Abduction as a strategy for concept formation in mathematics: Cardano postulating a negative.

Albrecht Heeffer
Center for Logic and Philosophy of Science
Ghent University, Belgium

Abstract

When dealing with abductive reasoning in scientific discovery, historical case studies are focused mostly on the physical sciences, as with the discoveries of Kepler, Galilei and Newton. We will present a case study of abductive reasoning in early algebra. Two new concepts introduced by Cardano in his *Ars Magna*, imaginary numbers and a negative solution to a linear problem, can be explained as a result of a process of abduction. We will show that the first appearance of these new concepts fits very well Peirce's original description of abductive reasoning. Abduction may be regarded as one important strategy for the formation of new concepts in mathematics.

Peirce on abduction: a recapitulation

Peirce's definition of abduction has been understood and explained in two different meanings, coined by Magnani (2001) as *selective* and *creative* abduction. Selective abduction is the process of finding the right explanatory hypothesis from a given set of possible explanations. A common example of selection is medical diagnosis. In contrast, creative abduction *generates* the (right) explanatory hypothesis. As we are discussing concept formation, the abductive reasoning in our case study is of the creative type. We propose to view the formation of a new concept in mathematics as the hypothesis abduced to explain an anomaly. Peirce does not discuss concept formation as such but comes very close when he describes abduction as "the only logical operation which introduces new ideas" (1958, 5, 171). The existence of an anomaly is crucial in the cases we have studied, and is expressed by Peirce in similar wordings. He describes abduction as a form of inference motivated by the observation of a "surprising fact" (1958, 5, 188-9), reasoning initiated by "genuine doubt" and "genuine surprise" (1958, 5, 524), a motivation to break away from our habits "due to some novel experience" (ibid.). Abduction takes place when "we find ourselves confronted with some experience contrary to our expectations" (1958, 7, 36). We will show below that Peirce's depiction of the mental state associated with abduction aptly describes the historical cases under discussion.

A last relevant aspect of Peirce's account of abduction is that the generated hypothesis must follow specific conditions to become the explanatory hypothesis: "Namely, the hypothesis cannot be admitted, even as a hypothesis, unless it be supposed that it would account for the facts or some of them" (1958, 5, 188-9). This necessitates some necessary connection between the novel hypothesis and the observed anomaly. The generated hypothesis delivers, at that time, the most adequate explanation for the "surprising fact".

We will now sketch the historical background in which the two original concepts emerged in algebra and why they can be considered as explanations for the observed anomalies.

The historical context

In 1545 Cardano published his *Ars Magna*, which is considered to be one of the landmarks in early algebra. The main cause for this recognition is the publication of the rules for solving certain types of cubic equations. While this is regarded as one of the major achievements of sixteenth-century algebra we believe Cardano's work deserves even greater credit for introducing three important conceptual innovations. All three have contributed to the emergence of symbolic algebra. The first is the introduction of operations between two linear equations in order to eliminate unknowns. Until then operations were only performed on a single equation. The second is the acceptance of negative solutions to linear and quadratic equations. The third is the first formulation of what would become imaginary numbers. These last two concepts are the subject of our analysis. They are introduced in one chapter called *de regula falsum ponendis* (the rule of postulating a negative). We can therefore assume that Cardano believes they share a common mechanism. This historical case allows us to expound the idea that this mechanism is one of abduction and plays a major role in the formation of these two new mathematical concepts.

Negative solutions in early algebra

Firstly, we have to underline that negative solutions to linear equations and negative roots of quadratic equations where completely unacceptable before Cardano. Those who have studied so-called negative solutions in early algebra (Sesiano 1985, Gericke 1996) failed to recognize that by introducing a debt in linear problems, Renaissance abacus masters explicitly deny negative solutions. Negative solutions were then a major oddity for which no sensible explanation could be given.

Algebraic problem solving in Renaissance Italy was imbedded in a rigid rhetorical structure, being both its major strength and its weakness. The rhetorical structure compensated for the lack of symbolism but prohibited, for example, multiple solutions to quadratic problems. Where two positive solutions for certain type of quadratic problems were accepted in Arabic algebra, their recognition gradually fades within the abacus tradition (Heeffer, 2006). The reason is that the analytical structure of the problem solving process poses one specific arithmetical value as the *cosa*, the unknown. When one poses the *cosa* for the smaller part of a division problem which reduces to a quadratic one, it makes no sense to arrive at a value for both parts.

Negative values in an algebraic solution to a problem first turn up in Europe in the work of Fibonacci. Apart from his *Liber Abbaci*, his best known work, Fibonacci wrote several other texts. In the *Flos*, several linear problems are discussed at length. A typical problem is that of men comparing their money:[1]

> There are four man each having a number of bezants. The first, with half of the bezants of the three others, has 33, the second, with a third part of the bezants of the others, has 35, again, the third, with one fourth of the bezants of the others, has 36, and the fourth, with one fifth of the bezants of the first, second and third, has 37. Asked is how much each of them has.
>
> .

This is a very old problem. Known commonly as the problem of men who want to buy a horse, its history is discussed by Tropfke (1980, 608) and Singmaster (2004, 7R.2). Early versions appear in Diophantus' *Arithmetica* and in Chinese and Hindu texts. A possible source of Fibonacci is Al-Karkhī (Woepcke 1853). However, the use of *bisanti* suggests an influence from a Byzantine source. Fibonacci was the first in the western world to apply algebra to the solution of the problem. In a modern reading, the problem can be represented by a set of linear equations using the unknown *x*, *y*, *z* and *t* for the shares of the four persons:

$$x + \frac{1}{2}(y + z + t) = 33$$

$$y + \frac{1}{3}(x + z + t) = 35$$

$$z + \frac{1}{4}(x + y + t) = 36$$

$$t + \frac{1}{5}(x + y + z) = 37$$

Such representation allows us to grasp the problem at once. With four equations and four unknowns, the problem is a determinate one which must lead to a unique solution. However, this symbolic representation is misleading for Fibonacci does not use these equations at all and it is inadequate to consider his solution method a resolution of equations. Furthermore, the solution method of the original text does not use four unknowns but a single one. At no point in the text are the shares of the four persons considered as algebraic unknowns. The problem is introduced with the following words:[2]

> I have fixed these numbers with great consideration, with the purpose that the solution of this problem falls in entire numbers; and I will show that with the given conditions it is insoluble.

Fibonacci starts by warning cardinal Ranerio Capocci, to whom the book is dedicated, that given the values, the problem has no proper solution, it is unsolvable. Looking for an integral solution he arrived at a negative value for one of the shares and this was difficult to accept at that time. Problems with similar solutions from Fibonacci's *Flos* and *Liber Abbaci* have been discussed both by Sesiano (1985) and Gericke (1996). Fibonacci must have been puzzled when he arrived at the value of a man's share being 33 – 36. He probably recalculated the whole procedure to double check for possible errors. In the end he found an interpretation for it. The man started with a debt to the other persons of 3 bezants. Sesiano and Gericke have no trouble interpreting the text as an instance of a solution with a negative number. We believe that a more cautious approach is appropriate here. The full acceptance of negative solutions to an equation was not established before the seventeenth century. Fibonacci explicitly avoids a negative solution in this text. He calls the problem unsolvable "unless one considers the first man having a debt". The interpretation as a debt actually disaffirms a negative solution. It must have bothered Fibonacci to such degree that he felt the need to apology before introducing the problem to the cardinal. He also looked for larger values to avoid assuming a debt. Probably by trial and error, Fibonacci tweaked the problem values to get rid of the "impossible solution":[3]

And if we will say that the first man together [with the part of the others] has the required 181, the second 183, the third party 184, the fourth 185; all arranged as written, we will find that the first man has 1, the second 94, the third party 125, and he fourth 141.

$$x + \frac{1}{2}(y + z + t) = 181$$
$$y + \frac{1}{3}(x + z + t) = 183$$
$$z + \frac{1}{4}(x + y + t) = 184$$
$$t + \frac{1}{5}(x + y + z) = 185$$

In fact, the revised problem as given by Fibonacci is the smallest integral solution for the given shares and the values in arithmetical progression n, $n + 2$, $n + 3$, $n + 4$, as shown by Sesiano (1985). Fibonacci must have spend a long time increasing the values until he arrived at four positive quantities. Such cases of negative values, and their interpretation as a debt, appear in isolated instances throughout the abacus texts of the next centuries. However, writing the history of negative numbers, Sesiano and Gericke look for the elements that support their story, and fall into the trap of biased observations. At the contrary, these cases support our claim that negative solutions were not accepted before the sixteenth century.

Before turning to Cardano's approach to negative values, we first sketch the context of using geometrical demonstrations for solving quadratic problems. This will allow us to understand Cardano's reasoning within the context of sixteenth-century algebra.

Geometrical demonstrations and quadratic problems

The algebraic solution of quadratic problems in medieval Europe was fully determined by the procedures of Arabic algebra. This knowledge was introduced in Europe through three Latin translations of al-Khwārizmī's *Algebra*, extant in sixteen manuscripts (Hughes 1982). These translations have been identified as from Robert of Chester (c. 1145), Gerard of Cremona (c. 1150) and Guglielmo de Lunis (c. 1215), although there is still discussion if the latter translation was to Latin or to Italian. What became available to the West was only the first part of al-Khwārizmī's treatise. Here, al-Kwārizmī gives solutions to algebraic problems by applying proven procedures in an algorithmic way. Early Arabic algebra recognized three types of quadratic problems, each having a given solution procedure. The validity of these solutions is demonstrated by geometrical diagrams. In contrast with Babylonian algebra the solution method is not geometrical in nature, only the demonstration is. In order to understand Cardano's approach we briefly discuss the geometrical demonstration of the case in which the *census* (or *subtancia* for Robert of Chester) and number equals a number of roots. The traditional example in symbolic representation is the equation

$$x^2 + 21 = 10x$$

This example appears in al-Khwārizmī's treatise (Robert of Chester, c. 1145, Karpinski 85; Hughes 1989, 39-41) as well as in the algebra of Abū Kamil (Levey 1966, 40; Sesiano 1993, 332) and many abacus texts of the fourteenth and fifteenth centuries. The geometrical demonstration is based on the representation in Figure 1.

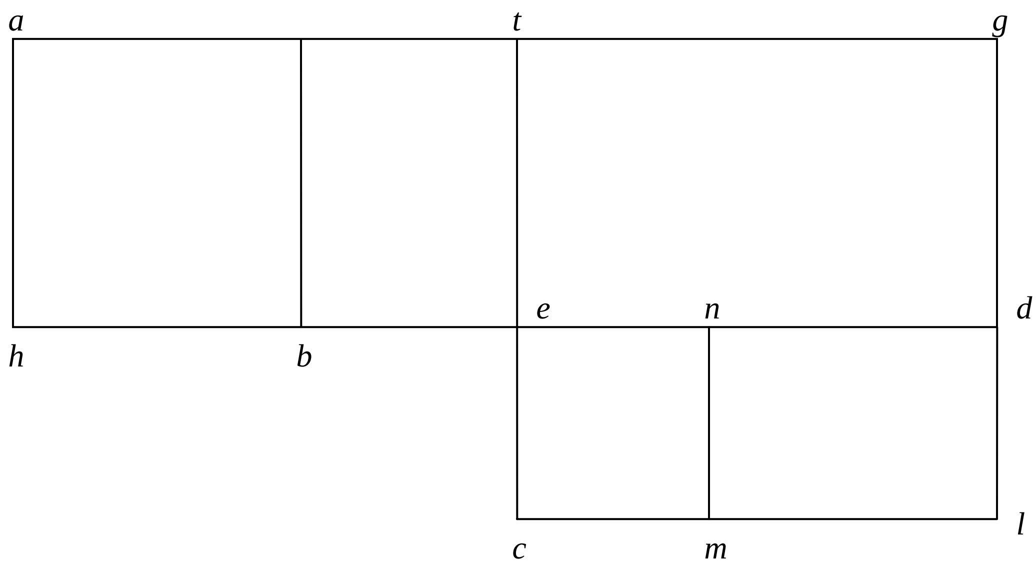

Figure 1: a geometrical proof for a quadratic problem from Arab algebra

The proof runs as follows:

- let x^2 be *ab* and 10*x* be *ad*
- therefore *ah* equals *x*, *hd* equals 10 and the area of *bg* is 21
- bisect *hd* at point *e*
- construct the square *tl* with side *tg*; its area is equal to 25
- subtract from *tl* the areas *tb* and *td* which results in the square *cn*
- this square has area 25 – 21 = 4, its side *ec* therefore is 2
- then *x* is *tc* minus *ec* or 5 – 2 equal to 3

So, for this type of quadratic problems, two positive solutions are recognized.

With this background, we can now turn to the chapter in which Cardano introduces some important new concepts in algebra.

Cardano posing a negative

Chapter 37 of the *Ars Magna* is named *de regula falsum ponendis*, or the "Rule of Postulating a Negative". Cardano introduces the subject by giving three reasons why it may be useful to postulate a negative.

- One can "assume a negative". The term "assuming" here refers to the rhetorical structure of algebraic problem solving. Every solution in abacus texts commences with the assumption that the rhetorical unknown equals some unknown quantity in the problem formulation. Assuming a negative thus means that one can use a negative *cosa*, or $-x$ instead of x, for the unknown quantity. This application is illustrated with three problems.
- One "seeks the square root of a negative".[4] After a failed geometrical demonstration Cardano solves two problems using imaginary numbers, for the first time in the history of mathematics.
- One can "seek what is not". With this cryptic formulation, Cardano refers to a negative which is neither of the first category nor of the second. However, his result stems from a wrong manipulation of imaginary numbers.[5]

We will discuss the first two applications of postulating a negative.

A negative solutions to a linear problem

Cardano illustrates the first use of postulating a negative with a problem which appears as a quadratic one, but reduces to a linear one (Cardano, *Opera Omnia* III, 286-7; Witmer 1968, 218):

> The dowry of Francis' wife is 100 aurei more than Francis' own property, and the square of the dowry is 400 more than the square of his property. Find the dowry and the property.
> We assume that Francis has $-x$; therefore the dowry of his wife is $100 - x$. Square the parts, making x^2 and
>
> $$10000 + x^2 - 200x$$
>
> The difference between these is 400 aurei. Therefore
>
> $$x^2 + 400 + 200x = 10000 + x^2$$
>
> Subtract the common terms and you will have 9600 equal to $200x$, wherefore x equals 48 and so much he has in the negative, i.e., is lacking, and the dowry will be the residue of 100, namely 52. Therefore Francis has -48 aurei, without any capital or property, and the dowry of his wife is 52 aurei. By working this way, you can solve the most difficult and inextricable problems.

A straightforward transliteration of the problem into algebra would lead to

$$x^2 + 200x + 10000 = x^2 + 400$$
$$200x = -9600$$

in which – 48 is a valid solution to the equation. As this was still considered an anomaly in the early sixteenth century, Cardano assumes that the property is – *x* instead of *x*. By doing so, he arrives at a positive value 48 for the unknown quantity. As he posed that the quantity of the dowry was – *x*, the value thus becomes – 48. The Latin text reads "igitur Franciscus habuit 48 aureos debiti", which sounds alike the interpretation of a debt by Fibonacci to the 'negative solution' of a linear problem. However, now we do see the acceptance of a negative solution. In this and the next problem solutions, Cardano uses '1 *res* m.' for – *x*, and '4 m.' for the result – 4. The reason why negative values now become acceptable is related to the problem formulation. In order to make this more clear, let us go back to the traditional quadratic problems. As argued before, the rhetorical structure of abacus texts limits the solution of a quadratic problem to a single value.
The most common problem, known already in Babylonian algebra, is to find two numbers, given their sum (or difference) and their product. In modern symbolism this amounts to the general problem:

$$x + y = a$$
$$xy = b$$

In the abacus tradition, with some rare exceptions, all problems are solved using a single unknown. Depending on the initial assumption in the problem solution, the unknown represents one of the two unknown quantities. For example, the problem with the difference between two numbers being 7 and their product 60, could be represented by the symbolic equations

$$b = a + 7$$
$$ab = 60$$

If the *cosa*, x is used for the value *b*, then the quadratic equation is

$$x^2 - 7x - 60 = 0$$

with solution $x = 12$. On the other hand, if we use *x* for the value *b*, then the equation becomes

$$x^2 + 7x - 60 = 0$$

with the single solution $x = 5$. Arguing that a quadratic equation has two roots, therefore has little meaning in problem solving in the abacus tradition. The reformulation of the problem with a hypothetical unknown determines the solution than one will find.

Now, Cardano uses – *x* as the rhetorical unknown. By "postulating a negative" he evidently arrives at a negative value. Embedded within the rhetorical structure, a negative solution now becomes perfectly acceptable. The acceptance of negative solutions is made possible by postulating negative unknowns.

Within two decades after the publication of Cardano's *Ars Magna*, isolated negative terms appear more frequently in algebra text books. The concept of a negative solution and consequently of negative algebraic terms are the result of an abductive reasoning step. Negative solutions were considered an anomaly since their first appearance in European algebra with Fibonacci. However, because of an epistemological trust in the value of the algebraic procedures leading to these solutions, an explanation urges itself on the practitioners of early algebra. Cardano looks for an explanatory hypothesis to make a negative solution acceptable. He finds this abductive hypothesis in the analytical reasoning of algebra and the rhetorical structure of abacus texts. Both start by defining a hypothetical unknown for an unknown quantity of the problem. By hypothesizing a negative unknown, a negative value for the quantity becomes perfectly acceptable.

Imaginary numbers

With the second motivation for postulating a negative, Cardano again writes history. A similar hypothesis allows Cardano to apply the rules of signs to negative values inside the root and in this way be the first to partially formulate the operations on imaginary numbers. As pointed out by Leibniz (1712), imaginary numbers can best be defined by enumerating the symbolic calculations that are possible on those objects. This coincides with the historical emergence of the imaginary numbers. Incidentally, this is also the most common way in which secondary school students are first exposed to them.

The root of minus fifteen was an even greater abnormality than negative solutions. Cardano arrives at $\sqrt{-15}$ using standard problem solving methods. Devising a geometrical demonstration for the problem, the best qualification Cardano could come up with is "a missing surface". Confronted with this strange new concept, Cardano arrives at a strategy to make it, to a large extend, acceptable. He does so by finding the right abductive hypothesis which leads to the acceptance of the new concept, posing a negative value.

adeo est subtile, ut sit inutile

Cardano was the first to perform a calculation with - what was later called - imaginary numbers. In the chapter we have just discussed Cardano (*Opera Omnia* III, 287; Witmer 1968, 219) writes:

> The second species of negative assumption involves the square root of a negative. I will give an example: If it should be said, divide 10 into two parts the product of which is 30 or 40, it is clear that this case is impossible. Nevertheless, we will work thus: We divide 10 into two equal parts, making each 5. These we square, making 25. Subtract 40, if you will, from the 25 thus produced, as I showed you in the chapter on operations in the sixth book, leaving a remainder of – 15, the square root of which added to or subtracted from 5 gives parts the product of which is 40. These will be
>
> $$5+\sqrt{-15} \text{ and } 5-\sqrt{-15}.$$

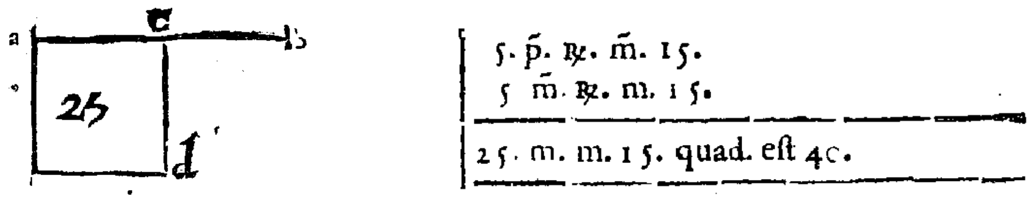

Figure 2: First occurrence of imaginary numbers from Cardano's *Ars Magna* (1545)

Using a geometrical demonstration he tries to get a grasp on the new concept (Figure 2). He proceeds as with the standard demonstration of the rule for solving quadratic problems. Let *AB* be the 10 to be divided. Divide the line at *C* into two equal parts. Square *AC* to *AD*. Since 40 is four times 10 this corresponds with the rectangle 4*AB*. Now $\sqrt{-15}$ corresponds with the subtraction of *AD* by the larger 4*AB*. Thus, Cardano finds that this strange new object is a negative or a missing surface. This makes no sense to him and therefore he writes that the problem is impossible.

Cardano still struggled with the interpretation of $\sqrt{-15}$, but was "putting aside the mental tortures involved" and performed the operation $(5+\sqrt{-15})(5-\sqrt{-15})$ correctly to arrive at 25 + 15 or 40. Multiplying the two binomials produces four terms. The first is evidently 25. The second and third are cancelled out by their signs.

$$5\left(\sqrt{-15}\right)+5\left(-\sqrt{-15}\right)$$

The innovation lies in the fourth.

$$\left(\sqrt{-15}\right)\left(-\sqrt{-15}\right)=-(-15)$$

Remark that the only way to accept these operations for the Renaissance mind is to approach them as purely symbolical. By replacing the – 15 with *x* as in "posing a negative" and you get a common operation:

$$\left(\sqrt{x}\right)\left(-\sqrt{x}\right)=-x$$

Cardano is able to accept the operations as valid *on the basis of their symbolic structure*, although he does not know what it means. Cardano closes the discussion with the words that this is a kind of arithmetic subtlety, "as refined as it is useless" ("adeo est subtile, ut sit inutile"). However, he is convinced that the operations are correct.

Concept formation by abduction

In both cases Cardano was confronted with an anomaly which appeared for several centuries of algebraic practice. Rather than avoiding such problems or keeping them in the dark, Cardano succeeded in finding a satisfactory solution, which he called "postulating a negative". He constructed an explanatory hypothesis which made these “surprising facts” acceptable. By doing so he created new concepts, such as imaginary numbers, which eventually lead to a whole new branch within mathematics. Both cases provide historical evidence of the emergence of new concepts in mathematics by abductive reasoning.

The question we can now ask is if there would have been other ways to deal with these anomalies at the time of Cardano. Most probably there were. In some way, the interpretation of Fibonacci of a negative solution as a debt is a form of abductive reasoning which also resolves the issue. It is, however, not an interesting one from a mathematical point of view. Also for imaginary numbers there may have been alternatives. In a recent paper, Jean Paul van Bendegem argued that a geometrical interpretation operating on missing surfaces could have provided an alternative to the symbolic model as it has emerged.[6] van Bendegem applies his geometrical interpretation to the so-called *casus irreducibilis*

$$x^3 = px + q \quad \text{with} \quad \left(\frac{1}{3}p\right)^3 > \left(\frac{1}{2}q\right)^2, \qquad \text{e.g. } x^3 = 15x + 4,$$

the case studied by Bombelli (1572). Though he was well aware of the problem, Cardano quietly dismissed the irreducible cases in the first chapters of the *Ars Magna*. In that way, imaginary numbers are irrelevant for Cardano’s approach to the cubic. But the geometrical interpretation proposed by van Bendegem is perfectly applicable for the quadratic problem discussed above.

However, we would like to argue that Cardano’s approach was indeed the best one available at that time. The period of 1515 tot 1545 was one of the increasing use of symbols in mathematics. In fact, the plus and minus sign were introduced within the context of algebra in early sixteenth-century German manuscripts.[7] Once algebraic terms are affected by symbols, they turn into tools much more powerful. The use of symbols allows to abstract from the arithmetical content they represent. New operations become possible because they can be applied symbolically, without accounting for the values they stand for. Even if the idea of a negative value or the root of a negative value is still ridiculous, it now becomes possible to reason with negative terms on a symbolic level. As shown above, symbolic abstraction plays an important role in both cases of abductive reasoning by Cardano.

Acknowledgements

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Notes

[1] This and the following quotations are my translation from the Italian by Picutti 1983, 324-6, 365-6: “Vi sono dunque quattro uomini che hanno dei bisanti; il primo di loro con la metà dei bisanti dei rimanenti tre uomini, ne abbia 33; il secondo con la terza parte dei bisanti dei rimanenti tre ne abbia 35; ancora, il terzo

con ¼ dei bisanti dei rimanenti ne abbia 36; il quarto poi, con 1/5 dei bisanti del primo e del secondo e del terzo uomo, ne abbia 37. Si domanda quanto abbia avuto ciascuno di essi".

[2] Ibid. "Ho fissato questi numeri meditatamente, allo scopo che la soluzione di questo problema cada in numeri interi; e mostrerò che con le condizioni poste esso è insolubile".

[3] Picutti 1983, 326: "E se diremo che il primo uomo abbia assieme al chiesto 181, il secondo 183, il terzo 184, il quarto 185, disposto tutto come sopra scritto, troveremo che il primo uomo ha 1, il secondo 94, il terzo 125, il quarto 141".

[4] Witmer (1968, 215) wrongly translates this as "one seeks a negative square root". Cardano 1663, III, 286, writes " ℜ m", meaning $\sqrt{-x}$.

[5] In the last problem Cardano wrongly assumes that $\left(\frac{1}{4}\right)\left(-\sqrt{-\frac{1}{4}}\right)=\frac{1}{8}$. Operations with imaginary numbers can be tricky as shown by Euler (1770) in the example of the fallacy

$$-2=\left(\sqrt{-2}\right)\left(\sqrt{-2}\right)=\sqrt{\left(-2\right)\left(-2\right)}=\sqrt{4}=2$$

[6] van Bendegem, unpublished, "What could an 'alternative' mathematics be like, if anything?", presented at the ninth VlaPoLo conference, 6-8 May 2005, Ghent.

[7] The first appearance of the + sign in the Dresden codex C80, f. 350^v, written around 1486. Exemplary is the anonymous Vienna codex 5277, written between 1500 and 1518. Here the rules of signs are introduced with symbols as operations on polynomials.

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