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A winding problem for a resonator driven by a white noise

By

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1. Introduction.

Given a standard 1-dimensional Brownian motion with sample paths $t \to e(t)$ ($e(0) \equiv 0$), let $P_{ab}(B)$ be the chance that the solution $\mathfrak{x}: t \to (u, v) \in R^2$ of

1a. $\mathrm{D}[u] \equiv \ddot{u} + c_1(u)\dot{u} + c_2(u) = \dot{e}$

1b. $v = \dot{u}$

2a. $u(0) = a$

2b. $v(0) = b$

experiences the event B, interpreting 1a as $v + \int_0^t [c_1(u)v + c_2(u)]ds = b + e$. $[\mathfrak{x}, P.]$ is a (singular) diffusion in the plane winding clockwise about the origin, governed by

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial b^2} + b\frac{\partial p}{\partial a} - [c_1(a)b + c_2(a)]\frac{\partial p}{\partial b};$$

it should be viewed as *the response of the resonator* **D** *to the white noise* $\dot{e}$.

J. Potter [5] found that for a spring ($uc_2 \geq 0$) with no damping ($c_1 \equiv 0$), the *energy* $\mathfrak{e} = (1/2)v^2 + \int_0^u c_2$ is a martingale and used this fact to obtain the bounds

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$$c_1 t / lg_2 t < \max_{s \leq t} \mathfrak{e}(s) < c_2 t \, lg_2 t \qquad (t \uparrow \infty)$$
$$c_1 > 0, \ c_2 > 1, \ lg_2 t \equiv lg(lg\, t).$$

Potter also proved that the sample path hits each disc *i.o.* $(t \uparrow \infty)$ if $\int_0^\infty \left(1 + \int_0^u c_2\right)^{-1/2} du < \infty$.

M. Kac [4] studied the damped spring $\mathbf{D}[u] = \ddot{u} + c_1 \dot{u} + c_2 u$ $(0 < c_1, c_2 = \text{constant})$: in that case $[\mathfrak{x}, P.]$ is Gaussian having a stable distribution $p(da \times db)$ of total mass $+1$, and letting E denote the integral (expectation) based on $P = \int p(da \times db) P_{ab}$ and $\mathfrak{t}_1$ the time between roots of $u = 0$, the total angle $\theta = \theta(t)$ swept out between times 1 and $t > 1$ is found to be about $2\pi t / E(\mathfrak{t}_1)$ $(t \uparrow \infty)$. S. O. Rice [6] had evaluated $E(\mathfrak{t}_1)$ and now Kac finds a minimum principle for $E(\mathfrak{t}_1^2)$ similar to Thompson's principle for Newtonian electrostatic capacities ; the actual distribution of $\mathfrak{t}_1$ is still unknown.

The purpose of the present note is to give a complete description of the winding of the phase path about the origin in the simplest case $(c_1 = c_2 \equiv 0)$; the joint distribution of the $1/2$ winding time $\mathfrak{t}_1 = \min(t : t > 0, u(t) = 0)$ and the hitting place $\mathfrak{h}_1 = |v(\mathfrak{t}_1)|$ is evaluated for paths starting on the line $a = 0$, and the following strong laws for the speed of winding are established :

$$P_{ab}[\lim_{t \uparrow \infty} (lg\, t)^{-1} \theta(t) = -\sqrt{3}/8] = 1$$
$$P_{00}[\lim_{t \downarrow 0} (lg\, 1/t)^{-1} \theta(t) = +\sqrt{3}/8] = 1.$$

2. Winding times and hitting places $(c_1 = c_2 \equiv 0)$.

Before it is possible to talk about winding about $\mathfrak{x} = 0$, it must be proved that the sample path does not hit $\mathfrak{x} = 0$ at positive times.

$\mathbf{D}[u] = \ddot{u}$ implies $v = b + \int_0^t \mathfrak{e} ds$, so $\mathfrak{x}$ is Gaussian and it is a simple matter to evaluate the probabilities

1. $P_{ab}[u(t) \in d\xi, v(t) \in d\eta] \equiv p(t, a, b, \xi, \eta) d\xi d\eta$
$$= (\sqrt{3}/\pi t^2) \exp\left[-\frac{(\xi - a - bt)^2}{t^3/6} + \frac{(\xi - a - bt)(\eta - b)}{t^2/6} - \frac{(\eta - b)^2}{t/2} \right] d\xi d\eta$$

of coming from $\mathfrak{x}(0)=(a, b)$ into $d\xi\times d\eta$ in time t and to check the that the Green function

2. $$G(a, b)\equiv\int_0^\infty p(t, a, b, 0, 0)dt$$
$$=\int_0^\infty \frac{\sqrt{3}}{\pi t^2}\exp\left(-\frac{(a+bt)^2}{t^3/6}+\frac{(a+bt)b}{t^2/6}-\frac{b^2}{t/2}\right)dt$$

has the following properties:

3a. $G<\infty \qquad a^2+b^2>0$

3b. $\lim_{a^2+b^2\downarrow 0} G=\infty$.

$G(u, v, 0, 0)$ is now a continuous supermartingale, its sample paths are bounded on bounded time intervals if $\mathfrak{x}(0)\neq 0$, and the result follows from the fact that $P_{ab}(\mathfrak{x}(t)=0)=0$ at each positive time.

Given a sample path $\mathfrak{x}$ starting at $\mathfrak{x}(0)=(a, b)\neq 0$, the 1/2 *winding time* $\mathfrak{t}_1\equiv(t: t>0,\ u(t)=0)$ statisfies $P_{ab}(0<\mathfrak{t}_1<\infty)=1$.

$P_{ab}(0<\mathfrak{t}_1)=1$ is immediate.

$\mathfrak{t}_1=\infty$ implies that $\mathfrak{x}$ moves in a 1/2 plane for all positive times, or, what is the same, that $\int_0^t e ds$ is bounded above or below for all positive times. But such Brownian (tail) events have probabilities 0 or 1 and so the obvious bound

$$P_{ab}(\mathfrak{t}_1=\infty)\leq\lim_{d\uparrow\infty}\lim_{t\uparrow\infty}P\left(\int_0^t e ds< d\right)=1/2$$

implies the desired $P_{ab}(\mathfrak{t}_1<\infty)=1$.

Consider now the 1/2 winding time $\mathfrak{t}_1$ and the corresponding *hitting place* $\mathfrak{h}_1\equiv|v(\mathfrak{t}_1)|>0$ for sample paths $\mathfrak{x}$ starting on the line $a=0$ $(v(0)=b\neq 0)$. Because the Brownian scaling $e\to ce(t/c^2)$ $(c>0)$ takes e into a new standard Brownian motion, the 1/2 winding time $\mathfrak{t}_1=\min(t:\left(t>0,\ bt+\int_0^t e ds=0\right)$ is identical in law to

$$\min\left((t: t>0,\ bt+\int_0^t ce(s/c^2)ds=0\right)$$
$$=\min\left((t: t>0,\ c^2bt/c^2+c^3\int_0^{t/c^2} e(s)ds=0\right)$$
$$=c^2\min\left((t: t>0,\ c^2b+c^3\int_0^t e ds=0\right)$$
$$=b^2\min\left((t: t>0,\ \pm 1+\int_0^t e ds=0\right)\qquad c\equiv|b|,$$

i.e., $\mathfrak{t}_1$ *is identical in law to* $b^2 \times$ *the* 1/2 *winding time for paths starting at* (0, 1), and the same trick applied to $v=b+e$ verifies that *the hitting place* $\mathfrak{h}_1$ *for paths starting at* (0, b) *is identical in law to* $b \times$ *the hitting place for paths starting at* (0, 1), indeed, since the motion starts afresh at its passage time to the line $a=0$, it follows that the series of 1/2 winding times and hitting places

4a. $\mathfrak{t}_n = \min(t : t > t_{n-1}, u(t) = 0) - \mathfrak{t}_0 \qquad n \geq 1,$

$\mathfrak{t}_0 \equiv \min(t : t \geq 0, u(t) = 0)$

4b. $\mathfrak{h}_n = |v(\mathfrak{t}_n)| \qquad n \geq 1$

for paths starting at $\mathfrak{x}(0) = (a, b) \neq 0$ is identical in law to the series

5a. $c^2 t_1, c^2(t_1 + h_1^2 t_2), c^2(t_1 + h_1^2 t_2 + (h_1 h_2)^2 t_3)$, *etc.*

5b. $ch_1, ch_1 h_2, ch_1 h_2 h_3$, *etc.*,

in which $c \equiv |v(\mathfrak{t}_0)|$ and the pairs (t_1, h_1), (t_2, h_2), *etc.* are independent with common distribution $P_{01}(\mathfrak{t}_1 < t, \mathfrak{h}_1 < h)$.

3. Computing the joint distribution $P_{01}(\mathfrak{t}_1 < t, \mathfrak{h}_1 < h)$.

Because $\mathfrak{x}$ winds clockwise about the origin and begins afresh at the 1/2 winding time $\mathfrak{t}_1$, the Gauss function p of 2. 1[2] satisfies

1. $p(t, 0, 1, 0, b)$

$$= \int_0^t \int_0^\infty P_{01}(\mathfrak{t}_1 \in ds, h_1 \in da) p(t-s, 0, -a, 0, b)$$

$t > 0, \ b > 0,$

and, using the Laplace transform

2. $$\int_0^\infty e^{-\alpha t} p(t, 0, a, 0, b) dt$$

$$= \textit{a constant depending on } \alpha \textit{ alone} \times \frac{K_{-1}(\sqrt{8\alpha(a^2+ab+b^2)})}{\sqrt{a^2+ab+b^2}}$$

$\alpha, a, b > 0,$[3]

1 becomes

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[2] $n \cdot m$ means formula m of section n.
[3] [1(2) : 146(29)]. K_{-1} is the usual modified Bessel function.

3a. $$\frac{K_{-1}(\sqrt{8\alpha(1+b+b^2)})}{\sqrt{1+b+b^2}} = \int_0^\infty E_{01}(e^{-\alpha t_1}, \mathfrak{h}_1 \in da)\frac{K_{-1}(\sqrt{8\alpha(a^2-ab+b^2)})}{\sqrt{a^2-ab+b^2}}$$

$\alpha > 0,\ b > 0.$[4]

3a is now multiplied by $K_\gamma(\sqrt{8\alpha}\ \ b)$ ($|\gamma| < 1$) and integrated (db) over $[0, +\infty)$: the result is

3b. $$\frac{K_\gamma(\sqrt{8\alpha})}{2\cos(\pi\gamma/3)} = \int_0^\infty E_{01}(e^{-\alpha t_1}, \mathfrak{h}_1 \in da) K_\gamma(\sqrt{8\alpha}\ \ a)/a \qquad |\gamma| > 1,$$ [5]

and now using the Lebedev transform pair[6]

4a. $$\hat{f}(\gamma) = \int_0^\infty f(a) K_{i\gamma}(a)\frac{da}{a}$$

4b. $$f(a) = \int_0^\infty \hat{f}(\gamma) K_{i\gamma}(a)do \qquad do \equiv 2\pi^{-2}\gamma \sinh \pi\gamma\, d\gamma,$$

3b is solved to obtain

5. $$E_{01}(e^{-\alpha t_1}, \mathfrak{h}_1 \in da) = \int_0^\infty \frac{K_{i\gamma}(\sqrt{8\alpha})K_{i\gamma}(\sqrt{8\alpha}\ \ a)}{2\cosh(\pi\gamma/3)}\, do\, da,$$

which in turn can be inverted as a Laplace transform to obtain the joint distribution of t_1 and $\mathfrak{h}_1$:

6. $$P_{01}(t_1 \in dt, \mathfrak{h}_1 \in da) = \frac{1}{2t}e^{-2(1+a^2)/t}\int_0^\infty \frac{K_{i\gamma}(4a/t)}{2\cosh(\pi\gamma/3)}\, do\, dt\, da = \frac{3a}{\pi\sqrt{2}\,t^2}e^{-2/t(1-a+a^2)}\int_0^{4a/t}\frac{e^{-3\theta/2}}{\sqrt{\pi\theta}}\,d\theta.$$ [7]

6 can be integrated to obtain

7. $$P_{01}(\mathfrak{h}_1 \in dh) = \frac{3}{2\pi}\frac{h^{3/2}}{1+h^3}dh$$

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[4] E_{01} is the integral (expectation) based on P_{01}.

[5] [1(2) : 377(34)].

[6] [1(2) : 173].

[7] [1(1) : 285(64)] justifies line 2, while line 3 follows from the classical formula $K_{i\gamma}(a) = \int_0^\infty \exp(-a\cosh t)\cos \gamma t\, dt$.

and

8a. $E_{01}(lg\,\mathfrak{h}_1) = \frac{3}{2\pi}\int_0^\infty lg\,h \frac{h^{3/2}}{1+h^3}dh = \frac{4\pi}{\sqrt{3}}$

8b. $E_{01}[(lg\,\mathfrak{t}_1)^2] < \infty$.

8 is needed below. I could not perform the integrals needed to find $P_{01}(\mathfrak{t}_1 \in dt)$.

4. Speed of winding.

Given $a^2+b^2>0$ and using 2. 5b, 3. 8a, the strong law of large numbers, and the fact that $\mathfrak{x}$ starts afresh each time it hits the line $a=0$, one finds

1. $P_{ab}[\lim_{n\uparrow\infty} n^{-1}\,lg\,\mathfrak{h}_n = 4\pi/\sqrt{3}] = 1$.

Recall the series 2. 5a and the bound 3. 8b. Because t_1, t_2, etc. are independent with common distribution $P_{01}(\mathfrak{t}_1<t)$, it follows from the Borel-Cantelli lemma that $|lg\,t_n|<n\delta$ as $n\uparrow\infty$ $(\delta>0)$, and this bound applied to 2. 5a implies that as $n\uparrow\infty$, $n^{-1}\,lg\,\mathfrak{t}_n$ behaves like $n^{-1}\,lg\,h_1^2h_2^2\cdots h_{n-1}^2$, whence

2. $P_{ab}[\lim_{n\uparrow\infty} n^{-1}\,lg\,\mathfrak{t}_n = 8\pi/\sqrt{3}] = 1$.

2 in turn implies that if $\theta=\theta(t)$ is the total algebraic angle swept out up to time t, then

3. $P_{ab}[\lim_{t\uparrow\infty} (lg\,t)^{-1}\,\theta(t) = -\sqrt{3}/8] = 1$

since $\mathfrak{t}_{n-1}\le t<\mathfrak{t}_n$ is the same as $-(n-1)\pi\ge\theta-\theta(\mathfrak{t}_0)>-n\pi$ and $lg\,\mathfrak{t}_n\sim 8\pi n/\sqrt{3}$ as $n\uparrow\infty$.

5. Winding for paths beginning at $\mathfrak{x}=0$.

Given a sample path beginning at $\mathfrak{x}(0)=0$, it follows from 3. 6, the scaling established in 2, and the starting afresh of $\mathfrak{x}$ at passage times that the *forward chain*:

1. $\mathfrak{t}_1^+ = \min(t:\, t>1,\, u(t)=0)$, $\quad \mathfrak{h}_1^+ = |v(\mathfrak{t}_1^+)|$
   $\mathfrak{t}_2^+ = \min(t:\, t>\mathfrak{t}_1^+,\, u(t)=0)$, $\quad \mathfrak{h}_2^+ = |v(\mathfrak{t}_2^+)|$
   etc.

of 1/2 winding times and hitting places is Markovian with transition probabilities

2. $P_{00}(\mathfrak{t}_n^+ \in dt,\ \mathfrak{h}_n^+ \in dh \mid \mathbf{B}_{n-1}^+)$
$$\equiv p^+(\mathfrak{t}_{n-1}^+,\ \mathfrak{h}_{n-1}^+,\ dt\times dh)$$
$$= \frac{3h}{\pi\sqrt{2}\,(t-\mathfrak{t}_{n-1}^+)^2}\exp\left(-2(h^2-h\mathfrak{h}_{n-1}^+ + \mathfrak{h}_{n-1}^{+2})/(t-\mathfrak{t}_{n-1}^+)\right)$$
$$\times\int_0^{4h\mathfrak{h}_{n-1}^+/(t-\mathfrak{t}_{n-1}^+)} \frac{e^{-3\theta/2}}{\sqrt{\pi\theta}}d\theta\, dt\, dh \qquad (t\geq \mathfrak{t}_{n-1}^+)$$
$$= 0 \qquad (t< \mathfrak{t}_{n-1}^+),$$

where $\mathbf{B}_{n-1}^+ =$ *the field of* $\mathfrak{t}_1^+, \mathfrak{h}_1^+, \cdots, \mathfrak{t}_{n-1}^+, \mathfrak{h}_{n-1}^+$.

Consider now the *backward chain*:

3. $\mathfrak{t}_1^- = \max(t:\ t<1,\ u(t)=0), \qquad \mathfrak{h}_1^- = |v(\mathfrak{t}_-^1)|$
$\mathfrak{t}_2^- = \max((t:\ t<\mathfrak{t}_1^-,\ u(t)=0), \qquad \mathfrak{h}_2^- = |v(\mathfrak{t}_2^-)|$
etc.

of 1/2 winding times and hitting places as the path spirals back toward the origin as $t\downarrow 0$. Both $\mathfrak{t}_n^-$ and $\mathfrak{h}_n^-$ are positive and $\downarrow 0$ as $n\uparrow\infty$ as is evident from the fact that $\int_0^t e ds$ experiences an infinite number of changes of sign as $t\downarrow 0$, and taking advantage of the scaling properties of winding times and hitting places, a little computation reveals that the backward chain is Markovian with transition probabilities:

4. $P_{00}(\mathfrak{t}_n^- \in dt,\ \mathfrak{h}_n^- \in dh \mid \mathbf{B}_{n-1}^-)$
$$\equiv p^-(\mathfrak{t}_{n-1}^-,\ \mathfrak{h}_{n-1}^-,\ dt\times dh)$$
$$= \frac{p(dt\times dh)p^+(t,\ h,\ d\mathfrak{t}_{n-1}^-\times d\mathfrak{h}_{n-1}^-)}{p(d\mathfrak{t}_{n-1}^-\times d\mathfrak{h}_{n-1}^-)},$$

where $\mathbf{B}_{n-1}^- =$ *the field of* $\mathfrak{t}_1^-, \mathfrak{h}_1^-, \cdots, \mathfrak{t}_{n-1}^-, \mathfrak{h}_{n-1}^-$, and $p(dt\times dh)$ stands for the (infinite) stable mass distribution

5. $p(dt\times dh) = \exp(-2h^2/t)t^{-2}dt\, h\, dh$

for the forward chain. 4 states *that the backward chain has the same transition probabilities as the dual* $[\mathfrak{t}_{-n}, \mathfrak{h}_{-n} : n = \cdots, -1, 0,$ *etc.*$]$ *of the two-sided forward chain* $[\mathfrak{t}_{+n}, \mathfrak{h}_{+n} : n = \cdots, -1, 0,$ *etc.*$]$ *with stable distribution* $p(dt\times dh)$, *i.e., with* (*infinite*) *shift-invariant distribution*

6. $Q[\mathfrak{t}_m \in dt_m, \mathfrak{h}_m \in dh_m, \cdots, \mathfrak{t}_n \in dt_n, \mathfrak{h}_n \in dh_n]$
$= p(dt_m \times dh_m) p^+(t_m, h_m, dt_{m+1} \times dh_{m+1})$
$\cdots p^+(t_{n-1}, h_{n-1}, dt_n \times dh_n)$
$n, m = \cdots, -1, 0, \textit{etc.}, n < m$

(see G. Hunt [2] or [3] for such dual chains).

But now

7a. $Q[\mathfrak{h}_m \in dh_m, \cdots, \mathfrak{h}_n \in dh_n]$
$= \int_0^\infty (1/2h_m) dh_m P_{0h_m}(\mathfrak{h}_1 \in dh_{m+1}) \cdots P_{0h_{n-1}}(\mathfrak{h}_1 \in dh_n)$

and

7b. $P_{0a}(\mathfrak{h}_1 \in db) \equiv p^+(a, db) = \frac{3}{2\pi} \frac{(ab)^{3/2}}{a^3 + b^3} \frac{db}{a},$

so that the (Markovian) dual chain of hitting places $[\mathfrak{h}_{-n} : n = \cdots, -1, 0, \textit{etc.}, Q]$ has as its transition probabilities

8. $$\begin{aligned} p^-(b, da) &= \frac{a^{-1} da\, p^+(a, db)}{b^{-1} db} \\ &= \frac{3}{2\pi} \frac{(ba)^{3/2}}{a^3 + b^3} \frac{b\, da}{a^2} \\ &= \frac{3}{2\pi} \frac{(ab)^{-3/2}}{a^{-3} + b^{-3}} \frac{da^{-1}}{b^{-1}} \\ &= p^+(b^{-1}, da^{-1}), \end{aligned}$$

i.e., the dual hitting chain has the same transition probabilities as the reciprocal $[\mathfrak{h}_n^{-1} : n = \cdots, -1, 0, \textit{etc.}, Q]$ *of the original* (*Markovian*) *forward chain of hits*, and it follows that

9. $P_{00}[\min_{n \uparrow \infty} n^{-1} \lg \mathfrak{h}_n^- = -4\pi/\sqrt{3}] = 1.$

As to the 1/2 winding times $[\mathfrak{t}_n : n = \cdots, -1, 0, \textit{etc.}, Q]$, it is immediate that the pairs $t_n \equiv (\mathfrak{t}_n - \mathfrak{t}_{n-1})/\mathfrak{h}_{n-1}^2$ and $h_n \equiv \mathfrak{h}_n / \mathfrak{h}_{n-1}$ ($n = \cdots, -1, 0, \textit{etc.}$) are independent with common distribution 3. 6, so with the aid of the expression $\mathfrak{t}_n = \sum_{m \leq n} \mathfrak{h}_{m-1}^2 t_m$ ($n \leq 0$), the bound $|\lg t_n| < n^\delta$ ($n \uparrow \infty$) leads at once to the strong law

10. $P_{00}[\lim_{n \uparrow \infty} n^{-1} \lg \mathfrak{t}_n^- = -8\pi/\sqrt{3}] = 1$

for the backward chain of 1/2 winding times and to the strong law

11. $P_{00}[\lim_{t\downarrow 0} (lg\, 1/t)^{-1}\theta(t) = +\sqrt{3}/8] = 1$

for the total angle θ swept out between times 1 and $t<1$.

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Note added in proof: K. Itô (private communication) showed me the following rapid proof of the strong laws 4.3 and 5.11. Because $c^{-1/2}e(ct)$ $(t\geq 0)$ is a standard Brownian motion if $c>0$, the law of the pair

$$\mathfrak{x}^* = [u^*, v^*] : u^*(t) = t^{-3/2}\int_0^t e(s)ds,\ v^*(t) = t^{-1/2}e(t)$$

is unchanged by the substitution $t\to ct$, so the angle $\theta^*=\theta^*(t)$ swept out by $\mathfrak{x}^*$ between times 1 and t is identical in law to $\theta^*(ct)-\theta^*(c)$. But this means that the law of the functional $d\theta^*(e^t)/dt(\varphi) = -\int \theta^*(e^t)d\varphi$ is unchanged by an additive shift of the time scale, and it follows by the strong law of large numbers that

$$\lim_{t\uparrow\infty} t^{-1}\theta^*(e^t) = \lim_{t\uparrow\infty} (lg\, t)^{-1}\theta^*(t) = \mathit{constant},$$

using the fact that Brownian tail events are trivial. Also, $|\theta^*-\theta|\leq\pi/2$ so that $(lg\, t)^{-}\theta(t)$ tends to the same constant as $t\uparrow\infty$. A similar proof leads to 5.11.