0$.*

Suppose that $\omega \in \tilde{A}^I_{p/\gamma'}(R_+)$ with $p \geq \gamma'$, where γ' is the dual exponent to γ. Then

$$\|T_{\Gamma,b}(f)\|_{L^p(\omega)} \leq C\|f\|_{L^p(\omega)} \tag{1.5}$$

provided $\Omega \in H^1(S^{n-1})$, where C is independent of f.

If Γ is a strictly decreasing function, we also have a similar result.

THEOREM 2. *Let $b \in \Delta_\gamma$ for $\gamma \geq 2$, $1 < p < \infty$. Let Γ be a nonnegative C^1 function on $(0, \infty)$ satisfying:*

(a′) *Γ is strictly decreasing and $\Gamma(t) \geq \lambda\Gamma(2t)$ for all t and some $\lambda > 1$,*

(b′) *$\Gamma(t) \leq c\Gamma(2t)$ for all $t > 0$ and some $c > 0$,*

(c′) *$|\Gamma'(t)| \geq C_1\Gamma(t)/t$ for all t and some $C_1 > 0$.*

Suppose that $\omega \in \tilde{A}^I_{p/\gamma'}(R_+)$ with $p \geq \gamma'$, where γ' is the dual exponent to γ. Then $T_{\Gamma,b}$ is bounded on $L^p(\omega)$ provided $\Omega \in H^1(S^{n-1})$.

REMARKS. (1) In both T and T_b, the singularity appears along the diagonal $\{x = y\}$. Recently many problems in analysis have led one to consider singular integrals with singularity along more general sets, some of which are in the form of $\{x = \Psi(y)\}$ (see [St]). Here we focus our attention on singular integrals $T_{\Gamma,b}$ which have singularity along sets of the form $\{x = \Gamma(|y|)y'\}$.

(2) The precise conditions on the constants in Theorems 1–2 should be $c \geq \lambda > 1$ and $C_1 \in (0, \log_2 c]$. Model functions for the Γ in Theorem 1 are $\Gamma(t) = t^d$ with $d > 0$, and

their linear combinations with positive coefficients. Model functions for the Γ in Theorem 2 are $\Gamma(t) = t^{\sigma}$ with $\sigma < 0$, and their linear combinations with positive coefficients.

(3) By the relationship $L \log^+ L(S^{n-1}) \subset H^1(S^{n-1})$ remarked above, one sees that, even in the special case $\Gamma(t) \equiv t$ and $b = 1$ ($T_{\Gamma,b} = T$), Theorem 1 represents an improvement of Theorem B in the case of $\omega \in \tilde{A}^I_p(\boldsymbol{R}_+)$.

(4) The method used in proving Theorem B depends heavily on the rotation method of Calderón and Zygmund. More precisely, one first considers the case that Ω is odd and reduces the operator T to the one-dimensional Hilbert transform whose $L^p(\omega)$ boundedness is well-known. For an even function $\Omega \in L \log^+ L$, one can compose Ω with suitable Riesz transforms to reduce it to an odd function. Clearly this method is no longer applicable if an extra rough function b appears in the kernel. To prove our theorem, we will use the atomic decomposition of the Hardy space, as well as some estimates about the atoms obtained in a previous paper ([FP1]).

We will review the definition of Hardy space and give some simple lemmas in section 2. The proofs of the theorems can be found in the third section. The truncated maximal operator will be studied in the fourth section. In Sections 5–8, we will give several applications of our theorems.

Throughout this paper, the letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Lemmas related to the H^1 space. We start this section by reviewing the definition of the H^1 space in the unit sphere. Recall that the Poisson kernel on S^{n-1} is defined by

$$P_{ry'}(x') = (1 - r^2)/|ry' - x'|^n ,$$

where $0 \le r < 1$ and $x', y' \in S^{n-1}$.

For any $f \in L^1(S^{n-1})$, we define the radial maximal function $P^+ f(x')$ by

$$P^+ f(x') = \sup_{0 \le r < 1} \left| \int_{S^{n-1}} f(y') P_{rx'}(y') d\sigma(y') \right| .$$

The Hardy space $H^1(S^{n-1})$ is the linear space of all $f \in L^1(S^{n-1})$ with the finite norm $\|f\|_{H^1(S^{n-1})} = \|P^+ f\|_{L^1(S^{n-1})} < \infty$. The space $H^1(S^{n-1})$ was well-studied in [Co] (see also [CTW]). In particular, it was shown that $H^1(S^{n-1})$ has the atomic decomposition property, which will be reviewed below.

A q-atom is an L^q ($1 < q \le \infty$) function $a(\cdot)$ that satisfies the following conditions (2.1)–(2.3).

(2.1) $\operatorname{supp}(a) \subset S^{n-1} \cap B(x_0, \rho)$, where $B(x_0, \rho)$ is the ball with center $x_0 \in S^{n-1}$ and radius $\rho \in (0, 2]$,

$$\int_{S^{n-1}} a(\xi') d\sigma(\xi') = 0 , \tag{2.2}$$

$$\|a\|_q \le \rho^{(n-1)(1/q-1)} . \tag{2.3}$$

From [Co] or [CTW], we find that any $\Omega \in H^1(S^{n-1})$ with the mean zero property (1.1) has an atomic decomposition $\Omega = \sum c_j a_j$, where the a_j's are ∞-atoms and $\sum |c_j| \leq C\|\Omega\|_{H^1(S^{n-1})}$.

If Γ is the function in Theorem 1, we let $\gamma_k = \Gamma(2^k)$. If Γ is the function in Theorem 2, we let $\gamma_k = 1/\Gamma(2^k)$. Then, by the conditions of Γ, it is easy to see $\inf_{k\in Z} \gamma_{k+1}/\gamma_k \geq \lambda > 1$ so that $\{\gamma_k\}$ is a lacunary sequence of positive numbers. For any $\Omega \in L^1(S^{n-1})$, we define the operator $\sigma_{\Omega,\Gamma,k}$ by

$$\sigma_{\Omega,\Gamma,k} * f(x) = \int_{2^k \leq |y| < 2^{k+1}} b(|y|)\Omega(y')|y|^{-n} f(x - \Gamma(|y|)y')dy \tag{2.4}$$

and $\sigma_{\Omega,k} = \sigma_{\Omega,\Gamma,k}$ if $\Gamma(t) = t$.

The maximal operator $\sigma^*_{\Omega,\Gamma}$ is defined by, for locally integrable functions f,

$$\sigma^*_{\Omega,\Gamma} f(x) = \sup_{k\in \boldsymbol{Z}} \int_{2^k \leq |y| < 2^{k+1}} |b(|y|)\Omega(y')|\,|y|^{-n}|f(x - \Gamma(|y|)y')|dy \tag{2.5}$$

and $\sigma^*_{\Omega} = \sigma^*_{\Omega,\Gamma}$ if $\Gamma(t) = t$.

For any $\rho > 0$, we define the linear transforms B_ρ and L_ρ on $\boldsymbol{R}^n$ by

$$B_\rho\xi = (\rho^2\xi_1, \rho\xi_2, \dots, \rho\xi_n), \qquad L_\rho\xi = (\rho\xi_1, \dots, \rho\xi_{n-1}, \rho^2\xi_n), \tag{2.5}$$

where $\xi = (\xi_1, \dots, \xi_n) \in \boldsymbol{R}^n$.

We have the following estimates for $\sigma_{a,\Gamma,k}$ if $\Omega = a$ is an ∞-atom.

LEMMA 2.1. *Let $b \in \Delta_\gamma$, $\gamma \geq 2$. Suppose that $a(\cdot)$ is an ∞-atom on S^{n-1} with* supp$(a) \subseteq S^{n-1} \cap B(\mathbf{1}, \rho)$, *where $\mathbf{1} = (1, 0, 0, \dots, 0) \in S^{n-1}$. Then there exist positive constants α and β such that if Γ is as in Theorem* 1, *then*

$$|\hat{\sigma}_{a,\Gamma,k}(\xi)| \leq C \min\{|B_\rho\xi|^\alpha \gamma_k^\alpha, \gamma_k^{-\beta}|B_\rho\xi|^{-\beta}\}; \tag{2.6}$$

if Γ is as in Theorem 2, *then*

$$|\hat{\sigma}_{a,\Gamma,k}(\xi)| \leq C \min\{|B_\rho\xi|^\alpha \gamma_k^{-\alpha}, \gamma_k^{\beta}|B_\rho\xi|^{-\beta}\}, \tag{2.6$'$}$$

where C is a constant independent of k, ξ and ρ.

PROOF. The lemma is a modification of Theorem B in [DR]. The proofs for (2.6) and (2.6′) are essentially the same as that in Section 3 of [FP1]. For completeness, we state the proof of (2.6′).

First, we consider the case $n > 2$. For any fix $\xi \neq 0$, we choose a rotation O such that $O(\xi) = |\xi|\mathbf{1}$. Let $y' = (s, y'_2, y'_3, \dots, y'_n)$. Then it is easy to see that

$$\hat{\sigma}_{a,\Gamma,k}(\xi) = \int_{2^k}^{2^{k+1}} b(t)t^{-1} \int_{S^{n-1}} a(O^{-1}(y'))e^{-i\Gamma(t)|\xi|\langle\mathbf{1},y\rangle} d\sigma(y')dt,$$

where O^{-1} is the inverse of O. Now $a(O^{-1}(y'))$ is again an ∞-atom with support in $B(\xi', \rho) \cap S^{n-1}$, where $\xi' = \xi/|\xi|$. For simplicity, we still denote it by $a(y')$. Now we have

$$\hat{\sigma}_{a,\Gamma,k}(\xi) = \int_{2^k}^{2^{k+1}} b(t)t^{-1} \int_{\boldsymbol{R}} F_a(s, \xi')e^{-i\Gamma(t)|\xi|s} dsdt,$$

where

$$F_a(s,\xi') = (1-s^2)^{(n-3)/2}\chi_{(-1,1)}(s)\int_{S^{n-2}} a(s,(1-s^2)^{1/2}\tilde{y})d\sigma(\tilde{y}),$$

$\chi_{(-1,1)}$ is the characteristic function of the interval $(-1,1)$ and $d\sigma(\tilde{y})$ is the Lebesgue measure on the unit sphere S^{n-2}. By Lemma 2.1 of [FP1], we know that the function F_a satisfies

(i) $\mathrm{supp}(F_a) \subseteq (\xi' - 2r(\xi), \xi' + 2r(\xi'))$;

(ii) $\|F_a\|_\infty \le C/r(\xi')$;

(iii) $\int_{\boldsymbol{R}} F_a(s,\xi')ds = 0$,

where C is a constant independent of the atom a and $r(\xi') = |\xi|^{-1}|B_\rho\xi|$. By (iii), we have

$$\hat{\sigma}_{a,\Gamma,k}(\xi) = \int_{2^k}^{2^{k+1}} t^{-1}b(t)\int_{\boldsymbol{R}} F_a(s,\xi')(e^{-i\Gamma(t)|\xi|s} - e^{-i\Gamma(t)|\xi|\xi'})dsdt.$$

By (ii), (iii), the definition of $b \in \Delta_\gamma$ and the conditions on Γ, we obtain

$$|\hat{\sigma}_{a,\Gamma,k}(\xi)| \le C\Gamma(2^{k+1})|\xi|r(\xi') \le C\gamma_k^{-1}|B_\rho\xi|. \tag{2.7}$$

On the other hand, using Hölder's inequality, we have

$$|\hat{\sigma}_k(\xi)| \le C\|b\|_{\Delta_2}2^{-k/2}\mathscr{I}_k,$$

where

$$\mathscr{I}_k = \left\{\int_{2^k}^{2^{k+1}}\left|\int_{\boldsymbol{R}} e^{-i\Gamma(t)r(\xi')|\xi|s}A(s)ds\right|^2 dt\right\}^{1/2}$$

and $A(s) = r(\xi')F_a(r(\xi')s,\xi')$ is an L^∞ function supported in the interval $(\xi_1|B_\rho\xi| - 2, \xi_1|B_\rho\xi|+2)$. By shifting variables, we may assume that $A(s)$ is an L^∞ function supported in the interval $(-1,1)$. To estimate $\mathscr{I}_k$, we choose a function $\psi \in C^\infty(\boldsymbol{R})$ satisfying

$$\psi(t) \equiv 1 \quad \text{for } |t| \le 1, \qquad \psi(t) \equiv 0 \quad \text{for } |t| \ge 2.$$

Let $I_k = (2^k, 2^{k+1})$ and define T_k by

$$(T_kf)(t) = \chi_{I_k}(t)\int_{\boldsymbol{R}} e^{-is\Gamma(t)r(\xi')|\xi|}\psi(s)f(s)ds.$$

Then

$$T_kT_k^*f(t) = \int_{\boldsymbol{R}} L(t,s)f(s)ds,$$

where

$$L(t,s) = \int_{\boldsymbol{R}} e^{iv(\Gamma(s)-\Gamma(t))r(\xi')|\xi|}\psi^2(v)dv\chi_{I_k}(t)\chi_{I_k}(s).$$

We easily see that

$$|L(t,s)| \le C\chi_{I_k}(s)\chi_{I_k}(t).$$

On the other hand, using integration by parts, we have

$$|L(t,s)| \le C\{|\Gamma(s) - \Gamma(t)|r(\xi')|\xi|\}^{-1}\chi_{I_k}(t)\chi_{I_k}(s).$$

So

$$|L(t,s)| \le C\{|\Gamma(s) - \Gamma(t)|r(\xi')|\xi|\}^{-1/2}\chi_{I_k}(t)\chi_{I_k}(s).$$

Now by the mean value theorem and conditions (a′)–(c′) on Γ, we have

$$|L(t,s)| \leq C2^{k/2}\{r(\xi')|\xi|\,|s-t|\Gamma(2^k)\}^{-1/2}\chi_{I_k}(s)\chi_{I_k}(t).$$

Therefore,

$$\sup_{s>0}\int_{\mathbf{R}} |L(t,s)|dt \cong \sup_{t>0}\int_{\mathbf{R}} |L(t,s)|ds \leq C2^k(\Gamma(2^k)r(\xi')|\xi|)^{-1/2}.$$

This shows

$$\|T_k f\|_2 \leq C2^{k/2}(r(\xi')|\xi|)^{-1/4}\gamma_k^{1/4}\|f\|_2,$$

which leads to

$$|\hat{\sigma}_k(\xi)| \leq C(|B_\rho \xi|)^{-1/4}\gamma_k^{1/4}. \tag{2.7'}$$

By (2.7) and (2.7′), we obtain (2.6′) for $n > 2$. Using the same argument and Lemma 2.2 in [FP1], we can prove the lemma for the case $n = 2$.

Similar to Lemma 2.1, we can obtain the following lemma.

LEMMA 2.2. *Let $b \in \Delta_\gamma$, $\gamma \geq 2$. Suppose that a is an ∞-atom on S^{n-1} with* supp$(a) \subseteq S^{n-1} \cap B(\tilde{\mathbf{1}}, \rho)$, *where $\tilde{\mathbf{1}} = (0,0,\dots,1) \in S^{n-1}$. Then there exist positive constants α and β such that if Γ is as in Theorem* 1 *then*

$$|\hat{\sigma}_{a,\Gamma,k}(\xi)| \leq C\min\{|L_\rho\xi|^\alpha\gamma_k^\alpha, |L_\rho\xi|^{-\beta}\gamma_k^{-\beta}\}; \tag{2.8}$$

if Γ is as in Theorem 2 then

$$|\hat{\sigma}_{a,\Gamma,k}(\xi)| \leq C\min\{|L_\rho\xi|^\alpha\gamma_k^{-\alpha}, |L_\rho\xi|^{-\beta}\gamma_k^{\beta})\}; \tag{2.8'}$$

where C is a constant independent of k, ξ and $\rho > 0$.

Next we treat the atom supported in $S^{n-1} \cap B(x_0, \rho)$ with an arbitrary x_0 on S^{n-1}. Let $SO(n)$ be the rotation group on $\mathbf{R}^n$. There exist $O_1, O_2 \in SO(n)$ such that $O_1x_0 = \mathbf{1}$ and $O_2x_0 = \tilde{\mathbf{1}}$. For a function $f(x)$ we define $f_i(x) = f(O_ix)$, $i = 1, 2$.

LEMMA 2.3. *Let a be an ∞-atom supported in $S^{n-1} \cap B(x_0, \rho)$. Then*

$$(\sigma_{a,\Gamma,k} * f)(x) = (\sigma_{a_i,\Gamma,k} * f_i)(O_i^{-1}x), \quad i = 1, 2, \tag{2.9}$$

where a_1 is an ∞-atom supported in $S^{n-1} \cap B(\mathbf{1}, \rho)$, a_2 is an ∞-atom supported in $S^{n-1} \cap B(\tilde{\mathbf{1}}, \rho)$ and O_i^{-1} is the inverse of O_i, $i = 1, 2$.

PROOF. By changing variables, we have

$$\begin{aligned}(\sigma_{a,\Gamma,k} * f)(x) &= \int_{2^k \leq |y| < 2^{k+1}} b(|y|)a_i(y')f(x - O_i\Gamma(|y|)y')dy \\ &= \int_{2^k \leq |y| < 2^{k+1}} b(|y|)a_i(y')f_i(O_i^{-1}x - \Gamma(|y|)y')dy,\end{aligned}$$

which proves the lemma.

For the maximal function $\sigma^*_{a,\Gamma}$, we have the following $L^p(\omega)$ boundedness result.

LEMMA 2.4. *Let $b \in \Delta_\gamma$, $\gamma > 1$, $p > \gamma'$, and a an ∞-atom. Let Γ be the function either in Theorem 1 or in Theorem 2 and $\omega \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$. Then*

$$\|\sigma^*_{a,\Gamma} f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}, \tag{2.10}$$

where C is a constant independent of a and f.

PROOF. Using the spherical coordinate and Hölder's inequality, we have

$$\begin{aligned}
&\int_{2^k\le|y|<2^{k+1}} |b(|y|)a(y')|\,|y|^{-n}|f(x-\Gamma(|y|)y')|dy \\
&\quad = C\int_{2^k}^{2^{k+1}} t^{-1}|b(t)|\int_{S^{n-1}} |a(y')|\,|f(x-\Gamma(t)y')|d\sigma(y')dt \\
&\quad \le C\left\{\int_{2^k}^{2^{k+1}} t^{-1}\left(\int_{S^{n-1}} |a(y')|\,|f(x-\Gamma(t)y')|d\sigma(y')\right)^{\gamma'} dt\right\}^{1/\gamma'} \\
&\quad \le C\left\{\int_{2^k}^{2^{k+1}} t^{-1}\int_{S^{n-1}} |a(y')|\,|f(x-\Gamma(t)y')|^{\gamma'} d\sigma(y')dt\right\}^{1/\gamma'}.
\end{aligned}$$

Let $g = |f|^{\gamma'}$ and $s = \Gamma(t)$. From (c) in Theorem 1 or (c$'$) in Theorem 2, we have $t^{-1}dt \le cs^{-1}ds$. So, by a change of variable, it is easy to see that the last integral above is bounded by

$$C\left|\int_{\Gamma(2^k)}^{\Gamma(2^{k+1})} s^{-1}\int_{S^{n-1}} |a(y')|g(x-sy')d\sigma(y')ds\right|^{1/\gamma'}.$$

Thus we have

$$\sigma^*_{a,\Gamma} f(x) \le C\left\{\int_{S^{n-1}} |a(y')|M_{y'}g(x)d\sigma(y')\right\}^{1/\gamma'}, \tag{2.11}$$

where

$$M_{y'}g(x) = \sup_{R>0} R^{-1}\int_0^R |g(x-sy')|ds$$

is the Hardy-Littlewood maximal function of g in the direction y'. Since $p > \gamma'$, we have

$$\begin{aligned}
\left\|\left\{\int_{S^{n-1}} |a(y')|M_{y'}g(\cdot)d\sigma(y')\right\}^{1/\gamma'}\right\|^{\gamma'}_{L^p(\omega)} &= \left\|\int_{S^{n-1}} |a(y')|M_{y'}g(\cdot)d\sigma(y')\right\|_{L^{p/\gamma'}(\omega)} \\
&\le \int_{S^{n-1}} |a(y')|\,\|M_{y'}g\|_{L^{p/\gamma'}(\omega)}d\sigma(y').
\end{aligned}$$

By $\omega \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$ and (8) in [Du], we know

$$\|M_{y'}g\|_{L^{p/\gamma'}(\omega)} \le C\|g\|_{L^{p/\gamma'}(\omega)}$$

with C independent of y'. So the lemma is proved by noting

$$\|g\|_{L^{p/\gamma'}(\omega)} = \|f\|^{\gamma'}_{L^p(\omega)}.$$

3. Proof of theorems. Since the proofs of Theorem 1 and Theorem 2 are essentially the same, we shall prove Theorem 2 only. In our proof we will apply the machinery developed by Duoandikoetxea and Rubio de Francia in [DR].

First we consider the case $p > \gamma'$. Note that $T_{\Gamma,b}(f)$ is equal to

$$\int_{\boldsymbol{R}^n} |y|^{-n} b(|y|)\Omega(y')f(x-\Gamma(|y|)y')dy\,, \tag{3.1}$$

where $\Omega = \sum c_j a_j$, $\sum |c_j| \le C\|\Omega\|_{H^1(S^{n-1})}$ and each a_j is an ∞-atom. We have

$$\|T_{\Gamma,b}f\|_{L^p(\omega)} \le C\sum |c_j|\,\|T^j_{\Gamma,b}(f)\|_{L^p(\omega)}\,,$$

where

$$T^j_{\Gamma,b}(f)(x) = \int_{\boldsymbol{R}^n} b(|y|)|y|^{-n}a_j(y')f(x-\Gamma(|y|)y')dy\,.$$

Therefore, it suffices to show

$$\|T^j_{\Gamma,b}(f)\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}\,, \tag{3.3}$$

where C is independent of the atoms $a_j(\cdot)$ and f.

For simplicity of the notation, we shall denote $a_j(\cdot)$ by $a(\cdot)$ and $T^j_{\Gamma,b}(f)$ by $T_{\Gamma,b}(f)$. In the following we assume that supp(a) is contained in the ball $B(x_0,\rho)\cap S^{n-1}$. Let I_k be the interval $(2^k, 2^{k+1})$. Then $T_{\Gamma,b}(f)(x)$ is equal to

$$\int_{\boldsymbol{R}^n} b(|y|)|y|^{-n}a(y')\sum_{k=-\infty}^{\infty}\chi_{I_k}(|y|)f(x-\Gamma(|y|)y')dy = \sum_{k=-\infty}^{\infty}\sigma_{a,\Gamma,k}*f(x)\,.$$

Let $\{\Phi_j\}_{-\infty}^{\infty}$ be a smooth partition of the unity in $(0,\infty)$ adapted to the intervals $(\gamma_{j-1},\gamma_{j+1})$, that is,

$$\Phi_j \in C^\infty(0,\infty)\,,\quad 0\le\Phi_j\le 1\,,\quad \sum_{j=-\infty}^{\infty}\Phi_j(t)^2 = 1\quad \text{for all } t\,,$$
$$\operatorname{supp}(\Phi_j)\subseteq(\gamma_{j-1},\gamma_{j+1})\,.$$

Define the multiplier operators S_j in $\boldsymbol{R}^n$ by $(S_jf)\hat{}(\xi) = \hat{f}(\xi)\Phi_j(|B_\rho\xi|)$. Following the proof of Lemma in [DR], we decompose the operator $T_{\Gamma,b}(f)$ by

$$T_{\Gamma,b}(f) = \sum_j\left(\sum_k S_{j+k}(\sigma_{a,\Gamma,k}*S_{j+k}f)\right) = \sum_j \tilde{T}_j f\,. \tag{3.5}$$

We first estimate the L^2 norm of $\tilde{T}_j$. By Lemma 2.3, we have for $i=1,2$,

$$\begin{aligned}\|\tilde{T}_j f\|_2^2 &\le C\sum_k \int_{\boldsymbol{R}^n} |\sigma_{a,\Gamma,k} * (S_{j+k}f)(y)|^2 dy \\ &= C\sum_k \int_{\boldsymbol{R}^n} |\sigma_{a_i,\Gamma,k} * (S_{j+k}f)_i(O_i^{-1}y)|^2 dy \\ &= C\sum_k \int_{\boldsymbol{R}^n} |\hat{\sigma}_{a_i,\Gamma,k}(\xi)|^2 |(S_{j+k}f)\hat{_i}(\xi)|^2 d\xi\,.\end{aligned}$$

By Lemma 2.1, it is easy to see that

$$\|\tilde{T}_j f\|_2^2 \le C\sum_k \int_{\boldsymbol{R}^n} |B_\rho\xi|^{2\alpha}\gamma_k^{-2\alpha} |(S_{j+k}f)\hat{_1}(\xi)|^2 d\xi\,. \tag{3.6}$$

So we have

$$\|\tilde{T}_j f\|_2^2 \le C\sum_k \int_{\boldsymbol{R}^n} \{|B_\rho\xi|/\gamma_k\}^{2\alpha} |(S_{j+k}f)\hat{_1}(\xi)|^2 d\xi\,. \tag{3.7}$$

Similarly, we have

$$\|\tilde{T}_j f\|_2^2 \le C\sum_k \int_{\boldsymbol{R}^n} \{|B_\rho\xi|/\gamma_k\}^{-2\beta} |(S_{j+k}f)\hat{_1}(\xi)|^2 d\xi\,. \tag{3.8}$$

Now an easy computation shows that

$$(S_{j+k}f)\hat{_1}(\xi) = \Phi_{j+k}(|B_\rho\xi|)\hat{f}(O_1^{-1}\xi)\,.$$

Thus if $j > 0$, we use (3.8) and the choice of Φ_{j+k} to obtain that

$$\|\tilde{T}_j f\|_2^2 \le C\sum_k \int_{D_{j+k}} |\hat{f}(O_1^{-1}\xi)|^2 (|B_\rho\xi|/\gamma_k)^{-2\beta} d\xi\,,$$

where

$$D_j = \{\xi : \gamma_{j-1} \le |B_\rho\xi| \le \gamma_{j+1}\}\,.$$

This shows that if $j \ge 0$,

$$\|\tilde{T}_j f\|_2 \le C\lambda^{-j\beta}\|f\|_2\,. \tag{3.9}$$

Similarly, using (3.7) we have for $j < 0$

$$\|\tilde{T}_j f\|_2 \le C\lambda^{j\alpha}\|f\|_2\,. \tag{3.10}$$

Next, we estimate the $L^p(\omega)$ norm of $\tilde{T}_j$ for $\omega \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$. We shall prove the following proposition.

PROPOSITION 3.1. *If $\omega \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$, $p > \gamma'$, then*

$$\|\tilde{T}_j f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}\,, \tag{3.11}$$

where C is independent of the atom a.

If (3.11) holds, then we can easily prove the theorems for $p > \gamma'$. In fact, interpolating between (3.9)–(3.10) and (3.11) with $\omega \equiv 1$, we have for some $\theta > 0$,

$$\|\tilde{T}_j f\|_p \le C\lambda^{-|j|\theta}\|f\|_p, \tag{3.12}$$

where C is independent of the atoms a.

For any $\omega \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$, there is an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$. Thus by (3.11), we have

$$\|\tilde{T}_j f\|_{L^p(\omega^{1+\varepsilon})} \le C\|f\|_{L^p(\omega^{1+\varepsilon})}, \tag{3.13}$$

where C is independent of the atom a. Therefore, using Stein and Weiss' interpolation theorem with change of measures [SW], we may interpolate between (3.11) and (3.13) to obtain a positive number ν such that

$$\|\tilde{T}_j f\|_{L^p(\omega)} \le C 2^{-\nu|j|}\|f\|_{L^p(\omega)}, \tag{3.14}$$

which implies

$$\|T_{\Gamma,b}(f)\|_{L^p(\omega)} \le C\sum_j \|\tilde{T}_j f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}. \tag{3.15}$$

Thus to prove the theorem for $p > \gamma'$, we only need to prove (3.11) in Proposition 3.1. For $k \in \boldsymbol{Z}$, we define the operators E_k by

$$E_k f(x) = \int_{2^k \le |y| < 2^{k+1}} |b(|y|)a(y')|\,|y|^{-n} f(x - \Gamma(|y|)y')dy, \tag{3.16}$$

and invoke the following lemma in [Du].

LEMMA 3.1. *Let $\omega \in A_p(\boldsymbol{R}^n)$. If the vector-valued inequality*

$$\left\|\left(\sum_{k=-\infty}^{\infty} |E_k f_k|^2\right)^{1/2}\right\|_{L^p(\omega)} \le C\left\|\left(\sum_{k=-\infty}^{\infty} |f_k|^2\right)^{1/2}\right\|_{L^p(\omega)} \tag{3.17}$$

is true, then (3.11) *holds.*

PROOF. The proof can be found in the proof of Lemma 1 in [Du].

Now it remains to prove (3.17) for any $\omega \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+) \subseteq A_p(\boldsymbol{R}^n)$. By Hölder's inequality, it is easy to see that

$$|E_k f_k(x)|^{\gamma'} \le C\int_{2^k}^{2^{k+1}} \int_{S^{n-1}} g_k(x - \Gamma(t)y')|a(y')|d\sigma(y')t^{-1}dt,$$

where $g_k = |f_k|^{\gamma'}$ and C is independent of a. Let $r = p/\gamma'$. Since $r > 1$, for $\omega \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$, there exists a nonnegative function u with unit norm in $L^{r'}(\omega^{1-r'})$ such that

$$\left\| \left(\sum_k |E_k f_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\omega)}^{\gamma'} \leq C \left| \sum_k \int_{\boldsymbol{R}^n} \int_{2^k}^{2^{k+1}} \int_{S^{n-1}} g_k(x - \Gamma(t)y')|a(y')|d\sigma(y')t^{-1}dt\, u(x)dx \right| \leq C \int_{\boldsymbol{R}^n} \sum_k g_k(x)(M_{a,\Gamma}u)(x)dx\,, \tag{3.18}$$

where

$$(M_{a,\Gamma}u)(x) = \sup_{k \in \boldsymbol{Z}} \int_{2^k \leq |y| < 2^{k+1}} u(x + \Gamma(|y|)y')|a(y')|\,|y|^{-n}dy\,.$$

Thus by Hölder's inequality, we have

$$\left\| \left(\sum_k |E_k f_k|^{\gamma'} \right)^{1/\gamma'} \right\|_{L^p(\omega)}^{\gamma'} \leq C \left\| \left\{ \sum_k |f_k|^{\gamma'} \right\}^{1/\gamma'} \right\|_{L^p(\omega)}^{\gamma'} \left\{ \int_{\boldsymbol{R}^n} |(M_{a,\Gamma}u)(x)|^{r'} \omega(|x|)^{1-r'} dx \right\}^{1/r'}. \tag{3.19}$$

Recalling $r = p/\gamma'$, it is easy to check that $\omega \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$ if and only if $\omega^{1-r'} \in \tilde{A}_{r'}(\boldsymbol{R}_+)$. Thus, by the same proof as that of Lemma 2.4, we have

$$\left\{ \int_{\boldsymbol{R}^n} |M_{a,\Gamma}u(x)|^{r'} \omega(|x|)^{1-r'} dx \right\}^{1/r'} \leq C\|u\|_{L^{r'}(\omega^{1-r'})} \leq C\,, \tag{3.20}$$

where C is independent of a. By (3.19) and (3.20), we obtain

$$\left\| \left\{ \sum_k |E_k f_k|^{\gamma'} \right\}^{1/\gamma'} \right\|_{L^p(\omega)} \leq C \left\| \left\{ \sum_k |f_k|^{\gamma'} \right\}^{1/\gamma'} \right\|_{L^p(\omega)}. \tag{3.21}$$

On the other hand, it is trivial to see

$$\sup_{k \in \boldsymbol{Z}} |E_k f_k(x)| \leq \sigma_{a,\Gamma}^* \left(\sup_{k \in \boldsymbol{Z}} |f_k| \right)(x)\,.$$

Thus by Lemma 2.4, we have

$$\left\| \sup_{k \in \boldsymbol{Z}} |E_k f_k| \right\|_{L^p(\omega)} \leq \left\| \sup_{k \in \boldsymbol{Z}} |f_k| \right\|_{L^p(\omega)}. \tag{3.22}$$

Since $\gamma' \in (1, 2]$, (3.17) easily follows by (3.21), (3.22) and the Riesz-Thorin interpolation theorem ([GR, page 481]). This proves the theorem if $p > \gamma'$. In the endpoint case $p = \gamma'$, since $T_{\Gamma,b}$ is bounded on $L^r(\omega)$ for any $\omega \in \tilde{A}_1(\boldsymbol{R}_+)$ and $r > \gamma'$, and on L^s for any $1 <$

$s \le \gamma'$ (by Theorem A), we obtain that $T_{\Gamma,b}$ is bounded on $L^{\gamma'}(\omega)$ for any $\omega \in \tilde{A}_1(\boldsymbol{R}_+)$ by interpolating with change of measures (see [BL, page 119]). The theorem is proved.

By Theorems 1 and 2, we can easily obtain the following two corollaries.

COROLLARY 1. *Let* $1 < p \le \gamma, \gamma \ge 2, p \ne \infty$, *and* $\omega^{1/(1-p)} \in \tilde{A}_{p'/\gamma'}(\boldsymbol{R}_+)$. *Let* Γ, b, Ω *be the same as either in Theorem* 1 *or in Theorem* 2, *where* p' *is the dual exponent to* p. *Then* $T_{\Gamma,b}$ *is bounded in* $L^p(\omega)$.

PROOF. Corollary 1 follows easily by duality and Theorems 1–2.

COROLLARY 2. *Let* Γ, b, Ω *be the same as in either Theorem* 1 *or in Theorem* 2. *For* $p_0 \ge \gamma'$, $p_1 \in (1, \gamma')$ *and* $t \in [0, 1]$, *let* $r(t) = tp_0/(p_1(1-t) + tp_0)$ *and* $p_t = p_0p_1/(p_1(1-t) + tp_0)$. *Suppose* $\omega \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$. *Then* $T_{\Gamma,b}$ *is bounded on* $L^{p_t}(\omega^{r(t)})$.

PROOF. Interpolating with change of measures between Theorems 1 and 2 and Theorem A, we obtain this corollary.

We remark that Theorem 1 and Theorem 2 remain ture for weights in $\tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$ if $b(x) \equiv 1$.

4. The maximal singular integrals. In this section, we will study the truncated maximal functions $T^*_{\Gamma,b}$ of $T_{\Gamma,b}$. For any $\varepsilon > 0$, we define

$$T^{\varepsilon}_{\Gamma,b}f(x) = \int_{|y|>\varepsilon} b(|y|)|y|^{-n}\Omega(y')f(x - \Gamma(|y|)y')dy\,,$$

and

$$T^*_{\Gamma,b}f(x) = \sup_{\varepsilon>0} |T^{\varepsilon}_{\Gamma,b}f(x)|\,.$$

It is well-known that the boundedness of $T^*_{\Gamma,b}$ on $L^p(\omega)$ implies the almost everywhere existence of $\lim_{\varepsilon\to 0} T^{\varepsilon}_{\Gamma,b}f(x)$, the principal value defining $T_{\Gamma,b}$ for $f \in L^p(\omega)$.

THEOREM 3. *Suppose* $\Omega \in H^1(S^{n-1})$ *satisfy* (1.1), $1 < p < \infty$ *and* $b \in \Delta_\gamma$ *with* $\gamma \ge 2$. *Let* Γ *be the function satisfying the conditions either in Theorem* 1 *or in Theorem* 2. *If* $\omega \in \tilde{A}^I_{p/\gamma'}(\boldsymbol{R}_+)$ *with* $p > \gamma'$, *then* $T^*_{\Gamma,b}$ *is bounded on* $L^p(\omega)$.

PROOF. We only prove the case that Γ satisfies the conditions in Theorem 1, since the proof of the other case is similar, with minor modifications. Similar to the proof of $T_{\Gamma,b}$, we may assume that $\Omega(y') = a(y')$ is an ∞-atom supported in $B(x_0, \rho) \cap S^{n-1}$. Since $T_{\Gamma,b}f = \sum_k \sigma_{a,\Gamma,k} * f$, for any $\varepsilon > 0$ there is an integer k such that $2^{k-1} \le \varepsilon < 2^k$. So we have

$$T^*_{\Gamma,b}f \le \sigma^*_{a,\Gamma}|f| + \sup_{k\in\boldsymbol{Z}} \left|\sum_{j=k}^{\infty} \sigma_{a,\Gamma,j} * f\right| = \sigma^*_{a,\Gamma}|f| + \sup_{k\in\boldsymbol{Z}} |I_k(f)|\,.$$

By Lemma 2.4, we only need to prove the $L^p(\omega)$-boundedness for $I^*(f) = \sup_{k\in\boldsymbol{Z}} |I_k(f)|$.

Let δ be the Dirac delta function. For the λ in the conditions of Theorem 1, we take a radial function $\varphi \in \mathscr{S}(\boldsymbol{R}^n)$ such that $\varphi(\xi) = 1$ when $|\xi| < 1/\lambda$ and $\varphi(\xi) = 0$ when $|\xi| > \lambda$.

Let $\varphi_k(\xi) = \varphi(\Gamma_k |B_\rho \xi|)$, where $\Gamma_k = \Gamma(2^k)$, and let $\hat{\Phi}_k(\xi) = \varphi_k(\xi)$. Now

$$\begin{aligned} I_k(f) &= (\delta - \Phi_k) * \sum_{j=k}^{\infty} \sigma_{a,\Gamma,j} * f + \Phi_k * (T_{\Gamma,b} f) - \Phi_k * \sum_{j=-\infty}^{k-1} \sigma_{a,\Gamma,j} * f \\ &= I_{k,1}(f) + I_{k,2}(f) + I_{k,3}(f) . \end{aligned}$$

Clearly, by Theorem 1,

$$\left\| \sup_{k \in \boldsymbol{Z}} |I_{k,2}(f)| \right\|_{L^p(\omega)} \le C \|M(T_{\Gamma,b} f)\|_{L^p(\omega)} \le C \|f\|_{L^p(\omega)} . \tag{4.1}$$

Next,

$$\begin{aligned} \sup_{k \in \boldsymbol{Z}} |I_{k,3}(f)| &= \sup_{k \in \boldsymbol{Z}} \left| \sum_{j=1}^{\infty} \sigma_{a,\Gamma,k-j} * \Phi_k * f \right| \\ &\le \sum_{j=1}^{\infty} \left(\sup_{k \in \boldsymbol{Z}} |\sigma_{a,\Gamma,k-j} * \Phi_k * f| \right) = \sum_{j=1}^{\infty} G_j(f) . \end{aligned}$$

By Lemma 2.4 we have

$$\|G_j(f)\|_p \le C \|f\|_p , \quad 1 < p < \infty , \tag{4.2}$$

and for any $\omega \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$, there is an $\varepsilon > 0$ such that

$$\|G_j(f)\|_{L^p(\omega^{1+\varepsilon})} \le C \|f\|_{L^p(\omega^{1+\varepsilon})} , \quad \gamma' < p < \infty . \tag{4.2$'$}$$

On the other hand,

$$G_j(f) \le \left(\sum_k |\sigma_{a,\Gamma,k-j} * \Phi_k * f|^2 \right)^{1/2} .$$

Thus by Plancherel's theorem and by inspecting the proof of Theorem 2, it is easy to see that

$$\|G_j(f)\|_2^2 \le C \sup_{\xi \in \boldsymbol{R}^n \setminus \{0\}} \sum_{\Gamma_{k-1} \le |B_\rho \xi|^{-1}} (\Gamma_k |B_\rho \xi|)^{2\alpha} \lambda^{-2j\alpha} \|f\|_2^2 ,$$

where we used the support condition of $\hat{\Phi}_k$. Thus we have

$$\|G_j(f)\|_2 \le C \lambda^{-j\alpha} \|f\|_2 . \tag{4.3}$$

Interpolating between (4.2) and (4.3), we obtain a $\theta > 0$ such that for any $p \in (1, \infty)$

$$\|G_j(f)\|_p \le C \lambda^{-j\theta} \|f\|_p . \tag{4.4}$$

Interpolating between (4.2′) and (4.4), we find a $\nu > 0$ such that

$$\|G_j(f)\|_{L^p(\omega)} \le C \lambda^{-j\nu} \|f\|_{L^p(\omega)} \tag{4.5}$$

for any $\omega \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$ and $p > \gamma'$. Therefore,

$$\left\| \sup_{k \in \boldsymbol{Z}} |I_{k,3}(f)| \right\|_{L^p(\omega)} \le \sum_{j=1}^{\infty} \|G_j(f)\|_{L^p(\omega)} \le C \|f\|_{L^p(\omega)} . \tag{4.6}$$

Using a similar argument, we can prove

$$\left\| \sup_{k \in Z} |I_{k,1}(f)| \right\|_{L^p(\omega)} \leq C \|f\|_{L^p(\omega)} . \tag{4.7}$$

The theorem is proved.

5. The weights $|x|^\alpha$. A special case of radial weights is the power weights $|x|^\alpha$, $\alpha \in \boldsymbol{R}$. The following theorem was proved by Muckenhoupt and Wheeden (see [MW]) and by Duoandikoetxea with a different proof (see Theorem 6 of [Du]).

THEOREM C. *Let $1 < q \leq \infty$, $1 < p < \infty$. Suppose that T is the operator defined in Section 1 with $\Omega \in L^q(S^{n-1})$ satisfying (1.1) ($T = T_{\Gamma,b}$ when $\Gamma(t) = t$ and $b(t) = 1$). Then T is bounded on $L^p(|x|^\alpha)$ if*

$$\max(-n, -1-(n-1)p/q') < \alpha < \min(n(p-1), p-1+(n-1)p/q') , \tag{5.1}$$

where q' is the dual exponent of q. Also the range in (5.1) is optimal.

In the limit case $q = 1$, the range (5.1) becomes $\alpha \in (-1, p-1)$. It is well-known that the theorem fails for some $\Omega \in L^1(S^{n-1})$ even in the non-weighted case $\alpha = 0$. But Theorem C is still true if $\alpha \in (-1, p-1)$ and $\Omega \in L \log^+ L$ (see page 880 in [Du]). As H^1 is a natural substitution of L^1, we obtain the following theorem.

THEOREM 4. *Let $T_{\Gamma,b}$ be the operator satisfying either the conditions in Theorem 1 or the conditions in Theorem 2. Let $b \in \Delta_\gamma$ with $\gamma \geq 2$, and $p \in (1, \infty)$ satisfying $p > \gamma'$. Then $T_{\Gamma,b}$ is bounded in $L^p(|x|^\alpha)$ if $\alpha \in (-1, p/\gamma' - 1)$.*

PROOF. For any $\alpha \in (-1, p/\gamma' - 1)$, we choose a $\beta \in (-1, 0)$ such that $\alpha > \beta$ and $\alpha - \beta < p/\gamma' - 1$. Then we write

$$|x|^\alpha = \nu_1(|x|)\nu_2(|x|)^{1-p/\gamma'} , \tag{5.2}$$

where $\nu_1(t) = t^\beta$ and $\nu_2(t) = t^{(\alpha-\beta)/(1-p/\gamma')}$. Clearly, both ν_1 and ν_2 are decreasing. On the other hand, it is well known that $t^\mu \in A_1(\boldsymbol{R}_+)$ if $-1 < \mu \leq 0$. Thus we easily check that $\nu_1 \in A_1(\boldsymbol{R}_+)$ and $\nu_2 \in A_1(\boldsymbol{R}_+)$, which implies $|x|^\alpha \in \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$. Now Theorem 4 follows from Theorem 1 and Theorem 2, because $|x|^\alpha \in A_p^I(\boldsymbol{R}^n)$ if $\alpha \in (-1, p-1)$.

Using interpolation with change of measures between Theorem 4 and Theorem A, we can further obtain the following

COROLLARY 3. *Let Γ, Ω, b are the same as either in Theorem 1 or in Theorem 2. For $p_0 > \gamma'$, $p_1 \in (1, \gamma']$ and $t \in [0, 1]$, let $p_t = p_0 p_1/(p_1(1-t) + tp_0)$. Then*

$$\int_{\boldsymbol{R}^n} |T_{\Gamma,b} f(x)|^{p_t} |x|^\alpha dx \leq C \int_{\boldsymbol{R}^n} |f(x)|^{p_t} |x|^\alpha dx$$

provided that $\alpha p_0/(p_1(1-t) + tp_0)$ is in the interval $(-1, p_0/\gamma' - 1)$.

6. Boundedness in Morrey spaces. We are now going to study the boundedness of $T_{\Gamma,b}$ on the Morrey spaces. The classical Morrey spaces were introduced in [Mo] by Morrey in order to study the local behavior of solutions to second order elliptic partial differential equations. In [Mi], Mizuhara introduced the following generalized Morrey spaces.

Let ϕ be a positive increasing function on $(0, \infty)$ and satisfy that for any $r > 0$, $\phi(2r) \leq D\phi(r)$, where $D \geq 1$ is a constant independent of r.

DEFINITION 6.1. Let $1 \leq p < \infty$. We denote by $L^{p,\phi} = L^{p,\phi}(\boldsymbol{R}^n)$ the space of locally integrable functions f for which

$$\int_{B_r(x_0)} |f(x)|^p dx \leq C^p \phi(r) \tag{6.1}$$

for all $x_0 \in \boldsymbol{R}^n$ and $r > 0$, where $B_r(x_0)$ is the ball with center x_0 and radius r. We denote the smallest constant C satisfying (6.1) by $\|f\|_{L^{p,\phi}}$.

THEOREM 5. *Let $1 \leq D < 2^n$, $1 < p < \infty$, $b \in \Delta_\gamma$ with $p \geq \gamma'$ and $\gamma \geq 2$. Suppose that $T_{\Gamma,b}$ are the singular integrals satisfying the conditions either in Theorem* 1 *or in Theorem* 2. *Then the operators $T_{\Gamma,b}$ are bounded on $L^{p,\phi}$.*

PROOF. Let χ_B be the characteristic function of the ball $B_r(x_0)$. Clearly we have $\chi_B(x) \leq (M\chi_B)(x)$ for almost all $x \in \boldsymbol{R}^n$, where $M\chi_B$ is the Hardy-Littlewood maximal function of χ_B. By a simple computation one easily sees

$$(M\chi_B)(x) \cong (r/(r + |x - x_0|))^n . \tag{6.2}$$

Let $\omega(x) = (r/(r + |x|))^{\theta n}$ for any fixed number $\theta \in (0, 1)$. Since χ_B is a positive Borel measure such that $M\chi_B(x) < \infty$ for all $x \in \boldsymbol{R}^n$. By the property for weights [GR, page 436], it is easy to check that $\omega(t) \in A_1(\boldsymbol{R}_+)$. Since $\omega(t)$ is decreasing, we have $\omega(|x|) \in \tilde{A}_1(\boldsymbol{R}_+) \subseteq \tilde{A}_{p/\gamma'}(\boldsymbol{R}_+)$, for all $p \in (1, \infty)$. Define $f_{x_0}(x)$ by $f_{x_0}(x) = f(x + x_0)$. We choose a θ such that $1 < 1/\theta < \log 2^n / \log D$. Then using Theorem 1 or Theorem 2, we have

$$\begin{aligned}
&\int_{B_r(x_0)} |T_{\Gamma,b} f(x)|^p dx \\
&\quad = \int_{\boldsymbol{R}^n} |T_{\Gamma,b} f(x)|^p \chi_B(x)^\theta dx \\
&\quad \leq C \int_{\boldsymbol{R}^n} |T_{\Gamma,b} f_{x_0}(x)|^p \omega(x) dx \leq C \int_{\boldsymbol{R}^n} |f(x)|^p \omega(x - x_0) dx \\
&\quad = C \left\{ \int_{B_{2r}(x_0)} |f(x)|^p \omega(x - x_0) dx + \sum_{j=1}^\infty \int_{B_{2^{j+1}r}(x_0) \setminus B_{2^j r}(x_0)} |f(x)|^p \omega(x - x_0) dx \right\} \\
&\quad \leq C \left\{ \int_{B_{2r}(x_0)} |f(x)|^p dx + \sum_{j=1}^\infty 2^{-jn\theta} \int_{B_{2^{j+1}r}(x_0)} |f(x)|^p dx \right\} \\
&\quad \leq C \|f\|_{L^{p,\phi}}^p \left\{ \phi(2r) + \sum_{j=1}^\infty 2^{-jn\theta} \phi(2^{j+1} r) \right\} \\
&\quad \leq C \|f\|_{L^{p,\phi}}^p \phi(r) \sum_{j=0}^\infty D^{j+1} 2^{-j\theta n} \leq C \|f\|_{L^{p,\phi}}^p \phi(r) .
\end{aligned}$$

The theorem is proved.

Under a stronger condition, a special case of Theorem 5 was recently obtained by Yong Ding [Di], in which $\Gamma(t) = t$, $b \in L^\infty(0,\infty)$, and $\Omega \in L^q(S^{n-1})$ with $1 < p < q$ or $q' \le p < \infty$.

We remark that the Morrey space and the generalized Morrey space are recently used, respectively, by Fazio and Ragusa in [FR] and Huang in [Hu] to measure the regularity of the solution to an elliptic second order equation with discontinuous coefficients; see Theorems 3.3 and 3.4 in [FR] and Theorems 2.1, 2.2 and 2.5 in [Hu]. Moreover, by means of an integral representation formula of the second derivatives of the solution to the above mentioned equation in [CFL], Fazio and Ragusa in [FR] obtained the regularity in Morrey spaces of the solution to these equations just by first establishing the boundedness on Morrey spaces of some singular integral operators; see Theorem 2.3 in [FR].

7. Boundedness in Herz spaces. To study the convolution algebra, A. Beurling [Br] first introduced some primordial form of non-homogeneous Herz spaces which are also called Beurling algebras. Later, C. Herz [He] introduced versions of the spaces defined below in a slightly different setting. Since then, the theory of Herz spaces has been significantly developed and these spaces have turned out to be quite useful in harmonic analysis. For example, they were used by Baernstein and Sawyer [BS] to characterize the multipliers on the standard Hardy spaces.

For simplicity, we only discuss the boundedness of $T_{\Gamma,b}$ in homogeneous Herz spaces here. Let $B_k = \{x \in \boldsymbol{R}^n : |x| \le 2^k\}$ and $C_k = B_k \setminus B_{k-1}$ for $k \in \boldsymbol{Z}$. We also denote by χ_k the characteristic function of the set C_k.

DEFINITION 7.1. Let $\alpha \in \boldsymbol{R}$ and $0 < p, q < \infty$. The homogeneous Herz spaces $\dot{K}_p^{\alpha,q}(\boldsymbol{R}^n)$ is defined by

$$\dot{K}_p^{\alpha,q}(\boldsymbol{R}^n) = \{f \in L^p_{\mathrm{loc}}(\boldsymbol{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_p^{\alpha,q}(\boldsymbol{R}^n)} < \infty\},$$

where

$$\|f\|_{\dot{K}_p^{\alpha,q}(\boldsymbol{R}^n)} = \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha q}\|f\chi_k\|^q_{L^q(\boldsymbol{R}^n)}\right)^{1/q}.$$

It is worth pointing out that the norms of these spaces in [He] are different, but equivalent. T. M. Flett [Ft] gave a characterization of these spaces which is easily seen to be equivalent to Definition 7.1.

Obviously, $\dot{K}_p^{0,p}(\boldsymbol{R}^n) = L^p(\boldsymbol{R}^n)$ and $\dot{K}_p^{\alpha/p,p}(\boldsymbol{R}^n) = L^p(|x|^\alpha)$ for any $\alpha \in \boldsymbol{R}$ and $0 < p < \infty$. Moreover, in [HLY], Hu, Lu and Yang established the following general theorem on the relation of boundedness of sublinear operators between $L^p(|x|^\alpha)$ and $\dot{K}_p^{\alpha,q}(\boldsymbol{R}^n)$.

THEOREM D. *Let T be a sublinear operator and T be bounded on $L^p(|x|^\beta)$ for all $\beta \in (\beta_1, \beta_2)$ and certain $p \in (1,\infty)$, where $\beta_1, \beta_2 \in \boldsymbol{R}$. Then T is bounded on $\dot{K}_p^{\alpha,q}(\boldsymbol{R}^n)$ provided $\alpha \in (\beta_1/p, \beta_2/p)$ and $q \in (0,\infty)$.*

As a simple corollary of Theorem 4 and Theorem D, we have

THEOREM 6. *Let $T_{\Gamma,b}$ be the operator satisfying either the conditions in Theorem* 1 *or those in Theorem* 2. *Let $b \in \Delta_\gamma$ with $\gamma \geq 2$, and $p \in (\gamma', \infty)$. Then $T_{\Gamma,b}$ is bounded on $\dot{K}_p^{\alpha,q}(\boldsymbol{R}^n)$ if $0 < q < \infty$ and $\alpha \in (-1/p, 1/\gamma' - 1/p)$.*

8. Commutators. Let $h(r) \in L^1_{\text{loc}}(\boldsymbol{R}_+)$. We say $h(r) \in \text{BMO}(\boldsymbol{R}_+)$ if

$$\|h\|_* = \sup_{I \subseteq \boldsymbol{R}_+} |I|^{-1} \int_I |h(r) - h_I| dr < \infty, \tag{8.1}$$

where $h_I = |I|^{-1} \int_I h(r) dr$.

Let $h_j(r) \in \text{BMO}(\boldsymbol{R}_+)$, $j = 1, 2, \dots, m$. We define the higher order commutator about T_b by

$$T_b^m f(x) = \text{p.v.} \int_{\boldsymbol{R}^n} \Omega(x - y) b(|x - y|) |x - y|^{-n} \prod_{j=1}^m \{h_j(|x|) - h_j(|y|)\} f(y) dy.$$

The main result in this section is the following theorem.

THEOREM 7. *If $\Omega \in H^1(S^{n-1})$ satisfies* (1.1) *and $b \equiv 1$. Suppose that $h_j(x)$ is a radial function such that $h_j(r) \in \text{BMO}(\boldsymbol{R}_+)$, $j = 1, 2, \dots, m$, and $\omega \in \bar{A}_p(\boldsymbol{R}_+)$. Then we have*

$$\|T_b^m f\|_{L^p(\omega)} \leq C_p \prod_{j=1}^m \|h_j\|_* \|f\|_{L^p(\omega)}, \tag{8.2}$$

provided $\infty > p > 1$, where $C_p \cong p^m$ is a constant depending only on p.

PROOF. For any fixed $p > 1$, we write

$$\prod_{j=1}^m (h_j(x) - h_j(y)) = p^m \prod_{j=1}^m \|h_j\|_* \{(h_j(x) - h_j(y)) \|h_j\|_*^{-1} p^{-1}\}.$$

Thus without loss of generality we may assume all $\|h_j\|_* = p^{-1}$. We now use the induction argument on m to prove the theorem. By Theorem 1, we know that the theorem holds if $m = 0$. Now we assume that the conclusion of the theorem holds for $m - 1$ and prove the conclusion for m.

For simplicity of the notation, we write $h = h_m$. For any $\omega \in \bar{A}_p(\boldsymbol{R}_+)$, there is an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in \bar{A}_p(\boldsymbol{R}_+)$. Thus by the proof of Theorem 3 in [DL, page 440], without loss of generality we may assume $e^{tph(x)(1+\varepsilon)/\varepsilon} \in \bar{A}_p(\boldsymbol{R}_+)$ for all $|t| \leq 1$. Recalling the definition of $\bar{A}_p(\boldsymbol{R}_+)$, we have a constant $C > 0$ such that for any interval $I \subseteq \boldsymbol{R}_+$,

$$\left(|I|^{-1} \int_I \omega^{2+2\varepsilon}(r) dr\right) \left(|I|^{-1} \int_I \omega^{-2(1+\varepsilon)/(p-1)}(r) dr\right)^{p-1} \leq C,$$

$$\left(|I|^{-1} \int_I e^{2tph(r)(1+\varepsilon)/\varepsilon} dr\right) \left(|I|^{-1} \int_I e^{-2tph(r)(1+\varepsilon)/\varepsilon(p-1)} dr\right)^{p-1} \leq C.$$

Thus by Hölder's inequality we can find another constant $C_1 > 0$ such that for any interval $I \subseteq \boldsymbol{R}_+$

$$\left(|I|^{-1}\int_I \omega^2(r)e^{2tph(r)}dr\right)\left(|I|^{-1}\int_I \omega^{-2/(p-1)}e^{-2tph(r)/(p-1)}dr\right)^{p-1} \le C_1 .$$

This shows that, by the definition of $\bar{A}_p(\boldsymbol{R}_+)$, $\omega(x)e^{tph(x)} \in \bar{A}_p(\boldsymbol{R}_+)$ for all $|t| \le 1$. Now by the assumption of the induction we have

$$\|T_b^{m-1}f\|_{L^p(\omega e^{ph(\cdot)\cos\theta})} \le C\|f\|_{L^p(\omega e^{ph(\cdot)\cos\theta})} \tag{8.3}$$

for all real θ. Recall the well-known formula

$$h(x) - h(y) = (2\pi)^{-1}\int_0^{2\pi} e^{e^{i\theta}\{h(x)-h(y)\}}e^{-i\theta}d\theta .$$

Let $g_\theta(y) = f(y)e^{-e^{i\theta}h(y)}$, then it is easy to see that the commutator $T_b^m f(x)$ is equal to

$$(2\pi)^{-1}\int_0^{2\pi} e^{-i\theta}\int_{\boldsymbol{R}^n}\frac{\Omega(x-y)b(|x-y|)}{|x-y|^n}\prod_{j=1}^{m-1}\{h_j(x)-h_j(y)\}g_\theta(y)dye^{e^{i\theta}h(x)}d\theta .$$

Thus by Minkowski's inequality, we have

$$\begin{aligned}\|T_b^m f\|_{L^p(\omega)} &\le (2\pi)^{-1}\int_0^{2\pi}\|T_b^{m-1}(g_\theta)\|_{L^p(\omega e^{ph(x)\cos\theta})}d\theta \\ &\le (2\pi)^{-1}\int_0^{2\pi}\|g_\theta\|_{L^p(\omega e^{ph(x)\cos\theta})}d\theta \le C\|f\|_{L^p(\omega)} .\end{aligned}$$

The theorem is proved.

In [DL], Ding and Lu studied certain commutators with an oscillatory factor $e^{iP(x,y)}$ in their kernels, where $P(x, y)$ is a real polynomial on $\boldsymbol{R}^n \times \boldsymbol{R}^n$. More precisely, they studied the operators

$$T_b^{m,P}f(x) = \text{p.v.}\int_{\boldsymbol{R}^n} e^{iP(x,y)}\frac{\Omega(x-y)}{|x-y|^n}b(|x-y|)\{h(|x|)-h(|y|)\}^m dy$$

and proved the following theorem.

THEOREM E ([DL, Theorem 1]). *Under the conditions $b(t) \in BV(\boldsymbol{R}_+)$, $h(t) \in \mathrm{BMO}(\boldsymbol{R}_+)$, $\omega \in \bar{A}_p(\boldsymbol{R}_+)$ and $\Omega \in L\log^+ L(S^{n-1})$ with (1.1). If T_b is bounded in $L^p(\omega)$ then $T_b^{m,P}$ is also bounded in $L^p(\omega)$ for any $m \in \boldsymbol{Z}_+$ and any real polynomial $P(x, y)$.*

By our Theorem 1 and the above Theorem E, we now have

THEOREM 8. *Let h, Ω and ω be the same as in Theorem E and $b \equiv 1$. Then the higher order commutator $T_b^{m,P}$ is bounded in $L^p(\omega)$, $1 < p < \infty$.*

The special case $m = 0$ and $\omega \equiv 1$ of Theorem 8 was obtained in [JL2] under a stronger condition $b \equiv 1$.

Define the maximal operator $T_b^{m,*}$ by

$$T_b^{m,*}f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} b(|x-y|)|x-y|^{-n}\Omega(x-y)\prod_{j=1}^{m}(h_j(|x|)-h(|y|))f(y)dy\right| .$$

Then, using the same proof as that of Theorem 8 and the result for $m = 0$ proved in Section 4, we have

THEOREM 9. *Let b, Ω, ω, be the same as in Theorem 8. Then we have*

$$\|T_b^{m,*}f\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}$$

provided $\infty > p > \gamma'$.

A FINAL REMARK. It is possible to prove that the commutators in this section are also bounded on the Morrey spaces and on the Herz spaces. We omit the details.

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