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ABELIAN THEOREMS FOR A CLASS OF PROBABILITY DISTRIBUTIONS IN R^d AND THEIR APPLICATION

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A class of multidimensional absolutely continuous distributions is considered. Each of them has a moment-generating function that is finite in a bounded set S and, therefore, generates a family of so-called conjugate or associated distributions. At the focus of our attention are the limiting distributions for this family that appear as the conjugating parameter tends to the boundary of S. As in the one-dimensional case, each such limiting distribution can be obtained as a consequence of an Abelian theorem.

1. Introduction

Let P be a probability measure defined on the Borel sets of R^d, $d > 1$, and $f(s)$ be its moment-generating function, that is,

$$f(s) = \int_{R^d} e^{\langle s,x\rangle}\, P(dx).$$

By $\langle\cdot,\cdot\rangle$ we denote the inner product. Suppose that the set

$$S = (s \in R^d\colon\ f(s) < \infty)$$

is not empty and its dimensionality equals d.

The moment-generating function plays a role of great importance in the large-deviation theory. Its basic properties are discussed in [1–5]. The present paper aims to make a contribution toward the further development of this theory. At the focus of our attention is the case when S is, being always convex, bounded.

Let S_0 be the interior of S. If S is bounded and $0 \in S_0$, then S_0 can be represented as

$$S_0 = \{s\colon\ s = te,\ 0 \le t < h(e),\ e \in S^{d-1}\}.$$

It is convenient to call $h(e)$ the *shape function* of S or simply the *shape* of S.

Obviously, for $0 < t < h(e)$, $u > 0$ the Markov inequality holds, that is,

$$P(x\colon\ \langle e,x\rangle \ge u) \le f(te)e^{-tu}. \tag{1.1}$$

In what follows, we assume that P is absolutely continuous. Denote its density by $p(x)$.

Further, assume that

$$p(x) = b(x)e^{-|x|a(e_x)}, \tag{1.2}$$

where $e_x = |x|^{-1}x$ and

$$0 < \inf_{e\in S^{d-1}} a(e) \le \sup_{e\in S^{d-1}} a(e) < \infty.$$

If $p(x)$ is of the form (1.2) and $b(x)$ does not grow too fast as $|x| \to \infty$, then $f(s)$ is finite for some S with $0 \in S_0$. Intuitively, it is $a(e)$ that determines the shape of S. The following proposition justifies this conjecture.

Proposition 1.1. *Assume that in* (1.2)

$$c_-(1+|x|)^{-\beta} \le b(x) \le c_+(1+|x|)^{\beta}, \quad \beta > 0, \quad c_\pm > 0. \tag{1.3}$$

Then

1°.

$$h(e) = \inf_{(\varepsilon\in S^{d-1}\colon\ \langle e,\varepsilon\rangle>0)} \frac{a(\varepsilon)}{\langle e,\varepsilon\rangle}.$$

2°. *For the shape function $h(e)$ of any bounded open convex set S_0 that contains 0, there exists $p(x)$ of the form* (1.2) *such that the interior of $S = (s\colon\ f(s) < \infty)$ is S_0. As $a(e)$ in* (1.2) *one may take*

$$a(e) = \sup_{(\varepsilon\in S^{d-1}\colon\ \langle e,\varepsilon\rangle>0)} h(\varepsilon)\langle e,\varepsilon\rangle.$$

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The question arises: *What can be said about the asymptotic behavior of $f(s)$ as $s \to \partial S$?* The answer requires additional restrictions imposed on both $a(e)$ and $b(x)$ in (1.1). Our goal is to establish a multidimensional analog of the following fact.

Let $d = 1$ and

$$p(x) = e^{-s_+ x} r_\alpha(x), \tag{1.4}$$

where $s_+ > 0$ and $r_\alpha(x)$ is of regular variation as $x \to \infty$ with the exponent $\alpha > -1$. When $\tau \downarrow 0$,

$$f(s_+ - \tau) \sim \Gamma(1+\alpha)\tau^{-1} r_\alpha(\tau^{-1}). \tag{1.5}$$

This is one of the simplest forms of the so-called Abelian theorem (see, e.g., [12]).

First, we need a relevant multidimensional analog of (1.4) and (1.5). Having it in mind, we introduce the following notion of regular variation that, in essence, coincides with that given in [11, Sec. 5.4.2].

Let $\lambda(e)$ be a nonnegative function defined on S^{d-1}.

Definition 1.2. We say that $b(x)$, $x \in R^d$, is the function of (α, λ)-regular variation in the cone $C_\lambda = (x \in R^d\colon \lambda(e_x) > 0)$ if

$$b(x) = r_\alpha(|x|)(\lambda(e_x) + u(x)), \tag{1.6}$$

where $r_\alpha(t)$ is of regular, in Karamata's sense, variation as $t \to \infty$ with the exponent α while

$$\lim_{|x|\to\infty} \sup_{e_x \in C_\lambda} |u(x)| = 0.$$

Denote $\Delta(\varepsilon) = a(\varepsilon) - h(e)\langle e, \varepsilon\rangle$. We need the following assumptions:

(A) For a given direction e, the set $\arg\min_{(\varepsilon \in S^{d-1}\colon \langle e,\varepsilon\rangle > 0)} a(\varepsilon)/\langle e, \varepsilon\rangle$ consists of a single point $\varepsilon' = \varepsilon'(e)$.

(B) $\Delta(\varepsilon)$ in a neighborhood of ε' admits the representation

$$\Delta(\varepsilon) = \frac{1}{2}(\varepsilon - \varepsilon')^T \Lambda (\varepsilon - \varepsilon') + o(|\varepsilon - \varepsilon'|^2).$$

Here Λ is a nonnegative definite matrix, and its rank equals $d-1$. Furthermore, $\Lambda\varepsilon' = 0$.

(C) For all sufficiently small δ,

$$\inf_{|\varepsilon - \varepsilon'| > \delta} \Delta(\varepsilon) = c(\delta) > 0.$$

By λ_j, $j = 1, \ldots, d-1$, we denote the nonzero eigenvalues of Λ.

Consider the class of densities of the form (1.2), where $b(x)$ is of (α, λ)-regular variation in $(x\colon |e_x - e| < \delta)$, while in $(x\colon |e_x - e| \geq \delta)$ we have

$$b(x) \leq (1 + |x|)^\beta$$

for some $\beta > 0$.

Theorem 1.3. *Let $p(x)$ be of the form* (1.2) *with $\alpha > -(d+1)/2$. Assume that $\lambda(e)$ is continuous for $|e - \varepsilon'| < \delta$ and* (A), (B), *and* (C) *hold. Then as $\tau \downarrow 0$ (cf.* (1.5)),

$$f((h(e) - \tau)e) = c_\alpha g(e) \tau^{-(d+1)/2} r_\alpha(\tau^{-1})(1 + o(1)),$$

where

$$c_\alpha = \Gamma\left(\alpha + \frac{d+1}{2}\right)(2\pi)^{(d-1)/2}$$

and

$$g(e) = \lambda(\varepsilon'(e))\langle \varepsilon'(e), e\rangle^{-\alpha-(d+1)/2}(\lambda_1 \ldots \lambda_{d-1})^{-1/2}.$$

The question arises: *How does $f(s)$ behave as s approaches ∂S alongside some other direction?* It turns out that there exists a cone of admissible directions in which the form of the Abelian theorem is, in essence, preserved.

Theorem 1.4. *If the conditions of Theorem 1.3 hold, then, for any arbitrarily small $\eta > 0$ and $\tau \downarrow 0$,*

$$f(h(e)e - \tau\hat{e}) = c_\alpha g(e, \hat{e}) \tau^{-(d+1)/2} r_\alpha(\tau^{-1})(1 + o(1))$$

uniformly in $\hat{e}$, $\langle \varepsilon', \hat{e}\rangle \geq \eta$. Here

$$g(e, \hat{e}) = \lambda(\varepsilon'(e))\langle \varepsilon'(e), \hat{e}\rangle^{-\alpha-(d+1)/2}(\lambda_1 \ldots \lambda_{d-1})^{-1/2}.$$

Consider the measures $P_s(A)$, $s \in S$, defined as

$$P_s(A) = \frac{\int\limits_{R^d} \chi_A(x) e^{\langle s,x \rangle} p(x)\, dx}{f(s)}.$$

These measures are called *conjugate* or *associated* to P. They proved to be useful in the large-deviation theory. When one deals with large deviations of arbitrarily large order it is required to learn much about the asymptotic properties of P_s as $s \to \partial S$ (see, e.g., [8–10]). The following theorem is devoted to such a case. Before stating it, we consider an orthogonal matrix C that reduces Λ to a diagonal matrix that is $C^T \Lambda C = \Lambda_0$, where

$$\Lambda_0 = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda_{d-1} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Set

$$\overline{\Lambda}_0 = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{d-1} \end{pmatrix}$$

and $\overline{x} = (x_1, \dots, x_{d-1})$ for $x = (x_1, \dots, x_d) \in R^d$.

THEOREM 1.5. *Assume that the conditions of Theorem 1.3 hold and $s = (h(e) - \tau)e$, $\tau \to 0$. Then $(\tau|\xi|, \tau^{-1/2} C^T(e_\xi - \varepsilon'))$ converges in P_s-distribution to a random vector (ρ, ζ) such that $\mathbf{P}(\zeta_d = 0) = 1$ and*

$$\mathbf{P}(\rho \in A;\ \overline{\zeta} \in B) = \int\limits_A q_\alpha(r) \Big(\int\limits_B r^{(d-1)/2} \varphi_{\overline{\Lambda}_0}(z r^{1/2})\, dz \Big)\, dr,$$

where

$$q_\alpha(r) = \frac{r^{\alpha+(d-1)/2} \langle \varepsilon', e \rangle^{\alpha+(d+1)/2} e^{-r\langle \varepsilon', e \rangle}}{\Gamma(\alpha + (d+1)/2)}, \quad r > 0,$$

while

$$\varphi_{\overline{\Lambda}_0}(z) = (2\pi)^{-(d-1)/2} (\lambda_1 \dots \lambda_{d-1})^{1/2} \exp\Big(-\frac{1}{2} z^T \overline{\Lambda}_0 z \Big), \quad z \in R^{d-1}.$$

Thus $|\xi|$ as a limiting, with respect to P_s, distribution has the gamma distribution, while the limiting distribution for $e_\xi - \varepsilon'$ is a mixture of normal distributions. That mixture can also be represented in the following form:

$$\mathbf{P}(\overline{\zeta} \in B) = (2\pi)^{-(d-1)/2} \frac{\Gamma(\alpha + d)}{\Gamma(\alpha + (d+1)/2)} (\lambda_1 \dots \lambda_{d-1})^{1/2} \int\limits_B \Big(1 + \frac{1}{2} z^T \overline{\Lambda}_0 z \Big)^{-\alpha-d} dz.$$

It is a nonstandard multidimensional Student distribution (see, e.g., [7, p. 134]).

The paper is organized as follows. Sections 2 and 3 contain the proofs of Proposition 1.1 and Theorem 1.3, respectively. The proofs of Theorems 1.4 and 1.5 are sketched in Sec. 4.

2. Proof of Proposition 1.1

From now on, c denotes any positive constant whose concrete value is of no importance. This means that $c + c = c$, $c^2 = c$, etc. By $\omega(t)$, we denote any nonnegative function such that $\lim_{t\to\infty} \omega(t) = 0$ while θ varies within $[-1, 1]$.

Set for $e \in S^{d-1}$

$$P^{(e)}(u) = \mathbf{P}(\langle \xi, e \rangle \geq u), \quad u \in R^1.$$

We should show that

$$\lim_{u\to\infty} \frac{-\log P^{(e)}(u)}{u} = h(e) \tag{2.7}$$

as $u \to \infty$ (see (1.1)).

From (1.2) and (1.3) it follows that for all sufficiently large u

$$P_- = \frac{c_-}{2} \int\limits_{|x|\langle e,e_x\rangle \geq u} |x|^{-\beta} e^{-|x|a(e_x)}\, dx \leq P^{(e)}(u) \leq 2c_+ \int\limits_{|x|\langle e,e_x\rangle \geq u} |x|^{\beta} e^{-|x|a(e_x)}\, dx = P_+.$$

The change of variables

$$x_i = r\varepsilon_i, \qquad i = 1, \dots, d-1, \qquad x_d = \operatorname{sign}(x_d) r (1 - \varepsilon_1^2 - \cdots - \varepsilon_{d-1}^2)^{1/2},$$

having Jacobian $r^{d-1}|\varepsilon_d|^{-1}$ with $|\varepsilon_d| = (1 - \varepsilon_1^2 - \cdots - \varepsilon_{d-1}^2)^{1/2}$, leads to

$$P_\pm = c \int\limits_{r\langle \varepsilon, e\rangle \geq u} r^{\pm\beta+d-1} e^{-ra(\varepsilon)}\, dr\, \chi_{d-1}(d\varepsilon),$$

where $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)^T$, $|\varepsilon| = 1$. From now on, χ_{d-1} is the normalized Haar measure on S^{d-1}.

Since for any $c > 0$ and β as $z \to \infty$

$$\int\limits_z^\infty r^\beta e^{-cr}\, dr \sim c^{-1} z^\beta e^{-cz},$$

we have as $u \to \infty$

$$P_\pm \sim cu^{\pm\beta+d-1} \int\limits_{\langle \varepsilon, e\rangle > 0} \frac{e^{-ua(\varepsilon)/\langle \varepsilon, e\rangle}}{a(\varepsilon)\langle \varepsilon, e\rangle^{\pm\beta+d-1}}\, \chi_{d-1}(d\varepsilon). \tag{2.8}$$

It is readily seen that both integrals converge. Denote them, respectively, by $I_\pm$.

Let $\delta' > 0$ be arbitrarily small. Set

$$E' = \left(\varepsilon \in S^{d-1}\colon\ h(e) \leq \frac{a(\varepsilon)}{\langle \varepsilon, e\rangle} \leq h(e) + \delta'\right).$$

Obviously,

$$I_- \geq e^{-u(h(e)+\delta')} \int\limits_{E'} a(\varepsilon)^{-1} \langle \varepsilon, e\rangle^{\beta-d+1}\, \chi_{d-1}(d\varepsilon). \tag{2.9}$$

Estimate I_+ as

$$I_+ \leq \int\limits_{E'} + \int\limits_{\langle \varepsilon, e\rangle > 0, \varepsilon \notin E'} = I_1 + I_2.$$

It is clear that

$$I_1 \leq e^{-uh(e)} \int\limits_{E'} a(\varepsilon)^{-1} \langle \varepsilon, e\rangle^{-\beta-d+1}\, \chi_{d-1}(d\varepsilon) = ce^{-uh(e)}.$$

Since for $\varepsilon \notin E'$

$$\inf_{\varepsilon \notin E'} (a(\varepsilon) - h(e)\langle e, \varepsilon\rangle) \geq c \inf_{\varepsilon \notin E'} (1 - h(e)\langle e, \varepsilon\rangle / a(\varepsilon)) \geq c\left(1 - \frac{h(e)}{h(e) + \delta'}\right),$$

we obtain for all sufficiently large u

$$I_2 \leq e^{-uh(e)} \int\limits_{\langle \varepsilon, e\rangle > 0, \varepsilon \notin E'} a(\varepsilon)^{-1} \langle \varepsilon, e\rangle^{-\beta-d+1} \exp(-(\langle \varepsilon, e\rangle)^{-1})\, \chi_{d-1}(d\varepsilon) < ce^{-uh(e)} \sup_{t>0} e^{-t} t^{\beta+d-1} = ce^{-uh(e)}.$$

Thus

$$I_+ < ce^{-uh(e)}. \tag{2.10}$$

Uniting (2.8)–(2.10) yields

$$cu^{-\beta+d-1} e^{-uh(e)} \leq P^{(e)}(u) \leq cu^{\beta+d-1} e^{-uh(e)},$$

whence (2.7) follows immediately. The first statement of Proposition 1.1 is proved.

Let S be a bounded open convex set. Consider its support function

$$\sigma(x) = \sup_{y \in S} \langle x, y \rangle, \quad x \in R^d,$$

and the so-called Minkowski function

$$m(x) = \inf(t:\ t > 0,\ t^{-1}x \in S), \quad x \in R^d.$$

It is well known that (see, e.g., [12])

$$m(x) = \sup_{(y\ \langle x,y\rangle > 0)} \frac{\langle x, y\rangle}{\sigma(y)}$$

and, therefore,

$$m(e) = \sup_{(\varepsilon\ \langle e,\varepsilon\rangle > 0)} \frac{\langle e, \varepsilon\rangle}{\sigma(\varepsilon)}, \quad e \in S_{d-1}.$$

Since $m(e) = 1/h(e)$, it follows that

$$h(e) = \inf_{(\varepsilon\ \langle e,\varepsilon\rangle > 0)} \frac{\sigma(\varepsilon)}{\langle e, \varepsilon\rangle}.$$

In other words, in order to come to the form $h(e)$, we should choose $a(e) = \sigma(e)$ in (1.2). Thus, the second statement is also proved.

3. Proof of Theorem 1.3

Assume for the time being that $\varepsilon' = \varepsilon'(e) = (0, \dots, 0, 1)^T$ and $\Lambda = \Lambda_0$.

Set for brevity $s = (h(e) - \tau)e$, and let

$$X_1 = (x:\ |x| > L,\ |e_x - \varepsilon'| < M\tau^{1/2}), \qquad X_2 = (x:\ |x| > L,\ M\tau^{1/2} \le |e_x - \varepsilon'| < \delta'),$$

$$X_3 = (x:\ |x| > L,\ |e_x - \varepsilon'| \ge \delta'), \qquad X_4 = (x:\ |x| \le L),$$

where $L > 0$ and $M > 0$ are arbitrarily large, while $0 < \delta' < \delta$ is arbitrarily small. Obviously,

$$f(s) = \sum_{i=1}^{4} f_i(s) \tag{3.11}$$

with

$$f_i(s) = \int\limits_{X_i} e^{\langle s, x\rangle} p(x)\, dx.$$

It is worth recalling that in $X_1 \cup X_2$ the function $b(x)$ admits representation (1.6). Consider the function $r_\alpha(t)$. It is of regular variation and, therefore, $r_\alpha(t) = t^\alpha l(t)$, where $l(t)$ slowly varies as $t \to \infty$. From the well-known Karamata representation (see, e.g., [6, Chap. VIII]) it follows that for any $\eta > 0$ there exists $L > 0$ such that for $\min(\rho, r) > L$

$$\frac{1}{2}\min((r/\rho)^\eta, (r/\rho)^{-\eta}) \le \frac{l(r)}{l(\rho)} \le 2\max((r/\rho)^\eta, (r/\rho)^{-\eta}). \tag{3.12}$$

Let us estimate $f_i(s)$ in (3.11) one after another. Represent X_1 as follows:

$$X_1 = X_{11} \cup X_{12} \cup X_{13}, \tag{3.13}$$

where

$$X_{11} = X_1 \cap (x:\ |x| < N^{-1}\tau^{-1}),$$

$$X_{12} = X_1 \cap (x:\ N^{-1}\tau^{-1} \le |x| < N\tau^{-1}),$$

$$X_{13} = X_1 \cap (x:\ |x| \ge N\tau^{-1}),$$

and $N > 0$ is arbitrarily large. Denote

$$f_{ik}(s) = \int\limits_{X_{ik}} e^{\langle s,x\rangle} p(x)\, dx. \tag{3.14}$$

Set $\overline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_{d-1})^T$. If $x \in X_1$, then for $\varepsilon = e_x$ we have $|\overline{\varepsilon}| = O(\tau^{1/2})$ and $|\varepsilon_d - 1| = O(\tau)$. Furthermore,

$$(\varepsilon - \varepsilon')^T \Lambda_0 (\varepsilon - \varepsilon') = \overline{\varepsilon}^T \overline{\Lambda}_0 \overline{\varepsilon}, \tag{3.15}$$

where as above $\overline{\Lambda}_0$ is the $(d-1) \times (d-1)$ diagonal matrix with the diagonal $(\lambda_1, \dots, \lambda_{d-1})$.

In view of (3.13) and (3.14), we have

$$f_{12}(s) = \lambda(\varepsilon') l(1/\tau) \int\limits_{X_{12}} |x|^\alpha \exp(-|x|(\Delta(\varepsilon) + \tau\langle e, \varepsilon\rangle))\, dx (1 + o(1)).$$

From (B) and (3.15), taking into account that $\varepsilon_d = 1 + O(\tau)$, we easily obtain

$$f_{12}(s) = \lambda(\varepsilon') l(1/\tau) \int\limits_{N^{-1} < r\tau < N} \int\limits_{|\overline{\varepsilon}| < M\tau^{1/2}} r^{\alpha+d-1} \exp\left(-\frac{\tau}{2} \overline{\varepsilon}^T \overline{\Lambda}_0 \overline{\varepsilon} - r\tau e_d\right) dr\, d\overline{\varepsilon} (1 + o(1)).$$

Making the change of variables $r\tau = u$, $\overline{z} = \tau^{-1/2}\overline{\varepsilon}$ yields

$$f_{12}(s) = \tau^{-\alpha-(d+1)/2} l(1/\tau)(c_\alpha g(e) + \theta\omega(\min(M, N))), \tag{3.16}$$

where

$$c_\alpha = \Gamma\left(\alpha + \frac{d+1}{2}\right)(2\pi)^{(d-1)/2}$$

and

$$g(e) = \lambda(\varepsilon')\langle \varepsilon', e\rangle^{-\alpha-(d+1)/2} (\det \overline{\Lambda}_0)^{-1/2} = \lambda(\varepsilon')\langle \varepsilon', e\rangle^{-\alpha-(d+1)/2} (\lambda_1 \dots \lambda_{d-1})^{-1/2}.$$

From (B), taking into account that $|\langle e, \varepsilon\rangle - e_d| = O(\tau^{1/2})$, we obtain

$$f_{11}(s) \le 2 \sup_\varepsilon \lambda(\varepsilon) \int\limits_{X_{11}} r_\alpha(|x|) \exp(-|x|(\Delta(\varepsilon) + \tau\langle e, \varepsilon\rangle))\, dx \le c \int\limits_{L}^{N^{-1}\tau^{-1}} r^{\alpha+d-1} l(r) \left(\int\limits_{|\overline{\varepsilon}| < M\tau^{1/2}} e^{-cr|\overline{\varepsilon}|^2}\, d\overline{\varepsilon} \right) dr.$$

In view of (3.12), we continue

$$f_{11}(s) \le c\tau^{-\alpha-(d+1)/2} \int\limits_0^{N^{-1}} r^{\alpha+(d-1)/2} l(r/\tau)\, dr \le c\tau^{-\alpha-(d+1)/2} l(1/\tau) \int\limits_0^{N^{-1}} r^{\alpha+(d-1)/2-\eta}\, dr \le c\tau^{-\alpha-(d+1)/2} l(1/\tau)\omega(N)$$

provided that $0 < \eta < \min(1, \alpha + (d+1)/2)$. Similarly,

$$f_{13}(s) \le 2 \sup_\varepsilon \lambda(\varepsilon) \int\limits_{X_{13}} r_\alpha(|x|) \exp(-|x|(\Delta(\varepsilon) + \tau\langle e, \varepsilon\rangle))\, dx \le c \int\limits_{N/\tau}^{\infty} r^{\alpha+d-1} l(r) e^{-cr\tau} \left(\int\limits_{|\overline{\varepsilon}| < M\tau^{1/2}} e^{-cr|\overline{\varepsilon}|^2}\, d\overline{\varepsilon} \right) dr.$$

Taking advantage of (3.12), we get

$$f_{13}(s) \le c\tau^{-\alpha-(d+1)/2} \int\limits_N^\infty r^{\alpha+(d-1)/2} l(r/\tau) e^{-cr}\, dr$$

$$\le c\tau^{-\alpha-(d+1)/2} l(1/\tau) \int\limits_N^\infty r^{\alpha+(d-1)/2+\eta} e^{-cr}\, dr \le c\tau^{-\alpha-(d+1)/2} l(1/\tau)\omega(N).$$

Thus,

$$f_1(s) = \tau^{-\alpha-(d+1)/2} l(1/\tau)(c_\alpha g(e) + \theta\omega(\min(M, N))). \tag{3.17}$$

Before estimating $f_2(s)$, note that for $x \in X_2$ we have $|\bar{\varepsilon}| < \delta'$ and $|\varepsilon_d - 1| < c\delta'^2$. Moreover,

$$(\varepsilon - \varepsilon')^T \Lambda_0 (\varepsilon - \varepsilon') \geq c|\bar{\varepsilon}|^2.$$

That is why

$$f_2(s) \leq 2 \sup_{\varepsilon} \lambda(\varepsilon) \int_{X_2} r_\alpha(|x|) e^{-c|x||\bar{\varepsilon}|^2 - c|x|\tau}\, dx \leq c \int_L^\infty r^{\alpha+d-1} l(r) e^{-cr\tau} \left(\int_{|\bar{\varepsilon}| \geq (1/2)M\tau^{1/2}} e^{-cr|\bar{\varepsilon}|^2}\, d\bar{\varepsilon} \right) dr.$$

Represent

$$\int_L^\infty r^{\alpha+d-1} l(r) e^{-cr\tau} \left(\int_{|\bar{\varepsilon}| \geq (1/2)M\tau^{1/2}} e^{-cr|\bar{\varepsilon}|^2}\, d\bar{\varepsilon} \right) dr = \int_L^{N^{-1}\tau^{-1}} + \int_{N^{-1}\tau^{-1}}^\infty .$$

The first integral on the right-hand side is estimated as $f_{11}(s)$. As to the second one, it is easily seen that

$$\int_{N^{-1}\tau^{-1}}^\infty r^{\alpha+d-1} l(r) e^{-cr\tau} \left(\int_{|\bar{\varepsilon}| \geq (1/2)M\tau^{1/2}} e^{-cr|\bar{\varepsilon}|^2}\, d\bar{\varepsilon} \right) dr$$

$$\leq \tau^{-\alpha-(d+1)/2} \int_{N^{-1}}^\infty r^{\alpha+d-1} l(r/\tau) e^{-cr} \left(\int_{|\bar{z}| \geq (1/2)M} e^{-(c/N)|\bar{z}|^2}\, d\bar{z} \right) dr.$$

Obviously,

$$\int_{N^{-1}}^\infty r^{\alpha+d-1} l(r/\tau) e^{-cr}\, dr \sim l(1/\tau),$$

while

$$\int_{|\bar{z}| \geq (1/2)M} e^{-(c/N)|\bar{z}|^2}\, d\bar{z} = M^{d-1}\omega(M^2/N).$$

Therefore,

$$f_2(s) = \tau^{-\alpha-(d+1)/2} l(1/\tau)\theta M^{d-1}\omega(M^2/N). \tag{3.18}$$

From (C) it follows that

$$f_3(s) \leq \int_{R^d} b(x) e^{-|x|(c(\delta) + \tau(e,\varepsilon))}\, dx \leq \int_{R^d} |x|^{\beta_1} e^{-c|x|}\, dx, \quad \beta_1 > \beta,$$

provided τ is sufficiently small. Therefore,

$$f_3(s) = O(1). \tag{3.19}$$

Finally,

$$f_4(s) = O(1). \tag{3.20}$$

Since L, M, N, and δ' are arbitrary from (3.11) and (3.17)–(3.20), it follows that

$$f(s) \sim c_\alpha g(e) \tau^{-\alpha-(d+1)/2} l(1/\tau). \tag{3.21}$$

Let us turn to the general case. Denote by C an orthogonal matrix such that $C^T \Lambda C = \Lambda_0$. It is obvious that $\varepsilon'(e) = C\varepsilon_0$ with $\varepsilon_0 = (0, \ldots, 0, 1)^T$. Set for $0 < \tau < h(e)$, $s = (h(e) - \tau)e$

$$f(s) = \int_{R^d} e^{(s, Cx)} p(Cx)\, dx.$$

From (B) it follows that

$$\Delta(C\varepsilon) = \frac{1}{2}(\varepsilon - \varepsilon_0)^T \Lambda_0 (\varepsilon - \varepsilon_0) + o(|\varepsilon - \varepsilon_0|^2).$$

It remains to apply (3.21). Theorem is proved.

4. On the Proofs of Theorems 1.4 and 1.5

Here we give a sketch of the proofs of these statements.

Proof of Theorem 1.4. Let $s = h(e)e - \tau\hat{e}$. Then

$$\langle s, x\rangle - |x|a(\varepsilon) = -|x|(\Delta(\varepsilon) + \tau\langle\hat{e}, \varepsilon\rangle).$$

Suppose that $\varepsilon' = (0, \dots, 0, 1)^T$ and $\Lambda = \Lambda_0$. If $x \in X_{12}$, then

$$\langle\hat{e}, \varepsilon\rangle = \langle\hat{e}, \varepsilon'\rangle + O(\tau^{1/2}) = \hat{e}_d + O(\tau^{1/2}).$$

Therefore, $f_{12}(s)$ acquires the form

$$f_{12}(s) = \lambda(\varepsilon')l(1/\tau) \int\limits_{N^{-1}<r\tau<N} \int\limits_{|\bar{\varepsilon}|<M\tau^{1/2}} r^{\alpha+d-1} \exp\Big(-\frac{\tau}{2}\bar{\varepsilon}^T\overline{\Lambda}_0\bar{\varepsilon} - \hat{e}_d r\tau\Big)\, dr\, d\bar{\varepsilon}(1 + o(1)),$$

and instead of (3.16) we obtain

$$f_{12}(s) = \tau^{-\alpha-(d+1)/2} l(1/\tau)(c_\alpha g(e, \hat{e}) + \theta\omega(\min(M, N))) \tag{4.22}$$

uniformly in $\hat{e}_d \geq \eta > 0$. The rest of the proof needs, in essence, no alteration. So, we get

$$f(s) = c_\alpha g(e, \hat{e})\tau^{-\alpha-(d+1)/2} l(1/\tau)(1 + o(1)).$$

If $\varepsilon' \neq (0, \dots, 0, 1)^T$, we should argue as in the proof of Theorem 1.3.

Proof of Theorem 1.5. As in the proof of Theorem 1.3, consider, first, the simplest case $\varepsilon' = (0, \dots, 0, 1)^T$, $\Lambda = \Lambda_0$. Let $t > 0$, $r_2 > r_1 > 0$ be arbitrary and

$$X_\tau = (x\colon\ r_1 < \tau|x| < r_2,\ |e_x - \varepsilon'| < t\tau^{1/2}).$$

Repeating the argument that led to (3.21), one easily obtains

$$\int\limits_{X_\tau} e^{\langle s,x\rangle} p(x)\, dx = \lambda(\varepsilon')\tau^{-\alpha-(d+1)/2} l(1/\tau) \int\limits_{r_1}^{r_2} r^{\alpha+d-1} e^{-re_d} \Big(\int\limits_{|\bar{z}|<t} e^{-(r/2)\bar{z}^T\overline{\Lambda}_0\bar{z}}\, d\bar{z}\Big)\, dr(1 + o(1))$$

or

$$\int\limits_{X_\tau} e^{\langle s,x\rangle} p(x)\, dx = (2\pi)^{(d-1)/2}(\lambda_1 \dots \lambda_{d-1})^{-1/2} \lambda(\varepsilon')\tau^{-\alpha-(d+1)/2} l(1/\tau)$$

$$\times \int\limits_{r_1}^{r_2} r^{\alpha+(d-1)/2} e^{-re_d} \Big(\int\limits_{|\bar{z}|<t} r^{(d-1)/2}\varphi_{\overline{\Lambda}_0}(\bar{z}r^{1/2})\, d\bar{z}\Big)\, dr(1 + o(1)).$$

It remains to recall the definition of the conjugate measure and to apply Theorem 1.3. It is easily seen that instead of the cubes $|e_x - \varepsilon'| < t\tau^{1/2}$ we could take, say, parallelograms or ellipses.

The general case $\varepsilon' \neq (0, \dots, 0, 1)^T$ requires obvious alterations (cf. the proof of Theorem 1.4).

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