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A WEIGHTED UNIFORM L^p–ESTIMATE OF BESSEL FUNCTIONS: A NOTE ON A PAPER OF GUO

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ABSTRACT. An improved Guo's uniform L^p estimate of Bessel functions is shown by using a uniform pointwise bound of Barceló and Córdoba.

Recently, Guo has shown, [Guo, Theorem 3.5], the following uniform L^p estimate:

$$\int_0^\infty |J_\nu(x)|^p x dx \le C(p-4)^{-1}, \qquad \nu \ge 0,\ p > 4. \tag{1}$$

Here $J_\nu(x)$ denotes the Bessel function of the first kind of order ν, cf. [W]. This estimate was proved first for $\nu = 0, 1, \dots$, by means of a dual form of a Fourier restriction theorem for the plane unit circle and then extended to an arbitrary $\nu \ge 0$. The estimate was crucial in proving the main result of [Guo], Theorem 4.1.

It was quite reasonable to expect a proof of (1) based on intrinsic properties of Bessel functions. Furthermore, it was natural to expect an estimate like (1) for a larger range of p's by adding an appropriate power weight in the integral on the left side of (1). More precisely, it was natural to look for an inequality of the form

$$\int_0^\infty |J_\nu(x)|^p x^a dx \le C(p,a), \qquad \nu \ge 0, \tag{2}$$

with a constant $C(p,a) > 0$ depending only on p and a (we did not care about making the constant $C(p,a)$ the best possible).

Since $J_\nu(x) = O(x^{-1/2})$, $x \to \infty$, the necessary assumption on a to make the integral in (2) convergent at infinity for every single $\nu \ge 0$ is $a < p/2 - 1$. On the other hand $J_\nu(x) = O(x^\nu)$, $x \to 0$; hence the necessary assumption on a to make the integral in (2) convergent at zero for every $\nu \ge 0$ is $a > -1$.

It is now interesting to note that Guo's result, (1), shows that the assumption $-1 < a < p/2 - 1$ is also sufficient for (2) to hold in the case $0 < p \le 4$. Indeed, assume

$$\int_1^\infty |J_\nu(x)|^q x dx \le C_q, \qquad \nu \ge 0,$$

holds true for every $q > 4$ and consider p and a such that $0 < p \le 4$ and $a < p/2-1$. Since $2(a+1) < p \le 4$, we can choose $s > 1$ satisfying $2(a+1)s < 4 < ps$. Then,

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because $2(a+1)s < 4$ implies $(a-1)s' + 1 < -1$, by Hölder's inequality we obtain

$$\int_1^\infty |J_\nu(x)|^p x^a dx \le \left(\int_1^\infty |J_\nu(x)|^{ps} x dx\right)^{1/s} \left(\int_1^\infty x^{(a-1)s'} x dx\right)^{1/s'}$$
$$\le A(C_{ps})^{1/s}.$$

In a similar way, assuming

$$\int_0^1 |J_\nu(x)|^q x dx \le D_q, \qquad \nu \ge 0,$$

is satisfied for every $q > 4$ and taking p and a such that $0 < p \le 4$ and $a > -1$, we obtain

$$\int_0^1 |J_\nu(x)|^p x^a dx \le B(D_{ps})^{1/s},$$

where, this time $s > 1$ is chosen in such a way that $ps > 4$ and $(a+1)s > 2$. The main result of this note claims that, under suitable restrictions on a, (2) is valid for any p, $0 < p < \infty$.

Proposition. *Let* $0 < p < \infty$ *and* $-1 < a < \frac{p}{2} - 1$ *when* $0 < p \le 4$ *or* $-1 < a < \frac{p}{3} - \frac{1}{3}$ *in the case* $4 < p < \infty$. *Then the uniform estimate*

$$\int_0^\infty |J_\nu(x)|^p x^a dx \le C(p,a), \qquad \nu \ge 0, \tag{3}$$

holds true.

The proof of the proposition is based on the following, uniform on $\nu \ge 2$, pointwise bounds for the Bessel functions (C and d are positive constants):

$$|J_\nu(x)| \le C \begin{cases} \exp(-d\nu), & 0 < x < \nu/2, \\ \nu^{-1/4}(|x-\nu| + \nu^{1/3})^{-1/4}, & \nu/2 < x < 2\nu, \\ x^{-1/2}, & 2\nu < x < \infty. \end{cases} \tag{4}$$

The estimate (4) on the interval $0 < x < \nu/2$ is a consequence of

$$\Gamma(\nu+1)(x/2)^{-\nu}|J_\nu(x)| \le 1, \qquad x > 0,$$

(and Stirling's formula) that holds for every $\nu \ge -1/2$ [W, p. 49 (1)], while on the two other intervals it is a consequence of bounds done by Barceló and Córdoba (see [BC, p. 66] or [C, p. 24]; cf. also [Va, p. 70]).

Proof of the Proposition. The left side of (3) is a continuous function of the variable $\nu \ge 0$; hence we can assume ν to be large, say $\nu \ge 2$. Given $\nu \ge 2$ we split the integration in (3) onto the intervals $(0, \nu/2)$, $(\nu/2, 2\nu)$ and $(2\nu, \infty)$. Then

$$\int_{2\nu}^\infty |J_\nu(x)|^p x^a dx \le C \int_{2\nu}^\infty x^{a-p/2} dx = C_1 \nu^{a-p/2+1} \le C_2$$

for $\nu \ge 2$ and p and a satisfying $a < p/2 - 1$. Also,

$$\int_0^{\nu/2} |J_\nu(x)|^p x^a dx \le C \exp(-pd\nu) \int_0^{\nu/2} x^a dx \le C_3$$

for $\nu \geq 2$ when a satisfies $a > -1$. On the interval $(\nu/2, 2\nu)$ we consider only the integration over $(\nu, 2\nu)$; the integration over $(\nu/2, \nu)$ can be treated analogously. We have

$$\int_{\nu}^{2\nu} |J_\nu(x)|^p x^a dx \leq C\nu^{a-p/4} \int_{\nu}^{2\nu} (x-\nu+\nu^{1/3})^{-p/4} dx. \tag{5}$$

If $0 < p \leq 4$, we evaluate the last integral and bound the right side of (5) by $C\nu^{a-p/2+1}$ when $p < 4$ or, by $C\nu^{a-1}\log\nu$ when $p = 4$. Both bounds are small for large ν by the assumption made on a. If $p > 4$, evaluating the last integral gives the bound $C\nu^{a-p/3+1/3}$ for the right side of (5) which is also correct by the assumption made on a. This finishes the proof of the proposition. □

Remark. In fact, using the asymptotics of [BC, p. 66] leads to precise asymptotics of weighted L^p norms of the Bessel functions. Let $1 \leq p \leq \infty$, $\alpha < \frac{1}{2} - \frac{1}{p}$ and $\nu \to \infty$. Then (for $p = \infty$ one has to take $\sup_{x>0} |J_\nu(x)x^\alpha|$ as the L^∞ norm)

$$\left(\int_0^\infty |J_\nu(x)x^\alpha|^p dx\right)^{1/p} \sim \begin{cases} \nu^{\alpha-1/2+1/p}, & 1 \leq p < 4, \\ \nu^{\alpha-1/4}(\log\nu)^{1/4}, & p = 4, \\ \nu^{\alpha-1/3+1/(3p)}, & 4 < p \leq \infty. \end{cases} \tag{6}$$

Here $f(\nu) \sim g(\nu)$ as $\nu \to \infty$ stands for $f(\nu) = O(g(\nu))$ and $g(\nu) = O(f(\nu))$ as $\nu \to \infty$. The upper bound in (6) is obtained, as in the proof of the proposition, by dividing $(0, \infty)$ into three different subintervals, majorizing the integrand and comparing the occurring bounds. The lower bound in (6) is a consequence of the aforementioned precise asymptotics of [BC].

References

[BC] J. A. Barceló, A. Córdoba, *Band–limited functions: L^p–convergence*, Trans. Amer. Math. Soc. **313** (1989), 655–669. MR **90g:**42017

[C] A. Córdoba, *The disc multiplier*, Duke Math. J. **58** (1989), 21–29. MR **90j:**42038

[Guo] K. Guo, *A uniform L^p estimate of Bessel functions and distributions supported on S^{n-1}*, Proc. Amer. Math. Soc. **125** (1997), 1329–1340. MR **97g:**46047

[Va] J. L. Varona, *Fourier series of functions whose Hankel transform is supported on [0,1]*, Constr. Approx. **10** (1994), 65–75. MR **94m:**42071

[W] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, 1966.

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