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A ZERO-ONE LAW FOR PLANAR RANDOM WALKS IN RANDOM ENVIRONMENT

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We solve the problem posed by S. A. Kalikow whether the event that the x-coordinate of a random walk in a two-dimensional random environment approaches ∞ has necessarily probability either zero or one. The answer is *yes* if we assume the environment to be i.i.d. and in general *no* if we allow the environment to be just stationary and ergodic.

0. Introduction and results. The main purpose of this work is the derivation of a zero-one law for random walks $(X_n)_n$ in i.i.d. random environments on $\mathbb{Z}^2$. We show that for any fixed direction $\ell \in \mathbb{R}^2\backslash\{0\}$ the event that the inner product $X_n\ell$ tends to $+\infty$ as $n \to \infty$ has probability either 0 or 1, thus solving Problem 2 out of [2].

Let us first present the precise model. We assign to the lattice sites $z \in \mathbb{Z}^d$ $(d \geq 1)$ identically distributed $2d$-dimensional vectors $(\omega(z, z+e))_{|e|=1, e\in\mathbb{Z}^d}$ with a common distribution μ and with strictly positive components which add up to one. That is, we assume that μ is supported on the set $\mathscr{P}_+$ of $2d$-vectors $(p(e))_{|e|=1,e\in\mathbb{Z}^d}$ with $p(e) > 0$ and $\sum_e p(e) = 1$. The random variables $\omega(z, z+e)$ can then be realized as the canonical projections on the product space $\Omega := \mathscr{P}_+^{\mathbb{Z}^d}$ endowed with the canonical product σ-algebra and a suitable probability measure $\mathbb{P}$ with marginals μ. We want to stress that unlike [2] and [4] we do not assume the existence of a so-called ellipticity constant $\kappa > 0$ such that $\mathbb{P}$-a.s. $\omega(z, z+e) \geq \kappa$ for all $z \in \mathbb{Z}^d$, $|e| = 1$. We only do not allow $\omega(z, z+e)$ to be zero.

Given such an environment ω, the values $\omega(z, z+e)$ serve as transition probabilities for the Markov chain $(X_n)_{n\geq 0}$, called random walk in random environment (RWRE). This walk moves on $\mathbb{Z}^d$ and is for fixed starting point $x \in \mathbb{Z}^d$ defined on the sample space $(\mathbb{Z}^d)^{\mathbb{N}}$ endowed with the so-called quenched measure $P_{x,\omega}$ which satisfies

$$P_{x,\omega}[X_0 = x] = 1$$

and

$$P_{x,\omega}[X_{n+1} = X_n + e \mid X_0, X_1, \dots, X_n] = \omega(X_n, X_n + e), \qquad P_{x,\omega}\text{-a.s.}$$

for all $e \in \mathbb{Z}^d$ with $|e| = 1$ and all $n \geq 0$. The so-called annealed measures P_x, $x \in \mathbb{Z}^d$, are then defined as the semi-direct products $P_x := \mathbb{P} \times P_{x,\omega}$

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on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$. The corresponding expectations are denoted by $E_{x,\omega}$ and E_x, respectively.

We are interested in the event

$$A_\ell := \left\{ \lim_{n\to\infty} X_n \ell = \infty \right\} \qquad (\ell \in \mathbb{R}^d \setminus \{0\})$$

that the walker tends in a rough sense into direction ℓ.

Suppose that $\mathbb{P}$ is the product measure $\mathbb{P} := \otimes_{\mathbb{Z}^d} \mu$, which means that the vectors of transition probabilities $(\omega(z, z+e))_{|e|=1, e\in\mathbb{Z}^d}$ are i.i.d. under $\mathbb{P}$. Under the further assumption of the existence of some ellipticity constant $\kappa > 0$ as described above, Kalikow ([2] Theorem 3) has shown that the event that $X_n \ell$ remains of constant sign for large n has probability either 0 or 1. As shown in [4], Lemma 1.1, this implies that the zero-one law

$$P_0[A_\ell \cup A_{-\ell}] \in \{0, 1\} \tag{1}$$

holds. Kalikow's question (see [2], Problem 2) was whether $P_0[A_{e_1}] \in \{0, 1\}$ if $d = 2$ and $\mathbb{P}$ is a product measure. Here e_i $(i = 1, \dots, d)$ are the canonical unit vectors in $\mathbb{Z}^d$. This question is answered positively without the assumption of ellipticity by the following result, which holds for general directions ℓ.

THEOREM 1. *If $d = 2$ and if $(\omega(x, x+e))_{|e|=1}$, $x \in \mathbb{Z}^2$, are i.i.d. under $\mathbb{P}$, then for any $\ell \in \mathbb{R}^2 \setminus \{0\}$,*

$$P_0[A_\ell] \in \{0, 1\}. \tag{2}$$

The general problem, whether (2) holds in arbitrary dimensions $d \geq 2$, posed, for example, in [4], page 1855, remains unsolved in this paper. The assumption of independence for the environment is in a sense essential as is shown by the following result.

PROPOSITION 2. *For $d = 2$, there is a probability measure $\mathbb{P}$ on Ω which is stationary and ergodic with respect to shifts in $\mathbb{Z}^2$ such that*

$$P_0\left[\lim_{n\to\infty} \frac{X_n}{n} = \frac{e_1 + e_2}{2}\right] = \frac{1}{2} = P_0\left[\lim_{n\to\infty} \frac{X_n}{n} = -\frac{e_1 + e_2}{2}\right] \tag{3}$$

and in particular $P_0[A_{e_1}] = 1/2$.

A different example of a time-shift invariant event with non-trivial probability in random environment was found by Burton and Madras; see [1], pages 42 and 43.

Let us describe how the present article is organized. Section 1 deals with the general d-dimensional case ($d \geq 1$). First we extend (1) to the non-elliptic case, see Proposition 3. After some basic lemma we describe the main idea of the proof of Theorem 1. The first part of this idea is essentially a coupling argument, which we use in Section 2 in the case of two dimensions to prove Theorem 1. In the final section we construct some $\mathbb{P}$ satisfying Proposition 2.

Throughout the paper we denote by $c_1, c_2, \dots$ strictly positive constants which may depend only the dimension d, on $\mathbb{P}$ and on the (fixed) direction ℓ. If a constant is to depend on some other quantity, this will be made explicit.

1. Preliminaries for general dimension. In this section we allow $d \geq 1$. Let us start with some definitions. We denote by θ_n $(n \geq 0)$ the canonical shift on $(\mathbb{Z}^d)^{\mathbb{N}}$ by n time steps and by $\mathscr{F}_n$ $(n \geq 0)$ the canonical filtration of $X_\cdot$, that is, $\mathscr{F}_n := \sigma(X_0, \dots, X_n)$. For fixed $\ell \in \mathbb{R}^d \backslash \{0\}$ we consider the $(\mathscr{F}_n)$-stopping times

$$T_u := \inf\{n \geq 0 : \ X_n\ell \geq u\}, \qquad \tilde{T}_u := \inf\{n \geq 0 : \ X_n\ell \leq u\},$$
$$D := \inf\{n \geq 0 : \ X_n\ell < X_0\ell\}, \qquad \tilde{D} := \inf\{n \geq 0 : \ X_n\ell > X_0\ell\}.$$

For $x \in \mathbb{Z}^d$ we denote by

$$H(x) := \inf\{n \geq 0 : \ X_n = x\}$$

the hitting time of x. Similarly we define for sets $M \subseteq \mathbb{Z}^d$ the entrance time $H(M) := \inf\{n \geq 0 : \ X_n \in M\}$ of the walk into the set M.

We first generalize [4], Lemma 1.1, which assumed the existence of some ellipticity constant, to the non-elliptic case.

PROPOSITION 3. *Assume that $(\omega(x, x+e))_{|e|=1}$, $x \in \mathbb{Z}^d$, are i.i.d. under the probability measure $\mathbb{P}$ on Ω and let $\ell \in \mathbb{R}^d \backslash \{0\}$. Then $P_0[A_\ell \cup A_{-\ell}] \in \{0, 1\}$.*

For the proof we need the following lemma which implies that it is P_0-a.s. impossible to visit a slab of finite width infinitely often without visiting both of its neighboring half spaces.

LEMMA 4. *Under the assumptions of Proposition 3 for all $u, v \in \mathbb{R}$ with $u < v$,*

$$P_0[\sharp\{n \geq 0 : \ X_n\ell \geq u\} = \infty, T_v = \infty] = 0. \tag{4}$$

PROOF. Without loss of generality we assume $\ell e_1 > 0$. Therefore there is some $N \in \mathbb{N}$ such that for all $x \in S := \{x : \ u \leq x\ell < v\}$ the vertex $y = x + Ne_1$ is to the right of S in the sense that $y\ell \geq v$. Now, if the walker visits the slab S infinitely often then it either visits some point $x \in S$ infinitely often or it visits infinitely many points in S or it does both. Therefore the left side of (4) is less than or equal to

$$\sum_{x \in S} P_0[\sharp\{n : \ X_n = x\} = \infty, \ T_v = \infty] \tag{5}$$

$$+ P_0[\sharp\{x \in S : \ H(x) < \infty\} = \infty, \ T_v = \infty]. \tag{6}$$

Fix $x \in S$. Observe that on the event $\{\sharp\{n : \ X_n = x\} = \infty\}$ the $\mathscr{F}_n$-stopping times σ_k, $k \geq 0$, defined by

$$\sigma_0 := H(x), \quad \sigma_k := \inf\{n > \sigma_{k-1} + N : \ X_n = x\} \qquad (k \geq 1)$$

are finite. Therefore for $x \in S$,

$$P_0[\sharp\{n: X_n = x\} = \infty,\ T_v = \infty] \le \mathbb{E}\left[\inf_{k\ge 1} P_{0,\omega}[\sigma_k < T_v]\right]. \tag{7}$$

However, by the strong Markov property,

$$\begin{aligned}\inf_{k\ge1} P_{0,\omega}[\sigma_k < T_v] &\le \inf_{k\ge1} P_{0,\omega}[\sigma_{k-1} < T_v] P_{x,\omega}[N < T_v]\\ &= \left(\inf_{k\ge1} P_{0,\omega}[\sigma_k < T_v]\right) P_{x,\omega}[N < T_v],\end{aligned} \tag{8}$$

using in the last step an index shift and the fact that $k \mapsto P_{0,\omega}[\sigma_k < T_v]$ decreases monotonically. Observe that due to the choice of N,

$$\begin{aligned}P_{x,\omega}[N < T_v] &\le 1 - P_{x,\omega}[X_i - X_{i-1} = e_1 (i = 1, \dots, N)]\\ &= 1 - \omega(x, x+e_1)\cdots\omega(x+(N-1)e_1, x+Ne_1),\end{aligned} \tag{9}$$

which is strictly less than 1. Consequently, the right hand sides of (8) and (7) vanish as well as the sum over $x \in S$ in (5). Now we are going to show that also the term in (6) is zero. To this end define for $y \in \mathbb{Z}^{d-1}$,

$$M_y := \{z = (x, y) \in \mathbb{Z}\times\mathbb{Z}^{d-1} = \mathbb{Z}^d : u \le z\ell < v\}.$$

The sets M_y, $y \in \mathbb{Z}^{d-1}$, have cardinality at most N and partition S. Hence the $\mathscr{F}_n$-stopping times defined by

$$\tau_0 := H(S), \quad \tau_k := \inf\{n > \tau_{k-1} + N : \exists y \in \mathbb{Z}^{d-1} : n = H(M_y)\} \quad (k \ge 1)$$

are finite on the event $\{\sharp\{x \in S: H(x) < \infty\} = \infty\}$. Note that at each finite τ_k the walker visits a set M_y that it has never visited before. Consequently the term in (6) is less than or equal to

$$\inf_{k\ge1} P_0[\tau_k < T_v] \le \inf_{k\ge1}\sum_x \mathbb{E}[P_{0,\omega}[\tau_{k-1} < T_v, X_{\tau_{k-1}} = x] P_{x,\omega}[N < T_v]]. \tag{10}$$

Notice that for $x \in M_y$, the right hand side of (9) depends only on the values of $\omega(z, \cdot)$, where $z \in M_y$ or $z\ell \ge v$, whereas $P_{0,\omega}[\tau_{k-1} < T_v, X_{\tau_{k-1}} = x]$ depends only on the values of $\omega(z, \cdot)$ with $z \notin M_y$ and $z\ell < v$. Therefore by independence and (9) we may estimate the right side of (10) from above by

$$\left(\inf_{k\ge1} P_0[\tau_{k-1} < T_v]\right)\left(1 - \mathbb{E}[\omega(0, e_1)]^N\right).$$

Since $\mathbb{E}[\omega(0, e_1)] > 0$ this shows that the terms in (10) and therefore also in (6) vanish. □

PROOF OF PROPOSITION 3. The proof goes along the same lines as the one of [4], Lemma 1.1. The only difference is that we use Lemma 4 where one uses ellipticity in the original proof. We therefore only give a sketch of the proof. Assume that $P_0[A_\ell] > 0$. First observe that the proof of [4], Proposition 1.2, (1.16), does not use ellipticity at all, and therefore

$$P_0[D = \infty] > 0 \tag{11}$$

(see also [2], page 765). Then one shows that $P_0[O_\ell] = 0$, which is the analogue to [2], Theorem 3, where O_ℓ is the event that $X_n\ell$ changes its sign infinitely often. To do this we consider $M := \sup_n X_n\ell$. For fixed $v > 0$ the event $O_\ell \cap \{M < v\}$ is a subset of the event considered in (4) with $u = 0$ and is therefore a P_0-nullset. Therefore for the proof of $P_0[O_\ell] = 0$ it suffices to show $P_0[O_\ell^\infty] = 0$ with $O_\ell^\infty := O_\ell \cap \{M = \infty\}$. This is done as follows: Define recursively the $(\mathscr{F}_n)$-stopping times S_k and R_k by setting $S_0 := 0$,

$$R_k := \inf\{n \geq S_k : \; X_n\ell < 0\}$$

and

$$S_{k+1} := \inf\{n \geq R_k : \; X_n\ell > \max\{X_m\ell : \; m < n\}\} \qquad (k \geq 0).$$

On O_ℓ^∞ all these stopping times are finite. Now observe that at each time S_k the walk has entered a half space which it has never touched before. Since the environment in this half space is independent of what the walker has seen before, the walker has at each time S_k the chance $P_0[D = \infty]$ never to backtrack again below its position at time S_k. Therefore by induction $P_0[R_k < \infty] \leq P_0[D < \infty]^k$, which tends to 0 as $k \to \infty$, thus showing $P_0[O_\ell] = 0$.

Now assume that (1) fails. Then there is some $v > 0$ such that the event that the walker visits the slab $\{x : \; -v \leq x\ell \leq v\}$ infinitely often has positive P_0-probability. Split this slab into two slabs according to the sign of $x\ell$ and apply Lemma 4 twice with $(\ell, -v, 0)$ and $(-\ell, -v, 0)$ instead of (ℓ, u, v) to see that on this event the event O_ℓ, which is a P_0-nullset, occurs P_0-almost surely. □

In the proof of Theorem 1, the probability

$$r(x, \omega) := P_{x,\omega}[A_\ell] \qquad (x \in \mathbb{Z}^d, \; \omega \in \Omega)$$

that a walker starting at x in the environment ω converges to infinity in direction ℓ plays a crucial role. The limiting behavior of the process $(r(X_n, \omega))_{n \geq 0}$ is described by the following lemma.

LEMMA 5. *Assume the conditions of Proposition* 3. *Then*

$$\lim_{n \to \infty} r(X_n, \omega) = 1_{A_\ell}, \qquad P_0\text{-}a.s.$$

PROOF. Note that for given ω, by the Markov property, $r(X_n, \omega) = P_{0,\omega}[A_\ell | \mathscr{F}_n]$ holds $P_{0,\omega}$-almost surely. Hence $r(\cdot, \omega)$ is a bounded harmonic function in the sense that $(r(X_n, \omega))_{n \geq 0}$ is a bounded martingale under $P_{0,\omega}$. Since $A_\ell \in \mathscr{F}_\infty := \sigma(\bigcup_{n \in \mathbb{N}} \mathscr{F}_n)$, the classical martingale convergence theorem implies

$$\lim_{n \to \infty} r(X_n, \omega) = 1_{A_\ell} \quad P_{0,\omega}\text{-a.s. for all environments } \omega,$$

which immediately implies the claim. □

We now try to describe intuitively the idea of the proof of Theorem 1. We assume that (2) fails. Consider a walker $(X_n^1)_n$ starting at the origin. Whenever it enters a vertex, it observes the transition probabilities at its present location. These probabilities are fixed by the environment it walks in. On the other hand, the walker has no information about the environment at points that it has not visited up to the present time. Another walker $(X_n^2)_n$ starts at some remote point y, which lies far to the right of the origin. The first walker goes with positive probability $P_0[A_\ell]$ to the right in the sense of $X_n^1\ell \to \infty$, and the second walker goes with positive probability $P_y[A_{-\ell}] = P_0[A_{-\ell}]$ to the left. Suppose that the paths of the walkers (not necessarily the walkers themselves) meet each other at some intermediate point x, which is both far away from the origin and from y. Observe that as long as the paths of the walkers have not crossed each other, the walkers move independently of each other even under the annealed law due to the i.i.d. structure of the environment: no matter whether the two walks take place in the same environment under the annealed law, or whether the two walkers evolve independently in independent environments under the annealed law, the probabilities for their two paths to meet at x for the first time are the same. Now consider $r(x, \omega)$. Since the first walker which started at the origin is going to the right and has already covered a long distance, $r(x, \omega)$ should be close to one according to Lemma 5. However, the same argument for the second walker suggests that $r(x, \omega)$ should be close to zero, which is a contradiction.

Now why should such a crossing point x exist? As in the case of two simple random walks in two dimensions with opposite drifts in a homogeneous medium we can choose y deterministically such that the paths have a high probability to collide. Here of course we heavily use the fact that $d = 2$. We remark that although large transversal fluctuations of the walker would favor the crossing of their paths we do not need any fluctuation estimates at all.

Since the dimension enters the proof only in the last part of the argument we summarize the first part in the following lemma, which holds in general dimension. It gives a sufficient condition for $P_0[A_\ell] \in \{0, 1\}$.

LEMMA 6. *Suppose that $(\omega(x, x+e))_{|e|=1}$, $x \in \mathbb{Z}^d$, are i.i.d. under the probability measure $\mathbb{P}$ on Ω and let $\ell \in \mathbb{R}^d\setminus\{0\}$. Furthermore assume that there are a sequence $(y_L)_L \in (\mathbb{Z}^d)^{\mathbb{N}}$ and a $c_1 > 0$ such that for all $L \in \mathbb{N}$,*

$$y_L\ell \geq 3L \tag{12}$$

and

$$c(y_L, L) \geq c_1, \tag{13}$$

where

$$c(y, L) := \sum_{a,b\in\mathbb{Z}^2} P_0[X_{T_L} = a, T_L \leq D]\, P_y\left[X_{\tilde{T}_{y\ell-L}} = b, \tilde{T}_{y\ell-L} \leq \tilde{D}\right]$$

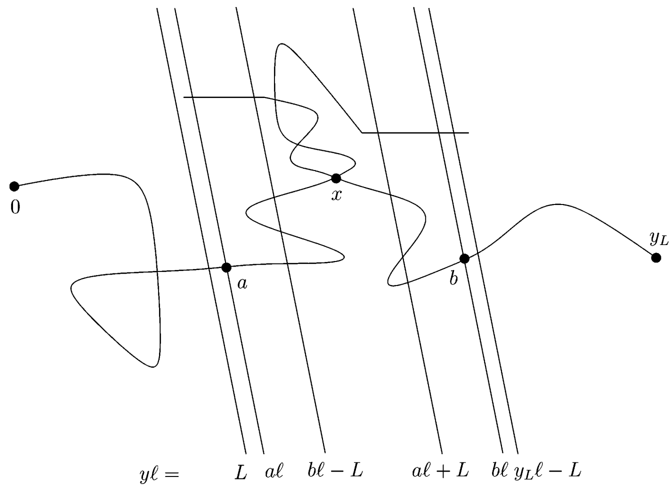

FIG. 1. *Sketch for Lemma* 6 *and the proof of Theorem* 1. *The paths of two walkers starting in* 0 *and* y_L, *respectively, meet in the middle slab* $\{y : a\ell \le y\ell \le b\ell\}$ *in some vertex* x *before leaving this slab on the side which is opposite to the side of their respective entrance.*

$$\times \sum_{\substack{\pi_1,\pi_2\subseteq\mathbb{Z}^d,\\ \pi_1\cap\pi_2\neq\varnothing}} P_a\left[\left\{X_0,\ldots,X_{\min\{D,T_{b\ell}\}}\right\}=\pi_1\right] \tag{14}$$

$$\times P_b\left[\left\{X_0,\ldots,X_{\min\{\tilde D,\tilde T_{a\ell}\}}\right\}=\pi_2\right].$$

Then $P_0[A_\ell] \in \{0,1\}$.

Here we introduce the convention that whenever a condition defining an event uses X_σ for some stopping time σ, we additionally assume in this event that σ is finite.

PROOF OF LEMMA 6. For $x_1, x_2 \in \mathbb{Z}^d$ and $\omega \in \Omega$ we denote by $P_{x_1,x_2,\omega}$ the probability measure governing the Markov chain $((X_n^1)_n, (X_n^2)_n)$ where $(X_n^1)_n$ and $(X_n^2)_n$ move independently of each other in the same environment ω according to $P_{x_1,\omega}$ and $P_{x_2,\omega}$, respectively. Stopping times referring to the walk $(X_n^i)_n$ will be marked by an upper index i ($i = 1, 2$). Now assume that the statement of the Lemma is false, that is $0 < P_0[A_\ell] < 1$, and consider the annealed probability

$$q_L := \mathbb{E}\left[P_{0,y_L,\omega}\left[\exists x \in \mathbb{Z}^d : L \le x\ell \le y_L\ell - L, H^1(x) < \infty, H^2(x) < \infty\right]\right].$$

We are going to show that $c(y_L, L) \le q_L \to 0$ as $L \to \infty$, which is a contradiction to (13). For the proof of $q_L \to 0$ first observe that $q_L \le q_{L,1} + q_{L,2}$ where

$$q_{L,1} := P_0[\exists x \in \mathbb{Z}^d : \ L \le x\ell, H(x) < \infty, r(x,\omega) \le 1/2]$$

and

$$\begin{aligned} q_{L,2} &:= P_{y_L}[\exists x \in \mathbb{Z}^d : \ x\ell \le y_L\ell - L, H(x) < \infty, r(x,\omega) \ge 1/2] \\ &= P_0[\exists x \in \mathbb{Z}^d : \ x\ell \le -L, H(x) < \infty, r(x,\omega) \ge 1/2]. \end{aligned}$$

To reach some x with $L \le x\ell$ when starting in the origin takes at least $c_2 L$ time steps where $c_2 > 0$ is only depending on ℓ. Therefore and by Proposition 3,

$$q_{L,1} \le P_0[\exists n \ge c_2 L : \ r(X_n,\omega) \le 1/2, A_\ell] + P_0[T_L < \infty, A_{-\ell}].$$

Obviously the second term in the above expression tends to 0 as $L \to \infty$. Due to Lemma 5 the first term also converges to 0, thus giving us $q_{L,1} \to 0$. Observe that due to Proposition 3 and $P_0[A_\ell] > 0$,

$$P_{x,\omega}[A_{-\ell}] = 1 - r(x,\omega) \tag{15}$$

holds for all $x \in \mathbb{Z}^d$ on a set of full of $\mathbb{P}$-measure. Hence one gets analogously $q_{L,2} \to 0$, which implies $q_L \to 0$.

For the proof of $q_L \ge c(y_L, L)$ first note that the event in the definition of q_L occurs, if there is a vertex x which is visited by both walks before they leave the middle slab $S_L := \{y : \ L \le y\ell \le y_L\ell - L\}$ for the first time. To handle boundary effects for directions ℓ which do not point into the direction of an axis we make this slab even smaller and replace it by $\{y : \ a\ell \le y\ell \le b\ell\}$ where a and b are the respective entrance points of the walks into S_L. For reasons which will become clear later [see the explanation after (30)], we also impose the condition that the walkers do not backtrack below the starting point before reaching S_L. This means that q_L is bigger than or equal to

$$\begin{aligned} \mathbb{E}\big[P_{0,y_L,\omega}\big[&\exists a,b,x \in \mathbb{Z}^d : X^1_{T^1_L} = a, T^1_L \le D^1, X^2_{\tilde{T}^2_{y_L\ell - L}} = b, \tilde{T}^2_{y_L\ell - L} \le \tilde{D}^2, \\ &H^1(x)\circ\theta_{T^1_L} \le \min\{D^1, T^1_{b\ell}\}\circ\theta_{T^1_L} < \infty, \\ &H^2(x)\circ\theta_{\tilde{T}^2_{y_L\ell - L}} \le \min\{\tilde{D}^2, \tilde{T}^2_{a\ell}\}\circ\theta_{\tilde{T}^2_{y_L\ell - L}} < \infty\big]\big]. \end{aligned}$$

By the strong Markov property, this expression equals

$$\begin{aligned} \sum_{a,b\in\mathbb{Z}^d} \mathbb{E}\Big[&P_{0,\omega}[X_{T_L} = a, T_L \le D] P_{y_L,\omega}[X_{\tilde{T}_{y_L\ell - L}} = b, \tilde{T}_{y_L\ell - L} \le \tilde{D}] \\ &\times P_{a,b,\omega}[\exists x : \ H^1(x) \le \min\{D^1, T^1_{b\ell}\} < \infty, H^2(x) \le \min\{\tilde{D}^2, \tilde{T}^2_{a\ell}\} < \infty]\Big]. \end{aligned}$$

Now observe that the three factors inside the above $\mathbb{E}$-expectation depend on disjoint regions of the environment. By the independence structure of the environment, the above expression equals

$$\begin{aligned} \sum_{a,b\in\mathbb{Z}^d} &P_0[X_{T_L} = a, T_L \le D] \ P_{y_L}[X_{\tilde{T}_{y_L\ell - L}} = b, \tilde{T}_{y_L\ell - L} \le \tilde{D}] \\ &\times \mathbb{E}[P_{a,b,\omega}[\exists x : \ H^1(x) \le \min\{D^1, T^1_{b\ell}\} < \infty, \\ &\qquad H^2(x) \le \min\{\tilde{D}^2, \tilde{T}^2_{a\ell}\} < \infty]]. \end{aligned} \tag{16}$$

The last factor in (16) equals

$$\begin{aligned}\mathbb{E}\Big[P_{a,b,\omega}\Big[\Big\{X_0^1,\dots,X^1_{\min\{D^1,T^1_{b\ell}\}}\Big\}\cap\Big\{X_0^2,\dots,X^2_{\min\{\tilde D^2,\tilde T^2_{a\ell}\}}\Big\}\neq\varnothing\Big]\Big]\\ =1-\sum_{\pi_1\cap\pi_2=\varnothing}\mathbb{E}\Big[P_{a,\omega}\big[\{X_0,\dots,X_{\min\{D,T_{b\ell}\}}\}=\pi_1\big]\\ \times P_{b,\omega}\Big[\Big\{X_0,\dots,X_{\min\{\tilde D,\tilde T_{a\ell}\}}\Big\}=\pi_2\Big]\Big].\end{aligned}$$

The disjointness of the paths π_1 and π_2 and again the independence in the environment imply that the above expression equals

$$\begin{aligned}1-&\sum_{\pi_1\cap\pi_2=\varnothing}P_a\big[\{X_0,\dots,X_{\min\{D,T_{b\ell}\}}\}=\pi_1\big]P_b\Big[\Big\{X_0,\dots,X_{\min\{\tilde D,\tilde T_{a\ell}\}}\Big\}=\pi_2\Big]\\ =&\sum_{\pi_1\cap\pi_2\neq\varnothing}P_a\big[\{X_0,\dots,X_{\min\{D,T_{b\ell}\}}\}=\pi_1\big]P_b\Big[\Big\{X_0,\dots,X_{\min\{\tilde D,\tilde T_{a\ell}\}}\Big\}=\pi_2\Big].\end{aligned}$$

This together with (16) proves $q_L\ge c(y_L,L)$. □

2. Proof of Theorem 1. We start with the following elementary lemma.

LEMMA 7. *Let Y_1,Y_2,Z_1,Z_2 be independent integer valued random variables on some probability space with probability measure P such that Y_1 and Y_2 have the same distribution and Z_1 and Z_2 have the same distribution. Then there exists some deterministic $\xi\in\mathbb{Z}$ such that*

$$P[Y_1+Z_1+Z_2\le\xi\le Z_1+Y_1+Y_2]\ge 1/256 \tag{17}$$

or

$$P[Z_1+Y_1+Y_2\le\xi\le Y_1+Z_1+Z_2]\ge 1/256. \tag{18}$$

Let us informally describe how this lemma is going to be used. Consider two walkers. The first one starts at the origin, the second one starts at some remote point y_L with e_2-coordinate $y_Le_2=\xi$. Then Y_1 stands for the e_2-coordinate of the entrance point a of the first walker into the halfspace with normal vector ℓ and distance L from the origin. Y_1+Y_2 denotes the e_2-coordinate of the entrance point a' into the halfspace with distance L from a for the first walker which resumes its walk at a. The corresponding quantities for the second walker travelling into the opposite direction are denoted by $\xi-Z_1$ and $\xi-Z_1-Z_2$. Although we have no information about the distribution of Y_i and Z_i the above lemma guarantees that the probability that the paths of the walkers cross each other after having entered the first halfspace is uniformly positive.

PROOF OF LEMMA 7. Let $-\infty=a_0<a_1\le a_2\le a_3<a_4=\infty$ denote integer $k/4$-quantiles of Y_1, $k=0,1,2,3,4$; that is, $P[Y_1\le a_k]\ge k/4$, and $P[Y_1\ge a_k]\ge 1-k/4$, and $a_k\in\mathbb{Z}\cup\{\pm\infty\}$. Similarly, let $-\infty=b_0<b_1\le b_2\le b_3<b_4=\infty$ denote integer $k/4$-quantiles of Z_1. We set for $k=1,2,3,4$: $A_k:=$

$[a_{k-1}, a_k]$ and $B_k := [b_{k-1}, b_k]$. (We remark that some of these intervals A_k may degenerate to one point, and that some of these intervals have one point in common.) Let $|A|$ denote the length of an interval A. We choose $i \in \{2, 3\}$ such that $|A_i| = \min\{|A_2|, |A_3|\}$, and $j \in \{2, 3\}$ such that $|B_j| = \min\{|B_2|, |B_3|\}$. Then $a_3 - a_1 = |A_2| + |A_3| \geq 2|A_i|$ and similarly $b_3 - b_1 = |B_2| + |B_3| \geq 2|B_j|$. Summing up these two inequalities implies

$$(a_3 - b_1) + (b_3 - a_1) \geq 2(|A_i| + |B_j|);$$

consequently at least one of the two inequalities

$$a_3 - b_1 \geq |A_i| + |B_j|, \tag{19}$$

$$b_3 - a_1 \geq |A_i| + |B_j|, \tag{20}$$

holds. We examine the first case (19): we define $\xi := a_i + b_j + b_1 \in \mathbb{Z}$ in this case. Then the following estimate holds on the event $E := \{Y_1 \in A_i, Y_2 \in A_4, Z_1 \in B_j, Z_2 \in B_1\}$:

$$\begin{aligned} Z_1 + Y_1 + Y_2 \geq b_{j-1} + a_{i-1} + a_3 &\overset{(19)}{\geq} a_i + b_j + b_1 = \xi \\ &\geq Y_1 + Z_1 + Z_2. \end{aligned} \tag{21}$$

The independence of Y_1, Y_2, Z_1, Z_2, the facts $Y_1 \sim Y_2$, $Z_1 \sim Z_2$, $P[Y_1 \in A_k] \geq 1/4$, and $P[Z_1 \in B_k] \geq 1/4$ $(k = 1, 2, 3, 4)$ together imply $P[E] \geq 1/256$. Hence (17) holds in the case (19). The second case [i.e., (20) holds, but not (19)] can be treated similarly: this time we may choose $\xi := a_i + b_j + a_1 \in \mathbb{Z}$; here we obtain (18). $\square$

PROOF OF THEOREM 1. The proof is by contradiction. Suppose that the statement (2) of Theorem 1 is false and therefore

$$0 < P_0[A_\ell] < 1 \quad \text{and} \quad 0 < P_0[A_{-\ell}] < 1 \tag{22}$$

according to Proposition 3. Without loss of generality we assume that $\ell = (\ell_1, \ell_2)$ fulfills

$$|\ell_1| \geq |\ell_2| \quad \text{and} \quad \ell_1 > 0 \tag{23}$$

because otherwise we rename the axes. Furthermore we may assume that ℓ has euclidean norm $\|\ell\|_2 = 1$, which together with (23) implies $\ell_1 \geq 1/\sqrt{2}$.

Let $L \in \mathbb{N}$ and let Y_1, Y_2, Z_1, Z_2 be independent random variables on some probability space with probability measure P such that Y_1 and Y_2 are distributed as $X_{T_L} e_2$ is under $P_0[\cdot | T_L \leq D]$ and $-Z_1$ and $-Z_2$ are distributed as $X_{\tilde{T}_{-L}} e_2$ under $P_0[\cdot | \tilde{T}_{-L} \leq \tilde{D}]$. For this choice of Y_i and Z_i let $\xi \in \mathbb{Z}$ be according to Lemma 7. This means that (17) or (18) holds. We assume (17). The other case (18) is treated analogously. Now define $y_L \in \mathbb{Z}^2$ by

$$y_L e_1 := \left\lceil \frac{3L + 2 - \xi \ell_2}{\ell_1} \right\rceil \quad \text{and} \quad y_L e_2 := \xi, \tag{24}$$

where $\lceil x \rceil$ is the smallest integer $\geq x$. We are now going to verify (12) and (13). Due to (24) we get

$$y_L \ell = y_L e_1 \ell_1 + y_L e_2 \ell_2 \geq 3L + 2 - \xi \ell_2 + \xi \ell_2 = 3L + 2 \tag{25}$$

and thus (12). For the proof of (13) we exploit the assumption $d = 2$. Consider $a, b \in \mathbb{Z}^2$ for which

$$0 < P_0[X_{T_L} = a, T_L \leq D]\ P_{y_L}\left[X_{\tilde{T}_{y_L \ell - L}} = b, \tilde{T}_{y_L \ell - L} \leq \tilde{D}\right], \tag{26}$$

which shows up in the definition (14) of $c(y_L, L)$. We introduce the following events for $\pi_1, \pi_2 \subseteq \mathbb{Z}^2$:

$$E(b, \pi_1) := \left\{\{X_0, \dots, X_{T_{b\ell}}\} = \pi_1, T_{b\ell} \leq D, X_{T_{b\ell}} e_2 \geq b e_2\right\},$$

$$\tilde{E}(a, \pi_2) := \left\{\{X_0, \dots, X_{\tilde{T}_{a\ell}}\} = \pi_2, \tilde{T}_{a\ell} \leq \tilde{D}, X_{\tilde{T}_{a\ell}} e_2 \geq a e_2.\right\}$$

The inner sum in the definition of $c(y_L, L)$ depends on a and b; we claim that for our choice of a and b, this inner sum is greater or equal to

$$\begin{aligned} &\sum_{\pi_1, \pi_2 \subseteq \mathbb{Z}^2} P_a[E(b, \pi_1)] P_b[\tilde{E}(a, \pi_2)] \\ &\qquad = P_a[T_{b\ell} \leq D, X_{T_{b\ell}} e_2 \geq b e_2] P_b\left[\tilde{T}_{a\ell} \leq \tilde{D}, X_{\tilde{T}_{a\ell}} e_2 \geq a e_2\right]. \end{aligned} \tag{27}$$

To prove this claim, we observe that for $\pi_1, \pi_2 \subseteq \mathbb{Z}^2$:

$$P_a[E(b, \pi_1)] \leq P_a\left[\{X_0, \dots, X_{\min\{D, T_{b\ell}\}}\} = \pi_1\right],$$

$$P_b[\tilde{E}(a, \pi_2)] \leq P_b\left[\left\{X_0, \dots, X_{\min\{\tilde{D}, \tilde{T}_{a\ell}\}}\right\} = \pi_2\right];$$

so it remains to show that $\pi_1 \cap \pi_2 \neq \varnothing$ whenever $P_a[E(b, \pi_1)] P_b[\tilde{E}(a, \pi_2)] > 0$. Under this assumption, π_1 contains an unique point a' with $a'\ell \geq b\ell$ and $a' e_2 \geq b e_2$, namely the value of $X_{T_{b\ell}}$ on the event $E(b, \pi_1)$. Similarly π_2 contains an unique point b' with $b'\ell \leq a\ell$ and $b' e_2 \geq a e_2$. With the possible exception of a' and b', $\pi_1 \cup \pi_2$ contains only points in the strip $\{y \in \mathbb{R}^2 : a\ell \leq y\ell \leq b\ell\}$. Through the points of π_1, there is a nearest neighbor lattice path (called $a \rightsquigarrow a'$) from a to a', namely any path $(X_0, \dots, X_{T_{b\ell}})$ on the event $E(b, \pi_1)$. Let (a'', a') denote the last step in that path. Then the line $]a'', a']$ contains an (unique) point a''' with $a'''\ell = b\ell$. We have $a''' e_2 \geq b e_2$. To see this, we examine three cases:

Case 1. If $a' e_2 > b e_2$, then $a' e_2 - b e_2 \geq 1$, hence $a''' e_2 - b e_2 \geq a''' e_2 - a' e_2 + 1 \geq 0$.

Case 2. If $a' = b$, then $a''' = b$, hence $a''' e_2 = b e_2$.

Case 3. If $a' e_2 = b e_2$, but $a' \neq b$, then $(a' - b) e_1 \geq 1$, and the last step in the path $a \rightsquigarrow a'$ must have been in the $\pm e_2$-direction: $a' - a'' \in \{\pm e_2\}$. This case leads to a contradiction: $0 > a''\ell - b\ell = (a' - b)\ell + (a'' - a')\ell = (a' - b) e_1 \ell_1 + (a'' - a') e_2 \ell_2 \geq \ell_1 - \ell_2 \geq 0$.

Let $a \rightsquigarrow a'''$ denote the part of the path $a \rightsquigarrow a'$ (viewed as a piecewise linear path in $\mathbb{R}^2$) that runs inside the strip $S_{a,b} := \{y \in \mathbb{R}^2 : a\ell \leq y\ell \leq b\ell\}$.

Similarly, there exists a lattice path $b \rightsquigarrow b'$ from b to b' with vertices π_1 and a part $b \rightsquigarrow b'''$ of this path that runs in $S_{a,b}$; furthermore the end point $b''' \in \mathbb{R}^2$ of the path $b \rightsquigarrow b'''$ should fulfill $b'''\ell = a\ell$ and $b'''e_2 \geq ae_2$. The facts $(a \rightsquigarrow a''') \subseteq S_{a,b}$, $(b \rightsquigarrow b''') \subseteq S_{a,b}$, $b\ell = a'''\ell$, $a'''e_2 \geq be_2$, $a\ell = b'''\ell$, and $b'''e_2 \geq ae_2$ imply that the two paths $a \rightsquigarrow a'''$ and $b \rightsquigarrow b'''$ have to intersect each other in at least one point for topological reasons (recall $d = 2$); consequently the paths $a \rightsquigarrow a'$ and $b \rightsquigarrow b'$ intersect each other *in a lattice point*; this means $\pi_1 \cap \pi_2 \neq \varnothing$, which proves the above claim (27).

Observe that due to (26) and $\|\ell\|_2 = 1$ we have

$$L \leq a\ell \ \leq \ L + 1$$

and

$$y_L\ell - L - 1 \leq b\ell \ \leq \ y_L\ell - L.$$

Therefore we can estimate $b\ell - (a\ell + L)$ from below using (25) by

$$b\ell - (a\ell + L) \geq y_L\ell - L - 1 - (2L + 1) \geq 0 \tag{28}$$

and from above by

$$\begin{aligned} b\ell - (a\ell + L) &\leq y_L\ell - L - 2L \\ &\leq \left(\frac{3L + 2 - \xi\ell_2}{\ell_1} + 1\right)\ell_1 + \xi\ell_2 - 3L \leq 3. \end{aligned} \tag{29}$$

Now consider the event $\{T_{b\ell} \leq D, X_{T_{b\ell}}e_2 \geq be_2\}$, which appears in (27). One possible strategy for the walker starting at a to let this event occur is to fulfill $T_{a\ell+L} \leq D$ with $X_{T_{a\ell+L}}e_2 \geq be_2$ and then to go in the next six steps after $T_{a\ell+L}$ into direction e_1. Indeed, due to (28) we have $T_{a\ell+L} \leq T_{b\ell}$ and by (29) and $\ell_1 \geq 1/\sqrt{2}$ we see that

$$(X_{T_{a\ell+L}} + 6e_1)\ell \geq a\ell + L + 6\ell_1 \geq b\ell - 3 + 6/\sqrt{2} \geq b\ell.$$

Consequently, the first factor on the right-hand side of (27) is greater or equal to

$$P_a[T_{a\ell+L} \leq D, X_{T_{a\ell+L}}e_2 \geq be_2, X_{i+1} - X_i = e_1 \ (i \in \{T_{a\ell+L}, \dots, T_{a\ell+L} + 5\})].$$

Using the strong Markov property, this can be rewritten as

$$\begin{aligned} \sum_{x \in \mathbb{Z}^2} \mathbb{E}[P_{a,\omega}[T_{a\ell+L} \leq D, X_{T_{a\ell+L}} = x, xe_2 \geq be_2] \\ \times P_{x,\omega}[X_{i+1} - X_i = e_1 \text{ for } i \in \{0, \dots, 5\}]]. \end{aligned}$$

Observe that the two factors inside the above expectation are independent as they depend on disjoint regions of the environment. Therefore the above expression equals

$$\begin{aligned} P_a[T_{a\ell+L} \leq D, X_{T_{a\ell+L}}e_2 \geq be_2]P_0[X_1 = e_1]^6 \\ = c_3 P_0[T_L \leq D, X_{T_L}e_2 \geq (b - a)e_2]. \end{aligned}$$

Analogously we handle the second factor in (27) and thus estimate (27) from below by

$$c_4 P_0[T_L \le D, X_{T_L}e_2 \ge (b-a)e_2] P_0[\tilde{T}_{-L} \le \tilde{D}, X_{\tilde{T}_{-L}}e_2 \ge (a-b)e_2].$$

Inserting this into the definition of $c(y_L, L)$ yields

$$\begin{aligned} c(y_L, L) \ge \sum_{a,b\in\mathbb{Z}^2} & P_0[X_{T_L} = a \mid T_L \le D] P_0[X_{T_L}e_2 \ge (b-a)e_2 \mid T_L \le D] \\ & \times P_0\left[X_{\tilde{T}_{-L}} = b - y_L \mid \tilde{T}_{-L} \le \tilde{D}\right] \\ & \times P_0\left[X_{\tilde{T}_{-L}}e_2 \ge (a-b)e_2 \mid \tilde{T}_{-L} \le \tilde{D}\right] \\ & \times c_4 P_0[T_L \le D]^2 P_0[\tilde{T}_{-L} \le \tilde{D}]^2. \end{aligned} \tag{30}$$

Here it becomes clear why we required the walkers not to backtrack at their starting point and at their entrance point into the slab: We did not allow backtracking at the entrance point because this way we could use the strong Markov property and the independence in the environment to decouple the part of the path which is outside the slab from the part inside the slab under the annealed measure. Therefore, since we want that both parts of the path have the same distribution we did not permit backtracking at the beginning either.

To make the spatial random variables involved in (30) one-dimensional like Y_1, Y_2 and Z_1, Z_2, we estimate (30) from below by

$$\begin{aligned} \sum_{a_2,b_2\in\mathbb{Z}} & P_0[X_{T_L}e_2 = a_2 \mid T_L \le D] P_0[X_{T_L}e_2 \ge b_2 - a_2 \mid T_L \le D] \\ & \times P_0\left[X_{\tilde{T}_{-L}}e_2 = b_2 - y_L e_2 \mid \tilde{T}_{-L} \le \tilde{D}\right] P_0\left[X_{\tilde{T}_{-L}}e_2 \ge a_2 - b_2 \mid \tilde{T}_{-L} \le \tilde{D}\right] \\ & \times c_4 P_0[D = \infty]^2 P_0[\tilde{D} = \infty]^2. \end{aligned}$$

Like in (11) it follows from (22) that both quantities $P_0[D = \infty]$ and $P_0[\tilde{D} = \infty]$ are strictly positive. Therefore and from the definition of Y_1, Y_2, Z_1, Z_2 the last expression equals

$$\begin{aligned} & c_5 \sum_{a_2,b_2\in\mathbb{Z}} P[Y_1 = a_2] P[Y_2 \ge b_2 - a_2] P[-Z_1 = b_2 - \xi] P[-Z_2 \ge a_2 - b_2] \\ & = c_5 \sum_{a_2,b_2\in\mathbb{Z}} P[Y_1 = a_2, Y_2 \ge (\xi - Z_1) - Y_1, -Z_1 = b_2 - \xi, -Z_2 \ge Y_1 - (\xi - Z_1)] \\ & = c_5 P[Y_1 + Y_2 + Z_1 \ge \xi \ge Y_1 + Z_1 + Z_2] \ \ge \ c_5/256 =: c_1 > 0 \end{aligned}$$

by (17). Since c_1 does not depend on L, this proves (13). Therefore by Lemma 6, $P_0[A_\ell] \in \{0, 1\}$, which gives us the desired contradiction to Assumption (22). $\square$

3. A stationary and ergodic counterexample. In the proof of Theorem 1 we used the observation that two different walkers in the same environment move independently of each other under the annealed law as long as their paths have not crossed each other. Therefore, the idea for the following proof of Proposition 2 is to arrange for local reflections which prevent the paths from crossing each other.

PROOF OF PROPOSITION 2. In the first step we will define a stationary and ergodic measure $\mathbb{P}'$ fulfilling (3) with the semi-direct product $P_0' := \mathbb{P}' \times P_{0,\omega}$ instead of P_0 but with degenerate transition probabilities $\omega(x, y)$ which are 1 for exactly one neighbor y of x and 0 otherwise. Therefore $\mathbb{P}'$ is not a probability measure on Ω but on $\Omega' := \mathscr{P}^{\mathbb{Z}^2}$ where $\mathscr{P}$ is the set of 4-dimensional vectors $(p(e))_{|e|=1}$ with $p(e) \geq 0$ and $\sum_e p(e) = 1$. In a second step we will construct $\mathbb{P}$ from $\mathbb{P}'$ by a transformation which preserves stationarity, ergodicity and (3).

1. Let $\mathbb{P}_0'$ be the measure on Ω' under which the matrices

$$(\omega(2x, \cdot),\ \omega(2x+e_1, \cdot),\ \omega(2x+e_2, \cdot),\ \omega(2x+e_1+e_2, \cdot)), \qquad x \in \mathbb{Z}^2,$$

are i.i.d. such that the probability of such a matrix to be of Type 1 or Type 2 (see Figure 2), respectively, is 1/2.

To the best of our knowledge, a similar construction appeared first in the work of Arratia (see [3] and the references therein) in the context of coalescing random walks on $\mathbb{Z}$ and the corresponding dual process. Observe that for $\mathbb{P}_0'$-almost all $\omega' \in \Omega'$ and all $x \in \mathbb{Z}^2$, the process $(X_{2n}(e_1-e_2)/2)_n$ is under $P_{2x,\omega'}$ a simple symmetric random walk on $\mathbb{Z}$ with start in $x(e_1-e_2)$ whereas $X_{2n}(e_1+e_2)/2$ equals deterministically $x(e_1+e_2)+n$. From this it follows that X_n/n tends $P_{2x,\omega'}$-a.s. to $v := (e_1+e_2)/2$. Similarly, one sees that $P_{2x+e_1+e_2,\omega'} - a.s.$ for all $x \in \mathbb{Z}^2$, $X_n/n \to -v$. Since $\mathbb{P}_0'$ is not stationary we are going to average four shifted versions of $\mathbb{P}_0'$ as follows. Denote by τ_x ($x \in \mathbb{Z}^2$) the shift on Ω' defined by $\tau_x(\omega(y, \cdot)) := \omega(y+x, \cdot)$ and set

$$\mathbb{P}' := \frac{1}{4}\sum_{i=0}^{3} \mathbb{P}_i' \qquad \text{where} \quad \mathbb{P}_i' := \mathbb{P}_0' \circ \tau_{v_i}^{-1} \quad \text{and}$$

$$v_0 := 0, \quad v_1 := e_1, \quad v_2 := e_2, \quad v_3 := e_1+e_2.$$

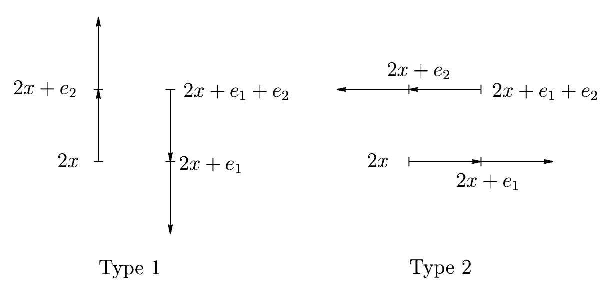

FIG. 2. *The two types of transition probabilities used in the counterexample. Here an edge from y to z marked with an arrow means that a particle which is currently located at y jumps in the next step to z with probability one.*

By construction, $\mathbb{P}'$ is stationary with respect to all shifts τ_x, $x \in \mathbb{Z}^2$. We claim that $\mathbb{P}'$ is also ergodic.

To prove this, denote by $\mathscr{I}'$ and $\mathscr{I}'_i$ the σ-algebras of all measurable subsets of Ω' which are shift-invariant up to null sets under all shifts τ_x with respect to $\mathbb{P}'$ and $\mathbb{P}'_i$, respectively. For $\mathscr{I}'$ this means that for all $A \in \mathscr{I}'$ the symmetric difference $\tau_x[A]\Delta A$ of $\tau_x[A]$ and A is a $\mathbb{P}'$-nullset for all $x \in \mathbb{Z}^2$. Consequently, by definition of $\mathbb{P}'$, $\tau_x[A]\Delta A$ is also a $\mathbb{P}'_i$-nullset for all $A \in \mathscr{I}'$ and $x \in \mathbb{Z}^2$, saying that $\mathscr{I}' \subseteq \mathscr{I}'_i$ for all $i \in \{0, 1, 2, 3\}$.

Now fix $A \in \mathscr{I}'$ with $\mathbb{P}'[A] > 0$. To show ergodicity of $\mathbb{P}'$ we have to show $\mathbb{P}'[A] = 1$. Since $\mathbb{P}'[A] > 0$ there is some $i \in \{0, 1, 2, 3\}$ such that $\mathbb{P}'_i[A] > 0$. From $A \in \mathscr{I}'_i$ we get

$$0 < \mathbb{P}'_i[A] = \mathbb{P}'_i[\tau_{v_i}[A]] = \mathbb{P}'_0[\tau_{v_i}^{-1}[\tau_{v_i}[A]]] = \mathbb{P}'_0[A]. \tag{31}$$

Now observe that as a product measure $\mathbb{P}'_0$ is ergodic with respect to shifts of the form τ_{2x}, $x \in \mathbb{Z}^2$, which implies that $\mathscr{I}'_0$ is trivial. Hence from (31) and $A \in \mathscr{I}'_0$,

$$1 = \mathbb{P}'_0[A] = \mathbb{P}'_0[\tau_{-v_j}[A]] = \mathbb{P}'_j[A]$$

for all $j \in \{0, 1, 2, 3\}$, which implies $\mathbb{P}'[A] = 1$. We remark, that although ergodic, $\mathbb{P}'$ is not mixing.

2. Now we are going to turn $\mathbb{P}'$ into a probability measure $\mathbb{P}$ on Ω. Observe that for $\mathbb{P}'$ almost every realization ω' the environment forms two trees, see Figure 3. One tree consists of the edges directed upwards or to the right and the other tree is built up by the edges which are directed downwards or to the left. The branches of the trees are directed random walk paths as described above which are independent of each other until they eventually meet and coalesce. Any point $x \in \mathbb{Z}^2$ has $\mathbb{P}'$-a.s. a unique successor $s(x, \omega') \in \mathbb{Z}^2$ with $\omega'(x, s(x, \omega')) = 1$. Furthermore, for any x the subtree for which x is the root is $\mathbb{P}'$-a.s. finite. Indeed, because otherwise by stationarity the $\mathbb{P}'$-probability for a point $x \in \mathbb{Z}^2$ to be the root of an infinite subtree would be positive and the same for all points x. This would imply by ergodicity the $\mathbb{P}'$-a.s. existence of two disjoint infinite subbranches of the same tree. However, for topological reasons these two infinite subbranches would cut the other tree into two disjoint pieces which does $\mathbb{P}'$-a.s. not occur. Therefore the height $h(x, \omega') \geq 0$ of the subtree in x, that is, the length of the longest branch in this subtree, is $\mathbb{P}'$-a.s. finite for all $x \in \mathbb{Z}^2$. Now define $\omega \in \Omega$ as a function of ω' by

$$\omega(x, x+e) := \begin{cases} 1 - 3/(h^2(x, \omega') + 4), & \text{if } \omega'(x, x+e) = 1, \\ 1/(h^2(x, \omega') + 4), & \text{if } \omega'(x, x+e) = 0, \end{cases}$$

where $x \in \mathbb{Z}^2$, $|e| = 1$. Call the pushforward of $\mathbb{P}'$ under this map $\mathbb{P}$. Clearly, stationarity and ergodicity are preserved under this map. Relation (3) follows from the observation that the probability under ω that a walker currently located at x strictly follows the path consisting of the successive successors

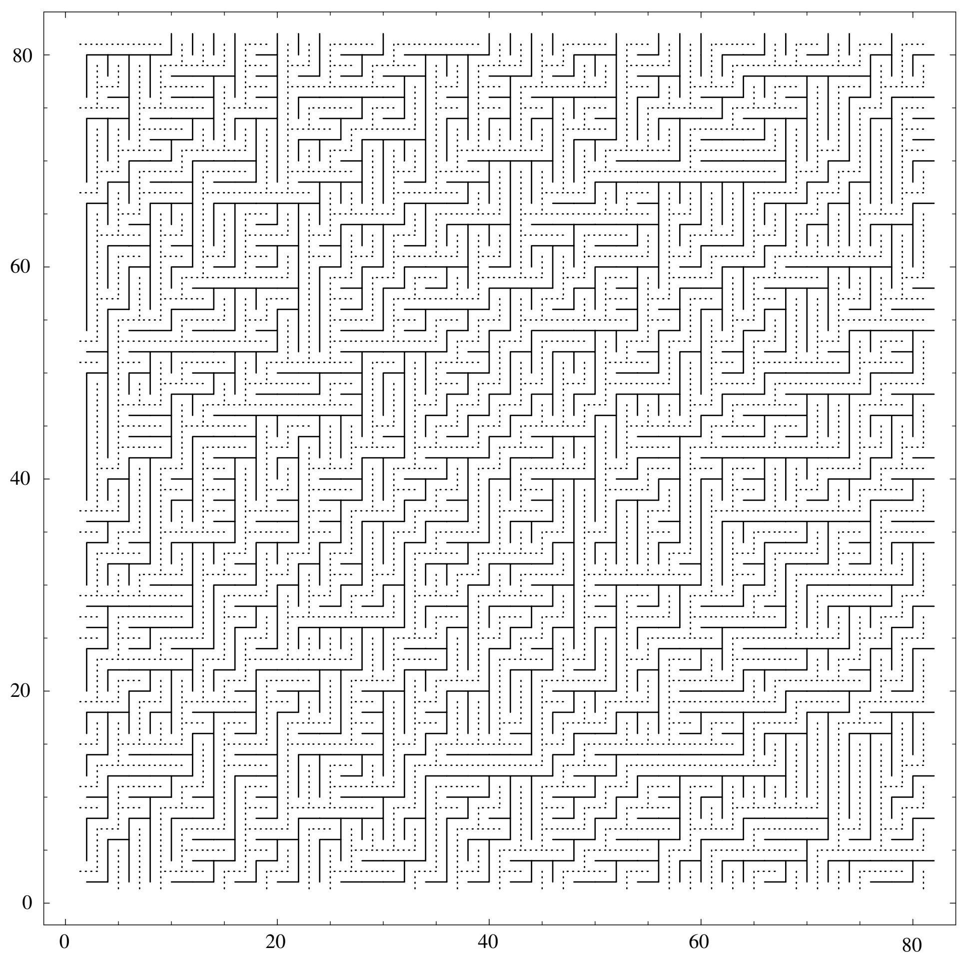

FIG. 3. *A realization of the counterexample shown on a* 80×80 *fragment. The thick lines are coalescing and converging northeast, the dashed lines are coalescing and converging southwest.*

$s(x, \omega'), s^2(x, \omega'), \dots$ is

$$\prod_{n\geq 0} \omega(s^n(x, \omega'), s^{n+1}(x, \omega')) = \prod_{n\geq 0} \left(1 - \frac{3}{h^2(s^n(x, \omega')) + 4}\right)$$
$$\geq \prod_{n\geq 0} \left(1 - \frac{3}{n^2 + 4}\right)$$

which is strictly positive. □

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While preparing this manuscript we heard of an announcement by O. Adelman of a zero-one law which might have some overlap with our result.

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