# ABELIAN MAPS, BRACE BLOCKS, AND SOLUTIONS TO THE YANG-BAXTER EQUATION 

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#### Abstract

Let $G$ be a finite nonabelian group. We show how an endomorphism of $G$ with abelian image gives rise to a family of binary operations $\left\{\circ_{n}: n \in \mathbb{Z} \geq 0\right\}$ on $G$ such that $\left(G, \circ_{m}, \circ_{n}\right)$ is a skew left brace for all $m, n \geq 0$. A brace block gives rise to a number of non-degenerate settheoretic solutions to the Yang-Baxter equation. We give examples showing that the number of solutions obtained can be arbitrarily large.


## 1. Introduction

Rump [Rum07] introduced the notion of braces to construct involutive, non-degenerate settheoretic solutions to the Yang-Baxter equation, an equation whose solutions have numerous, wellknown applications (see, e.g., [Che12, Jim89a, Jim89b, Jim94, LR97]). This construction was then generalized to skew left braces in [GV17] to provide solutions which are not necessarily involutive. In [Koc20] we show how one can construct many examples of skew left braces using what we call abelian maps, i.e., endomorphisms of a finite group $G=(G, \cdot)$ with abelian image; our description also allows us to easily generate a pair of set-theoretic solutions to the YBE if $G$ is nonabelian.

Here, we significantly improve upon our abelian map construction. Namely, we prove that a single abelian map $\psi: G \rightarrow G$ gives a family of skew left braces; we call this family a brace block. A brace block consists of $G$ together with a collection $\left\{\circ_{n}\right\}$ of binary operations such that $\left(G, \circ_{m}, \circ_{n}\right)$ is a skew left brace for all $m, n$.

The skew left braces we construct are examples of bi-skew braces. In [Chi19] Childs introduces bi-skew braces: a bi-skew brace is a skew left brace which is also a skew left brace if the operations are reversed. The skew left braces in our block are evidently bi-skew because both $\left(G, \circ_{m}, \circ_{n}\right)$ and $\left(G, \circ_{n}, \circ_{m}\right)$ are skew left braces for all $m, n$. We will tend to view our objects as skew left as opposed to bi-skew with the goal of simplifying the statements of our results.

Starting with an abelian map $\psi: G \rightarrow G$ we generate a family of abelian maps $\left\{\psi_{n}: n \in \mathbb{Z} \geq 0\right\}$ on $G$. We then use [Koc20, Th. 1] to construct the group ( $G, \circ_{n}$ ) and produce the skew left brace $\left(G, \cdot, \circ_{n}\right)$. It turns out that $\left(G, \circ_{m}, \circ_{n}\right)$ is a skew left brace for every choice of $m, n \geq 0$, hence we get a brace block. Each skew left brace $\left(G, \circ_{m}, \circ_{n}\right)$ provides a non-degenerate set-theoretic solution $R$ to the YBE; if $\left(G, \circ_{m}\right)$ is abelian this solution is involutive, otherwise we obtain a second solution $R^{\prime}$, inverse in the sense that $R R^{\prime}=R^{\prime} R=\mathrm{id}$.

The paper is structured as follows. After providing background definitions and setting notation, we construct our abelian maps $\psi_{n}$ for $n \geq 0$ and give their properties. We show how $\psi_{n}$ gives us a

[^0]binary operation $\circ_{n}$ on $G$. We then introduce the concept of brace blocks and arrive at main result, Theorem 4.7, which proves that $\left(G, \circ_{m}, \circ_{n}\right)$ is a skew left brace for each $m, n \geq 0$, and hence the set of all such braces forms a brace block. We then explicitly construct a pair of inverse solutions to the Yang-Baxter equation from each $\left(G, \circ_{m}, \circ_{n}\right)$; as stated previously, the two solutions will coincide if and only if $\left(G, \circ_{m}\right)$ is abelian. The solutions coming from $\left(G, \circ_{m}, \circ_{n}\right)$ will be entirely in terms of $\psi_{m}, \psi_{n}$, and the underlying group operation on $G$. We then do some first examples of brace blocks, focusing on cases where $\psi: G \rightarrow G$ is fixed point free in the sense of [Chi13]; these examples tend to produce a small number of solutions to the YBE. Finally, we examine a set of examples which allow us to produce large numbers of solutions; in fact, we will see that there is no bound to the number of solutions obtainable using abelian maps.

Throughout, $G$ is a nonabelian group: while the constructions to follow will work if $G$ is abelian, the brace block will produce only one brace, namely the trivial brace on $G$, making the block construction unnecessary. Since we will put multiple group structures on $G$, we will denote center of $G$ with respect to the operation $*$ by $Z(G, *)$ and write $Z(G)=Z(G, \cdot)$.

## 2. Braces braces and the Yang-Baxter equation

Here we shall give a short overview of the connection between different types of braces and the Yang-Baxter equation. Recall that a set-theoretic solution to the Yang-Baxter equation consists of a set $B$ together with a function $R: B \times B \rightarrow B \times B$ such that

$$
(R \times \mathrm{id})(\mathrm{id} \times R)(R \times \mathrm{id})=(\mathrm{id} \times R)(R \times \mathrm{id})(\mathrm{id} \times R): B \times B \times B \rightarrow B \times B \times B
$$

When $B$ is understood we simply denote the solution by $R$. Given a solution $R$, write $R(x, y)=$ $\left(f_{y}(x), g_{x}(y)\right)$. We say $R$ is non-degenerate if both $f_{y}: B \rightarrow B$ and $g_{x}: B \rightarrow B$ are invertible. Additionally, $R$ is involutive if $R^{2}=\mathrm{id}$. An example of a non-degenerate, involutive solution is constructed by taking $B$ to be any set and defining $R(x, y)=(y, x)$ for all $x, y \in B$.

Set-theoretic solutions were suggested by Drinfeld [Dri92] as a way to obtain solutions to the Yang-Baxter equation on a vector space $V$, i.e., endomorphisms $V \otimes V \rightarrow V \otimes V$ satisfying a "twisting" condition analogous to the one above. Given a set-theoretic solution with underlying set $B$, the corresponding vector space solution arises by letting $V$ be the vector space with basis $B$. With this is mind, we will say that two (set-theoretic) solutions are equivalent if they induce the same vector space solution up to a choice of basis.

Herein, we will use "solution" to mean a non-degenerate, set-theoretic solution. In general, our solutions will not be involutive.

Skew left braces are a useful tool for finding solutions to the Yang-Baxter equation. While there have been numerous papers on skew left braces recently, we feel a quick definition is in order to set notation. A skew left brace is a triple $(B, \cdot, \circ)$ consisting of a set and two binary operations such that $(B, \cdot)$ and $(B, \circ)$ are groups and

$$
x \circ(y \cdot z)=(x \circ y) \cdot x^{-1} \cdot(x \circ z)
$$

holds for all $x, y, z \in B$, where $x^{-1} \cdot x=1_{B}$, the identity common to both operations. We refer to the condition above as the brace relation. We will follow the typical (but not universal) practice of writing $x \cdot y$ as $x y$ when no confusion will arise, and $\bar{x}$ for the inverse of $x$ in $(G, \circ)$.

A very simple example of a brace is $(G, \cdot, \cdot)$ where $(G, \cdot)$ is any group. We call this the trivial brace on $G$, or just the trivial brace for short.

Skew left braces are generalizations of left braces, found in [Rum07] where $(B, \cdot)$ is assumed to be abelian. For simplicity, we will refer to a skew left brace simply as a brace.

Every brace $(B, \cdot, \circ)$ gives a solution to the Yang-Baxter equation [GV17, Th. 3.1], namely

$$
\begin{equation*}
R(x, y)=\left(x^{-1}(x \circ y), \overline{x^{-1}(x \circ y)} \circ x \circ y\right), x, y \in B \tag{2.1}
\end{equation*}
$$

There is also the notion of an opposite brace, considered independently by the author and Truman in [KT20a] and Rump in [Rum19]. The opposite brace to $(B, \cdot, \circ)$, which is simply $\left(B,,^{\prime}, \circ\right)$ with $x \cdot^{\prime} y=y \cdot x$, provides an additional solution to the YBE:

$$
\begin{equation*}
R^{\prime}(x, y)=\left((x \circ y) x^{-1}, \overline{(x \circ y) x^{-1}} \circ x \circ y\right) \tag{2.2}
\end{equation*}
$$

Note that $R=R^{\prime}$ if and only if $(B, \cdot)$ is abelian. By [KT20a, Th. 4.1] we have that $R R^{\prime}=R^{\prime} R=\mathrm{id}$.
We say that two braces $(B, \cdot, \circ)$ and $\left(B^{\prime}, .^{\prime}, \circ^{\prime}\right)$ are isomorphic if there exists a function $B \rightarrow B$ which is an isomorphism $(B, \cdot) \rightarrow\left(B^{\prime}, .^{\prime}\right)$ and also an isomorphism $(B, \circ) \rightarrow\left(B^{\prime}, \circ^{\prime}\right)$. Isomorphic braces will give equivalent solutions to the Yang-Baxter equation.

## 3. Endomorphisms from an abelian map

The motivation for the construction to follow can be found in [Koc20], where an abelian map $\psi:(G, \cdot) \rightarrow(G, \cdot)$ gives rise to a brace $(G, \cdot, \circ)$. It is not hard to show that $\psi$ respects $\circ$ as well, and that $\psi(G)$ is an abelian subgroup of $(G, \circ)$. Viewing $\psi:(G, \circ) \rightarrow(G, \circ)$ then allows for the construction of another brace, say $(G, \circ, \diamond)$. This can then be repeated it ad infinitum, of course, generating a "chain" of braces. It turns out that any two operations generated in this manner can be put together in a brace as well, e.g., $(G, \cdot, \diamond)$. The simplest way to describe this chain of operations is through the abelian maps $\psi_{n}$ introduced below.

Let $G$ be a group, written multiplicatively, and write $\operatorname{Map}(G)$ for the set of all functions $\phi: G \rightarrow$ $G$. We write $1 \in \operatorname{Map}(G)$ to denote the identity map and $0 \in \operatorname{Map}(G)$ for the trivial map. For $\phi, \psi \in \operatorname{Map}(G)$ define

$$
\begin{aligned}
(\phi+\psi)(g) & =\phi(g) \psi(g) \\
-\phi(g) & =\phi\left(g^{-1}\right) \\
\phi \psi(g) & =\phi(\psi(g)) .
\end{aligned}
$$

Then $\operatorname{Map}(G)$ carries the structure of a right near-ring: $(\operatorname{Map}(G),+)$ is a group (necessarily nonabelian), the multiplication is associative and the right distributive law holds. We define $\phi^{n}$ in the usual way for $n \geq 0$.

We are interested primarily in the subset $\operatorname{End}(G) \subset \operatorname{Map}(G)$ consisting of endomorphisms of $G$, particularly the set $\operatorname{Ab}(G) \subset \operatorname{End}(G)$ of maps $\psi: G \rightarrow G$ with $\psi(G)$ abelian: we call such endomorphisms abelian maps. A crucial fact about abelian maps is that they are constant on conjugacy classes: we will see how this observation greatly simplifies some calculations.

Neither $\operatorname{Ab}(G)$ nor $\operatorname{End}(G)$ are, in general, closed under the group operation, though both contain the identity and $\operatorname{Ab}(G)$ contains its inverses. Notice that if $\psi \in \operatorname{Ab}(G)$ then $\psi \phi \in \operatorname{Ab}(G)$ for all $\phi \in \operatorname{End}(G)$; in particular, $\psi^{n} \in \operatorname{Ab}(G)$ for all $n \geq 0$.

We shall now introduce a collection of abelian maps that can be constructed from a single abelian map. Let $\psi \in \operatorname{Ab}(G)$. For $n \geq 0$, define

$$
\psi_{n}=-(1-\psi)^{n}+1
$$

Note that $\psi_{0}=0$ and $\psi_{1}=\psi$. While the binomial formula does not work in $\operatorname{Map}(G)$ generally, we do have the following useful result.

Lemma 3.1. Let $\psi \in \operatorname{Ab}(G)$. For $n \geq 1$ we have

$$
\psi_{n}=\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i} \psi^{i}
$$

i.e.,

$$
\begin{equation*}
\psi_{n}(g)=\prod_{i=1}^{n} \psi^{i}\left(g^{(-1)^{i}\binom{n}{i}}\right)=\psi\left(g^{\binom{n}{1}}\right) \psi^{2}\left(g^{-\binom{n}{2}}\right) \cdots \psi^{n}\left(g^{(-1)^{n}\binom{n}{n}}\right), g \in G . \tag{3.1}
\end{equation*}
$$

Thus, $\psi_{n} \in \operatorname{Ab}(G)$ for all $n \geq 0$.
Proof. First, we will prove that

$$
\begin{equation*}
(1-\psi)^{n}(g)=g \prod_{i=1}^{n} \psi^{i}\left(g^{(-1)^{i}\binom{n}{i}}\right) \tag{3.2}
\end{equation*}
$$

holds for all $g \in G, n \geq 0$ by induction on $n$. It is clear that Equation 3.2 holds for $n=0$ as both sides reduce to $g$. Now assume Equation 3.2 holds for $n=k$. Then, since $\operatorname{Map}(G)$ has a right distributive law and $\psi$ is abelian,

$$
\begin{aligned}
(1-\psi)^{k+1}(g) & =(1-\psi)(1-\psi)^{k}(g) \\
& =(1-\psi)^{k}(g)-\psi\left((1-\psi)^{k}(g)\right) \\
& =g \prod_{i=1}^{k} \psi^{i}\left(g^{(-1)^{i}\binom{k}{i}}\right) \psi\left(g \prod_{i=1}^{k} \psi^{i}\left(g^{(-1)^{i}\binom{k}{i}}\right)\right)^{-1} \\
& =g \prod_{i=1}^{k} \psi^{i}\left(g^{(-1)^{i}\binom{k}{i}}\right) \cdot\left(\prod_{i=1}^{k} \psi^{i+1}\left(g^{(-1)^{i+1}\binom{k}{i}}\right)\right) \psi\left(g^{-1}\right) \\
& =g \prod_{i=1}^{k} \psi^{i}\left(g^{(-1)^{i}\binom{k}{i}}\right) \cdot\left(\psi\left(g^{-1}\right) \prod_{i=2}^{k+1} \psi^{i}\left(g^{(-1)^{i}\binom{k}{i-1}}\right)\right) \\
& =g \prod_{i=1}^{k} \psi^{i}\left(g^{(-1)^{i}\binom{k}{i}}\right) \cdot\left(\prod_{i=1}^{k+1} \psi^{i}\left(g^{\left.(-1)^{i}\binom{k}{i-1}\right)}\right)\right. \\
& \left.=\left(\prod_{i=1}^{k} \psi^{i}\left(g^{(-1)^{i}\binom{k}{i}}\right)+(-1)^{i}\binom{k}{i-1}\right)\right) \psi^{k+1}\left(g^{(-1)^{k+1}\binom{k+1}{k+1}}\right) \\
& =\prod_{i=1}^{k+1} \psi^{i}\left(g^{(-1)^{i}\binom{k+1}{i}}\right)
\end{aligned}
$$

and Equation 3.2 is true for all $n$.

Next,

$$
\psi_{n}(g)=\left(-(1-\psi)^{n}+1\right)(g)=\left(g \prod_{i=1}^{n} \psi^{i}\left(g^{(-1)^{i}\binom{n}{i}}\right)\right)^{-1} g=\prod_{i=1}^{n} \psi^{i}\left(g^{(-1)^{i}\binom{n}{i}}\right)
$$

and so Equation 3.1 holds. That $\psi_{n}$ is an endomorphism follows from the fact that each of $\psi^{i}$ are endomorphisms and that

$$
\psi^{i}\left(g^{k}\right) \psi^{j}\left(h^{\ell}\right)=\psi\left(\psi^{i-1}\left(g^{k}\right) \psi^{j-1}\left(h^{\ell}\right)\right)=\psi\left(\psi^{j-1}\left(h^{\ell}\right) \psi^{i-1}\left(g^{k}\right)\right)=\psi^{j}\left(h^{\ell}\right) \psi^{i}\left(g^{k}\right)
$$

for all $i, j, k, \ell$ with $i, j \geq 1$. Finally, $\psi_{n}(g)$ is a product of elements in $\psi(G)$, hence $\psi_{n}$ is abelian for $n \geq 1$, and since $\psi_{0}$ is trivial we get $\psi_{0} \in \operatorname{Ab}(G)$ and the lemma is proved.

For future reference we make note of some useful properties.
Lemma 3.2. For $\psi \in \operatorname{Ab}(G)$ we have, for all $m, n \geq 0$,
(1) $\psi_{n}(G) \leq \psi(G)$
(2) $\psi_{m}(g) \psi_{n}(h)=\psi_{n}(h) \psi_{m}(g)$
(3) $\psi_{m}+\psi_{n}=\psi_{n}+\psi_{m}$
(4) $(1-\psi)^{n}=1-\psi_{n}$
(5) $\psi_{n+1}=\psi+\psi_{n}(1-\psi)$.
(6) $\left(\psi_{m}\right)_{n}=\psi_{m n}$.

Proof. First, (1) should be clear from Equation 3.2, as is (2), and (3) is a special case of (2). The formulation in (4) follows quickly from the definition of $\psi_{n}$. To show the recursive formula in (5), we have

$$
\begin{align*}
\psi_{n+1}(g) & =\left(-(1-\psi)^{n+1}(g)\right) g \\
& =-\left(\left(1-\psi_{n}\right)(1-\psi)(g)\right) g  \tag{4}\\
& =\left(\left(1-\psi_{n}\right)\left(g \psi\left(g^{-1}\right)\right)^{-1} g\right. \\
& =\left(g \psi\left(g^{-1}\right) \psi_{n}\left(g \psi\left(g^{-1}\right)^{-1}\right)\right)^{-1} g \\
& =\psi_{n}\left(g \psi\left(g^{-1}\right)\right) \psi(g) g^{-1} g \\
& =\psi_{n}(1-\psi)+\psi \\
& =\psi+\psi_{n}(1-\psi) \tag{2}
\end{align*}
$$

Finally, (6) follows from a quick computation:

$$
\left(\psi_{m}\right)_{n}=-\left(1-\psi_{m}\right)^{n}+1=-\left((1-\psi)^{m}\right)^{n}+1=(1-\psi)^{m n}+1=\psi_{m n}
$$

Remark 3.3. In light of the introduction to this section, Lemma 3.2 (6) should be reasonable. For example, the sixth map starting with $\psi$ should be the same as the third map starting with $\psi_{2}$ or the second map starting with $\psi_{3}$.

## 4. Brace blocks and solutions to the Yang-Baxter equation

Here we will introduce our main object of study: the brace block. We will show that $\psi \in \operatorname{Ab}(G)$ can be used to generate a brace block, and we will provide explicit solutions to the Yang-Baxter equation that are obtained from $\psi$.
Definition 4.1. A brace block is a set $B$ together with binary operations $\left\{o_{n}: n \geq 0\right\}$ such that $B_{m, n}:=\left(B, \circ_{m}, \circ_{n}\right)$ is a brace for all $m, n \geq 0$.

Note that $B_{n, n}$ is the trivial brace on $\left(B, \circ_{n}\right)$.
As mentioned in the introduction, the braces in a brace block are necessarily bi-skew, a concept developed by Childs in [Chi19]. A bi-skew brace is a set $B$ with binary operations • and o such that both $(B, \cdot, \circ)$ and $(B, \circ, \cdot)$ are (skew left) braces. Given a brace block, both $\left(B, \circ_{m}, \circ_{n}\right)$ and $\left(B, \circ_{n}, \circ_{m}\right)$ are braces, hence $\left(B, \circ_{m}, \circ_{n}\right)$ is bi-skew.
Example 4.2. Any brace $(B, \cdot, \circ)$ can be made into a brace block with $x \circ_{0} y=x \cdot y, x \circ_{n} y=x \circ y$ for all $x, y \in B, n \geq 1$. Since each brace is the trivial brace on $(B, \cdot)$ we will call this a trivial brace block.

Example 4.3. Suppose $(B, \cdot, \circ)$ is a nontrivial bi-skew brace. We can then form a nontrivial brace block with

$$
x \circ_{n} y= \begin{cases}x \cdot y & n \text { even } \\ x \circ y & n \text { odd }\end{cases}
$$

Starting with a (nonabelian) group $G=(G, \cdot)$ and a $\psi \in \mathrm{Ab}(G)$ we will construct a brace block. We start by constructing the necessary binary operations. Let $\psi \in \operatorname{Ab}(G)$. For each $n \geq 0$ define a binary operation $\circ_{n}$ on $G$ by

$$
g \circ_{n} h=g \psi_{n}\left(g^{-1}\right) h \psi_{n}(g), g, h \in G .
$$

Notice that $\circ_{0}$ is the usual group operation in $G$, and $\circ_{1}$ is the operation denoted by $\circ$ in [Koc20] since $\psi_{1}=\psi$. We will frequently write $\cdot$ and $\circ$ for $\circ_{0}$ and $\circ_{1}$ respectively. Since $\psi_{n} \in \operatorname{Ab}(G)$ we have, by [Koc20, Th. 1],

Lemma 4.4. For all $n \geq 0,\left(G, \circ_{n}\right)$ is a group.
For future reference, the identity in $\left(G, \circ_{n}\right)$ is $1_{G}$, the identity in $(G, \cdot)$, and the inverse to $g \in\left(G, \circ_{n}\right)$ is $\psi_{n}(g) g^{-1} \psi_{n}\left(g^{-1}\right)$, as can be readily checked.

We also have the following recursive formulation.
Proposition 4.5. Let $\psi \in \operatorname{Ab}(G)$. Then for all $n \geq 0$ we have

$$
g \circ_{n+1} h=\left(\left(g \psi\left(g^{-1}\right)\right) \circ_{n} h\right) \psi(g), g, h \in G .
$$

Proof. Lemma $3.2(5)$ gives $\psi_{n+1}(g)=\psi(g) \psi_{n}(1-\psi)(g)$ for all $g \in G$. Applying this we get

$$
\begin{align*}
g \circ_{n+1} h & =g \psi_{n+1}\left(g^{-1}\right) h \psi_{n+1}(g) \\
& =g \psi_{n}\left(g \psi\left(g^{-1}\right)\right)^{-1} \psi\left(g^{-1}\right) h \psi(g) \psi_{n}\left(g \psi\left(g^{-1}\right)\right) \\
& =g \psi\left(g^{-1}\right) \psi_{n}\left(g \psi\left(g^{-1}\right)\right)^{-1} h \psi_{n}\left(g \psi\left(g^{-1}\right)\right) \psi(g)  \tag{2}\\
& =\left(\left(g \psi\left(g^{-1}\right)\right) \circ_{n} h\right) \psi(g) .
\end{align*}
$$

From this we get
Corollary 4.6. Let $\psi \in \operatorname{Ab}(G)$. If $\left(G, \circ_{n}\right)$ is abelian, then $\left(G, \circ_{n+1}\right)=\left(G, \circ_{n}\right)$, hence $\left(G, \circ_{m}\right)=$ $\left(G, \circ_{n}\right)$ for all $m \geq n$.

Proof. Suppose $\left(G, \circ_{n}\right)$ is abelian. Then for all $g, h \in G$ we have

$$
\begin{align*}
g \circ_{n+1} h & =\left(\left(g \psi\left(g^{-1}\right)\right) \circ_{n} h\right) \psi(g) \\
& =\left(h \circ_{n} g \psi\left(g^{-1}\right)\right) \psi(g) \\
& =h \psi_{n}\left(h^{-1}\right) g \psi\left(g^{-1}\right) \psi_{n}(h) \psi(g) \\
& =h \psi_{n}\left(h^{-1}\right) g \psi_{n}(h)  \tag{2}\\
& =h \circ_{n} g \\
& =g \circ_{n} h .
\end{align*}
$$

Thus the group operations are identical, and $\left(G, \circ_{n+1}\right)=\left(G, \circ_{n}\right)$.
Having established this collection of groups given by $\psi \in \mathrm{Ab}(G)$ we arrive at our main result, which states that any pair of the groups constructed above form a brace.

Theorem 4.7. Let $\psi \in \operatorname{Ab}(G)$. Then for all $m, n \geq 0$ we have $\left(G, \circ_{m}, \circ_{n}\right)$ is a brace.
Proof. We simply need to show that $\left(G, \circ_{m}, \circ_{n}\right)$ satisfies the brace relation for all $m, n \geq 0$, that is,

$$
g \circ_{n}\left(h \circ_{m} k\right)=\left(g \circ_{n} h\right) \circ_{m} \tilde{g} \circ_{m}\left(g \circ_{m} k\right)
$$

for all $g, h, k \in G$, where $\tilde{g}=\psi_{m}(g) g^{-1} \psi_{m}\left(g^{-1}\right)$ is the inverse to $g$ in $\left(G, \circ_{m}\right)$.
Recall that $\psi_{m}$ and $\psi_{n}$ are constant on conjugacy classes. We have

$$
\begin{aligned}
\left(g \circ_{n} h\right) \circ_{m} \tilde{g} \circ_{m}\left(g \circ_{m} k\right) & =\left(g \psi_{n}\left(g^{-1}\right) h \psi_{n}(g)\right) \circ_{m} \psi_{m}(g) g^{-1} \psi_{m}\left(g^{-1}\right) \circ_{m}\left(g \psi_{n}\left(g^{-1}\right) k \psi_{n}(g)\right) \\
& =\left(\left(g \psi_{n}\left(g^{-1}\right) h \psi_{n}(g)\right) \psi_{m}\left(h^{-1} g^{-1}\right) \psi_{m}(g) g^{-1} \psi_{m}\left(g^{-1}\right) \psi_{m}(g h)\right) \circ_{m}\left(g \psi_{n}\left(g^{-1}\right) k \psi_{n}(g)\right) \\
& =\left(\left(g \psi_{n}\left(g^{-1}\right) h \psi_{n}(g)\right) \psi_{m}\left(h^{-1}\right) g^{-1} \psi_{m}(h)\right) \circ_{m}\left(g \psi_{n}\left(g^{-1}\right) k \psi_{n}(g)\right) \\
& =\left(\left(g \psi_{n}\left(g^{-1}\right) h \psi_{n}(g)\right) \psi_{m}\left(h^{-1}\right) g^{-1} \psi_{m}(h)\right) \psi_{m}\left(h^{-1}\right)\left(g \psi_{n}\left(g^{-1}\right) k \psi_{n}(g)\right) \psi_{m}(h) \\
& =\left(g \psi_{n}\left(g^{-1}\right) h \psi_{n}(g)\right) \psi_{m}\left(h^{-1}\right) \psi_{n}\left(g^{-1}\right) k \psi_{n}(g) \psi_{m}(h) \quad \text { (Lemma 1.2 (2)) } \\
& =g \psi_{n}\left(g^{-1}\right) h \psi_{m}\left(h^{-1}\right) \psi_{n}(g) \psi_{n}\left(g^{-1}\right) k \psi_{m}(h) \psi_{n}(g) \quad \\
& =g \psi_{n}\left(g^{-1}\right) h \psi_{m}\left(h^{-1}\right) k \psi_{m}(h) \psi_{n}(g) \\
& =g \circ_{n}\left(h \circ_{m} k\right) .
\end{aligned}
$$

Thus the brace relation is satisfied and we are done.
From this we immediately obtain the following.
Corollary 4.8. Let $\psi \in \operatorname{Ab}(G)$. Then $\psi$ gives a brace block.

In theory we can construct an unlimited number of binary operations using $\psi$, but of course only a finite number of them will be distinct. We have the following result, which is closely related to [Koc20, Prop. 3.3].

Proposition 4.9. With the notation as above, the operations $\circ_{m}$ and $\circ_{n}$ agree if and only if $\left(\psi_{m}-\psi_{n}\right)(G) \subset Z(G)$.

Proof. Suppose $g \circ_{m} h=g \circ_{n} h$ for all $g, h \in G$. Then

$$
\begin{aligned}
g \psi_{m}\left(g^{-1}\right) h \psi_{m}(g) & =g \psi_{n}\left(g^{-1}\right) h \psi_{n}(g) \\
\psi_{m}\left(g^{-1}\right) h \psi_{m}(g) & =\psi_{n}\left(g^{-1}\right) h \psi_{n}(g) \\
h & =\psi_{m}(g) \psi_{n}\left(g^{-1}\right) h \psi_{n}(g) \psi_{m}\left(g^{-1}\right),
\end{aligned}
$$

hence $\psi_{m}(g) \psi_{n}\left(g^{-1}\right)=\left(\psi_{m}-\psi_{n}\right)(g) \in Z(G)$. The converse is trivial.
As a result, we see patterns emerging in the binary operations.
Corollary 4.10. If $g \circ_{m} h=g \circ_{n} h$ for all $g, h \in G$, then $g \circ_{m+1} h=g \circ_{n+1} h$ for all $g, h \in G$.
Proof. It suffices to show that if $\left(\psi_{m}-\psi_{n}\right) \subset Z(G)$ then $\left(\psi_{m+1}-\psi_{n+1}\right) \subset Z(G)$. Using Lemma 3.2 (2), (5) and the right distributive property in the near-ring we have

$$
\psi_{m+1}-\psi_{n+1}=\psi+\psi_{m}(1-\psi)-\psi-\psi_{n}(1-\psi)=\left(\psi_{m}-\psi_{n}\right)(1-\psi)
$$

and clearly $\left(\psi_{m}-\psi_{n}\right)(1-\psi)(G) \subset\left(\psi_{m}-\psi_{n}\right)(G) \subset Z(G)$.
For future reference, we also note the following.
Proposition 4.11. Let $\psi \in \operatorname{Ab}(G)$. Then $(1-\psi)(G) \leq G$. Furthermore, $(G, \circ)$ is abelian if and only if $((1-\psi)(G), \cdot)$ is abelian.

Proof. To show that $(1-\psi)(G) \leq G$ it suffices to show that $(1-\psi)(G)$ is closed under the usual operation on $G$, which follows from the identity

$$
((1-\psi)(g))((1-\psi)(h))=(1-\psi)\left(g \psi\left(g^{-1}\right) h \psi(g)\right)
$$

which can be readily verified. Now suppose $g \circ h=h \circ g$. Then

$$
\begin{aligned}
g \psi\left(g^{-1}\right) h \psi(g) & =h \psi\left(h^{-1}\right) g \psi(h) \\
g \psi\left(g^{-1}\right) h \psi\left(h^{-1} g h\right) & =h \psi\left(h^{-1}\right) g \psi\left(g^{-1} h g\right) \\
g \psi\left(g^{-1}\right) h \psi\left(h^{-1}\right) \psi(g h) & =h \psi\left(h^{-1}\right) g \psi\left(g^{-1}\right) \psi(g h) \\
g \psi\left(g^{-1}\right) h \psi\left(h^{-1}\right) & =h \psi\left(h^{-1}\right) g \psi\left(g^{-1}\right) .
\end{aligned}
$$

Thus, $(G, \circ)$ is abelian if and only if $\left(g \psi\left(g^{-1}\right)\right)\left(h \psi\left(h^{-1}\right)\right)=\left(h \psi\left(h^{-1}\right)\right)\left(g \psi\left(g^{-1}\right)\right)$, that is, $(1-\psi)(G)$ is abelian.

The above will be most useful when applied to $\psi_{n}$.
Corollary 4.12. Let $\psi \in \operatorname{Ab}(G)$. Then $\left(1-\psi_{n}\right)(G) \leq G$. Furthermore, $\left(G, \circ_{n}\right)$ is abelian if and only if $\left((1-\psi)^{n}(G), \cdot\right)$ is abelian.

Proof. Apply the proposition to $\psi_{n}$, and recall that $1-\psi_{n}=(1-\psi)^{n}$.
Example 4.13. Let $G=D_{4}=\left\langle r, s: r^{4}=s^{2}=r s r s=1_{G}\right\rangle$ be the dihedral group of order 8 . Let $\psi: G \rightarrow G$ be given by $\psi(r)=1, \psi(s)=r^{2} s$. It is easy to show that $\psi \in \operatorname{Ab}(G)$ (see also [Koc20, $\S 6])$. Since $(1-\psi)(r)=r$ and $(1-\psi)(s)=r^{2}$ we see that $(1-\psi)(G)=\langle r\rangle$ is abelian. Thus, $(G, \circ)$ is abelian.

Example 4.14. Let $G=\operatorname{Aff}\left(\mathbb{F}_{5}\right)$, the affine group of the finite field with five elements. Then $G \cong C_{5} \rtimes \operatorname{Aut}\left(C_{5}\right)$ where $\operatorname{Aut}\left(C_{5}\right)$ acts on $C_{5}$ in the obvious way. If we let $C_{5}=\langle g\rangle$ and let $\alpha \in \operatorname{Aut}\left(C_{5}\right)$ be given by $\alpha(g)=g^{2}$ then

$$
G=\left\langle g, \alpha: g^{5}=\alpha^{4}=g \alpha g^{2} \alpha^{3}=1_{G}\right\rangle
$$

Define $\psi: G \rightarrow G$ by $\psi(g)=1_{G}, \psi(\alpha)=\alpha^{-1}$. Then $\psi$ satisfies the relations above, and since $\psi(G)=\langle\alpha\rangle$ it follows that $\psi \in \operatorname{Ab}(G)$. We have

$$
(1-\psi)(g)=g,(1-\psi)(\alpha)=\alpha^{2}
$$

hence $(1-\psi)(G)=\left\langle g, \alpha^{2}\right\rangle$ which is nonabelian since $\alpha^{2} g=g^{-1} \alpha$, hence $(G, \circ)$ is nonabelian. (In fact, $(1-\psi)(G) \cong D_{5}$, the dihedral group of order 10.) However,

$$
(1-\psi)^{2}(g)=g,(1-\psi)^{2}(\alpha)=(1-\psi)\left(\alpha^{2}\right)=1_{G}
$$

so $\left(G, \circ_{2}\right)$ is abelian. It is not hard to show that $\left(G, \circ_{2}\right) \cong C_{5} \times C_{4} \cong C_{20}$.
By applying the techniques of obtaining solutions from braces, we obtain:
Theorem 4.15. Let $\psi \in \operatorname{Ab}(G)$. Then for all $m, n$ we have the following solutions to the YangBaxter equation:

$$
\begin{aligned}
& R_{m, n}(g, h)=\left(\psi_{m}(g) \psi_{n}\left(g^{-1}\right) h \psi_{n}(g) \psi_{m}\left(g^{-1}\right), \psi_{m}(g) \psi_{n}\left(g^{-1} h\right) h^{-1} \psi_{n}(g) \psi_{m}\left(g^{-1}\right) g \psi_{n}\left(g^{-1}\right) h \psi_{n}\left(g h^{-1}\right)\right) \\
& R_{m, n}^{\prime}(g, h)=\left(g \psi_{n}\left(g^{-1}\right) h \psi_{n}(g) \psi_{m}\left(h^{-1}\right) g^{-1} \psi_{m}(h), \psi_{n}(h) \psi_{m}\left(h^{-1}\right) g \psi_{m}(h) \psi_{n}\left(h^{-1}\right)\right)
\end{aligned}
$$

The proof is a straightforward computation using Equations (2.1) and (2.2). In the case $m=$ $0, n=1$ one can quickly recover the solutions $R_{1, \psi}$ and $R_{2, \psi}$ from [Koc20, Cor. 5.4], formulas which also appear in [KST20, Th. 5.1] where $\psi$ is subject to a "fixed point free" condition (as explained below).

Note that if $m=n$ then we have $R_{n, n}=R_{n, n}^{\prime}$.

## 5. Preliminary Examples

The motivation of the construction of brace blocks was to construct large families of braces emanating from a single abelian map $\psi$. Unfortunately, many simple examples of abelian maps found in [Chi13], [KST20], and [Koc20] do not give large families. The maps considered in [Chi13] and [KST20] are subject to an additional condition: that of being fixed point free. An endomorphism $\psi: G \rightarrow G$ is said to be fixed point free if $\psi(g)=g$ if and only if $g=1_{G}$.

Example 5.1. Let $G=D_{n}=\left\langle r, s: r^{n}=s^{2}=r s r s=1\right\rangle$. Any $\psi \in G$ must send $r$ and $s$ to elements of order dividing 2. Thus, $2 \psi=0$. We will consider three cases.

The first case is when $\psi=0$ is the trivial map. Of course, then $\psi_{n}=-(1-0)^{n}+1=0$ is also trivial, and we get a trivial brace block.

Now suppose $\psi$ is fixed point free and nontrivial. Then, by [Chi13, $\S 6], \psi(G)$ is a group of order 2 , say $\psi(G)=\left\{1_{G}, x\right\}$. Since $\psi$ is fixed point free, $\psi(x)=1_{G}$, hence $\psi(\psi(g))=1_{G}$ for all $g \in G$, i.e., $\psi^{2}=0$. Thus, for $k \geq 2$ we have

$$
\psi_{k}=\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i} \psi^{i}=k \psi= \begin{cases}\psi & k \text { odd } \\ 0 & k \text { even }\end{cases}
$$

Furthermore, both $(G, \cdot, \cdot)$ and $(G, \circ, \circ)$ are the trivial brace on $G$, hence $(G, \cdot, \cdot)=(G, \circ, \circ)$. It is known (see, e.g., [KT20b, Ex. 3.4]) that a trivial brace is not isomorphic to a non-trivial brace; furthermore we will see that $(G, \circ, \cdot) \cong(G, \cdot, \circ)$ (see Corollary 5.6). Consequently, this brace block has two nonisomorphic braces, namely $(G, \cdot, \cdot)$ and $(G, \cdot, \circ)$. Each provides two solutions to the Yang-Baxter equation, giving us four solutions in total.

Now suppose $\psi$ has fixed points. Again, $\psi(G)$ is a group of order 2, as shown in [Koc20, §6]. If we write $\psi(G)=\{1, x\}$ then $x$ must be the fixed point. Thus, $\psi^{2}(g)=\psi(g)$ for all $g \in G$ so $\psi^{k}=\psi$ for all $k \geq 1$. Thus,

$$
\psi_{k}=\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i} \psi^{i}=\left(\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}\right) \psi=\psi
$$

Therefore, $\left(G, \circ_{m}, \circ_{n}\right)=(G, \circ, \circ)$ for $m, n \geq 1$. By [Koc20, §6] we know that $(G, \circ) \not \neq D_{n}-$ it is, in fact, either $C_{n} \times C_{2}$ or $D_{n / 2} \times C_{2}$ depending on the parity of $n$ and the choice of $x$-so we have four nonisomorphic braces: $(G, \cdot, \cdot),(G, \cdot, \circ),(G, \circ, \cdot)$, and $(G, \circ, \circ)$. If $(G, \circ) \cong D_{n / 2} \times C_{2}$ then each gives two solutions to the Yang-Baxter equation, giving us a total of 8 solutions, whereas if $(G, \circ) \cong C_{n} \times C_{2}$ we obtain 6 solutions.

Example 5.2. Let $G=S_{n}, n \geq 5$. Here each abelian map sends all even permutations to the identity and all odd permutations to an element of order 2 , say $\xi$. As above we have $2 \psi=0$. If $\xi \in A_{n}$ then $\psi$ is fixed point free and clearly $\psi^{2}=0$. If $\xi \notin A_{n}$ then $\psi(G)=\langle\xi\rangle$ and $\psi^{2}=\psi$.

Thus we get the same two cases as above. In the fixed point free case we get the two braces $(G, \cdot, \cdot)$ and $(G, \cdot, \circ)$, and in the case with fixed points we get $(G, \cdot, \cdot),(G, \cdot, \circ),(G, \circ, \cdot)$, and $(G, \circ, \circ)$. In the latter case, $(G, \circ) \cong A_{n} \times C_{2}$ hence we will always get 8 solutions.

Example 5.3. Suppose $G=H K=H \rtimes K$ with $K$ abelian. Then $\psi(h k)=k$ is an abelian map. Since $\psi^{2}(h k)=k$ and $\psi\left((h k)^{2}\right)=k^{2}$. we get $\psi^{2}=\psi$ and $\psi_{k}=\psi$ as above. Our brace block produces braces $(G, \cdot, \cdot),(G, \cdot, \circ),(G, \circ, \cdot)$, and $(G, \circ, \circ)$, and as $(G, \circ) \cong H \times K$ we get 6 or 8 solutions depending on whether $H$ is abelian.

We conclude this section with a closer look at the case where $\psi \in \operatorname{Ab}(G)$ is fixed point free. First, we note:

Proposition 5.4. Let $\psi \in \operatorname{Ab}(G)$. Then $\psi$ is fixed point free if and only if $\psi_{n}$ is fixed point free for all $n \geq 1$.

Proof. First, suppose $\psi(g)=g$ for some $g \in G, g \neq 1_{G}$. Note that $g^{-1}$ is also a fixed point of $\psi$. Then $(1-\psi)\left(g^{-1}\right)=1_{G}$ and

$$
\psi_{n}(g)=\left(-(1-\psi)^{n}+1\right)(g)=\left((1-\psi)^{n}\left(g^{-1}\right)\right) g=g
$$

and hence $g$ is a fixed point of $\psi_{n}$ for all $n$.
Conversely, suppose $\psi$ is fixed point free. Let $n$ be the smallest positive integer such that $\psi_{n}$ has nontrivial fixed points, say $\psi_{n}(g)=g$ for $g \neq 1_{G}$. By Lemma 3.2, (5) we have

$$
g=\psi_{n}(g)=\left(\psi+\psi_{n-1}(1-\psi)\right)(g)=\psi(g) \psi_{n-1}\left(g \psi\left(g^{-1}\right)\right)
$$

which we may rewrite as

$$
\psi\left(g^{-1}\right) g=\psi_{n-1}\left(g \psi\left(g^{-1}\right)\right)=\psi_{n-1}\left(\psi\left(g^{-1}\right) g\right)
$$

and so $\psi\left(g^{-1}\right) g$ is a fixed point of $\psi_{n-1}$. But as $\psi_{n-1}$ is assumed to have no nontrivial fixed points it follows that $\psi\left(g^{-1}\right) g=1_{G}$, i.e., $g=\psi(g)$, hence $g=1_{G}$, a contradiction. Thus, $\psi_{n}$ is fixed point free for all $n \geq 1$.

The next result we will apply to fixed point free maps, however we state it more generally.
Proposition 5.5. Let $\psi \in \operatorname{Ab}(G)$, and for $n \geq 1$ let $\phi=(1-\psi) \in \operatorname{Map}(G)$. Then $\phi:\left(G, \circ_{n}\right) \rightarrow$ $\left(G, \circ_{n-1}\right)$ is a homomorphism.

Proof. We shall prove this by induction on $n$. Since

$$
\begin{aligned}
\phi(g \circ h) & =(g \circ h) \psi(g \circ h)^{-1} \\
& =g \psi\left(g^{-1}\right) h \psi(g) \psi\left(g \psi\left(g^{-1}\right) h \psi(g)\right) \\
& =g \psi\left(g^{-1}\right) h \psi(g) \psi\left(h^{-1} g^{-1}\right) \\
& =g \psi\left(g^{-1}\right) h \psi\left(h^{-1}\right) \\
& =\phi(g) \cdot \phi(g)
\end{aligned}
$$

we see that $\phi:\left(G, \circ_{1}\right) \rightarrow\left(G, \circ_{0}\right)$ is a homomorphism. Now suppose $\phi:\left(G, \circ_{k}\right) \rightarrow\left(G, \circ_{k-1}\right)$ is a homomorphism. Using Lemma 3.2 (2),(5) we get

$$
\begin{aligned}
\phi\left(g \circ_{k+1} h\right) & =\left(g \circ_{k+1} h\right) \psi\left(g \circ_{k+1} h\right)^{-1} \\
& =g \psi_{k+1}\left(g^{-1}\right) h \psi_{k+1}(g) \psi\left(h^{-1} g^{-1}\right) \\
& =g \psi\left(g^{-1}\right) \psi_{k}\left(g^{-1} \psi(g)\right) h \psi(g) \psi_{k}\left(g \psi\left(g^{-1}\right)\right) \psi\left(g^{-1} h^{-1}\right) \\
& =g \psi\left(g^{-1}\right) \psi_{k}\left(\left(g \psi\left(g^{-1}\right)\right)^{-1}\right) h \psi\left(h^{-1}\right) \psi_{k}\left(g \psi\left(g^{-1}\right)\right) \\
& =\phi(g) \circ_{k} \phi(h)
\end{aligned}
$$

and hence $\phi:\left(G, \circ_{k+1}\right) \rightarrow\left(G, \circ_{k}\right)$ is a homomorphism.
Once the above is established, we quickly get
Corollary 5.6. Let $\psi \in \operatorname{Ab}(G)$ be fixed point free, and let $n>m \geq 0$. Then $\left(G, \circ_{m}, \circ_{n}\right) \cong$ $\left(G, \cdot, \circ_{n-m}\right)$.

Proof. The map $\phi: G \rightarrow G$ as defined above is bijective if and only if $\psi$ is fixed point free [Gor68, Lemma 10.1.1]; hence $\phi^{n-m}:\left(G, \circ_{m}, \circ_{n}\right) \rightarrow\left(G \cdot, \circ_{n-m}\right)$ is an isomorphism in this case.

We will see in the next section that one can sometimes use fixed point free abelian maps to generate relatively large brace blocks.

Example 5.7. Let $n$ be a odd integer, and let $D_{n}$ denote the dihedral group of order $2 n$. Consider the group $D_{n} \times D_{n}$, presented as

$$
G=\left\langle r, s: r^{n}=s^{2}=r s r s=1_{G}\right\rangle \times\left\langle t, u: t^{n}=u^{2}=t u t u=1_{G}\right\rangle .
$$

Define $\psi: G \rightarrow G$ by $\psi(r)=\psi(t)=1_{G}, \psi(s)=u, \psi(u)=s$. Then $\psi \in \operatorname{Ab}(G)$. Since $s u \in \psi(G)$ and $\psi(G)$ is abelian, for all $g \in G$ we have

$$
g \circ s u=g \psi\left(g^{-1}\right) s u \psi(g)=g s u, s u \circ g=(s u)(u s) g(u s)=g s u,
$$

hence $s u \in Z(G, \circ)$. Since $Z(G, \cdot)=Z\left(D_{n}\right) \times Z\left(D_{n}\right)$ is trivial we get that $(G, \circ) \neq(G, \cdot)$. Also, $(G, \circ)$ is nonabelian since $r \circ u=r u, u \circ r=r^{-1} u$. In fact, it can be shown that $(G, \circ)=$ $\langle u s\rangle \times((\langle r\rangle \times\langle t\rangle) \rtimes\langle u\rangle) \cong C_{2} \times\left(\left(C_{n} \times C_{n}\right) \rtimes C_{2}\right)$ where the semidirect product arises from the map $C_{2} \rightarrow \operatorname{Aut}\left(C_{n} \times C_{n}\right)$ sending the nontrivial element to the inverse map.

Now we compute $\psi_{2}: G \rightarrow G$. It is easy to see that $\psi_{2}(r)=\psi_{2}(t)=1_{G}$; furthermore

$$
\psi_{2}(s)=\left(2 \psi-\psi^{2}\right)(s)=\psi\left(1_{G}\right) \psi^{2}(s)=s
$$

and $\psi_{2}(u)=u$ similarly. Since $G=G_{0} G_{1}$ where $G_{0}=\operatorname{ker} \psi_{2}$ and $G_{1}$ is the subgroup of fixed points of $\psi_{2}$, by [Koc20, Prop. 6.3] we have that $\left(G, \circ_{2}\right) \cong G_{0} \times G_{1} \cong\left(C_{n} \times C_{n}\right) \times\left(C_{2} \times C_{2}\right) \cong C_{2 n} \times C_{2 n}$. (Of course, that $\left(G, \circ_{2}\right)$ is abelian also follows easily by observing $(1-\psi)^{2}(G)=\langle r, t\rangle \cong C_{h} \times C_{h}$.)

Since $\left(G, \circ_{2}\right)$ is abelian, we have that $\left(G, \circ_{n}\right)=\left(G, \circ_{2}\right)$ for all $n \geq 2$ by Corollary 4.6. Thus we have 9 braces, namely $\left(G, \circ_{i}, \circ_{j}\right)$ for $0 \leq i, j \leq 2$, all pairwise nonisomorphic. If $i \neq 2$ then $\left(G, \circ_{i}, \circ_{j}\right)$ gives us two solutions to the Yang-Baxter equation, whereas $\left(G, \circ_{1}, \circ_{j}\right)$ gives us one. In total, we have 15 solutions.

## 6. SEmidirect products of two cyclic groups

Finally, we present a class of examples which provide brace blocks containing many nonisomorphic braces. We gratefully acknowledge Lindsay Childs for pointing out this class of examples.

We will follow the notation in [CC07]. Pick an integer $h \geq 3$, and let $F(h, k, b)=\left\langle s, t: s^{h}=t^{k}=\right.$ $\left.t s t^{-1} s^{-b}=1_{G}\right\rangle$ where $k \mid \phi(h)$ and $b$ has multiplicative order $k(\bmod h)$. We also have $F\left(h, k, b^{n}\right)$ for any $n \geq 0$ : while $b^{n}$ may not have multiplicative order $k$, there is some $c$ of multiplicative order $k$ such that $F\left(h, k, b^{n}\right)=F(h, k, c)$. As we shall see, it will be useful to refer to such groups using powers of $b$.

We start by addressing isomorphism questions among these groups.
Lemma 6.1. Let $h, k, b$ be as above. Let $n \in \mathbb{Z}$, and let $d=\operatorname{gcd}(k, n)$. Then $F\left(h, k, b^{n}\right) \cong$ $F\left(h, k, b^{d}\right)$.

Proof. Pick $e \in \mathbb{Z}$ such that $e n \equiv d(\bmod k)$ : such an $e$ exists since $\operatorname{gcd}(n / d, k)=1$. We define $\gamma: F\left(h, k, b^{d}\right) \rightarrow F\left(h, k, b^{n}\right)$ by $\gamma\left(s^{u} t^{v}\right)=s^{u} t^{v e}$. To see this is well-defined, observe that

$$
\gamma(t s)=\gamma\left(s^{b^{d}} t\right)=s^{b^{d}} t^{e}=s^{b^{e n}} t^{e}=t^{e} s=\gamma(t) \gamma(s)
$$

As $e$ is invertible $\bmod k$ it follows that $\gamma$ an isomorphism.
When $h, k$ and $b$ are understood we will write $F_{n}=F\left(h, k, b^{n}\right)$ for brevity.
Lemma 6.2. Let $n, d$ be as above, and suppose $h$ is prime. If $d \neq k$ then $\left|Z\left(F_{n}\right)\right|=d$; otherwise, $F_{n}$ is abelian.

Proof. In light of the previous result we may assume $n=d$. Suppose $s^{u} t^{v} \in Z\left(F_{d}\right)$. Then

$$
s^{u+b^{d v}} t=\left(s^{u} t^{v}\right) s=s\left(s^{u} t^{v}\right)=s^{u+1} t^{v},
$$

from which it follows that $b^{d v} \equiv 1(\bmod k)$, i.e., $k \mid d v$. Furthermore,

$$
s^{u} t^{v+1}=\left(s^{u} t^{v}\right) t=t\left(s^{u} t^{v}\right)=s^{u b^{d}} t^{v+1},
$$

and hence $u \equiv u b^{d}(\bmod h)$. Since $h$ is prime, it follows that either $u=0$ or $b^{d} \equiv 1(\bmod h)$. But if $b^{d} \equiv 1(\bmod h)$ then $k \mid d$, so $k=d$ and $F_{d}=F_{k} \cong C_{h} \times C_{k}$. If $k \neq d$ then $u=0$. Thus,

$$
Z\left(F_{d}\right)= \begin{cases}\left\langle t^{k / d}\right\rangle & k<d \\ G & k=d\end{cases}
$$

Combining the two previous results gives us
Proposition 6.3. Suppose $h$ is prime. then $F_{m} \cong F_{n}$ if and only if $\operatorname{gcd}(m, k)=\operatorname{gcd}(n, k)$.
For the remainder of this section, we assume $h$ is prime.
We will now construct brace blocks starting with $G=F_{1}$. Pick $j \in \mathbb{Z}$ and define $\psi: G \rightarrow G$ by $\psi(s)=1_{G}, \psi(t)=t^{1-j}$. This map clearly respects the relations in $G$, and as $\psi(G) \leq\langle t\rangle \cong C_{k}$ we see that $\psi \in \operatorname{Ab}(G)$. Note that if $j \equiv 1(\bmod k)$ then we will get a trivial brace block.

We shall now explicitly compute $\psi_{n}$. Note that

$$
(1-\psi)(s)=s \psi\left(s^{-1}\right)=s,(1-\psi)(t)=t \psi\left(t^{-1}\right)=t \cdot t^{j-1}=t^{j} .
$$

Thus,

$$
\begin{aligned}
& \psi_{n}(s)=\left(-(1-\psi)^{n}+1\right)(s)=s^{-1} \cdot s=1_{G} \\
& \psi_{n}(t)=\left(-(1-\psi)^{n}+1\right)(t)=t^{-j^{n}} \cdot t=t^{1-j^{n}} .
\end{aligned}
$$

Next, we compute the group $\left(G, \circ_{n}\right)$ for each $n$. Clearly, $s \circ_{n} g=s g$ for all $g \in G$; in particular, $s \circ_{n} t=s t$. Also,

$$
\begin{aligned}
& t \circ_{n} t=t \psi_{n}\left(t^{-1}\right) t \psi_{n}(t)=t \cdot t^{j^{n}-1} \cdot t \cdot t^{1-j^{n}}=t^{2} \\
& t \circ_{n} s=t \psi_{n}\left(t^{-1}\right) s \psi_{n}(y)=t \cdot t^{j^{n}-1} \cdot s \cdot t^{1-j^{n}}=s^{b^{j^{n}}} t=s^{b^{j^{n}}} \circ_{n} t .
\end{aligned}
$$

Thus, $\left(G, \circ_{n}\right)=\left\langle s, t: s^{\circ} h=t^{\circ} k=1_{G}, t \circ_{n} s=s \circ_{n} t^{b^{j^{n}}} \circ t\right\rangle$ where " $\circ_{n} m$ " in the exponent is the application of $\circ_{n}$ to the element $m$ times. In other words, $\left(G, o_{n}\right)=F_{j^{n}}$.

Notice that $(1-\psi)^{n}(s)=s$ and $(1-\psi)^{n}(t)=t^{j^{n}}$. Thus, $\left(G, \circ_{n}\right)=F_{j^{n}}$ is abelian if and only if $t^{j^{n}} \in Z(G)$. In particular, if $k \mid j^{n}$ then $\left(G, \circ_{n}\right)$ is abelian.

Example 6.4. Suppose $k$ is prime. Then $h \equiv 1(\bmod k)$ and $G=F_{1}$ is the unique nonabelian group of order $h k$. The abelian maps on $G$ appear in [Koc20, Ex. 3.8] using slightly different notation. If we pick $j$ such that $k \mid j$ then $\left(G, \circ_{j}\right) \cong C_{h} \times C_{k}$ and we get a specific instance of Example 5.3.

For all other choices of $j$ we have $\operatorname{gcd}\left(j^{n}, k\right)=1$ for all $n$, hence $\left(G, \circ_{n}\right) \cong G$ for all $n$. By [KT20b, Prop. 8.17] we see that $\left(G, \cdot, \circ_{m}\right) \cong\left(G, \cdot, \circ_{n}\right)$ if and only if $1-j^{m} \equiv 1-j^{n}(\bmod k)$, i.e., $j^{m} \equiv j^{n}(\bmod k)$. Thus, if $j$ has multiplicative order $\ell(\bmod k)$ then $\left\{\left(G, \cdot, \circ_{n}\right): 0 \leq n \leq \ell-1\right\}$ consist of $\ell-1$ distinct braces (note that $\left.\left(G, \cdot, \circ_{\ell}\right)=(G, \cdot, \cdot)\right)$. By Corollary 5.6 we see that all other braces in this brace block are isomorphic to one of braces in this set.

Since $(G, \cdot)=F_{1}$ is nonabelian we get $2 \ell$ solutions to the Yang-Baxter equation. In fact, if we pick $j$ to be a primitive root $\bmod k$ then we obtain all $2(k-1)$ solutions to the Yang-Baxter equation arising from a brace with both groups isomorphic to the metacyclic group. Indeed, by [KT20b] each brace with both groups metacyclic is either in the block above or is the opposite to a brace in the block above. Thus, we have constructed all solutions to the Yang-Baxter equation coming from braces with metacyclic group structures of order $h k$.

Example 6.5. Let $N$ be a large integer. By Dirichlet's Theorem, there exist prime numbers of the form $\ell \cdot 2^{N}+1$; let $h$ be one such prime, and let $k=2^{N}$. Pick $j=2$, i.e., $\psi(t)=t^{-1}$. Then $\left(G, \circ_{m}\right)=F_{2^{m}}$ for all $m$, and since $\operatorname{gcd}\left(2^{m}, 2^{N}\right)=2^{\min \{m, N\}}$ we get that $\left(G, \circ_{m}\right) \not \approx\left(G, \circ_{n}\right)$ for $0 \leq m<n \leq N$ and $\left(G, \circ_{m}\right) \cong\left(G, \circ_{N}\right) \cong C_{\ell \cdot 2^{N}+1} \times C_{2^{N}} \cong C_{(\ell+1) \cdot 2^{N}+1}$ for all $m \geq N$. Thus, we get a brace block consisting of $N$ pairwise nonisomorphic groups.

It follows that our brace block contains $(N+1)^{2}$ different braces, of which $N^{2}$ consist of nonabelian groups (obtained by requiring $m, n \neq N$ ), giving $4 N^{2}$ solutions to the Yang-Baxter equation; $N$ braces containing exactly one abelian group, giving another $2 N$ solutions; and 1 brace with both groups abelian, giving one more solution. In total, we get $4 N^{2}+2 N+1=(2 N+1)^{2}$ solutions to the Yang-Baxter equation.

These examples show that there are brace blocks with an arbitrarily number of pairwise nonisomorphic groups, and that the number of possible solutions to the YBE using an abelian map is unbounded.

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