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Abelian Ideals of a Borel Subalgebra and Long Positive Roots

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Let $\mathfrak{b}$ be a Borel subalgebra of a simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{Ab}$ denote the set of all Abelian ideals of $\mathfrak{b}$. It is easily seen that any $\mathfrak{a} \in \mathfrak{Ab}$ is actually contained in the nilpotent radical of $\mathfrak{b}$. Therefore, $\mathfrak{a}$ is determined by the corresponding set of roots. More precisely, let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ lying in $\mathfrak{b}$ and let Δ be the root system of the pair $(\mathfrak{g}, \mathfrak{t})$. Choose Δ^+, the system of positive roots, so that the roots of $\mathfrak{b}$ are positive. Then $\mathfrak{a} = \oplus_{\gamma \in I} \mathfrak{g}_\gamma$, where I is a suitable subset of Δ^+ and $\mathfrak{g}_\gamma$ is the root space for $\gamma \in \Delta^+$. It follows that there are finitely many Abelian ideals and that any question concerning Abelian ideals can be stated in terms of combinatorics of the root system.

An amazing result of D. Peterson says that the cardinality of $\mathfrak{Ab}$ is $2^{\operatorname{rk}\mathfrak{g}}$. His approach uses a one-to-one correspondence between the Abelian ideals and the so-called *minuscule* elements of the affine Weyl group $\widehat{W}$. An exposition of Peterson's results is found in [5]. Peterson's work appeared to be the point of departure for active recent investigations of Abelian ideals, ad-nilpotent ideals, and related problems of representation theory and combinatorics [1, 2, 3, 4, 5, 6, 7, 8]. We consider $\mathfrak{Ab}$ as poset with respect to inclusion, the zero ideal being the unique minimal element of $\mathfrak{Ab}$. Our goal is to study this poset structure. It is easily seen that $\mathfrak{Ab}$ is a ranked poset; the rank function attaches to an ideal its dimension. It was shown in [8] that there is a one-to-one correspondence between the maximal Abelian ideals and the long simple roots of $\mathfrak{g}$. (For each simple Lie algebra, the maximal Abelian ideals were determined in [10].) This correspondence possesses a number of nice properties, but the very existence of it was demonstrated in

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a case-by-case fashion. Here, we give a conceptual explanation of that empirical observation. More generally, we prove that

(i) there is a natural mapping $\tau : \overset{\circ}{\mathfrak{Ab}} \to \Delta^+_l$, where $\overset{\circ}{\mathfrak{Ab}}$ is the set of all nontrivial Abelian ideals and Δ^+_l is the set of long positive roots, see Proposition 2.5. We say that $\tau(I)$ is the rootlet of I;

(ii) each fibre $\mathfrak{Ab}_\mu := \tau^{-1}(\mu)$ is a poset in its own right, and we prove that $\mathfrak{Ab}_\mu$ contains a unique maximal and a unique minimal element, see Theorem 3.1;

(iii) if I is a maximal Abelian ideal, then $\tau(I)$ is a (long) simple root. Restricting τ to $\mathfrak{Ab}_{max}$, the set of maximal Abelian ideals, yields the above correspondence.

The uniqueness of maximal and minimal elements suggests that they can have a nice description. For any $\mu \in \Delta^+_l$, we explicitly describe the minimal ideal in $\mathfrak{Ab}_\mu$ and the corresponding minuscule element of $\widehat{W}$ (see Theorem 4.2). Let $I(\mu)_{min}$ denote the minimal element of $\mathfrak{Ab}_\mu$. The collection of these ideals has a transparent characterization. Given $I \in \overset{\circ}{\mathfrak{Ab}}$, we have $I = I(\mu)_{min}$ for some μ if and only if all roots of I are not orthogonal to θ, the highest root (see Theorem 4.3). We also determine the generators of the ideals $I(\mu)_{min}$.

In Section 5, the structure of posets $\mathfrak{Ab}_\mu$ is considered. It is shown that $\#(\mathfrak{Ab}_\mu) > 1$ if and only if $(\mu, \theta) = 0$. A criterion is also given for $\#(\mathfrak{Ab}_\mu) > 2$. In fact, we can give a general description of $\mathfrak{Ab}_\mu$ and, in particular, of the maximal element $I(\mu)_{max} \in \mathfrak{Ab}_\mu$. This description is in accordance with (actually, is inspired by) our computations for all simple Lie algebras, but we cannot give yet a general case-free proof. This description shows that any $\mathfrak{Ab}_\mu$ is isomorphic to the poset of all ideals sitting inside of an Abelian nilpotent radical. More precisely, there are a regular[1] simple subalgebra $\mathfrak{g}_{(\mu)} \subset \mathfrak{g}$ and a maximal parabolic subalgebra $\mathfrak{p}_{(\mu)} \subset \mathfrak{g}_{(\mu)}$ with Abelian nilpotent radical $\mathfrak{p}^{nil}_{(\mu)}$ such that $\mathfrak{Ab}_\mu$ is isomorphic to the poset of all Abelian $\mathfrak{b}_{(\mu)}$-ideals in $\mathfrak{p}^{nil}_{(\mu)}$. As is well known, the latter is isomorphic to the weight poset of a fundamental representation of the Langlands dual Lie algebra $\mathfrak{g}^\vee_{(\mu)}$ [9, 11]. Since this fundamental representation is minuscule, the weight poset of it is isomorphic to the Bruhat poset $W^{(\mu)}/W^{(\mu)}_\varphi$. Here, $W^{(\mu)}$ is the Weyl group of $\mathfrak{g}_{(\mu)}$ (or $\mathfrak{g}^\vee_{(\mu)}$) and $W^{(\mu)}_\varphi$ is the stabilizer of the fundamental weight in question. Such posets are also called *minuscule*. This completely solves the problem of describing the structure of $\mathfrak{Ab}_\mu$.

In Section 6, the general theory developed so far is illustrated with examples related to all simple Lie algebras. We compute $\#(\mathfrak{Ab}_\mu)$ for each $\mu \in \Delta^+_l$. For $\mathfrak{sl}_n$, $\mathfrak{sp}_{2n}$, $\mathbf{G}_2$, and $\mathbf{F}_4$, an explicit description of the posets $\mathfrak{Ab}_\mu$ is given. In case of $\mathfrak{sl}_n$, an algorithm is presented for writing out the minuscule element corresponding to an Abelian ideal.

[1]This means that the subalgebra is normalized by $\mathfrak{t}$.

Our proofs are based on the relationship between the Abelian ideals and the minuscule elements in the affine Weyl group. We repeatedly use the procedure of extension of Abelian ideals that follows from this relationship.

1 Preliminaries on Abelian ideals

1.1 Main notation

Let Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$ and let W be the usual Weyl group. For $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is the corresponding root space in $\mathfrak{g}$. Δ^+ is the set of positive roots and $\rho = (1/2)\sum_{\alpha\in\Delta^+}\alpha$. $\Pi = \{\alpha_1, \dots, \alpha_p\}$ is the set of simple roots in Δ^+.

We set $V := \mathfrak{t}_{\mathbb{Q}} = \oplus_{i=1}^{p}\mathbb{Q}\alpha_i$ and denote by $(\cdot,\cdot)$ a W-invariant inner product on V. As usual, $\mu^\vee = 2\mu/(\mu,\mu)$ is the coroot for $\mu \in \Delta$. Letting $\widehat{V} = V \oplus \mathbb{Q}\delta \oplus \mathbb{Q}\lambda$, we extend the inner product $(\cdot,\cdot)$ on $\widehat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta,\delta) = (\lambda,\lambda) = 0$ and $(\delta,\lambda) = 1$.

$\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}$ is the set of affine real roots and $\widehat{W}$ is the affine Weyl group.

Then $\widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geq 1\}$ is the set of positive affine roots and $\widehat{\Pi} = \Pi \cup \{\alpha_0\}$ is the corresponding set of affine simple roots. Here $\alpha_0 = \delta - \theta$, where θ is the highest root in Δ^+. The inner product $(\cdot,\cdot)$ on $\widehat{V}$ is $\widehat{W}$-invariant.

For α_i $(0 \leq i \leq p)$, we let s_i denote the corresponding simple reflection in $\widehat{W}$. If the index of $\alpha \in \widehat{\Pi}$ is not specified, then we merely write s_α. The length function on $\widehat{W}$ with respect to $s_0, s_1, \dots, s_p$ is denoted by l. For any $w \in \widehat{W}$, we set

$$\widehat{N}(w) = \left\{\alpha \in \widehat{\Delta}^+ \mid w(\alpha) \in -\widehat{\Delta}^+\right\}. \tag{1.1}$$

If $w \in W$, then $\widehat{N}(w) \subset \Delta^+$ and we also write $N(w) = \widehat{N}(w)$ in this case.

1.2 Abelian ideals

Let $\mathfrak{a} \subset \mathfrak{b}$ be an Abelian ideal. It is easily seen that $\mathfrak{a} \subset [\mathfrak{b},\mathfrak{b}]$. Therefore $\mathfrak{a} = \oplus_{\alpha\in I}\mathfrak{g}_\alpha$ for a subset $I \subset \Delta^+$, which is called the *set of roots of* $\mathfrak{a}$. As our exposition will be mostly combinatorial, an Abelian ideal will be identified with the respective set of roots, that is, I is said to be an Abelian ideal, too. Whenever we want to explicitly indicate the context, we say that $\mathfrak{a}$ is a *geometric* Abelian ideal, while I is a *combinatorial* Abelian ideal. In the combinatorial context, the definition of an Abelian ideal (subalgebra) can be stated as follows: $I \subset \Delta^+$ is an Abelian ideal if the following two conditions are satisfied:

(a) for any $\mu, \nu \in I$, we have $\mu + \nu \notin \Delta$;

(b) if $\gamma \in I$, $\nu \in \Delta^+$, and $\gamma + \nu \in \Delta$, then $\gamma + \nu \in I$.

If I satisfies only (a), then it is called an Abelian *subalgebra*.

Following Peterson, an element $w \in \widehat{W}$ is said to be *minuscule* if $\widehat{N}(w)$ is of the form $\{\delta - \gamma \mid \gamma \in I\}$, where I is a subset of Δ^+. It was shown by Peterson that such an I is a combinatorial Abelian ideal and, conversely, each Abelian ideal occurs in this way, see [3, Proposition 2.8] and [5]. Hence one obtains a one-to-one correspondence between the Abelian ideals of $\mathfrak{b}$ and the minuscule elements of $\widehat{W}$. If $w \in \widehat{W}$ is minuscule, then I_w (resp., $\mathfrak{a}_w$) is the corresponding combinatorial (resp., geometric) Abelian ideal. That is,

$$I_w = \{\gamma \in \Delta^+ \mid \delta - \gamma \in \widehat{N}(w)\}, \qquad \mathfrak{a}_w = \oplus_{\alpha \in I_w} \mathfrak{g}_\alpha. \tag{1.2}$$

Conversely, given $I \in \mathfrak{Ab}$, we write $w\langle I\rangle$ for the respective minuscule element. Notice that

$$\dim \mathfrak{a}_w = \#(I_w) = l(w). \tag{1.3}$$

Accordingly, being in combinatorial (resp., geometric) context, we speak about cardinality (resp., dimension) of an ideal. Throughout the paper, I or I_w stands for a combinatorial Abelian ideal.

2 Generators of Abelian ideals and long positive roots

Given an Abelian ideal I, we say that $\gamma \in I$ is a *generator* of I if $\gamma - \alpha \notin I$ for all $\alpha \in \Delta^+$. Clearly, this is equivalent to the fact that $I \setminus \{\gamma\}$ is still an Abelian ideal. Conversely, if $\varkappa$ is a maximal element of $\Delta^+ \setminus I$ (i.e., $(\varkappa + \Delta^+) \cap \Delta \subset I$) and $(\varkappa + I) \cap \Delta = \varnothing$, then $I \cup \{\varkappa\}$ is an Abelian ideal. These two procedures show that the following proposition is true.

Proposition 2.1. Suppose $I \subset J$ are two Abelian ideals. Then there is a chain of Abelian ideals $I = I_0 \subset I_1 \subset \cdots \subset I_m = J$ such that $\#(I_{i+1}) = \#(I_i) + 1$. In other words, $\mathfrak{Ab}$ is a ranked poset, with cardinality (dimension) of an ideal as the rank function. □

In the geometric setting, the set of generators has the following description. For an ideal $\mathfrak{a} = \oplus_{\gamma \in I} \mathfrak{g}_\gamma \subset \mathfrak{b}$, there is a unique $\mathfrak{t}$-stable space $\tilde{\mathfrak{a}} \subset \mathfrak{a}$ such that $\mathfrak{a} = [\mathfrak{b}, \mathfrak{a}] \oplus \tilde{\mathfrak{a}}$. Then γ is a generator of I if and only if it is a weight of $\tilde{\mathfrak{a}}$.

However, we need a description of generators of I in terms of the respective minuscule element. As usual, we write $\gamma > 0$ (resp., $\gamma < 0$), if $\gamma \in \widehat{\Delta}^+$ (resp., $\gamma \in -\widehat{\Delta}^+$). Let $w \in \widehat{W}$ be minuscule. Because $\alpha_i \notin \widehat{N}(w)$ $(i = 1, \dots, p)$, any reduced decomposition of w must end up with s_0. Let $w = s_{i_1} \cdot \cdots \cdot s_{i_r} s_0$ be a reduced decomposition. As is well known, one then has

$$\begin{aligned}\widehat{N}(w) &= \{\alpha_0, s_0(\alpha_{i_r}), s_0 s_{i_r}(\alpha_{i_{r-1}}), \dots, s_0 s_{i_r} \cdots s_{i_2}(\alpha_{i_1})\} \\ &=: \{\delta - \theta, \delta - \gamma_r, \dots, \delta - \gamma_1\}.\end{aligned} \tag{2.1}$$

Here $\gamma_i \in \Delta^+$ and $I_w = \{\theta, \gamma_r, \ldots, \gamma_1\}$. By construction, we have $\delta - \gamma_1 = s_0 s_{i_r} \cdots s_{i_2}(\alpha_{i_1})$ and hence $w(\delta - \gamma_1) = -\alpha_{i_1}$. Thus, any reduced decomposition of w induces a total ordering on the set $\widehat{N}(w)$. Moreover, w takes the last element in $\widehat{N}(w)$ to $-\widehat{\Pi}$, that is, $w(\delta - \gamma_1) = -\alpha_{i_1}$.

It follows that if we "shorten" w, that is, consider the element $w' = s_{i_1} w$, then $\widehat{N}(w) = \widehat{N}(w') \cup \{\delta - \gamma_1\}$ and $w'(\delta - \gamma_1) = \alpha_{i_1}$. In particular, w' is also minuscule.

Theorem 2.2. Suppose $\gamma \in I_w$. Then γ is a generator of I_w if and only if $w(\delta - \gamma) \in -\widehat{\Pi}$. □

Proof. "⇐". Suppose $w(\delta-\gamma) = -\alpha_i$. This means that $w^{-1}(\alpha_i) = \gamma - \delta < 0$. Therefore, there exists a reduced decomposition of w starting with s_i, that is, $w = s_i w'$, where $l(w') = l(w) - 1$. Hence $\widehat{N}(w') = \widehat{N}(w) \setminus \{\delta - \gamma\}$ and w' is still a minuscule element. Thus, $I_w \setminus \{\gamma\}$ is an Abelian ideal.

"⇒". Suppose $w(\delta - \gamma) \notin -\widehat{\Pi}$, that is, $w(\delta - \gamma) = -\kappa_1 - \kappa_2$, where $\kappa_i \in \widehat{\Delta}^+$. Then $w^{-1}(\kappa_1) + w^{-1}(\kappa_2) = -(\delta - \gamma) < 0$. Assume for definiteness that $w^{-1}(\kappa_2) < 0$. Since $w^{-1}(-\kappa_2) > 0$ and $w(w^{-1}(-\kappa_2)) < 0$, we have $w^{-1}(-\kappa_2) \in \widehat{N}(w)$, that is, $w^{-1}(-\kappa_2) = \delta - \gamma_2$ for some $\gamma_2 \in I_w \subset \Delta^+$. It follows that $w^{-1}(-\kappa_1) = \delta - \gamma - \delta + \gamma_2 \in \Delta$. As $w(\gamma_2 - \gamma) = -\kappa_1 < 0$ and w is minuscule, we must have $\gamma_2 - \gamma < 0$. Thus γ is not a generator of I_w. ■

Remark 2.3. By a result of Cellini and Papi [3, Theorem 2.6], to any ad-nilpotent ideal of $\mathfrak{b}$ (not necessarily Abelian), one may attach a unique element of $\widehat{W}$. Then one can extend Theorem 2.2 to this setting. However, the proof becomes more involved, since the procedure of shortening does not work for the corresponding elements of $\widehat{W}$. We hope to consider related problems in a subsequent publication.

Theorem 2.4. Let I_w be an Abelian ideal and $\gamma \in \Delta^+ \setminus I_w$. Then $I_w \cup \{\gamma\}$ is an Abelian ideal if and only if $w(\delta - \gamma) \in \widehat{\Pi}$. □

Proof. "⇐". Suppose $w(\delta - \gamma) = \alpha_i$. Then $l(s_i w) = l(w) + 1$ and $\widehat{N}(s_i w) = \widehat{N}(w) \cup \{\delta - \gamma\}$. That is, $s_i w$ is again minuscule and hence $I_w \cup \{\gamma\}$ is an Abelian ideal.

"⇒". It is clear that γ is a generator for $I_w \cup \{\gamma\} =: I_{\tilde{w}}$. By Theorem 2.2, we then have $\tilde{w}(\delta - \gamma) \in -\widehat{\Pi}$. Assume that it is $-\alpha_i$. Then $w = s_i \tilde{w}$ and $w(\delta - \gamma) = \alpha_i$. ■

Given a nontrivial minuscule $w \in \widehat{W}$, it was noticed before that $w(\alpha_i) > 0$, $i \in \{1, \ldots, p\}$, and $w(\alpha_0) < 0$. We study the last element. Let Δ_l^+ denote the subset of *long* roots in Δ^+. In the simply laced case, all roots are proclaimed to be long.

Proposition 2.5. If w is a nontrivial minuscule element, then $w(\alpha_0) + \delta \in \Delta_l^+$. □

Proof. Since $w(\alpha_0)$ is negative, we can write $w(\alpha_0) = -k\delta - \gamma_0$, where $k \in \{0, 1, 2, \ldots\}$ and $\gamma_0 \in \Delta$. Recall that $\alpha_0 = \delta - \theta$.

(a) Assume $k \geq 2$. Then $w(2\delta - \theta) = -(k-1)\delta - \gamma_0 < 0$. This contradicts the fact that w is minuscule.

(b) Assume $k = 0$. Then $w(\delta - \theta) = -\gamma_0$ and $\gamma_0 \in \Delta^+$. It is clear that $w \in \widehat{W} \setminus W$. Write the expression of θ through the simple roots $\theta = \sum_{i=1}^{p} n_i \alpha_i$ and set $\gamma_i = w(\alpha_i)$. Then $\sum_{i=1}^{p} n_i \gamma_i = \gamma_0 + \delta$. Since γ_i's are positive and $\gamma_0 \in \Delta$, there exists a unique $i_0 \in \{1, \dots, p\}$ such that $n_{i_0} = 1$, $\gamma_{i_0} \in \delta + \Delta$, and $\gamma_i \in \Delta$ for $i \neq i_0$. It follows that the elements $-\gamma_0, \gamma_j$ $(j \geq 1, j \neq i_0)$ form a basis for Δ. Hence there is $w' \in W$ which takes $-\gamma_0, \gamma_j$ $(j \neq i_0)$ to $\alpha_1, \dots, \alpha_p$. Because $w'(\gamma_{i_0}) \in \delta + \Delta$ and the elements $w'(\gamma_i)$ $(i = 0, 1, \dots, p)$ form a basis for $\widehat{\Delta}$, we see that $w'(\gamma_{i_0}) = \alpha_0$. Thus, $w'w$ takes $\widehat{\Pi}$ to itself and hence $w'w = 1$. This is however impossible since $w \notin W$.

Thus, $k = 1$ and $\mu := w(\alpha_0) + \delta = w(2\delta - \theta) \in \Delta$. Since δ is isotropic and θ is long, μ is long as well. Finally, since w is minuscule, $2\delta - \theta \notin \widehat{N}(w)$. Hence μ is positive. ■

Let $\overset{\circ}{\mathfrak{Ab}}$ denote the set of all nontrivial Abelian ideals. By Proposition 2.5, one obtains the mapping

$$\tau : \overset{\circ}{\mathfrak{Ab}} \longrightarrow \Delta_l^+, \tag{2.2}$$

which is given by

$$\tau(I_w) = w(\alpha_0) + \delta. \tag{2.3}$$

The long positive root $\tau(I_w)$ is said to be the *rootlet* of the Abelian ideal I_w. Note that the ideal $\{\theta\}$ is the unique minimal element of $\overset{\circ}{\mathfrak{Ab}}$ and, by Peterson's result, $\#(\overset{\circ}{\mathfrak{Ab}}) = 2^{\mathrm{rk}\,\mathfrak{g}} - 1$.

Theorem 2.6. (1) The mapping τ is onto.

(2) If the rootlet of I_w is not simple, that is, $w(\alpha_0) + \delta \in \Delta^+ \setminus \Pi$, then I_w is not maximal.

(3) If Δ is simply laced and $\tau(I_w)$ is not simple, then there are at least two maximal Abelian ideals containing I_w. □

Proof. (1) We perform a descending induction on the height of the rootlet of an ideal. The rootlet with maximal height is θ. Here one takes $w = s_0$. Then $I_{s_0} = \{\theta\}$ and $\tau(I_{s_0}) = \theta$. The induction step goes as follows. If $\mu = \tau(I_w)$ and $\mu \notin \Pi$, then there exists an $\alpha \in \Pi$ such that $(\alpha, \mu) > 0$. Then $\mu' = s_\alpha(\mu) = \mu - n_\alpha \alpha \in \Delta_l^+$ and $\mathrm{ht}(\mu') = \mathrm{ht}(\mu) - n_\alpha$. Notice that $n_\alpha = 1$ if and only if α is long. Set $\mu'' = \mu - \alpha$. It is again a positive root (not necessarily long).

We have $w(\delta - \theta) = -\delta + \mu'' + \alpha$. Hence $w^{-1}(\mu'') + w^{-1}(\alpha) = 2\delta - \theta$. It follows that $w^{-1}(\mu'') = (k+2)\delta - \mu_1$ and $w^{-1}(\alpha) = -k\delta - \mu_2$, for some $k \in \mathbb{Z}$ and $\mu_1, \mu_2 \in \Delta^+$ such that $\mu_1 + \mu_2 = \theta$.

As w is minuscule, neither of the elements in the right-hand side is negative (for instance, if $w^{-1}(\alpha)$ were negative, i.e., $k \geq 0$, then $w(\mu_2) = -k\delta - \alpha_2 < 0$, which contradicts the fact that w is minuscule). It follows that $k+2 > 0$ and $-k > 0$, hence $k = -1$. In particular, we have $w(\delta - \mu_2) = \alpha \in \Pi$. It then follows from Theorem 2.4 that $w' = s_\alpha w$ is again a minuscule element and $I_{w'} = I_w \cup \{\mu_2\}$. The previous formulae show that $\tau(I_{w'}) = s_\alpha(\mu) = \mu'$. Obviously, any positive long root can be obtained from θ through a suitable sequence of simple reflections. Hence the assertion.

(2) The previous argument also shows that if $\tau(I_w) \notin \Pi$, then I_w is contained in a larger Abelian ideal.

(3) As above, $\mu = \tau(I_w)$. Making use of the induction argument from part (1), we may reduce the problem to the case where $\mathrm{ht}(\mu) = 2$. Then $\mu = \alpha_1 + \alpha_2$, the sum of two *simple* roots. Again the argument from part (1) (with α_1 and α_2 in place of μ'' and α) shows that there are two different Abelian extensions of I_w; namely, $I_{w_1} = I_w \cup \{\mu_1\}$ and $I_{w_2} = I_w \cup \{\mu_2\}$, where $w^{-1}(\alpha_1) = \delta - \mu_1$ and $w^{-1}(\alpha_2) = \delta - \mu_2$. But $I_w \cup \{\mu_1, \mu_2\}$ is not Abelian since $\mu_1 + \mu_2 = \theta$. ■

Remark 2.7. In the doubly-laced case, it may happen that the rootlet of an Abelian ideal is not simple, but the ideal lies in a unique maximal one. For instance, let $\mathfrak{g}$ be the simple Lie algebra of type F_4. We use Vinberg-Onishchik's numbering of simple roots [13]. If $\mu = 2\alpha_2 + \alpha_3$, then $\tau^{-1}(\mu)$ consists of two ideals (of dimensions 7 and 8). In the notation of Table 6.1 $\tau^{-1}(\mu) = \{I_7'', I_8'\}$. The only maximal ideal containing these two is I_9. Denoting by Π_l the set of long simple roots in Π, we record an important consequence of the theorem.

Corollary 2.8. If I_w is a maximal Abelian ideal, then $w(\alpha_0) + \delta \in \Pi_l$. □

Thus, denoting by $\mathfrak{Ab}_{\max}$ the set of all maximal Abelian ideals, we obtain the mapping

$$\bar{\tau} : \mathfrak{Ab}_{\max} \longrightarrow \Pi_l, \tag{2.4}$$

which is the restriction of τ to $\mathfrak{Ab}_{\max}$. By Theorem 2.6, $\bar{\tau}$ is onto. We prove below that $\bar{\tau}$ is actually one-to-one. It turns out that the correspondence obtained between the maximal Abelian ideals and the long simple roots is precisely the one described in [8]. So that our present results provide an a priori proof for some empirical observations in [8].

3 Basic properties of posets $\mathfrak{Ab}_\mu$

Given $\mu \in \Delta_l^+$, let $\mathfrak{Ab}_\mu$ denote the fibre of μ for $\tau : \overset{o}{\mathfrak{Ab}} \to \Delta_l^+$. The following useful equality is a consequence of Peterson's result:

$$\sum_{\mu\in\Delta_l^+} \#(\mathfrak{Ab}_\mu) = 2^{\operatorname{rk}\mathfrak{g}} - 1. \tag{3.1}$$

Each $\mathfrak{Ab}_\mu$ is a poset in its own right, and it appears that cutting $\overset{o}{\mathfrak{Ab}}$ into pieces parametrized by Δ_l^+ has a number of good properties.

Theorem 3.1. For any $\mu \in \Delta_l^+$,

(i) the poset $\mathfrak{Ab}_\mu$ contains a unique maximal and a unique minimal element;

(ii) the dimension of the minimal Abelian ideal in $\mathfrak{Ab}_\mu$ is equal to $1 + (\rho, \theta^\vee - \mu^\vee)$;

(iii) if $I, J \in \mathfrak{Ab}_\mu$ and $I \subset J$, then any intermediate ideal also belongs to $\mathfrak{Ab}_\mu$. In particular, $\mathfrak{Ab}_\mu$ is a ranked poset. □

The proof of this result consists of several parts. The uniqueness of the minimal (resp., maximal) element will be proved in Proposition 3.5 (resp., Proposition 3.7), and the dimension formula for the minimal ideal is proved in Theorem 4.2. The latter is a by-product of an explicit description of the minimal ideal in $\mathfrak{Ab}_\mu$ obtained in Section 4. Part (iii) is proved in Corollary 3.3.

To prove the theorem, we look at the procedure of extension of Abelian ideals in more details. If $I, J \in \mathfrak{Ab}$, $\dim J = \dim I + 1$, and $I \subset J$, then we say that J is an (Abelian) *elementary extension* of I. Given $I = I_w$, it follows from Theorem 2.4 that an elementary extension of I_w is possible if and only if $w(\delta - \gamma) = \alpha_i \in \widehat{\Pi}$ for some $\gamma \in \Delta^+$. Then one can replace w with $w' = s_i w$ and I_w with $I_{w'} = I_w \cup \{\gamma\}$. The passage $w \to s_i w$ is also said to be an *elementary extension* (via the reflection s_i). We realize what happens with the rootlet under this procedure. Recall that Δ (or, more generally, the root lattice) has a standard partial order; one writes $\mu \preccurlyeq \nu$, if $\nu - \mu$ is a sum of positive roots.

Proposition 3.2. Suppose $I_{w'}$ is an elementary extension of I_w, as above. Then $\tau(I_{w'}) = s_i(\tau(I_w)) \preccurlyeq \tau(I_w)$. Moreover, if $w' = s_0 w$ (i.e., $i = 0$), then $\tau(I_w) = \tau(I_{w'})$. □

Proof. Set $\nu := w(\alpha_0) + \delta$, the rootlet of I_w. Then the rootlet of $I_{w'}$ is $s_i w(\alpha_0) + \delta = s_i(\nu - \delta) + \delta = s_i(\nu)$. We have two equalities:

$$w(\delta - \gamma) = \alpha_i, \qquad w(\delta - \theta) = \nu - \delta. \tag{3.2}$$

Consider two possibilities for i:

(a) $i \neq 0$. Here we have

$$(\alpha_i, \nu) = (\alpha_i, \nu - \delta) = (\delta - \gamma, \delta - \theta) = (\gamma, \theta) \geq 0, \tag{3.3}$$

as δ is isotropic. It follows that $s_i(\nu) = \nu - (\nu, \alpha_i^\vee)\alpha_i \preccurlyeq \nu$;

(b) $i = 0$. As $\alpha_0 = \delta - \theta$, we obtain

$$0 \le (\gamma, \theta) = (\nu - \delta, \delta - \theta) = -(\nu, \theta) \le 0. \tag{3.4}$$

Hence $(\gamma, \theta) = (\nu, \theta) = 0$ and $s_0 w(\alpha_0) + \delta = s_0(\nu) = \nu$. ■

Corollary 3.3. If $I, J \in \overset{o}{\mathfrak{Ab}}$ and $I \subset J$, then $\tau(J) \preccurlyeq \tau(I)$. In particular, if $I, J \in \mathfrak{Ab}_\mu$, then any intermediate ideal also belongs to $\mathfrak{Ab}_\mu$. □

Proof. Obviously, for any pair $I \subset J$ of Abelian ideals there is a sequence of elementary extensions that makes J from I. ■

The following result will be our main tool in induction arguments.

Proposition 3.4. Let $I = I_w$ be an Abelian ideal. Suppose I has two different elementary extensions $I_1 = I \cup \{\gamma_1\}$ and $I_2 = I \cup \{\gamma_2\}$. Write $s_i w$ for the minuscule element corresponding to I_i, $i = 1, 2$.

(1) If $\tilde{I} := I_1 \cup I_2$ is not Abelian, then $\tau(I_1) = \alpha_2$, $\tau(I_2) = \alpha_1$, and $\tau(I) = \alpha_1 + \alpha_2$. Moreover, $\alpha_1, \alpha_2 \in \Pi_l$.

(2) If $\tilde{I}$ is Abelian, then $s_1 s_2 = s_2 s_1$ and $w\langle \tilde{I} \rangle = s_1 s_2 w$.

(3) If $\tau(I) = \tau(I_1)$, then $\tilde{I}$ is Abelian as well and $\tau(I_2) = \tau(\tilde{I})$. □

Proof. The equalities $s_i w = w\langle I_i \rangle$ and $I_i = I \cup \{\gamma_i\}$ mean together that

$$w(\delta - \gamma_i) = \alpha_i \in \widehat{\Pi}, \quad i = 1, 2. \tag{3.5}$$

(1) Assume that $I_1 \cup I_2$ is not Abelian. Since both I_1 and I_2 are Abelian, the only possibility for this is that $\gamma_1 + \gamma_2 \in \Delta^+$.

If $\gamma_1 + \gamma_2 \ne \theta$, then there is an $\alpha \in \Pi$ such that $\gamma_1 + \gamma_2 + \alpha$ is a (positive) root. Then $\gamma_1 + \alpha \in \Delta$ or $\gamma_2 + \alpha \in \Delta$ (the proof is left to the reader). If, for instance, the second condition is satisfied, then $\gamma_2 + \alpha \in I$ and $\gamma_1 \in I_1$, which contradicts the fact that I_1 is Abelian. Hence $\gamma_1 + \gamma_2 = \theta$.

Now, taking the sum of equations (3.5) yields

$$\alpha_1 + \alpha_2 = w(2\delta - \gamma_1 - \gamma_2) = w(\delta - \theta) + \delta = \tau(I). \tag{3.6}$$

Since $\tau(I) \in \Delta_l^+$, we have $\alpha_1, \alpha_2 \in \Pi_l$. It follows that $\tau(I_1) = s_1(\alpha_1 + \alpha_2) = \alpha_2$ and $\tau(I_2) = s_2(\alpha_1 + \alpha_2) = \alpha_1$.

(2) The presence of the elementary extension $I_1 \mapsto I_1 \cup \{\gamma_2\} = \tilde{I}$ shows that $w\langle \tilde{I} \rangle = s_2 \cdot w\langle I_1 \rangle = s_2 s_1 w$ and $s_2 w(\delta - \gamma_1) \in \widehat{\Pi}$. The latter means that $s_2(\alpha_1)$ is a simple root. It follows that $s_2(\alpha_1) = \alpha_1$ and hence $s_2 s_1 = s_1 s_2$.

(3) Under the assumption $\tau(I) = \tau(I_1)$, the first case cannot occur. Hence $\tilde{I}$ is Abelian. Since s_1 and s_2 commute, we have $s_2 s_1 w(\alpha_0) + \delta = s_2(\nu) = s_2 w(\alpha_0) + \delta$, that is, $\tau(\tilde{I}) = \tau(I_2)$. ■

Proposition 3.5. For any $\mu \in \Delta_l^+$, the poset $\mathfrak{Ab}_\mu$ has a unique minimal element. □

Proof. Assume $\tilde{I}_1$ and $\tilde{I}_2$ are two different minimal elements of $\mathfrak{Ab}_\mu$. Clearly, $I := \tilde{I}_1 \cap \tilde{I}_2$ is again an Abelian ideal, but $\tau(I)$ is strictly less than μ.

The ideal $\tilde{I}_1$ can be obtained from I via a chain of elementary extensions, say

$$I \longrightarrow I \cup \{\varkappa_1\} \longrightarrow \cdots \longrightarrow I \cup \{\varkappa_1, \dots, \varkappa_n\} = \tilde{I}_1. \tag{3.7}$$

Similarly, let $I \to I \cup \{\eta_1\}$ be the first step in the chain of extensions leading from I to $\tilde{I}_2$. Set $I(k, 0) = I \cup \{\varkappa_1, \dots, \varkappa_k\}$ and $I(k, 1) = I \cup \{\varkappa_1, \dots, \varkappa_k, \eta_1\}$, $0 \le k \le n$. By construction, $I(0, 1)$ and $I(k, 0)$ are Abelian ideals. Consider the sequence of statements depending on k: $(\mathcal{C}_k)$ $I(k, 0) \ne \tilde{I}_1$, $I(k, 1)$ is Abelian, $\mu = \tau(\tilde{I}_1) \preccurlyeq \tau(I(k, 1))$.

Claim 3.6. For any $k \ge 0$, $(\mathcal{C}_k)$ implies $(\mathcal{C}_{k+1})$. □

Note that $(\mathcal{C}_0)$ is true. (The last inequality follows from the equality $\tau(\tilde{I}_1) = \tau(\tilde{I}_2)$ and Corollary 3.3.) Therefore, granting the claim, we conclude that $(\mathcal{C}_n)$ is also true. But this is nonsense since $I(n, 0) = \tilde{I}_1$. This contradiction shows that $\mathfrak{Ab}_\mu$ cannot have two minimal elements. Thus, it remains to prove the claim.

Proof of the claim. By assumption, we have two elementary extensions:

$$I(k, 0) \longrightarrow I(k + 1, 0), \qquad I(k, 0) \longrightarrow I(k, 1). \tag{3.8}$$

If $w := w\langle I(k, 0)\rangle$, then $w\langle I(k, 1)\rangle = s'w$ and $w\langle I(k+1, 0)\rangle = s''w$ for some simple reflections s' and s''.

(1) Assume that $I(k+1, 1)$ is not Abelian. Applying Proposition 3.4(1) to the above triplet of ideals, we obtain $\tau(I(k, 0)) = \alpha' + \alpha''$, $\tau(I(k+1, 0)) = \alpha'$, and $\tau(I(k, 1)) = \alpha''$, where $\alpha', \alpha'' \in \Pi_l$. Since $I(k + 1, 0) \subset \tilde{I}_1$, we have $\tau(\tilde{I}_1) = \alpha'$. On the other hand, our assumptions give $\tau(\tilde{I}_1) \preccurlyeq \tau(I(k, 1)) = \alpha''$. Whence $\alpha' \preccurlyeq \alpha''$. This contradiction shows that $I(k + 1, 1)$ is Abelian.

(2) Since $I(k + 1, 1)$ is Abelian, Proposition 3.4(2) says that

$$s's'' = s''s', \qquad w\langle I(k + 1, 1)\rangle = s's''w. \tag{3.9}$$

It follows that

$$\tau\big(I(k+1,0)\big) = s''\big(\tau\big(I(k,0)\big)\big) = \tau\big(I(k,0)\big) - n''\alpha'',$$
$$\tau\big(I(k,1)\big) = s'\big(\tau\big(I(k,0)\big)\big) = \tau\big(I(k,0)\big) - n'\alpha', \tag{3.10}$$

for some $n', n'' \geq 0$. By the hypothesis

$$\tau(\tilde{I}_1) \preccurlyeq \tau\big(I(k,1)\big) = \tau\big(I(k,0)\big) - n'\alpha', \tag{3.11}$$

and, since $I(k+1,0) \subset \tilde{I}_1$,

$$\tau(\tilde{I}_1) \preccurlyeq \tau\big(I(k+1,0)\big) = \tau\big(I(k,0)\big) - n''\alpha''. \tag{3.12}$$

Hence $\tau(\tilde{I}_1) \preccurlyeq \tau(I(k,0)) - n's' - n''s'' = \tau(I(k+1,1))$.

(3) If $I(k+1,0) = \tilde{I}_1$, then the inequalities in the previous part of the proof imply that

$$\tau\big(I(k,0)\big) - n''\alpha'' \preccurlyeq \tau\big(I(k,0)\big) - n'\alpha'. \tag{3.13}$$

Hence $n' = n'' = 0$. Then $\mu = \tau(\tilde{I}_1) = \tau(I(k,0))$. Thus, $I(k,0)$ is smaller than $\tilde{I}_1$ and has the same rootlet, which contradicts the minimality of $\tilde{I}_1$. Hence $I(k+1,0) \neq \tilde{I}_1$, and the claim is proved. ■

This completes the proof of Proposition 3.5. ■

In what follows, $I(\mu)_{\min}$ stands for the minimal element of $\mathfrak{Ab}_\mu$.

Proposition 3.7. For any $\mu \in \Delta_l^+$, the poset $\mathfrak{Ab}_\mu$ has a unique maximal element. □

Proof. By Proposition 3.5, any ideal $I \subset \mathfrak{Ab}_\mu$ can be obtained from $I(\mu)_{\min}$ via a chain of elementary extensions. Moreover, it follows from Corollary 3.3 that each ideal in this chain belongs to $\mathfrak{Ab}_\mu$. Another consequence is that if $I, J \in \mathfrak{Ab}_\mu$, then $I \cap J \in \mathfrak{Ab}_\mu$ as well.

Suppose $I_1, I_2 \in \mathfrak{Ab}_\mu$. We prove that $I_1 \cup I_2 \in \mathfrak{Ab}_\mu$. Consider the set $I_2 \setminus I_1$ and pick there a maximal element with respect to "$\preccurlyeq$," say γ_2. Arguing by induction, it suffices to prove that $I_1 \cup \{\gamma_2\}$ lies in $\mathfrak{Ab}_\mu$. Similarly, take a maximal element $\nu_1 \in I_1 \setminus I_2$. Applying Proposition 3.4(3) to the ideal $I = I_1 \cap I_2 \in \mathfrak{Ab}_\mu$ and the roots ν_1 and γ_2, we conclude that $I \cup \{\nu_1, \gamma_2\}$ is in $\mathfrak{Ab}_\mu$. If $I' := I \cup \{\nu_1\} \neq I_1$, then take a maximal element $\nu_2 \in I_1 \setminus I'$. Then one applies Proposition 3.4(3) to I' and ν_2, γ_2. We eventually obtain $I_1 \cup \{\gamma_2\} \in \mathfrak{Ab}_\mu$.

Since $I_1 \cup I_2 \in \mathfrak{Ab}_\mu$ for any pair $I_1, I_2 \in \mathfrak{Ab}_\mu$, we see that $\mathfrak{Ab}_\mu$ has a unique maximal element. ■

Corollary 3.8. The map $\bar{\tau}: \mathfrak{Ab}_{\max} \to \Pi_l$ is bijective. □

Proof. It follows from Corollary 2.8 and Proposition 3.7 that the maximal Abelian ideals are precisely the maximal elements of the posets $\mathfrak{Ab}_\alpha$, $\alpha \in \Pi_l$. ■

In what follows, $I(\mu)_{\max}$ stands for the maximal element of $\mathfrak{Ab}_\mu$. We also say that $I(\mu)_{\min}$ is the μ-*minimal* and $I(\mu)_{\max}$ is the μ-*maximal* ideal.

4 μ-minimal ideals and their properties

In this section, an explicit description of $I(\mu)_{\min}$ is given for any $\mu \in \Delta_l^+$. We also characterize the set of all μ-minimal ideals and find the generators of $I(\mu)_{\min}$.

Theorem 4.1. Let $w \in W$ be an element of minimal length such that $w(\theta) = \mu$. Then

(1) $l(w) = (\rho, \theta^\vee - \mu^\vee)$;

(2) $N(w^{-1}) = \{\gamma \in \Delta^+ \mid (\gamma, \mu^\vee) = -1\}$.

In particular, the set $\{u \in W \mid u(\theta) = \mu\}$ contains a unique element of minimal length. □

Proof. (1) Recall that $(\rho, \alpha^\vee) = 1$ for all $\alpha \in \Pi$. A straightforward calculation shows that $(\rho, s_\alpha(\nu)^\vee) = (\rho, \nu^\vee) - (\alpha, \nu^\vee)$ for $\nu \in \Delta$ and $\alpha \in \Pi$. Since μ is long and positive, we have $(\alpha, \mu^\vee) \geq -1$. Hence

$$\left(\rho, s_\alpha(\mu)^\vee\right) \leq \left(\rho, \mu^\vee\right) + 1. \tag{4.1}$$

If $w \in W$ is a minimal length element such that $w(\mu) = \theta$ and $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition, then $N(w) \supset N(s_{i_j} \cdots s_{i_k})$ for any $1 \leq j \leq k$. Hence, $s_{i_j} \cdots s_{i_k}(\mu) > 0$ for all j. Therefore, arguing by induction and using (4.1), we conclude that $k = l(w) \geq (\rho, \theta^\vee - \mu^\vee)$. On the other hand, if $\mu \in \Delta_l^+$ and $\mu \neq \theta$, then one can always find an $\alpha \in \Pi$ such that $(\alpha, \mu^\vee) = -1$. This means that starting with μ and moving up, one can reach θ after applying exactly $(\rho, \theta^\vee - \mu^\vee)$ simple reflections.

(2) Set $\Delta_\mu^+(i) = \{\gamma \in \Delta^+ \mid (\gamma, \mu^\vee) = i\}$. We are to show that $\Delta_\mu^+(-1) = N(w^{-1})$. Let us compare the cardinalities of these two sets. By the first part of the proof, $\#N(w^{-1}) = (\rho, \theta^\vee - \mu^\vee)$. On the other hand, one has the system of two equations

$$\begin{aligned} &\left(\rho, \mu^\vee\right) = 1 + \frac{1}{2}\left(\#\Delta_\mu^+(1) - \#\Delta_\mu^+(-1)\right), \\ &2\left(\rho, \theta^\vee\right) - 2 = \#\Delta_\theta(1) = \#\Delta_\mu(1) = \#\Delta_\mu^+(1) + \#\Delta_\mu^+(-1). \end{aligned} \tag{4.2}$$

The first equality stems from the very definition of ρ, whereas in the second equation we use the fact that θ is dominant and that μ and θ are W-conjugate. From the above system, we deduce that $\#\Delta^+_\mu(-1) = (\rho, \theta^\vee - \mu^\vee) = \#N(w^{-1})$.

On the other hand, if $\gamma \in \Delta^+_\mu(-1)$, then $(w^{-1}(\gamma), \theta^\vee) = -1$. Hence, $w^{-1}(\gamma)$ is negative and $N(w^{-1}) \supset \Delta^+_\mu(-1)$. ■

Notice that we also proved that if $u \in W$ is any element taking θ to μ, then $N(u^{-1}) \supset \Delta^+_\mu(-1)$. In what follows, we write w_μ for the unique element of minimal length in W that takes θ to μ.

Theorem 4.2. Set $\tilde{w}_\mu = w_\mu s_0 \in \widehat{W}$. Then

(1) $\tilde{w}_\mu(\alpha_0) + \delta = \mu$;

(2) $\tilde{w}_\mu$ is minuscule;

(3) the ideal $I_{\tilde{w}_\mu}$ is contained in $\{\gamma \in \Delta^+ \mid (\gamma, \theta) > 0\}$;

(4) $I_{\tilde{w}_\mu} = I(\mu)_{\min}$, the minimal element of $\mathfrak{Ab}_\mu$, and $\#(I_{\tilde{w}_\mu}) = (\rho, \theta^\vee - \mu^\vee) + 1$. □

Proof. (1) Obvious.

(2) Suppose $(\rho, \theta^\vee - \mu^\vee) = k \geq 1$ and let $w_\mu = s_{i_k} \cdots s_{i_1}$ be a reduced decomposition. We argue by induction on k. Set $u := s_{i_{k-1}} \cdots s_{i_1} \in W$ and $\nu := u(\theta)$. Then $l(u) = k-1$ and $s_{i_k}(\nu) = \mu$. Using Theorem 4.1, we obtain

$$k - 1 \geq (\rho, \theta^\vee - \nu^\vee) = (\rho, \theta^\vee - \mu^\vee) - (\alpha_{i_k}, \nu^\vee) = k - (\alpha_{i_k}, \nu^\vee). \tag{4.3}$$

Since ν is long, $(\alpha_{i_k}, \nu^\vee) = 1$. It follows that $(\rho, \theta^\vee - \nu^\vee) = k-1$ and hence $u = w_\nu$. Set $\tilde{w}_\nu = w_\nu s_0$. By the induction assumption, $\tilde{w}_\nu$ is minuscule. To prove that $\tilde{w}_\mu = s_{i_k}\tilde{w}_\nu$ is minuscule, one has to verify that $\tilde{w}_\nu(\delta - \gamma_{i_k}) = \alpha_{i_k}$ for some $\gamma_{i_k} \in \Delta^+$ (see Theorem 2.4). In other words, it should be proved that $\delta - \tilde{w}_\nu^{-1}(\alpha_{i_k}) \in \Delta^+$. We have $\tilde{w}_\nu^{-1}(\alpha_{i_k}) = s_0 w_\nu^{-1}(\alpha_{i_k})$ and

$$(\theta, w_\nu^{-1}(\alpha_{i_k})) = (w_\nu(\theta), \alpha_{i_k}) = (\nu, \alpha_{i_k}) > 0. \tag{4.4}$$

Equation (4.4) shows that $w_\nu^{-1}(\alpha_{i_k}) \in \Delta^+$ and $(\alpha_0, w_\nu^{-1}(\alpha_{i_k})) < 0$. Therefore, $s_0 w_\nu^{-1}(\alpha_{i_k}) = w_\nu^{-1}(\alpha_{i_k}) - \theta + \delta$. Thus, $\delta - \tilde{w}_\nu^{-1}(\alpha_{i_k}) = \theta - w_\nu^{-1}(\alpha_{i_k}) \in \Delta^+$, and we are done.

(3) Again, we argue by induction on $l(w_\mu)$. Using the notation of the previous part of the proof, it suffices to observe that $I_{\tilde{w}_\mu} = I_{\tilde{w}_\nu} \cup \{\theta - w_\nu^{-1}(\alpha_{i_k})\}$ and $(w_\nu^{-1}(\alpha_{i_k}), \theta^\vee) = 1$.

(4) If $I_{\tilde{w}_\mu}$ were not minimal in $\mathfrak{Ab}_\mu$, then one could shorten $\tilde{w}_\mu$, so that to obtain a minuscule element giving the ideal with the same rootlet. But this is impossible for length reason, as w_μ has minimal possible length among the elements taking θ to μ. The dimension of this ideal is already computed in Theorem 4.2. Finally, $\#(I_{\tilde{w}_\mu}) = l(\tilde{w}_\mu) = l(w_\mu) + 1$. ■

Thus, we have completed the proof of Theorem 3.1.

Set $\mathcal{H} = \{\gamma \in \Delta^+ \mid (\theta, \gamma) > 0\}$. It is the set of the roots for the standard Heisenberg subalgebra of $\mathfrak{g}$. That is, $\mathfrak{h} = \oplus_{\gamma \in \mathcal{H}} \mathfrak{g}_\gamma$ is a Heisenberg subalgebra of $\mathfrak{g}$. Clearly, $\mathfrak{h}$ is a non-Abelian ideal of $\mathfrak{b}$.

The previous exposition shows that one has a distinguished collection of Abelian ideals $\{I(\mu)_{\min} \mid \mu \in \Delta_l^+\}$ and the corresponding subset of minuscule elements of $\widehat{W}$. These sets admit the following characterizations.

Theorem 4.3. The following conditions are equivalent for $I_w \in \overset{\circ}{\mathfrak{Ab}}$:

(i) $I_w = I(\mu)_{\min}$ for some $\mu \in \Delta_l^+$;

(ii) $I_w \subset \mathcal{H}$;

(iii) $w = w' s_0$, where $w' \in W$. □

Proof. (i)⇒(ii). This is proved in Theorem 4.2.

(ii)⇒(iii). Assume that a reduced decomposition of w' contains s_0, say $w' = w_2 s_0 w_1$. Since $s_0 w_1 s_0$ is also minuscule (see Section 2), we may assume, without loss of generality, that $w_2 = 1$, that is, a reduced decomposition of w' begins with s_0. Hence, there is the elementary extension $w_1 s_0 \to s_0 w_1 s_0$. It was already shown that in this case one adds to the ideal $I_{w_1 s_0}$ a root which is orthogonal to θ, see Proposition 3.2(b).

(iii)⇒(i). We argue by induction on $l(w')$. Suppose a reduced decomposition of w' starts with s_i, that is, $w = s_i w'' s_0$ and $w'' s_0$ is also minuscule. By the induction hypothesis, $I_{w'' s_0} = I(\nu)_{\min}$, where $\nu = w''(\theta)$. Then $w'' = w_\nu$ and $l(w'') = (\rho, \theta^\vee - \nu^\vee)$. Set $\mu = s_i(\nu) = w''(\theta)$. Then $\mu = \tau(I_w)$ and our goal is to prove that $s_i w'' = w_\mu$. Since $w'' s_0 \to s_i w'' s_0$ is an elementary extension, we have $w'' s_0(\delta - \gamma) = \alpha_i \in \Pi$ for some $\gamma \in \Delta^+$. It follows that $s_0(\delta - \gamma) \neq \delta - \gamma$. This yields $(\theta^\vee, \gamma) = 1$ and $s_0(\delta - \gamma) = \theta - \gamma$. Hence $w''(\theta - \gamma) = \alpha_i$. Therefore,

$$(\alpha_i, \nu^\vee) = (w''(\theta - \gamma), w''(\theta^\vee)) = (\theta - \gamma, \theta^\vee) = 1. \tag{4.5}$$

This equality implies that $(\rho, \theta^\vee - \mu^\vee) = (\rho, \theta^\vee - \nu^\vee) + 1 = 1 + l(w'') = l(w')$. By Theorem 4.1, this means that $w' = s_i w'' \in W$ is the shortest element taking θ to μ, and we are done. ■

Corollary 4.4. There is a natural one-to-one correspondence between the Abelian $\mathfrak{b}$-ideals in the Heisenberg subalgebra and the long positive roots. □

The next result describes the order relation on the set of μ-minimal ideals.

Theorem 4.5. For any $\mu, \nu \in \Delta_l^+$, $I(\mu)_{\min} \subset I(\nu)_{\min}$ if and only if $\nu \preccurlyeq \mu$, that is, the poset of μ-minimal elements is anti-isomorphic to the poset $(\Delta_l^+, \preccurlyeq)$. □

Proof. "$\Rightarrow$". This is contained in Corollary 3.3.

"$\Leftarrow$'". We show that $w_\nu = w' w_\mu$, where $l(w') = (\rho, \mu^\vee - \nu^\vee)$. Indeed,

(a) the inequality $l(w') \geq l(w_\nu) - l(w_\mu) = (\rho, \mu^\vee - \nu^\vee)$ is clear;

(b) the opposite inequality can be proved by induction. Set $\mu - \nu = \sum_{\alpha \in \Pi} k_\alpha \alpha$, where $k_\alpha \geq 0$. Since $|\mu| = |\nu|$, we obtain $(\nu, \sum k_\alpha \alpha) < 0$. Hence, there exists an $\alpha \in \Pi$ such that $k_\alpha > 0$ and $(\alpha, \nu) < 0$. Then $\nu \preccurlyeq s_\alpha(\nu) = \nu + (|\mu|^2/|\alpha|^2)\alpha \preccurlyeq \mu$. (One should use here the fact that, since μ and ν are long, k_α is divisible by $|\mu|^2/|\alpha|^2$.)

Thus, the minuscule element $\tilde{w}_\nu$ is obtained from $\tilde{w}_\nu$ via a sequence of elementary extensions and hence $I(\mu)_{\min} \subset I(\nu)_{\min}$. ■

Finally, we give a description of the generators for μ-minimal ideals. If $w = s_0$, then $I_{s_0} = \{\theta\}$ and everything is clear. So that we may assume that $\mu \neq \theta$, that is, $\tilde{w}_\mu = w_\mu s_0$ and $w_\mu \neq 1$.

Proposition 4.6. For $\mu \neq \theta$, there is a bijection between the generators of $I(\mu)_{\min}$ and the roots $\alpha \in \Pi$ such that $\alpha + \mu \in \Delta$ (i.e., $(\alpha, \mu^\vee) = -1$). The generator corresponding to such an α is $w_\mu^{-1}(\alpha + \mu)$. □

Proof. By Theorem 2.2, $\gamma \in \Delta^+$ is a generator if and only if $w_\mu s_0(\delta - \gamma) = -\alpha \in \widehat{\Pi}$. By Theorem 4.2(3), $(\gamma, \theta) > 0$. Therefore, the left-hand side is equal to $w_\mu(\theta - \gamma) = \mu - w_\mu(\gamma)$ and $\mu + \alpha = w_\mu(\gamma) \in \Delta$. Hence $\alpha \in \Pi$ and $\mu + \alpha$ is a root.

This argument can be reversed. Given $\alpha \in \Pi$ such that $(\alpha, \mu^\vee) = -1$, we set $\gamma = w_\mu^{-1}(\alpha + \mu)$. As $(\alpha + \mu, \mu^\vee) \neq -1$, it follows from Theorem 4.1(2) that $\gamma > 0$. The rest is clear. ■

5 More on the structure of $\mathfrak{Ab}_\mu$

We already know that each $\mathfrak{Ab}_\mu$ contains a unique maximal and a unique minimal element. In this section, we first answer the question: when is the cardinality of $\mathfrak{Ab}_\mu$ equal to one? An important observation concerning cardinality stems from Proposition 3.2. It was proved there that the elementary extension via the reflection s_0 does not affect the rootlet; and in this case the rootlet of an ideal has to be orthogonal to θ. What we prove now is that this gives a necessary and sufficient condition for $\#(\mathfrak{Ab}_\mu) > 1$.

Theorem 5.1. (i) $\#(\mathfrak{Ab}_\mu) > 1$ if and only if $(\mu, \theta) = 0$ (i.e., $\mu \notin \mathcal{H}$).

(ii) If $(\mu, \theta) = 0$, then the nonempty poset $\mathfrak{Ab}_\mu \setminus \{I(\mu)_{\min}\}$ has a unique minimal element, say I'. Here $I' = I(\mu)_{\min} \cup \{\gamma\}$, where $\gamma = w_\mu^{-1}(\theta)$. The corresponding minuscule element is $s_0 \tilde{w}_\mu = s_0 w_\mu s_0$. □

Proof. (i) In view of Theorem 3.1, it is clear that $\#(\mathfrak{Ab}_\mu) > 1$ if and only if $I(\mu)_{min}$ has an elementary extension that does not change the rootlet. So, we stick to considering possible elementary extensions of $I(\mu)_{min}$. This is based on the explicit description in Theorem 4.2.

(1) Since μ is the rootlet, we have

$$\tilde{w}_\mu(\delta - \theta) = \mu - \delta. \tag{5.1}$$

Suppose there is an elementary extension of $I(\mu)_{min}$, that is, we have a $\gamma \in \Delta^+$ such that

$$\tilde{w}_\mu(\delta - \gamma) = \alpha \in \widehat{\Pi}. \tag{5.2}$$

There are two possibilities for α:

(a) $\alpha = \alpha_i \in \Pi$. Rewriting (5.2) as $s_0(\delta-\gamma) = w_\mu^{-1}\alpha_i$, we see that $s_0(\delta-\gamma) \in \Delta$. This can only happen if $(\alpha_0, \delta-\gamma) > 0$, that is, $(\theta, \gamma) > 0$ (and then $s_0(\delta-\gamma) = \theta-\gamma$). Combining (5.1) and (5.2), we obtain $(\mu, \alpha_i) > 0$ and hence $s_i(\mu) \neq \mu$. Thus, any elementary extension via a simple reflection from W changes the rootlet of $I(\mu)_{min}$;

(b) $\alpha = \alpha_0$. Here we get the following chain inequalities:

$$0 \le (\theta, \gamma) = (\delta-\theta, \delta-\gamma) = (\mu-\delta, \delta-\theta) = -(\mu, \theta) \le 0. \tag{5.3}$$

Thus, we have the conclusion: if $I(\mu)_{min}$ has an extension that does not change the rootlet, then this extension uses the reflection s_0, and the condition $(\mu, \theta) = 0$ should be satisfied. This proves the "only if" part.

(2) Suppose $(\theta, \mu) = 0$. We wish to find an elementary extension of $I(\mu)_{min}$ that does not change the rootlet μ. Recall that $\tilde{w}_\mu = w_\mu s_0$. Take $\gamma = w_\mu^{-1}(\theta)$. From the description of w_μ^{-1} (see Theorem 4.1(2)), it follows that $\theta \notin N(w_\mu^{-1})$, that is, $\gamma \in \Delta^+$. Furthermore, $(\gamma, \theta) = (w_\mu(\gamma), w_\mu(\theta)) = (\theta, \mu) = 0$. Hence $\tilde{w}_\mu(\delta-\gamma) = \delta-\theta = \alpha_0$ and $s_0(\mu) = \mu$. Thus, $I(\mu)_{min} \cup \{\gamma\}$ is an Abelian ideal lying in $\mathfrak{Ab}_\mu$.

(ii) This is essentially proved in the previous part of proof, since s_0 is the only possible reflection that can be used for constructing an elementary extension of $I(\mu)_{min}$ with the rootlet μ. ∎

Remark 5.2. We have proved that, for $I(\mu)_{min}$, there is at most one elementary extension which lies inside $\mathfrak{Ab}_\mu$, and, if exists, this extension always exploits the reflection s_0. But if $I \in \mathfrak{Ab}_\mu$ is not minimal, then there can exist an elementary extension via s_i $(i \neq 0)$ that does not change the rootlet.

Now, we accomplish the following step in describing cardinality of $\mathfrak{Ab}_\mu$, that is, a criterion will be given for $\#(\mathfrak{Ab}_\mu) > 2$. We already know that the condition $(\mu, \theta) = 0$ is necessary.

Proposition 5.3. Suppose $\mu \in \Delta_l^+$ and $(\mu, \theta) = 0$. Then $\#(\mathfrak{Ab}_\mu) > 2$ if and only if there exists $\alpha_i \in \Pi$ such that $(\alpha_i, \theta) > 0$ and $(\alpha_i, \mu) = 0$.

If these conditions are satisfied for α_i, then one more element of $\mathfrak{Ab}_\mu$, which covers $I' = I(\mu) \cup \{w_\mu^{-1}(\theta)\}$, is

$$I'' = I(\mu)_{\min} \cup \left\{w_\mu^{-1}(\theta), w_\mu^{-1}\left(\theta - \alpha_i\right)\right\}. \tag{5.4}$$

□

Proof. In view of Theorem 5.1(ii), it is clear that $\#(\mathfrak{Ab}_\mu) > 2$ if and only if $I' = I_{s_0 w_\mu s_0}$ has an elementary extension with the same rootlet. So, we stick to considering possible extensions of $I_{s_0 w_\mu s_0}$.

"⇐". We show that $s_i s_0 w_\mu s_0$ is again minuscule and the corresponding rootlet is again μ. The second condition is satisfied, since $(\alpha_i, \mu) = 0$ and hence $s_i(\mu) = \mu$. The condition that $s_i s_0 w_\mu s_0$ is minuscule is equivalent, in view of Theorem 2.4, to that $s_0 w_\mu s_0(\delta - \gamma) = \alpha_i$ for some $\gamma \in \Delta^+$, that is, $\delta - s_0 w_\mu^{-1} s_0(\alpha_i) \in \Delta^+$. Using the definition of w_μ and the assumptions, the last expression is equal to $w_\mu^{-1}(\theta - \alpha_i)$. Since $(\mu, \theta - \alpha_i) = 0$, we deduce from Theorem 4.1(2) that $\theta - \alpha_i \notin N(w_\mu^{-1})$, that is, $w_\mu^{-1}(\theta - \alpha_i)$ is positive.

"⇒". Suppose there is an elementary extension of $I_{s_0 w_\mu s_0}$ that does not affect μ, that is, there is a $\gamma \in \Delta^+$ such that

$$s_0 w_\mu s_0(\delta - \gamma) = \alpha_i \tag{5.5}$$

and $s_i(\mu) = \mu$. Clearly, $i \neq 0$, that is, $\alpha_i \in \Pi$. Since $s_i(\mu) = \mu$, we have $(\alpha_i, \mu) = 0$. Thus, it remains to prove that $(\alpha_i, \theta) > 0$. If not, then $(\alpha_i, \theta) = 0$ and hence $s_0(\alpha_i) = \alpha_i$. Then (5.5) can be written as $\delta - \gamma = s_0 w_\mu^{-1}(\alpha_i)$. As $(\theta, w_\mu^{-1}(\alpha_i)) = (\mu, \alpha_i) = 0$, the right-hand side is equal to $w_\mu^{-1}(\alpha_i) \in \Delta$. This contradiction proves that $(\alpha_i, \theta) > 0$. ■

Remark 5.4. If $\mathfrak{g} \neq \mathfrak{sl}_n$, then there is only one simple root that is not orthogonal to θ. In any case, this condition is easy to verify in practice.

Actually, we can give a description of $I(\mu)_{\max}$ and $\mathfrak{Ab}_\mu$, which is consistent with both the previous results and our computations in Section 6, but we cannot find a general case-free proof yet. In order to provide a stronger motivation and more evidences in favour of the following description, we look again at previous results of this section. We have proved that

(i) if $(\mu, \theta) = 0$, then $\mathfrak{Ab}_\mu = \{I(\mu)_{\min}\}$;

(ii) if $(\mu, \theta) > 0$ and there is no simple roots $\alpha \in \Pi$ such that $(\theta, \alpha) > 0$ and $(\alpha, \mu) = 0$, then $\mathfrak{Ab}_\mu = \{I(\mu)_{\min}, I'\}$, where $I' = I(\mu)_{\min} \cup \{\gamma\}$ and $\gamma = w_\mu^{-1}(\theta)$;

(iii) if $(\mu, \theta) > 0$ and $\alpha \in \Pi$ satisfies the conditions $(\theta, \alpha) > 0$ and $(\alpha, \mu) = 0$, then one can further extend I' as follows: $I'' = I' \cup \{\gamma'\}$, where $\gamma' = w_\mu^{-1}(\theta - \alpha)$.

These first steps of constructing extensions show that each time one adds to $I(\mu)_{\min}$ some roots that are orthogonal to μ. Moreover, the following proposition is true.

Proposition 5.5. Suppose $\alpha_1, \dots, \alpha_t$ is a chain of simple roots such that $(\theta, \alpha_1) > 0$, $(\alpha_i, \alpha_{i+1}) > 0$ $(i = 1, \dots, t-1)$, and $(\theta, \mu) = (\alpha_1, \mu) = \cdots = (\alpha_t, \mu) = 0$. Then $\#(\mathfrak{Ab}_\mu) \geq t + 1$. More precisely,

$$\{I^{(0)}, I^{(1)}, \dots, I^{(t)}\} \subset \mathfrak{Ab}_\mu, \tag{5.6}$$

where $I^{(0)} = I(\mu)_{\min}$ and $I^{(i+1)} = I^{(i)} \cup \{w_\mu^{-1}(\theta - \alpha_1 - \cdots - \alpha_i)\}$. □

Proof. Argue by induction on t. The induction step is the same as the proof of Proposition 5.3. ■

After this preparations, we can state a general description of $I(\mu)_{\max}$ and $\mathfrak{Ab}_\mu$. Let $\tilde{\Gamma}$ be the extended Dynkin diagram of $\mathfrak{g}$. It has the "usual" nodes that correspond to the roots in Π and the "extra" node corresponding to $-\theta$. Let us delete from $\tilde{\Gamma}$ all nodes such that the corresponding roots are not orthogonal to μ. The remaining graph can be disconnected. Let Γ_μ denote the connected component of it that contains the node corresponding to $-\theta$. For instance, if $(\mu, \theta) > 0$, then $\Gamma_\mu = \varnothing$. Clearly, Γ_μ is the Dynkin diagram of a regular simple Lie subalgebra of $\mathfrak{g}$. Call this subalgebra $\mathfrak{g}_{(\mu)}$. If $\alpha_1, \dots, \alpha_k$ are all simple roots of $\mathfrak{g}$ that correspond to the usual nodes of Γ_μ, then $\{\theta, -\alpha_1, \dots, -\alpha_k\}$ can be taken as a set of simple roots for $\mathfrak{g}_{(\mu)}$, and one can consider the respective set of positive roots. Let $\mathfrak{b}_{(\mu)}$ be the Borel subalgebra corresponding to the chosen set of positive roots, and let $\mathfrak{b}^-_{(\mu)}$ be the opposite Borel subalgebra. With this convention, let $\mathfrak{p}_{(\mu)} \supset \mathfrak{b}_{(\mu)}$ be the maximal parabolic subalgebra of $\mathfrak{g}_{(\mu)}$ determined by θ (i.e., θ is the only simple root of $\mathfrak{g}_{(\mu)}$ that is not a root of the Levi subalgebra of $\mathfrak{p}_{(\mu)}$). Let M_μ be the set of roots of $\mathfrak{p}^{nil}_{(\mu)}$, the nilpotent radical of $\mathfrak{p}_{(\mu)}$. It is obvious that the nilpotent radical constructed in this way is Abelian, that is, for any $\gamma \in M_\mu$ the coefficient of θ can be only 1. Thus,

$$M_\mu = \left\{\theta - \sum_{i=1}^{k} c_i \alpha_i \mid c_i \geq 0\right\} \cap \Delta. \tag{5.7}$$

Notice that $\{\alpha_1, \dots, \alpha_k\}$ is a proper subset of Π, since $\mu \neq 0$. Therefore $M_\mu \subset \Delta^+$. Explicit computations show that one always has

$$I(\mu)_{\max} = I(\mu)_{\min} \cup w_\mu^{-1}(M_\mu). \tag{5.8}$$

Furthermore, to get all (combinatorial) Abelian ideals in $\mathfrak{Ab}_\mu$, one should exploit in (5.8) arbitrary subsets $A \subset M_\mu$ such that the corresponding geometric subspace $\oplus_{\gamma \in A}\mathfrak{g}_\gamma \subset \mathfrak{p}^{\text{nil}}_{(\mu)}$ be $\mathfrak{b}^-_{(\mu)}$-stable. At this point, we can use the following general property of Abelian nilpotent radicals: Let $\mathfrak{b}$ and $\mathfrak{b}^-$ be opposite Borel subalgebras of $\mathfrak{g}$ (i.e., $\mathfrak{b} \cap \mathfrak{b}^- = \mathfrak{t}$). Suppose $\mathfrak{p} \supset \mathfrak{b}$ is a parabolic subalgebra such that $\mathfrak{p}^{\text{nil}}$ is Abelian, and let E be a $\mathfrak{b}^-$-stable subspace of $\mathfrak{p}^{\text{nil}}$. If $\bar{E}$ is the unique $\mathfrak{t}$-stable complement of E, then $\bar{E}$ is also $\mathfrak{b}$-stable. (The proof is straightforward and left to the reader.)

In our situation, this means that $\oplus_{\gamma \in A}\mathfrak{g}_\gamma$ is $\mathfrak{b}^-_{(\mu)}$-stable if and only if $\oplus_{\gamma \in \overline{A}}\mathfrak{g}_\gamma \subset \mathfrak{p}^{\text{nil}}_{(\mu)}$ is $\mathfrak{b}_{(\mu)}$-stable, where $\overline{A} = M_\mu \setminus A$. In other words, $A \subset M_\mu$ gives rise to an element of $\mathfrak{Ab}_\mu$ if and only if $\overline{A}$ is a combinatorial $\mathfrak{b}_{(\mu)}$-ideal. It follows that $\mathfrak{Ab}_\mu$ is anti-isomorphic to the poset of $\mathfrak{b}_{(\mu)}$-ideals in $\mathfrak{p}^{\text{nil}}_{(\mu)}$. The posets of ideals in Abelian nilpotent radicals are known as *minuscule* posets (see, e.g., [9, 11]). In particular, any minuscule poset is self-dual, that is, it has an order-reversing involution. Therefore, the prefix "anti" in the above statement can be removed.

Although we cannot provide a priori proofs for all results described after (5.8), some ingredients can be derived without case-by-case verification. First, since each root in M_μ is orthogonal to μ, we have, by Theorem 4.1(2), that $w_\mu^{-1}(M_\mu) \subset \Delta^+$. Second, using the definition of $I(\mu)_{\min}$, it is not hard to prove that any subset $I(\mu)_{\min} \cup w_\mu^{-1}(A)$ is an Abelian *subalgebra* of Δ^+. However, we cannot proof a priori that all these subsets are ideals in Δ^+ and these do lie in $\mathfrak{Ab}_\mu$. Still, a direct verification shows that this construction gives the correct description in all cases.

Remark 5.6. In a recent preprint [12], Suter also studies partition of $\mathfrak{Ab}$ into subposets parameterized by the long positive roots. But his technique is different from ours, and the proofs are based, to a great extent, on case-by-case considerations.

6 Examples

Here, we present our computations for all simple Lie algebras.

6.1 $\mathfrak{g} = \mathfrak{sl}_n$

We assume that $\mathfrak{b}$ is the space of upper triangular matrices. Then the positive roots are identified with the pairs (i, j), where $1 \le i < j \le n$. Here, $\alpha_i = (i, i+1)$ and $\theta = (1, n)$.

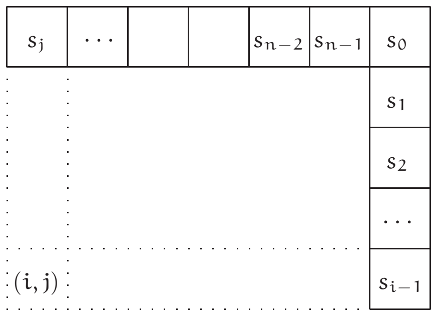

Figure 6.1 Filling of the hook.

An Abelian $\mathfrak{b}$-ideal is represented by a right-aligned Ferrers diagram such that the number of rows plus the number of columns is at most n. The unique northeast corner of the diagram corresponds to θ and the southwest corners give the generators of the corresponding ideal (see also [8, Theorem 3.3]). In this case, it is easy to explicitly describe the posets $\mathfrak{Ab}_\mu$. If $\mu = (i, j)$, then

$$\begin{aligned} I(i,j)_{\max} &= \{(p,q) \mid j \le q,\ p \le i\}, \\ I(i,j)_{\min} &= \{(1,q) \mid j \le q\} \cup \{(p,n) \mid 2 \le p \le i\}. \end{aligned} \tag{6.1}$$

In other words, $I(i,j)_{\max}$ is the rectangle with the low-left corner at (i,j) and $I(i,j)_{\min}$ is the "northeast" hook contained in this rectangle, see also Figure 6.1. Here $\#I(i,j)_{\max} = i(n+1-j)$ and $\#I(i,j)_{\min} = n+i-j$. It follows that $\#I(i,j)_{\max} = \#I(i,j)_{\min}$ if and only if $i = 1$ or $j = n$, that is, precisely for the roots that are not orthogonal to θ. It is not hard to compute that

$$\#\mathfrak{Ab}_{(i,j)} = \binom{n+i-j-1}{i-1}. \tag{6.2}$$

This shows again that $\#\mathfrak{Ab}_{(i,j)} = 1$ if and only if $i = 1$ or $j = n$. This equality is also in accordance with Proposition 5.3. It is curious to observe that the assignment $(i,j) \mapsto \#\mathfrak{Ab}_{(i,j)}$ gives exactly the Pascal triangle (rotated through the angle 45°).

There is an explicit algorithm for writing out the minuscule element for any $I \in \overset{o}{\mathfrak{Ab}}$, which can be interpreted as a filling of the respective hook (see Figure 6.1). Namely, the minuscule element corresponding to $I(i,j)_{\min}$ equals $(s_{i-1}\cdots s_2 s_1)(s_j \cdots s_{n-2}s_{n-1})s_0$.

Note that the products in parentheses, which correspond to the leg and the arm of the hook, commute, so that their order is irrelevant. For an arbitrary Abelian ideal, one should decompose the corresponding Ferrers diagram as the union of "northeast"

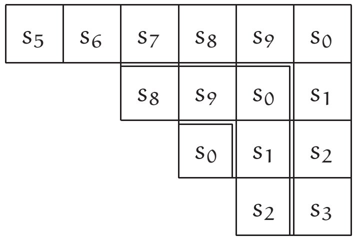

Figure 6.2 Decomposition and filling of the Ferrers diagram for an Abelian ideal in $\mathfrak{sl}_{10}$.

hooks, and then fill in each hook according to the above rule. The resulting minuscule element is the product of the corresponding hook elements; the first factor corresponds to the smallest hook, and so forth. The best way for understanding all this is to look at the concrete example.

Consider the Abelian ideal I in $\mathfrak{sl}_{10}$ with generators $(1,5)$, $(2,7)$, $(3,8)$, and $(4,9)$. Here, the Ferrers diagram is decomposed as the union of three hooks and the corresponding filling is depicted in Figure 6.2. Therefore, the respective minuscule element is $w\langle I\rangle = s_0(s_2s_1)\cdots s_0$.

6.2 $\mathfrak{g} = \mathfrak{so}_{2n+1}$ or $\mathfrak{so}_{2n}$

In the standard notation, the set of long positive roots is

$$\Delta_l^+ = \{\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le n\}. \tag{6.3}$$

Here, $\theta = \varepsilon_1 + \varepsilon_2$ and $\mathcal{H} \cap \Delta_l^+ = \{\varepsilon_i \pm \varepsilon_j \mid i = 1, 2, j \ge 3\} \cup \{\theta\}$. By Theorem 5.1, $\#(\mathfrak{Ab}_\mu) = 1$ for any $\mu \in \mathcal{H} \cap \Delta_l^+$. By Proposition 5.3, we obtain $\#(\mathfrak{Ab}_{\varepsilon_1-\varepsilon_2}) = 2$ and $\#(\mathfrak{Ab}_{\varepsilon_3\pm\varepsilon_j}) = 2$ $(j \ge 4)$. Straightforward computations for the other roots show that $\#(\mathfrak{Ab}_{\varepsilon_i\pm\varepsilon_j}) = 2^{i-2}$, if $i \ge 3$. We demonstrate how all this is related to the description of $\mathfrak{Ab}_\mu$ in Section 5.

Take, for instance, $\mu = \alpha_{n-2} = \varepsilon_{n-2} - \varepsilon_{n-1}$ for $\mathfrak{so}_{2n}$. Then

$$\mathfrak{g}_{(\mu)} = \begin{cases} 0, & \text{if } n = 4, \\ \mathfrak{sl}_2, & \text{if } n = 5, \\ \mathfrak{so}_{2n-6}, & \text{if } n \ge 6. \end{cases} \tag{6.4}$$

For $n \ge 6$, the Abelian nilpotent radical in $\mathfrak{g}_{(\mu)}$ corresponding to θ has dimension $(n-3)(n-4)/2$. This number is just the difference $\dim I(\alpha_{n-2})_{\max} - \dim I(\alpha_{n-2})_{\min}$. Hence,

$\dim I(\alpha_{n-2})_{\max} = (n-3)(n-4)/2+2n-3 = (n^2-3n+6)/2$ (cf. [8, Figure 3]). In this case, $\mathfrak{g}_{(\mu)} \simeq \mathfrak{g}^{\vee}_{(\mu)}$ and $\#(\mathfrak{Ab}_\mu)$ is the dimension of the half-spinor representation of $\mathfrak{so}_{2n-6}$, that is 2^{n-4}.

6.3 $\mathfrak{g} = \mathfrak{sp}_{2n}$

In this case, there is only a few long roots

$$\Delta^+_l = \{2\varepsilon_i \mid 1 \le i \le n\}, \tag{6.5}$$

and $\theta = 2\varepsilon_1$. We have $I(2\varepsilon_i)_{\min} = \{\varepsilon_1+\varepsilon_i, \dots, \varepsilon_1+\varepsilon_2, 2\varepsilon_1\}$ and $I(2\varepsilon_i)_{\max} = \{\varepsilon_k+\varepsilon_j \mid k \le j \le i\}$. The sole generator of $I(2\varepsilon_i)_{\min}$ (resp., $I(2\varepsilon_i)_{\max}$) is $\varepsilon_1+\varepsilon_i$ (resp., $2\varepsilon_i$). The minuscule element $w\langle I(2\varepsilon_i)_{\min}\rangle$ is $s_{i-1}\cdots s_2s_1s_0$. Using the matrix presentation of $\mathfrak{sp}_{2n}$ (see, e.g., [8, 3.3]), it is easily seen that there is a one-to-one correspondence between the ideals in $\mathfrak{Ab}_{2\varepsilon_i}$ and the Abelian ideals of $\mathfrak{sp}_{2i-2}$. Therefore, $\#(\mathfrak{Ab}_{2\varepsilon_i}) = 2^{i-1}$. It is also possible to give an algorithm for writing out the minuscule element corresponding to an Abelian ideal in terms of filling a shifted Ferrers diagram.

6.4 $\mathfrak{g} = \mathbf{F}_4$

Here, we have 12 long positive roots and 15 nontrivial Abelian ideals. The set $\mathcal{H}\cap\Delta^+_l$ consists of 9 roots. Hence, the fibre $\mathfrak{Ab}_\mu$ contains a unique ideal for these 9 roots and consists of two ideals for the other 3 roots. The computations of rootlets and minuscule elements are presented in Table 6.1. We follow the numbering of simple roots from [13, Table 1], and the root $\sum_{i=1}^4 c_i\alpha_i$ is denoted by $(c_1c_2c_3c_4)$. For instance, $\theta = (2432)$. The notation I_n means that the ideal has cardinality n. To distinguish different ideals with the same cardinality, we use "prime". The third, fourth, and fifth columns represent the ideal, the corresponding minuscule element, and the rootlet, respectively.

The maximal Abelian ideals are I_8''' and I_9.

6.5 $\mathfrak{g} = \mathbf{G}_2$

Here $\#(\overset{o}{\mathfrak{Ab}}) = \#(\Delta^+_l) = 3$, so that everything is easy. Let α (resp., β) be the short (resp., long) simple root. Then

$$\begin{aligned} &I_1 = \{3\alpha+2\beta\}, && w\langle I_1\rangle = s_0, && \tau(I_1) = 3\alpha+2\beta;\\ &I_2 = \{3\alpha+2\beta, 3\alpha+\beta\}, && w\langle I_2\rangle = s_\beta s_0, && \tau(I_2) = 3\alpha+\beta;\\ &I_3 = \{3\alpha+2\beta, 3\alpha+\beta, \beta\}, && w\langle I_3\rangle = s_\alpha s_\beta s_0, && \tau(I_3) = \beta. \end{aligned} \tag{6.6}$$

Table 6.1 The Abelian $\mathfrak{b}$-ideals in F_4.

No.	#I	I	$w\langle I\rangle$	$\tau(I)$
1	1	$\{\theta\}$	s_0	θ
2	2	$\{\theta, 2431\}$	$s_4 s_0$	2431
3	3	$\{\theta, 2431, 2421\}$	$s_3 s_4 s_0$	2421
4	4	$\{\theta, 2431, 2421, 2321\}$	$s_2 s_3 s_4 s_0$	2221
5	5	$I_5' = I_4 \cup \{2221\}$	$s_3 s_2 s_3 s_4 s_0$	2211
6	5	$I_5'' = I_4 \cup \{1321\}$	$s_1 s_2 s_3 s_4 s_0$	0221
7	6	$I_6' = I_5' \cup \{2211\}$	$w_6' = s_4 s_3 s_2 s_3 s_4 s_0$	2210
8	6	$I_6'' = I_5' \cup \{1321\} = I_5'' \cup \{2221\}$	$w_6'' = s_1 s_3 s_2 s_3 s_4 s_0$	0211
9	7	$I_7' = I_6' \cup \{2210\}$	$w_7' = s_0 w_6'$	2210
10	7	$I_7'' = I_6' \cup \{1321\} = I_6'' \cup \{2211\}$	$w_7'' = s_1 w_6' = s_4 w_6''$	0210
11	7	$I_7''' = I_6'' \cup \{1221\}$	$w_7''' = s_2 w_6''$	0011
12	8	$I_8' = I_7' \cup \{1321\} = I_7'' \cup \{2210\}$	$w_8' = s_1 w_7' = s_0 w_7''$	0210
13	8	$I_8'' = I_7'' \cup \{1221\} = I_7''' \cup \{2211\}$	$w_8'' = s_2 w_7'' = s_4 w_7'''$	0010
14	8	$I_8''' = I_7''' \cup \{0221\}$	$w_8''' = s_3 w_7'''$	0001
15	9	$I_9 = I_8' \cup \{1221\} = I_8'' \cup \{2210\}$	$w_9 = s_2 w_8' = s_0 w_8''$	0010

Table 6.2

	E_6	E_7	E_8
m_1	21	33	57
m_2	9	15	27
m_3	4	8	16
m_4	—	4	10
m_5	—	—	6
m_6	2	2	3
m_8	—	—	1
m_{12}	—	1	—
$\sum im_i$	$2^6 - 1$	$2^7 - 1$	$2^8 - 1$

6.6 $\mathfrak{g} = E_n, n = 6, 7, 8$

Set $\Delta^+_{(i)} = \{\mu \in \Delta^+ \mid \#(\mathfrak{Ab}_\mu) = i\}$ and $m_i = \#\Delta^+_{(i)}$. Note that $\Delta^+_{(1)} = \mathcal{H}$. The output of our calculations of numbers m_i is given in Table 6.2, where we include only the rows containing nonzero entries. The last row is the control one.

Table 6.3

	E_6	E_7	E_8
$\Delta^+_{(1)}$	$c_6 > 0$	$c_6 > 0$	$c_1 > 0$
$\Delta^+_{(2)}$	$c_6 = 0$ $c_3 > 0$	$c_6 = 0$ $c_5 > 0$	$c_1 = 0$ $c_2 > 0$
$\Delta^+_{(3)}$	$\{\alpha_1 + \alpha_2, \alpha_2, \alpha_4 + \alpha_5, \alpha_4\}$	$c_6 = c_5 = 0$ $c_4 > 0$	$c_1 = c_2 = 0$ $c_3 > 0$
$\Delta^+_{(4)}$	—	$c_6 = c_5 = c_4 = 0$ $c_7 > 0$ or $c_3 > 0$	$c_1 = c_2 = c_3 = 0$ $c_4 > 0$
$\Delta^+_{(5)}$	—	—	$c_1 = c_2 = c_3 = c_4 = 0$ $c_5 > 0$
$\Delta^+_{(6)}$	$\{\alpha_1, \alpha_5\}$	$\{\alpha_2, \alpha_1 + \alpha_2\}$	$\{\alpha_8, \alpha_6, \alpha_6 + \alpha_7\}$
$\Delta^+_{(8)}$	—	—	$\{\alpha_7\}$
$\Delta^+_{(12)}$	—	$\{\alpha_1\}$	—

An explicit description of the subsets $\Delta^+_{(i)}$ is also obtained (see Table 6.3). Again, we follow the numbering of simple roots from [13] and denote the root $\sum_{i=1}^n c_i\alpha_i$ by $(c_1 c_2 \cdots c_n)$. For instance, the highest root of E_6 (resp., E_7) is $(1\,2\,3\,2\,1\,2)$ (resp., $(1\,2\,3\,4\,3\,2\,2)$).

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