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A WIENER–HOPF MONTE CARLO SIMULATION TECHNIQUE FOR LÉVY PROCESSES

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We develop a completely new and straightforward method for simulating the joint law of the position and running maximum at a fixed time of a general Lévy process with a view to application in insurance and financial mathematics. Although different, our method takes lessons from Carr's so-called "Canadization" technique as well as Doney's method of stochastic bounds for Lévy processes; see Carr [*Rev. Fin. Studies* **11** (1998) 597–626] and Doney [*Ann. Probab.* **32** (2004) 1545–1552]. We rely fundamentally on the Wiener–Hopf decomposition for Lévy processes as well as taking advantage of recent developments in factorization techniques of the latter theory due to Vigon [Simplifiez vos Lévy en titillant la factorization de Wiener–Hopf (2002) Laboratoire de Mathématiques de L'INSA de Rouen] and Kuznetsov [*Ann. Appl. Probab.* **20** (2010) 1801–1830]. We illustrate our Wiener–Hopf Monte Carlo method on a number of different processes, including a new family of Lévy processes called hypergeometric Lévy processes. Moreover, we illustrate the robustness of working with a Wiener–Hopf decomposition with two extensions. The first extension shows that if one can successfully simulate for a given Lévy processes then one can successfully simulate for any independent sum of the latter process and a compound Poisson process. The second extension illustrates how one may produce a straightforward approximation for simulating the two-sided exit problem.

1. Introduction. Let us suppose that $X = \{X_t : t \geq 0\}$ is a general Lévy process with law $\mathbb{P}$ and Lévy measure Π. That is to say, X is a Markov process with paths that are right continuous with left limits such that the increments are stationary and independent and whose characteristic function at each time t is given by the Lévy–Khinchine representation

$$\mathbb{E}[e^{\mathrm{i}\theta X_t}] = e^{-t\Psi(\theta)}, \qquad \theta \in \mathbb{R}, \tag{1}$$

where

$$\Psi(\theta) = \mathrm{i}\theta a + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} \big(1 - e^{\mathrm{i}\theta x} + \mathrm{i}\theta x \mathbf{1}_{\{|x|<1\}}\big)\Pi(\mathrm{d}x). \tag{2}$$

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We have $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and Π is a measure supported on $\mathbb{R}$ with $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\Pi(\mathrm{d}x) < \infty$. Starting with the early work of Madan and Seneta [18], Lévy processes have played a central role in the theory of financial mathematics and statistics (see, e.g., the books [4, 8, 19, 20]). More recently, they have been extensively used in modern insurance risk theory (see, e.g., Klüppelberg, Kyprianou and Maller [12], Song and Vondraček [21]). The basic idea in financial mathematics and statistics is that the logarithm of the stock price or risky asset follows the dynamics of a Lévy process whilst in insurance mathematics, it is the Lévy process itself which models the surplus wealth of an insurance company until ruin. There are also extensive applications of Lévy processes in queuing theory, genetics and mathematical biology as well as through their appearance in the theory of stochastic differential equations.

In both financial and insurance settings, a key quantity of generic interest is the joint law of the current position and the running maximum of a Lévy process at a fixed time if not the individual marginals associated with the latter bivarite law. Consider the following example. If we define $\overline{X}_t = \sup_{s \leq t} X_s$, then the pricing of barrier options boils down to evaluating expectations of the form $\mathbb{E}[f(x + X_t)\mathbf{1}_{\{x+\overline{X}_t > b\}}]$ for some appropriate function $f(x)$ and threshold $b > 0$. Indeed if $f(x) = (K - e^x)^+$ then the latter expectation is related to the value of an "up-and-in" put. In credit risk, one is predominantly interested in the quantity $\widehat{\mathbb{P}}(\overline{X}_t < x)$ as a function in x and t, where $\widehat{\mathbb{P}}$ is the law of the dual process $-X$. Indeed it is as a functional of the latter probabilities that the price of a credit default swap is computed; see, for example, the recent book of Schoutens and Cariboni [20]. One is similarly interested in $\widehat{\mathbb{P}}(\overline{X}_t \geq x)$ in ruin theory as these probabilities are also equivalent to the finite-time ruin probabilities.

One obvious way to do Monte Carlo simulation of expectations involving the joint law of $(X_t, \overline{X}_t)$ that takes advantage of the stationary and independent increments of Lévy processes is to take a random walk approximation to the Lévy process, simulate multiple paths, taking care to record the maximum for each run. When one is able to set things up in this way so that one samples exactly from the distribution of X_t, the law of the maximum of the underlying random walk will not agree with the law of $\overline{X}_t$.

Taking account of the fact that all Lévy processes respect a fundamental path decomposition known as the Wiener–Hopf factorization, it turns out there is another very straightforward way to perform Monte Carlo simulations for expectations involving the joint law of $(X_t, \overline{X}_t)$ which we introduce in this paper. Our method allows for exact sampling from the law of $(X_{\mathbf{g}}, \overline{X}_{\mathbf{g}})$ where $\mathbf{g}$ is a random time whose distribution can be concentrated arbitrarily close around t.

There are several advantages of the technique. First, when it is taken in context with very recent developments in Wiener–Hopf theory for Lévy processes, for example, recent advances in the theory of scale functions for spectrally negative processes (see Kyprianou, Pardo and Rivero [16]), new complex analytical techniques

due to Kuznetsov [13] and Vigon's theory of philanthropy (see [22]), one may quickly progress the algorithm to quite straightforward numerical work. Second, our Wiener–Hopf method takes advantage of a similar feature found in the, now classical, "Canadization" method of Carr [7] for numerical evaluation of optimal stopping problems. The latter is generally acknowledged as being more efficient than appealing to classical random walk approximation Monte Carlo methods. Indeed, later in this paper, we present our numerical findings with some indication of performance against the method of random walk approximation. In this case, our Wiener–Hopf method appears to be extremely effective. Third, in principle, our method handles better the phenomena of discontinuities which can occur with functionals of the form $\mathbb{E}[f(x+X_t)\mathbf{1}_{\{x+\overline{X}_t>b\}}]$ at the boundary point $x=b$. It is now well understood that the issue of regularity of the upper and lower half line for the underlying Lévy process (see Chapter 6 of [14] for a definition) is responsible the appearance of a discontinuity at $x=b$ in such functions (cf. [1]). The nature of our Wiener–Hopf method naturally builds the distributional atom which is responsible for this discontinuity into the simulations.

Additional advantages to the method we propose include its simplicity with regard to numerical implementation. Moreover, as we shall also see in Section 4 of this paper, the natural probabilistic structure that lies behind our so-called Wiener–Hopf Monte Carlo method also allows for additional creativity when addressing some of the deficiencies of the method itself.

2. Wiener–Hopf Monte Carlo simulation technique. The basis of the algorithm is the following simple observation which was pioneered by Carr [7] and subsequently used in several contexts within mathematical finance for producing approximate solutions to free boundary value problems that appear as a result of optimal stopping problems characterizing the value of an American-type option.

Suppose that $\mathbf{e}_1, \mathbf{e}_2, \ldots$ are a sequence of i.i.d. exponentially distributed random variables with unit mean. Suppose they are all defined on a common product space with product law $\mathbf{P}$ which is orthogonal to the probability space on which the Lévy process X is defined. For all $t>0$, we know from the Strong Law of Large Numbers that

$$\sum_{i=1}^{n} \frac{t}{n}\mathbf{e}_i \to t \qquad \text{as } n \uparrow \infty \tag{3}$$

$\mathbf{P}$-almost surely. The random variable on the left-hand side above is equal in law to a Gamma random variable with parameters n and n/t. Henceforth, we write it $\mathbf{g}(n, n/t)$. Recall that $\mathbb{P}$ is our notation for the law of the Lévy process X. Then writing $\overline{X}_t = \sup_{s\le t} X_s$ we argue the case that, for sufficiently large n, a suitable approximation to $\mathbb{P}(X_t \in \mathrm{d}x, \overline{X}_t \in \mathrm{d}y)$ is $(\mathbf{P}\times\mathbb{P})(X_{\mathbf{g}(n,n/t)} \in \mathrm{d}x, \overline{X}_{\mathbf{g}(n,n/t)} \in \mathrm{d}y)$.

This approximation gains practical value in the context of Monte Carlo simulation when we take advantage of the fundamental path decomposition that applies

to all Lévy processes over exponential time periods known as the Wiener–Hopf factorization.

THEOREM 1. *For all $n \geq 1$ and $\lambda > 0$, define $\mathbf{g}(n,\lambda) := \sum_{i=1}^{n} \mathbf{e}_i/\lambda$. Then*

$$\big(X_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)}\big) \stackrel{d}{=} (V(n,\lambda), J(n,\lambda)), \tag{4}$$

where $V(n,\lambda)$ and $J(n,\lambda)$ are defined iteratively for $n \geq 1$ as

$$V(n,\lambda) = V(n-1,\lambda) + S_\lambda^{(n)} + I_\lambda^{(n)},$$
$$J(n,\lambda) = \max\big(J(n-1,\lambda), V(n-1,\lambda) + S_\lambda^{(n)}\big)$$

and $V(0,\lambda) = J(0,\lambda) = 0$. Here, $S_\lambda^{(0)} = I_\lambda^{(0)} = 0$, $\{S_\lambda^{(j)} : j \geq 1\}$ are an i.i.d. sequence of random variables with common distribution equal to that of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $\{I_\lambda^{(j)} : j \geq 1\}$ are another i.i.d. sequence of random variables with common distribution equal to that of $\underline{X}_{\mathbf{e}_1/\lambda}$.

PROOF. The Wiener–Hopf factorization tells us that $\overline{X}_{\mathbf{e}_1/\lambda}$ and $X_{\mathbf{e}_1/\lambda} - \overline{X}_{\mathbf{e}_1/\lambda}$ are independent and the second of the pair is equal in distribution to $\underline{X}_{\mathbf{e}_1/\lambda}$. This will constitute the key element of the proof.

Fix $n \geq 1$. Suppose we define $\overline{X}_{s,t} = \sup_{s \leq u \leq t} X_u$. Then it is trivial to note that

$$\begin{aligned}&\big(X_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)}\big)\\ &\quad = \big(X_{\mathbf{g}(n-1,\lambda)} + \big(X_{\mathbf{g}(n,\lambda)} - X_{\mathbf{g}(n-1,\lambda)}\big), \overline{X}_{\mathbf{g}(n-1,\lambda)} \vee \overline{X}_{\mathbf{g}(n-1,\lambda),\mathbf{g}(n,\lambda)}\big),\end{aligned} \tag{5}$$

where $\mathbf{g}(0,\lambda) := 0$. If we define $X_t^{(n)} = X_{\mathbf{g}(n-1,\lambda)+t} - X_{\mathbf{g}(n-1,\lambda)}$ and $\overline{X}_{\mathbf{e}_n/\lambda}^{(n)} = \sup_{s \leq \mathbf{e}_n/\lambda} X_s^{(n)}$, then from (5) it follows that

$$\big(X_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)}\big) = \big(X_{\mathbf{g}(n-1,\lambda)} + X_{\mathbf{e}_n/\lambda}^{(n)}, \overline{X}_{\mathbf{g}(n-1,\lambda)} \vee \big(X_{\mathbf{g}(n-1,\lambda)} + \overline{X}_{\mathbf{e}_n/\lambda}^{(n)}\big)\big).$$

Now noting that the process $X^{(n)}$ is independent of $\{X_s : s \leq \mathbf{g}(n-1,\lambda)\}$ and has law $\mathbb{P}$ and, moreover, recalling the distributional Wiener–Hopf decomposition described at the beginning of the proof, it follows that

$$\big(X_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)}\big) \stackrel{d}{=} \big(X_{\mathbf{g}(n-1,\lambda)} + S_\lambda^{(n)} + I_\lambda^{(n)}, \overline{X}_{\mathbf{g}(n-1,\lambda)} \vee \big(X_{\mathbf{g}(n-1,\lambda)} + S_\lambda^{(n)}\big)\big),$$

where $S_\lambda^{(n)}$ and $I_\lambda^{(n)}$ defined as in the statement of the theorem. The conclusion of the theorem now follows immediately. □

Note that the idea of embedding a random walk into the path of a Lévy process with two types of step distribution determined by the Wiener–Hopf factorization has been used in a different, and more theoretical context by Doney [9].

Given (3), it is clear that the pair $(V(n,n/t), J(n,n/t))$ converges in distribution to $(X_t, \overline{X}_t)$. This suggests that we need only to be able to simulate i.i.d.

copies of the distributions of $S_{n/t} := S^{(1)}_{n/t}$ and $I_{n/t} := I^{(1)}_{n/t}$ and then by a simple functional transformation we may produce a realisation of the random variables $(X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)})$. Given a suitably nice function F, using standard Monte Carlo methods one estimates for large k

$$\mathbb{E}[F(X_t, \overline{X}_t)] \simeq \frac{1}{k}\sum_{m=1}^{k} F\big(V^{(m)}(n, n/t), J^{(m)}(n, n/t)\big), \tag{6}$$

where $(V^{(m)}(n, n/t), J^{(m)}(n, n/t))$ are i.i.d. copies of $(V(n, n/t), J(n, n/t))$. Indeed the strong law of large numbers implies that the right-hand side above converges almost surely as $k \uparrow \infty$ to $\mathbf{E} \times \mathbb{E}(F(X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)}))$ which in turn converges as $n \uparrow \infty$ to $\mathbb{E}(F(X_t, \overline{X}_t))$.

3. Implementation. The algorithm described in the previous section only has practical value if one is able to sample from the distributions of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$. It would seem that this, in itself, is not that much different from the problem that it purports to solve. However, it turns out that there are many tractable examples and in all cases this is due to the tractability of their Wiener–Hopf factorizations.

Whilst several concrete cases can be handled from the class of spectrally one-sided Lévy processes thanks to recent development in the theory of scale functions, which can be used to described the laws of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$ (cf. [10, 17]), we give here two large families of two-sided jumping Lévy processes that have pertinence to mathematical finance to show how the algorithm may be implemented.

3.1. *β-class of Lévy processes.* The β-class of Lévy processes, introduced in [13], is a 10-parameter Lévy process which has characteristic exponent

$$\Psi(\theta) = \mathrm{i}a\theta + \frac{1}{2}\sigma^2\theta^2 + \frac{c_1}{\beta_1}\left\{\mathrm{B}(\alpha_1, 1-\lambda_1) - \mathrm{B}\left(\alpha_1 - \frac{\mathrm{i}\theta}{\beta_1}, 1-\lambda_1\right)\right\} + \frac{c_2}{\beta_2}\left\{\mathrm{B}(\alpha_2, 1-\lambda_2) - \mathrm{B}\left(\alpha_2 + \frac{\mathrm{i}\theta}{\beta_2}, 1-\lambda_2\right)\right\}$$

with parameter range $a, \sigma \in \mathbb{R}, c_1, c_2, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and $\lambda_1, \lambda_2 \in (0,3) \setminus \{1,2\}$. Here $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function (see [11]). The density of the Lévy measure is given by

$$\pi(x) = c_1 \frac{e^{-\alpha_1\beta_1 x}}{(1-e^{-\beta_1 x})^{\lambda_1}}\mathbf{1}_{\{x>0\}} + c_2 \frac{e^{\alpha_2\beta_2 x}}{(1-e^{\beta_2 x})^{\lambda_2}}\mathbf{1}_{\{x<0\}}.$$

Although Ψ takes a seemingly complicated form, this particular family of Lévy processes has a number of very beneficial virtues from the point of view of mathematical finance which are discussed in [13]. Moreover, the large number of parameters also allows one to choose Lévy processes within the β-class that have paths that are both of unbounded variation [when at least one of the conditions $\sigma \neq 0$,

$\lambda_1 \in (2,3)$ or $\lambda_2 \in (2,3)$ holds] and bounded variation [when all of the conditions $\sigma = 0$, $\lambda_1 \in (0,2)$ and $\lambda_2 \in (0,2)$ hold] as well as having infinite and finite activity in the jumps component [accordingly as both $\lambda_1, \lambda_2 \in (1,3)$ or not].

What is special about the β-class is that all the roots of the equation $\lambda + \Psi(\theta) = 0$ are analytically identifiable which leads to semi-explicit identities for the laws of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$ as the following result lifted from [13] shows.

THEOREM 2. *For $\lambda > 0$, all the roots of the equation*

$$\lambda + \Psi(\theta) = 0$$

are simple and occur on the imaginary axis. They can be enumerated by $\{\mathrm{i}\zeta_n^+ : n \geq 0\}$ on the positive imaginary axis and $\{\mathrm{i}\zeta_n^- : n \geq 0\}$ on the negative imaginary axis in order of increasing absolute magnitude where

$$\begin{aligned}
&\zeta_0^+ \in (0, \beta_2\alpha_2), \qquad \zeta_0^- \in (-\beta_1\alpha_1, 0),\\
&\zeta_n^+ \in \big(\beta_2(\alpha_2 + n - 1), \beta_2(\alpha_2 + n)\big) \qquad \text{for } n \geq 1,\\
&\zeta_n^- \in \big(\beta_1(-\alpha_1 - n), \beta_1(-\alpha_1 - n + 1)\big) \qquad \text{for } n \geq 1.
\end{aligned}$$

Moreover, for $x > 0$,

$$\mathbb{P}(\overline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}x) = -\left(\sum_{k\geq 0} c_k^- \zeta_k^- e^{\zeta_k^- x}\right)\mathrm{d}x, \tag{7}$$

where

$$c_0^- = \prod_{n\geq 1} \frac{1 + \zeta_0^-/(\beta_1(n - 1 + \alpha_1))}{1 - \zeta_0^-/\zeta_n^-}$$

and

$$c_k^- = \frac{1 + \zeta_k^-/(\beta_1(k - 1 + \alpha_1))}{1 - \zeta_k^-/\zeta_0^-} \prod_{n\geq 1, n\neq k} \frac{1 + \zeta_k^-/(\beta_1(n - 1 + \alpha_1))}{1 - \zeta_k^-/\zeta_n^-}.$$

A similar expression holds for $\mathbb{P}(-\underline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}x)$ with the role of $\{\zeta_n^- : n \geq 0\}$ being played by $\{-\zeta_n^+ : n \geq 0\}$ and α_1, β_1 replaced by α_2, β_2.

Note that when 0 is irregular for $(0, \infty)$ the distribution of $\overline{X}_{\mathbf{e}_1/\lambda}$ will have an atom at 0 which can be computed from (7) and is equal to $1 - \sum_{k\geq 0} c_k^-$. Alternatively, from Remark 6 in [13] this can equivalently be written as $\prod_{n\geq 0}(-\zeta_n^-)/\beta_1(n + \alpha_1)$. A similar statement can be made concerning an atom at 0 for the distribution of $-\underline{X}_{\mathbf{e}_1/\lambda}$ when 0 is irregular for $(-\infty, 0)$. Conditions for irregularity are easy to check thanks to Bertoin [3]; see also the summary in Kyprianou and Loeffen [15] for other types of Lévy processes that are popular in mathematical finance.

By making a suitable truncation of the series (7), one may easily perform independent sampling from the distributions $\overline{X}_{\mathbf{e}_1/\lambda}$ and $\underline{X}_{\mathbf{e}_1/\lambda}$ as required for our Monte Carlo methods.

3.2. *Philanthropy and general hypergeometric Lévy processes.* The forthcoming discussion will assume familiarity with classical excursion theory of Lévy processes for which the reader is referred to Chapter VI of [2] or Chapter 6 of [14].

According to Vigon's theory of philanthropy, a (killed) subordinator is called a *philanthropist* if its Lévy measure has a decreasing density on $\mathbb{R}_+$. Moreover, given any two subordinators H_1 and H_2 which are philanthropists, providing that at least one of them is not killed, there exists a Lévy process X such that H_1 and H_2 have the same law as the ascending and descending ladder height processes of X, respectively. (In the language of Vigon, the philanthropists H_1 and H_2 are *friends*.) Suppose we denote the killing rate, drift coefficient and Lévy measures of H_1 and H_2 by the respective triples (k, δ, Π_{H_1}) and $(\widehat{k}, \widehat{\delta}, \Pi_{H_2})$. Then [22] shows that the Lévy measure of X satisfies the following identity:

$$\overline{\Pi}_X^+(x) = \int_0^\infty \Pi_{H_1}(x + \mathrm{d}u)\overline{\Pi}_{H_2}(u) + \widehat{\delta}\pi_{H_1}(x) + \widehat{k}\,\overline{\Pi}_{H_1}(x), \qquad x > 0, \tag{8}$$

where $\overline{\Pi}_X^+(x) := \Pi_X(x, \infty)$, $\overline{\Pi}_{H_1}(u) := \Pi_{H_1}(u, \infty)$, $\overline{\Pi}_{H_2}(u) := \Pi_{H_2}(u, \infty)$ and π_{H_1} is the density of Π_{H_1}. By symmetry, an obvious analogue of (8) holds for the negative tail $\overline{\Pi}_X^-(x) := \Pi_X(-\infty, x)$, $x < 0$.

A particular family of subordinators which will be of interest to us is the class of subordinators which is found within the definition of Kuznetsov's β-class of Lévy processes. These processes have characteristics $(c, \alpha, \beta, \gamma)$ where $\gamma \in (-\infty, 0) \cup (0, 1)$, $\beta, c > 0$ and $1 - \alpha + \gamma > 0$. The Lévy measure of such subordinators is of the type

$$c\frac{e^{\alpha\beta x}}{(e^{\beta x} - 1)^{1+\gamma}} 1_{\{x>0\}}\,\mathrm{d}x. \tag{9}$$

From Proposition 9 in [13], the Laplace exponent of a β-class subordinator satisfies

$$\Phi(\theta) = \mathtt{k} + \delta\theta + \frac{c}{\beta}\{\mathrm{B}(1 - \alpha + \gamma, -\gamma) - \mathrm{B}(1 - \alpha + \gamma + \theta/\beta, -\gamma)\} \tag{10}$$

for $\theta \geq 0$ where δ is the drift coefficient and $\mathtt{k}$ is the killing rate.

Let H_1 and H_2 be two independent subordinators from the β-class where for $i = 1, 2$, with respective drift coefficients $\delta_i \geq 0$, killing rates $\mathtt{k}_i \geq 0$ and Lévy measure parameters $(c_i, \alpha_i, \beta, \gamma_i)$. Their respective Laplace exponents are denoted by Φ_i, $i = 1, 2$. In Vigon's theory of philanthropy, it is required that $\mathtt{k}_1\mathtt{k}_2 = 0$. Under this assumption, let us denote by X the Lévy process whose ascending and descending ladder height processes have the same law as H_1 and H_2, respectively. In other words, the Lévy process whose characteristic exponent is given by $\Phi_1(-\mathrm{i}\theta)\Phi_2(\mathrm{i}\theta)$, $\theta \in \mathbb{R}$. It is important to note that the Gaussian component of the process X is given by $2\delta_1\delta_2$; see [22]. From (8), the Lévy measure of X is such

that

$$\overline{\Pi}^{+}_{X}(x) = c_1 c_2 \int_x^\infty \frac{e^{\beta_1\alpha_1 u}}{(e^{\beta_1 u}-1)^{\gamma_1+1}} \int_{u-x}^\infty \frac{e^{\alpha_2\beta_2 z}}{(e^{\beta_2 z}-1)^{\gamma_2+1}}\,\mathrm{d}z\,\mathrm{d}u + \delta_2 c_1 \frac{e^{\beta_1\alpha_1 x}}{(e^{\beta_1 x}-1)^{\gamma_1+1}} + \mathrm{k}_2 c_1 \int_x^\infty \frac{e^{\beta_1\alpha_1 u}}{(e^{\beta_1 u}-1)^{\gamma_1+1}}\,\mathrm{d}x.$$

Assume first that $\gamma_2 < 0$, taking derivative in x and computing the resulting integrals with the help of [11] we find that for $x > 0$ the density of the Lévy measure is given by

$$\pi(x) = -\frac{c_1 c_2}{\beta}\mathrm{B}(\rho, -\gamma_2)e^{-\beta x(1+\gamma_1-\alpha_1)}{}_2F_1(1+\gamma_1, \rho; \rho-\gamma_2; e^{-\beta x}) + c_1\left(\mathrm{k}_2 + \frac{c_2}{\beta}\mathrm{B}(1+\gamma_2-\alpha_2, -\gamma_2)\right)\frac{e^{\alpha_1\beta x}}{(e^{\beta x}-1)^{1+\gamma_1}} - \delta_2 c_1 \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{e^{\alpha_1\beta x}}{(e^{\beta x}-1)^{1+\gamma_1}}\right],$$

where $\rho = 2+\gamma_1+\gamma_2-\alpha_1-\alpha_2$. The validity of this formula is extended for $\gamma_2 \in (0,1)$ by analytical continuation. The corresponding expression for $x < 0$ can be obtained by symmetry considerations.

We define a General Hypergeometric process to be the 13 parameter Lévy process with characteristic exponent given in compact form

$$\Psi(\theta) = \mathrm{d}\mathrm{i}\theta + \frac{1}{2}\sigma^2\theta^2 + \Phi_1(-\mathrm{i}\theta)\Phi_2(\mathrm{i}\theta), \qquad \theta \in \mathbb{R}, \tag{11}$$

where $\mathrm{d}, \sigma \in \mathbb{R}$. The two additional parameters d, σ are included largely with applications in mathematical finance in mind. Without these two additional parameters, it is difficult to disentangle the Gaussian coefficient and the drift coefficients from parameters appearing in the jump measure. Note that the Gaussian coefficient in (11) is now $\sigma^2/2 + 2\delta_1\delta_2$. The definition of General Hypergeometric Lévy processes includes previously defined Hypergeometric Lévy processes in Kyprianou, Pardo and Rivero [16], Caballero, Pardo and Pérez [5] and Lamperti-stable Lévy processes in Caballero, Pardo and Pérez [6].

Just as with the case of the β-family of Lévy processes, because Ψ can be written as a linear combination of a quadratic form and beta functions, it turns out that one can identify all the roots of the equation $\Psi(\theta) + \lambda = 0$ which is again sufficient to describe the laws of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$.

THEOREM 3. *For $\lambda > 0$, all the roots of the equation*

$$\lambda + \Psi(\theta) = 0$$

are simple and occur on the imaginary axis. They can be enumerated by $\{\mathrm{i}\xi_n^+ : n \geq 0\}$ *on the positive imaginary axis and* $\{\mathrm{i}\xi_n^- : n \geq 0\}$ *on the negative imaginary axis in order of increasing absolute magnitude where*

$$\begin{aligned}
&\xi_0^+ \in \big(0, \beta(1+\gamma_2-\alpha_2)\big), \qquad \xi_0^- \in \big(-\beta(1+\gamma_1-\alpha_1), 0\big),\\
&\xi_n^+ \in \big(\beta(\gamma_2-\alpha_2+n), \beta(1+\gamma_2-\alpha_2+n)\big) \qquad \textit{for } n \geq 1,\\
&\xi_n^- \in \big(-\beta(1+\gamma_1-\alpha_1+n), -\beta(\gamma_1-\alpha_1+n)\big) \qquad \textit{for } n \geq 1.
\end{aligned}$$

Moreover, for $x > 0$,

$$\mathbb{P}(\overline{X}_{\mathbf{e}_{1/\lambda}} \in \mathrm{d}x) = -\Bigg(\sum_{k\geq 0} c_k^- \xi_k^- e^{\xi_k^- x}\Bigg)\mathrm{d}x, \tag{12}$$

where

$$c_0^- = \prod_{n\geq 1} \frac{1+\xi_0^-/(\beta(\gamma_1-\alpha_1+n))}{1-\xi_0^-/\xi_n^-}$$

and

$$c_k^- = \frac{1+\xi_k^-/(\beta(\gamma_1-\alpha_1+k))}{1-\xi_k^-/\xi_0^-} \prod_{n\geq 1, n\neq k} \frac{1+\xi_k^-/(\beta(\gamma_1-\alpha_1+n))}{1-\xi_k^-/\xi_n^-}.$$

A similar expression holds for $\mathbb{P}(-\underline{X}_{\mathbf{e}_{1/\lambda}} \in \mathrm{d}x)$ *with the role of* $\{\xi_n^- : n \geq 0\}$ *replaced by* $\{-\xi_n^+ : n \geq 0\}$ *and* α_1, γ_2 *replaced by* α_2, γ_2.

PROOF. The proof is very similar to the proof of Theorem 10 in [13]. Formula (11) and reflection formula for the Beta function (see [11])

$$\mathrm{B}(-z;-\gamma) = \mathrm{B}(1+z+\gamma;-\gamma)\frac{\sin(\pi(z+\gamma))}{\sin(\pi z)} \tag{13}$$

tell us that $\Psi(\mathrm{i}\theta) \to -\infty$ as $\theta \to \beta(1+\gamma_2-\alpha_2)$, and since $\Psi(0) = 0$ we conclude that $\lambda + \Psi(\mathrm{i}\theta) = 0$ has a solution on the interval $\theta \in (0, \beta(1+\gamma_2-\alpha_2))$. Other intervals can be checked in a similar way [note that $\Phi_i(z)$ are Laplace exponents of subordinators, therefore they are positive for $z > 0$]. Next, we assume that $\sigma, \delta_1, \delta_2 > 0$. Using formulas (11), (13) and the asymptotic result

$$\frac{\Gamma(a+z)}{\Gamma(z)} = z^a + O(z^{a-1}), \qquad z \to +\infty,$$

which can be found in [11], we conclude that $\Psi(\mathrm{i}\theta)$ has the following asymptotics as $\theta \to +\infty$:

$$\begin{aligned}
\Psi(\mathrm{i}\theta) = &-\frac{1}{2}(\sigma^2 + 2\delta_1\delta_2)\theta^2 + O(\theta^{1+\gamma_2})\\
&- \frac{\delta_1\Gamma(-\gamma_2)}{\beta^{\gamma_2}} \frac{\sin(\pi(\alpha_2+\theta/\beta))}{\sin(\pi(\alpha_2-\gamma_2+\theta/\beta))}[\theta^{1+\gamma_2} + O(\theta^{\gamma_2+\gamma_1})].
\end{aligned}$$

Using the above asymptotic expansion and the same technique as in the proof of Theorem 5 in [13], we find that as $n \to +\infty$ there exists a constant C_1 such that

$$\xi_n^+ = \beta(n + 1 + \gamma_2 - \alpha_2) + C_1 n^{\gamma_2 - 1} + O(n^{\gamma_2 - 1 - \varepsilon}),$$

with a similar expression for ξ_n^-. Thus, we use Lemma 6 from [13] (and the same argument as in the proofs of Theorems 5 and 10 in [13]) to show that first there exist no other roots of meromorphic function $\lambda + \Psi(\mathrm{i}z)$ except for $\{\xi_n^{\pm}\}$, and secondly that we have a factorization

$$\frac{\lambda}{\lambda + \Psi(\theta)} = \frac{1}{1 + \mathrm{i}\theta/\xi_0^-} \prod_{n \geq 1} \frac{1 - \mathrm{i}\theta/(\beta(\gamma_1 - \alpha_1 + n))}{1 + \mathrm{i}\theta/\xi_n^-} \times \frac{1}{1 + \mathrm{i}\theta/\xi_0^+} \prod_{n \geq 1} \frac{1 + \mathrm{i}\theta/(\beta(\gamma_2 - \alpha_2 + n))}{1 + \mathrm{i}\theta/\xi_n^+}.$$

The Wiener–Hopf factoris $\phi_q^{\pm}(\theta)$ are identified from the above equation with the help of analytical uniqueness result, Lemma 2 in [13]. Formula (12) is obtained from the infinite product representation for $\phi_q^+(\theta)$ using residue calculus.

This ends the proof in the case $\sigma, \delta_1, \delta_2 > 0$, in all other cases the proof is almost identical, except that one has to do more work to obtain asymptotics for the roots of $\lambda + \Psi(\mathrm{i}\theta) = 0$. We summarize all the possible asymptotics of the roots below

$$\xi_n^+ = \beta(n - \alpha_2 + \omega_2) + C n^{\varrho_2} + O(n^{\varrho_2 - \varepsilon}) \qquad \text{as } n \to \infty,$$

where the coefficients ω_2, ϱ_2 and C are presented in Table 1. Corresponding results for ξ_n^- can be obtained by symmetry considerations. □

REMARK 1. Similar comments to those made after Theorem 2 regarding the existence of atoms in the distribution of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$ also apply here.

TABLE 1
Coefficients for the asymptotic expansion of ξ_n^+

Case	ω_2	C	ϱ_2
$\sigma^2, \delta_1, \delta_2 > 0$	$1 + \gamma_2$	$\frac{2\delta_1 c_2}{\beta\Gamma(1+\gamma_2)(\sigma^2 + 2\delta_1\delta_2)}$	$\gamma_2 - 1$
$\sigma = 0, \delta_1, \delta_2 > 0$	$1 + \gamma_2$	$\frac{c_2}{\beta\Gamma(1+\gamma_2)\delta_2}$	$\gamma_2 - 1$
$\sigma^2, \delta_2 > 0, \delta_1 = 0$	$1 + \gamma_2$	$\frac{2c_1c_2\Gamma(1-\gamma_1)}{\beta^{3+\gamma_1-\gamma_2}\Gamma(1+\gamma_2)\gamma_1\sigma^2}$	$\gamma_1 + \gamma_2 - 2$
$\sigma^2, \delta_1 > 0, \delta_2 = 0$	$1 + \gamma_2$	$\frac{2\delta_1 c_2}{\beta\Gamma(1+\gamma_2)\sigma^2}$	$\gamma_2 - 1$
$\delta_2 > 0, \sigma = \delta_1 = 0$	$1 + \gamma_2$	$\frac{c_2}{\beta\delta_2\Gamma(1+\gamma_2)}$	$\gamma_2 - 1$
$\delta_1 > 0, \sigma = \delta_2 = 0$	0	$\frac{\sin(\pi\gamma_2)}{\pi}\frac{\beta^2\gamma_2(\mu+\mathrm{d})}{\delta_1 c_2\Gamma(1-\gamma_2)}$	$-\gamma_2$
$\sigma^2 > 0, \delta_1 = \delta_2 = 0$	$1 + \gamma_2$	$\frac{2c_1c_2\Gamma(1-\gamma_1)}{\beta^{3+\gamma_1-\gamma_2}\Gamma(1+\gamma_2)\gamma_1\sigma^2}$	$\gamma_1 + \gamma_2 - 2$
$\sigma = \delta_1 = \delta_2 = 0$	1	$\frac{\beta^2\gamma_2}{c_2\Gamma(1-\gamma_2)}\frac{\sin(\pi\gamma_2)}{\pi}(\mathrm{k}_2 + \frac{c_2}{\beta}\mathrm{B}(1 + \gamma_2 - \alpha_2; -\gamma_2))$	$-\gamma_2$

REMARK 2. It is important to note that the hypergeometric Lévy process is but one of many examples of Lévy processes which may be constructed using Vigon's theory of philanthropy. With the current Monte Carlo algorithm in mind, it should be possible to engineer other favorable Lévy processes in this way.

4. Extensions.

4.1. *Building in arbitrary large jumps.* The starting point for the Wiener–Hopf Monte Carlo algorithm is the distribution of $\overline{X}_{\mathbf{e}_1/\lambda}$ and $\underline{X}_{\mathbf{e}_1/\lambda}$, and in Section 3 we have presented two large families of Lévy processes for which one can compute these distributions quite efficiently. We have also argued the case that one might engineer other fit-for-purpose Wiener–Hopf factorizations using Vigon's theory of philanthropy. However, below, we present another alternative for extending the application of the Wiener–Hopf Monte Carlo technique to a much larger class of Lévy processes than those for which sufficient knowledge of the Wiener–Hopf factorization is known. Indeed the importance of Theorem 4 below is that we may now work with any Lévy processes whose Lévy measure can be written as a sum of a Lévy measure from the β-family or hypergeometric family plus *any* other measure with finite mass. This is a very general class as a little thought reveals that many Lévy processes necessarily take this form. However, there are some obvious exclusions from this class, for example, cases of Lévy processes with bounded jumps.

THEOREM 4. *Let $Y = \{Y_t : t \geq 0\}$ be a sum of a Lévy process X and a compound Poisson process such that for all $t \geq 0$,*

$$Y_t = X_t + \sum_{i=1}^{N_t} \xi_i,$$

where $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity γ, independent of the i.i.d. sequence of random variables, $\{\xi_i : i \geq 1\}$, and X. Define iteratively for $n \geq 1$

$$V(n,\lambda) = V(n-1,\lambda) + S^{(n)}_{\lambda+\gamma} + I^{(n)}_{\lambda+\gamma} + \xi_n(1-\beta_n),$$

$$J(n,\lambda) = \max\bigl(V(n,\lambda), J(n-1,\lambda), V(n-1,\lambda) + S^{(n)}_{\lambda+\gamma}\bigr),$$

where $V(0,\lambda) = J(0,\lambda) = 0$, sequences $\{S^{(j)}_{\lambda+\gamma} : n \geq 1\}$ and $\{I^{(n)}_{\lambda+\gamma} : n \geq 1\}$ are defined in Theorem 1, and $\{\beta_n : n \geq 1\}$ are an i.i.d. sequence of Bernoulli random variables such that $\mathbb{P}(\beta_n = 1) = \lambda/(\gamma+\lambda)$. Then

$$\bigl(Y_{\mathbf{g}(n,\lambda)}, \overline{Y}_{\mathbf{g}(n,\lambda)}\bigr) \stackrel{d}{=} (V(T_n,\lambda), J(T_n,\lambda)), \tag{14}$$

where $T_n = \min\{j \geq 1 : \sum_{i=1}^{j} \beta_i = n\}$.

PROOF. Consider a Poisson process with arrival rate $\lambda+\gamma$ such that points are independently marked with probability $\lambda/(\lambda+\gamma)$. Then recall that the Poisson Thinning theorem tells us that the process of marked points is a Poisson process with arrival rate λ. In particular, the arrival time having index T_1 is exponentially distributed with rate λ.

Suppose that τ_1 is the first time that an arrival occurs in the process N, in particular τ_1 is exponentially distributed with rate γ. Let $\mathbf{e}_\lambda$ be another independent and exponentially distributed random variable, and fix $x\in\mathbb{R}$ and $y\geq 0$. Then making use of the Wiener–Hopf decomposition,

$$
\begin{aligned}
&(x+Y_{\tau_1\wedge\mathbf{e}_\lambda}, \max\{y, x+\overline{Y}_{\tau_1\wedge\mathbf{e}_\lambda}\})\\
&\quad=\begin{cases}
\big(x+S_\lambda^{(1)}+I_\lambda^{(1)}, \max\big\{y, x+S_\lambda^{(1)}\big\}\big), & \text{if } \mathbf{e}_\lambda<\tau_1,\\
\big(x+S_\gamma^{(1)}+I_\gamma^{(1)}+\xi_n, \max\big\{x+S_\gamma^{(1)}+I_\gamma^{(1)}+\xi_n, y, x+S_\gamma^{(1)}\big\}\big), & \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \tau_1\leq\mathbf{e}_\lambda.
\end{cases}
\end{aligned}
$$

If we momentarily set $(x,y)=(V(0,\lambda),J(0,\lambda))=(0,0)$, then by the Poisson Thinning theorem it follows that $(Y_{\tau_1\wedge\mathbf{e}_\lambda},\overline{Y}_{\tau_1\wedge\mathbf{e}_\lambda})$ is equal in distribution to $(V(1,\lambda),J(1,\lambda))$. Moreover, again by the Poisson Thinning theorem, $(Y_{\mathbf{e}_\lambda},\overline{Y}_{\mathbf{e}_\lambda})$ is equal in distribution to $(V(T_1,\lambda),J(T_1,\lambda))$. This proves the theorem for the case $n=1$.

In the spirit of the proof of Theorem 1, the proof for $n\geq 2$ can be established by an inductive argument. Indeed, if the result is true for $n=k-1$ then it is true for $n=k$ by taking $(x,y)=(V(k-1,\lambda),J(k-1,\lambda))$ then appealing to the lack of memory property, stationary and independent increments of Y and the above analysis for the case that $n=1$. The details are left to the reader. □

REMARK 3. A particular example where the use of the above theorem is of pertinence is a linear Brownian motion plus an independent compound Poisson process. This would include, for example, the so-called Kou model from mathematical finance in which the jumps of the compound Poisson process have a two-sided exponential distribution. In the case that X is a linear Brownian motion, the quantities $\overline{X}_{\mathbf{e}_1/\lambda}$ and $-\underline{X}_{\mathbf{e}_1/\lambda}$ are both exponentially distributed with easily computed rates.

4.2. *Approximate simulation of the law of* $(X_t,\overline{X}_t,\underline{X}_t)$. Next, we consider the problem of sampling from the distribution of the three random variables $(X_t,\overline{X}_t,\underline{X}_t)$. This is also an important problem for applications making use of the two-sided exit problem and, in particular, for pricing double barrier options. The following slight modification of the Wiener–Hopf Monte Carlo technique allows us to obtain two estimates for this triple of random variables, which in many cases can be used to provide upper and lower bounds for certain functionals of $(X_t,\overline{X}_t,\underline{X}_t)$.

THEOREM 5. *Given two sequences* $\{S_\lambda^{(n)} : n \geq 1\}$ *and* $\{I_\lambda^{(n)} : n \geq 1\}$ *introduced in Theorem 1 we define iteratively for* $n \geq 1$

$$
\begin{aligned}
V(n,\lambda) &= V(n-1,\lambda) + S_\lambda^{(n)} + I_\lambda^{(n)},\\
J(n,\lambda) &= \max\bigl(J(n-1,\lambda), V(n-1,\lambda) + S_\lambda^{(n)}\bigr),\\
K(n,\lambda) &= \min\bigl(K(n-1,\lambda), V(n,\lambda)\bigr),\\
\tilde{J}(n,\lambda) &= \max\bigl(\tilde{J}(n-1,\lambda), V(n,\lambda)\bigr),\\
\tilde{K}(n,\lambda) &= \min\bigl(\tilde{K}(n-1,\lambda), V(n-1,\lambda) + I_\lambda^{(n)}\bigr),
\end{aligned}
\tag{15}
$$

where $V(0,\lambda) = J(0,\lambda) = K(0,\lambda) = \tilde{J}(0,\lambda) = \tilde{K}(0,\lambda) = 0$. *Then for any bounded function* $f(x,y,z) : \mathbb{R}^3 \mapsto \mathbb{R}$ *which is increasing in* z*-variable we have*

$$
\mathbb{E}[f(V(n,\lambda), J(n,\lambda), K(n,\lambda))] \geq \mathbb{E}\bigl[f\bigl(X_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)}, \underline{X}_{\mathbf{g}(n,\lambda)}\bigr)\bigr], \tag{16}
$$

$$
\mathbb{E}[f(V(n,\lambda), \tilde{K}(n,\lambda), \tilde{J}(n,\lambda))] \leq \mathbb{E}\bigl[f\bigl(X_{\mathbf{g}(n,\lambda)}, \underline{X}_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)}\bigr)\bigr]. \tag{17}
$$

PROOF. From Theorem 1, we know that $(V(n,\lambda), J(n,\lambda))$ has the same distribution as $(X_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)})$, and, for each $n \geq 1$, $K(n,\lambda) = \min\{X_{\mathbf{g}(k,\lambda)} : k = 0,1,\ldots,n\} \geq \underline{X}_{\mathbf{g}(n,\lambda)}$. The inequality in (16) now follows. The equality in (17) is the result of a similar argument where now, for each $n \geq 1$, $\tilde{K}(n,\lambda) = \underline{X}_{\mathbf{g}(n,\lambda)}$ and $\tilde{J}(n,\lambda) = \max\{X_{\mathbf{g}(k,\lambda)} : k = 0,1,\ldots,n\} \leq \overline{X}_{\mathbf{g}(n,\lambda)}$. □

Theorem 5 can be understood in the following sense. Both triples of random variables $(V(n,\lambda), J(n,\lambda), K(n,\lambda))$ and $(V(n,\lambda), \tilde{J}(n,\lambda), \tilde{K}(n,\lambda))$ can be considered as estimates for $(X_{\mathbf{g}(n,\lambda)}, \overline{X}_{\mathbf{g}(n,\lambda)}, \underline{X}_{\mathbf{g}(n,\lambda)})$, where in the first case $K(n,\lambda)$ has a positive bias and in the second case $\tilde{J}(n,\lambda)$ has a negative bias. An example of this is handled in the next section.

5. Numerical results. In this section, we present numerical results. We perform computations for a process X_t in the β-family with parameters

$$
(a,\sigma,\alpha_1,\beta_1,\lambda_1,c_1,\alpha_2,\beta_2,\lambda_2,c_2) = (a,\sigma,1,1.5,1.5,1,1,1.5,1.5,1),
$$

where the linear drift a is chosen such that $\Psi(-\mathrm{i}) = -r$ with $r = 0.05$, for no other reason that this is a risk neutral setting which makes the process $\{\exp(X_t - rt) : t \geq 0\}$ a martingale. We are interested in two parameter sets. Set 1 has $\sigma = 0.4$ and Set 2 has $\sigma = 0$. Note that both parameter sets give us proceses with jumps of infinite activity but of bounded variation, but due to the presence of Gaussian component the process X_t has unbounded variation in the case of parameter Set 1.

As the first example, we compare computations of the joint density of $(\overline{X}_1, \overline{X}_1 - X_1)$ for the parameter Set 1. Our first method is based on the following Fourier inversion technique. As in the proof of Theorem 1, we use the fact that $\overline{X}_{\mathbf{e}_1/\lambda}$ and

$X_{\mathbf{e}_1/\lambda} - \overline{X}_{\mathbf{e}_1/\lambda}$ are independent, and the latter is equal in distribution to $\underline{X}_{\mathbf{e}_1/\lambda}$, to write

$$\begin{aligned}\mathbb{P}(\overline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}x)\mathbb{P}(-\underline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}y) &= \mathbb{P}(\overline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}x, \overline{X}_{\mathbf{e}_1/\lambda} - X_{\mathbf{e}_1/\lambda} \in \mathrm{d}y)\\ &= \lambda \int_{\mathbb{R}^+} e^{-\lambda t}\mathbb{P}(\overline{X}_t \in \mathrm{d}x, \overline{X}_t - X_t \in \mathrm{d}y)\,\mathrm{d}t.\end{aligned}$$

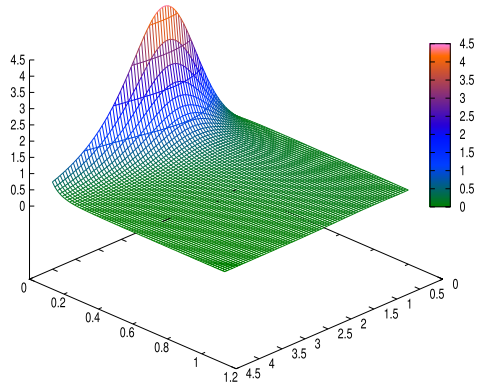
Writing down the inverse Laplace transform, we obtain

$$\begin{aligned}&\mathbb{P}(\overline{X}_t \in \mathrm{d}x, \overline{X}_t - X_t \in \mathrm{d}y)\\ &\quad = \frac{1}{2\pi \mathrm{i}} \int_{\lambda_0 + \mathrm{i}\mathbb{R}} \mathbb{P}(\overline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}x)\mathbb{P}(-\underline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}y)\lambda^{-1}e^{\lambda t}\,\mathrm{d}\lambda,\end{aligned} \tag{18}$$

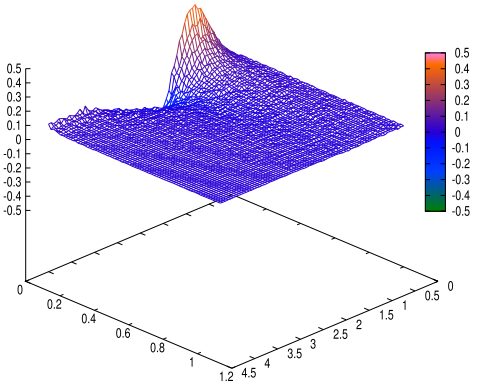
where λ_0 is any positive number. The values of analytical continuation of $\mathbb{P}(\overline{X}_{\mathbf{e}_1/\lambda} \in \mathrm{d}x)$ for complex values of λ can be computed efficiently using technique described in [13]. Our numerical results indicate that the integral in (18) can be computed very precisely, provided that we use a large number of discretization points in λ space coupled with Filon-type method to compute this Fourier type integral. Thus, first we compute the joint density of $(\overline{X}_1, \overline{X}_1 - X_1)$ using (18) and take it as a benchmark, which we use later to compare the Wiener–Hopf Monte Carlo method and the classical Monte Carlo approach. For both of these methods, we fix the number of simulations $M = 10^7$ and the number of time steps $N \in \{20, 50, 100\}$. For fair comparison, we use $2N$ time steps for the classical Monte Carlo, as Wiener–Hopf Monte Carlo method with N time steps requires simulation of $2N$ random variables $\{S_\lambda^{(j)}, I_\lambda^{(j)} : j = 1, 2, \ldots, N\}$. All the code was written in Fortran and the computations were performed on a standard laptop (Intel Core 2 Duo 2.5 GHz processor and 3 GB of RAM).

Figure 1 presents the results of our computations. In Figure 1(a), we show our benchmark, a surface plot of the joint probability density function of $(\overline{X}_1, \overline{X}_1 - X_1)$ produced using Fourier method (18), which takes around 40–60 seconds to compute. Figure 1(b)–(d) show the difference between the benchmark and the Wiener–Hopf Monte Carlo result as the number of time steps N increases from 20 to 50 to 100. The computations take around 7 seconds for $N = 100$, and 99% of this time is actually spent performing the Monte Carlo algorithm, as the precomputations of the roots $\zeta_n^{\pm}$ and the law of I_λ, S_λ take less than one tenth of a second. Figure 1(e) shows the result produced by the classical Monte Carlo method with $N = 100$ (which translates into 200 random walk steps according to our previous convention); this computation takes around 10–15 seconds since here we also need to compute the law of $X_{1/N}$, which is done using inverse Fourier transform of the characteristic function of X_t given in (1). Finally, Figure 1(f) shows the difference between the Monte Carlo result and our benchmark.

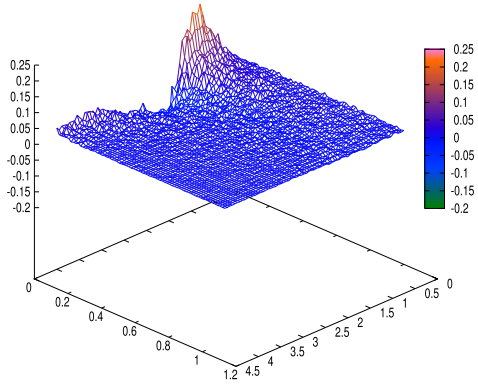
The results illustrate that in this particular example the Wiener–Hopf Monte Carlo technique is superior to the classical Monte Carlo approach. It gives a much more precise result, it requires less computational time, is more straightforward to



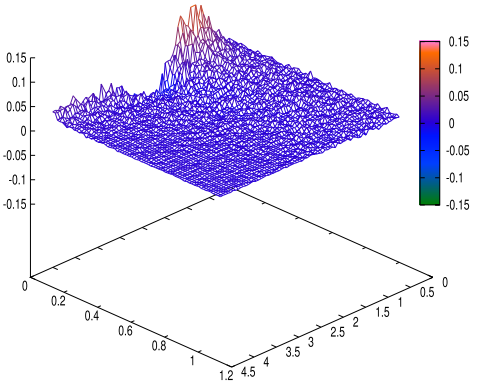
(a) Fourier method benchmark



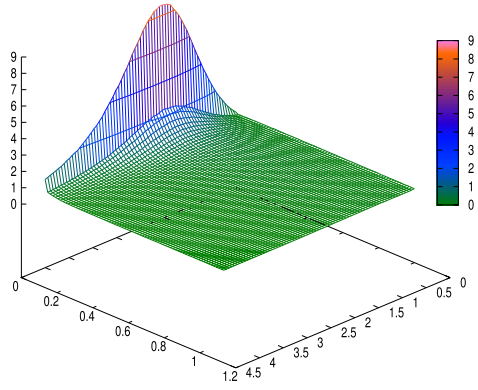
(b) $N = 20$, WH-MC error



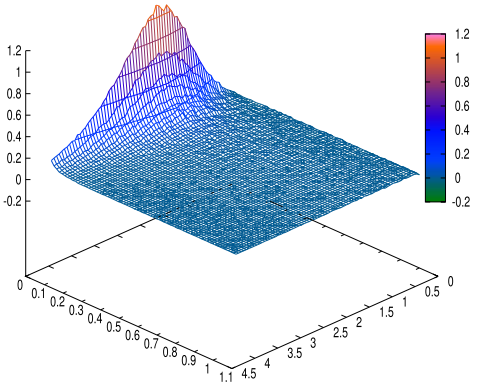
(c) $N = 50$, WH-MC error



(d) $N = 100$, WH-MC error



(e) $N = 100$, MC



(f) $N = 100$, MC error

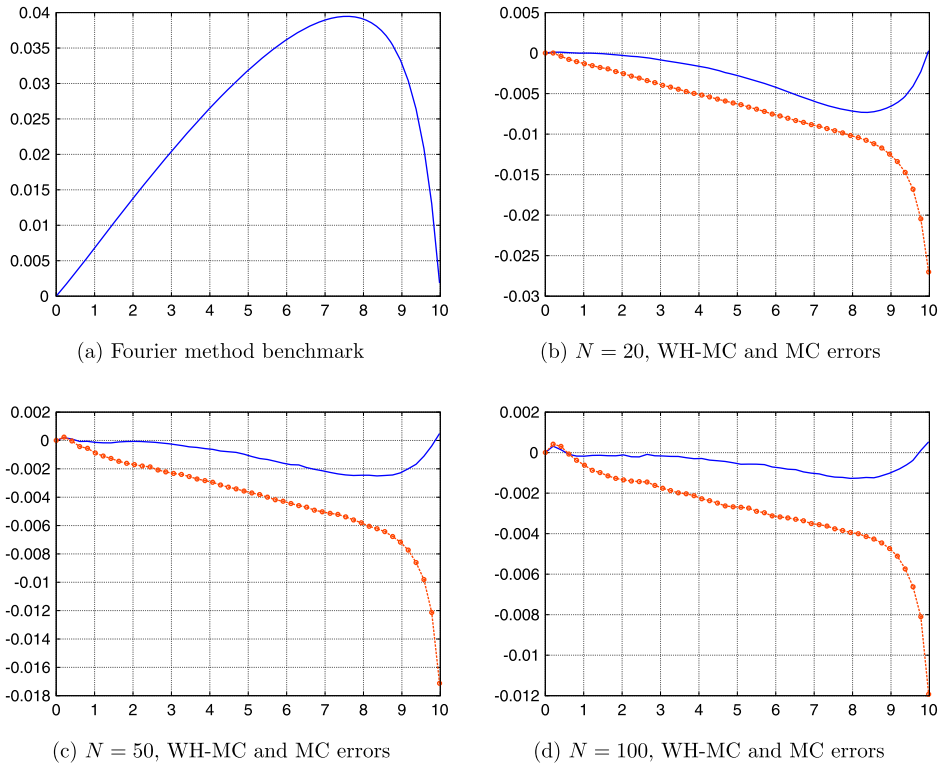
FIG. 1. *Computing the joint density of* $(\overline{X}_1, \overline{X}_1 - X_1)$ *for parameter Set* 1. *Here* $\overline{X}_1 \in [0, 1]$ *and* $\overline{X}_1 - X_1 \in [0, 4]$.

programme and does not suffer from some the issues that plague the Monte Carlo approach, such as the atom in distribution of $\overline{X}_1$ at zero, which is clearly visible in Figure 1(e).

Next, we consider the problem of pricing up-and-out barrier call option with maturity equal to one, which is equivalent to computing the following expectation:

$$\pi^{\mathrm{uo}}(s) = e^{-r}\mathbb{E}\big[(se^{X_1} - K)^+ \mathbf{1}_{\{s\exp(\overline{X}_1)<b\}}\big]. \tag{19}$$

Here $s \in [0, b]$ is the initial stock price. We fix the strike price $K = 5$, the barrier level $b = 10$. The numerical results for parameter Set 1 are presented in Figure 2. Figure 2(a) shows the graph of $\pi^{\mathrm{uo}}(s)$ as a function of s produced with Fourier method similar to (18), which we again use as a benchmark. Figure 2(b)–(d) show the difference between the benchmark and results produced by Wiener–Hopf Monte Carlo (blue solid line) and classical Monte Carlo (red line with circles) for $N \in \{20, 50, 100\}$. Again we see that Wiener–Hopf Monte Carlo method gives a better accuracy, especially when the initial stock price



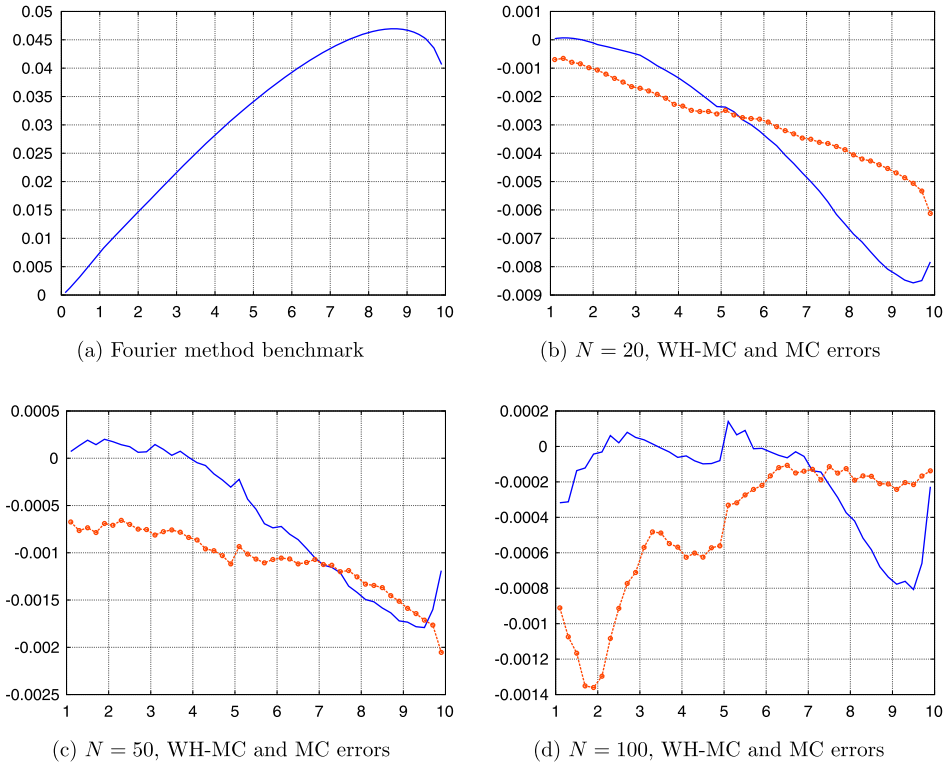
(a) Fourier method benchmark

(b) $N = 20$, WH-MC and MC errors

(c) $N = 50$, WH-MC and MC errors

(d) $N = 100$, WH-MC and MC errors

FIG. 2. *Computing the price of up-and-out barrier option for parameter Set* 1. *In figures* (b)–(d) *the graph of WH-MC error is solid line, the graph of MC error is line with circles.*



(a) Fourier method benchmark

(b) $N = 20$, WH-MC and MC errors

(c) $N = 50$, WH-MC and MC errors

(d) $N = 100$, WH-MC and MC errors

FIG. 3. *Computing the price of up-and-out barrier option for parameter Set* 2. *In figures* (b)–(d) *the graph of WH-MC error is solid line, the graph of MC error is line with circles.*

level s is close to the barrier b, as in this case the Monte Carlo approach produces an artificial atom in the distribution of $\overline{X}_1$ at zero which creates a large error.

Figure 3 shows corresponding numerical results for parameter Set 2. In this case, we have an interesting phenomenon of a discontinuity in $\pi^{\mathrm{uo}}(s)$ at the boundary b. The discontinuity should be there and occurs due to the fact that, for those particular parameter choices, there is irregularity of the upper half line. Irregularity of the upper half line is equivalent to there being an atom at zero in the distribution of $\overline{X}_t$ for any $t > 0$ (also at independent and exponentially distributed random times). We see from the results presented in Figures 2 and 3 that Wiener–Hopf Monte Carlo method correctly captures this phenomenon; the atom at zero is produced if and only if the upper half line is irregular, while the classical Monte Carlo approach always generates an atom. Also, analyzing Figure 3(b)–(d), we see that in this case classical Monte Carlo algorithm is also doing a good job and it is hard to find a winner. This is not surprising, as in the case of parameter Set 2 the process X_t has bounded variation, thus the bias produced in monitoring for supremum only at discrete times is smaller than in the case of process of unbounded variation.

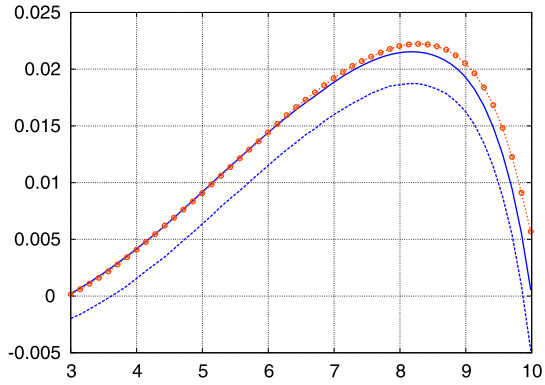


FIG. 4. *Computing the price of the double no-touch barrier option for parameter Set* 1. *The solid lines represent the upper/lower bounds produced by WH-MC method, the line with circles represents the MC result.*

Finally, we give an example of how one can use Theorem 5 to produce upper/lower bounds for the price of the double no-touch barrier call option

$$\pi^{\mathrm{dnt}}(s) = e^{-r}\mathbb{E}\big[(se^{X_1} - K)^+ \mathbf{1}_{\{s\exp(\overline{X}_1)<\overline{b};\, s\exp(\underline{X}_1)>\underline{b}\}}\big]. \tag{20}$$

First, we use identity $\mathbf{1}_{\{s\exp(\underline{X}_1)>\underline{b}\}} = 1 - \mathbf{1}_{\{s\exp(\underline{X}_1)<\underline{b}\}}$ and obtain

$$\pi^{\mathrm{dnt}}(s) = \pi^{\mathrm{uo}}(s) - e^{-r}\mathbb{E}\big[(se^{X_1} - K)^+ \mathbf{1}_{\{s\exp(\overline{X}_1)<\overline{b};\, s\exp(\underline{X}_1)<\underline{b}\}}\big].$$

Function $f(x, y, z) = -(se^x - K)^+ \mathbf{1}_{\{s\exp(y)<\overline{b};\, s\exp(z)<\underline{b}\}}$ is increasing in both variables y and z, thus using Theorem 5 we find that

$$\pi_1^{\mathrm{dnt}}(s) = \pi^{\mathrm{uo}}(s) - e^{-r}\mathbb{E}\big[\big(se^{V(n,n)} - K\big)^+ \mathbf{1}_{\{s\exp(\tilde{J}(n,n))<\overline{b};\, s\exp(\tilde{K}(n,n))<\underline{b}\}}\big],$$

$$\pi_2^{\mathrm{dnt}}(s) = \pi^{\mathrm{uo}}(s) - e^{-r}\mathbb{E}\big[\big(se^{V(n,n)} - K\big)^+ \mathbf{1}_{\{s\exp(J(n,n))<\overline{b};\, s\exp(K(n,n))<\underline{b}\}}\big]$$

are the lower/upper bounds for $\pi^{\mathrm{dnt}}(s)$. Figure 4 illustrates this algorithm for parameter Set 1, the other parameters being fixed at $K = 5$, $\underline{b} = 3$, $\overline{b} = 10$ and the number of time steps $N = 200$ (400 for the classical Monte Carlo). We see that the Monte Carlo approach gives a price which is almost always larger than the upper bound produced by the Wiener–Hopf Monte Carlo algorithm. This is not surprising, as in the case of Monte Carlo approach we would have positive (negative) bias in the estimate of infimum (supremum), and given that the payoff of the double no-touch barrier option is increasing in infimum and decreasing in supremum this amplifies the bias.

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REFERENCES

[1] ALILI, L. and KYPRIANOU, A. E. (2005). Some remarks on first passage of Lévy processes, the American put and pasting principles. *Ann. Appl. Probab.* **15** 2062–2080. MR2152253

[2] BERTOIN, J. (1996). *Lévy Processes*. *Cambridge Tracts in Mathematics* **121**. Cambridge Univ. Press, Cambridge. MR1406564

[3] BERTOIN, J. (1997). Regularity of the half-line for Lévy processes. *Bull. Sci. Math.* **121** 345–354. MR1465812

[4] BOYARCHENKO, S. I. and LEVENDORSKII, S. Z. (2002). *Non-Gaussian Merton–Black–Scholes Theory*. *Advanced Series on Statistical Science & Applied Probability* **9**. World Scientific, River Edge, NJ. MR1904936

[5] CABALLERO, M. E., PARDO, J. C. and PÉREZ, J. L. (2011). Explicit identities for Lévy processes associated to symmetric stable processes. *Bernoulli* **17** 34–59.

[6] CABALLERO, M. E., PARDO, J. C. and PÉREZ, J. L. (2010). On the Lamperti stable processes. *Probab. Math. Statist.* **30** 1–28.

[7] CARR, P. (1998). Randomization and the American Put. *Rev. Fin. Studies* **11** 597–626.

[8] CONT, R. and TANKOV, P. (2004). *Financial Modelling with Jump Processes*. Chapman & Hall/CRC, Boca Raton, FL. MR2042661

[9] DONEY, R. A. (2004). Stochastic bounds for Lévy processes. *Ann. Probab.* **32** 1545–1552. MR2060308

[10] HUBALEK, F. and KYPRIANOU, A. E. (2010). Old and new examples of scale functions for spectrally negative Lévy processes. In *Sixth Seminar on Stochastic Analysis, Random Fields and Applications* (R. Dalang, M. Dozzi and F. Russo, eds.). *Progress in Probability* **63** 119–145. Birkhäuser, Basel.

[11] JEFFREY, A., ed. (2007). *Table of Integrals*, *Series*, *and Products*, 7th ed. Academic Press, Amsterdam.

[12] KLÜPPELBERG, C., KYPRIANOU, A. E. and MALLER, R. A. (2004). Ruin probabilities and overshoots for general Lévy insurance risk processes. *Ann. Appl. Probab.* **14** 1766–1801. MR2099651

[13] KUZNETSOV, A. (2010). Wiener–Hopf factorization and distribution of extrema for a family of Lévy processes. *Ann. Appl. Probab.* **20** 1801–1830. MR2724421

[14] KYPRIANOU, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer, Berlin. MR2250061

[15] KYPRIANOU, A. E. and LOEFFEN, R. (2005). Lévy processes in finance distinguished by their coarse and fine path properties. In *Exotic Option Pricing and Advanced Lévy Models* (A. E. Kyprianou, W. Schoutens and P. Willmott, eds.) 1–28. Wiley, Chichester. MR2343206

[16] KYPRIANOU, A. E., PARDO, J. C. and RIVERO, V. (2010). Exact and asymptotic n-tuple laws at first and last passage. *Ann. Appl. Probab.* **20** 522–564. MR2650041

[17] KYPRIANOU, A. E. and RIVERO, V. (2008). Special, conjugate and complete scale functions for spectrally negative Lévy processes. *Electron. J. Probab.* **13** 1672–1701. MR2448127

[18] MADAN, D. and SENETA, E. (1990). The variance gamma (V.G.) model for share market returns. *Journal of Business* **63** 511–524.

[19] SCHOUTENS, W. (2003). *Lévy Processes in Finance*: *Pricing Financial Derivatives*. Wiley, Chichester.

[20] SCHOUTENS, W. and CARIBONI, J. (2009). *Lévy Processes in Credit Risk*. Wiley, Chichester.

[21] SONG, R. and VONDRAČEK, Z. (2008). On suprema of Lévy processes and application in risk theory. *Ann. Inst. H. Poincaré Probab. Statist.* **44** 977–986. MR2453779

[22] VIGON, V. (2002). Simplifiez vos Lévy en titillant la factorisation de Wiener–Hopf. Thèse. Laboratoire de Mathématiques de L'INSA de Rouen.

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