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ABOUT A DIOPHANTINE EQUATION

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Abstract

In this paper we study the Diophantine equation $x^4 - 6x^2y^2 + 5y^4 = 16F_{n-1}F_{n+1}$, where $(F_n)_{n\geq 0}$ is the Fibonacci sequence and we find a class of such equations having solutions which are determined.

1 Introduction

Let $(F_n)_{n\geq 0}$, $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$, be the Fibonacci sequence and $(L_n)_{n\geq 0}$, $L_0 = 2$, $L_1 = 1$, $L_{n+2} = L_{n+1} + L_n$, $n \geq 0$, be the Lucas sequence.
Sometimes the sequences are given under the forms:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right],$$

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

We find all solutions $(x, y, n) \in \mathbb{Z} \times \mathbb{Z} \times 3\mathbb{N}$ of the Diophantine equation

$$x^4 - 6x^2y^2 + 5y^4 = 16F_{n-1}F_{n+1},$$

when one of the Fibonacci numbers F_{n-1}, F_{n+1} is prime and another is prime or it is a product of two different prime numbers. There are such Fibonacci numbers, for example:

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$F_5 = 5$ and $F_7 = 13$; $F_{11} = 89$ and $F_{13} = 233$; $F_{17} = 1597$ and $F_{19} = 4181 = 37 \cdot 113$; $F_{29} = 514229$ and $F_{31} = 1346269 = 557 \cdot 2147$; $F_{41} = 165580141 = 2789 \cdot 59369$ and $F_{43} = 433494437$.

In this paper, we prove that:

All the solutions $(x, y, n) \in \mathbb{Z} \times \mathbb{Z} \times 3\mathbb{N}$ *of the Diophantine equation*

$$x^4 - 6x^2y^2 + 5y^4 = 16F_{n-1}F_{n+1}$$

with F_{n-1} *is a prime number and* $F_{n+1} = p_1 p_2$, *where* p_1, p_2 *are different prime natural numbers,are* $(x, y, n) = (\pm L_{6l}, \pm F_{6l}, 6l), l \in \mathbb{N}^*$, *when* $6l - 1$ *are prime numbers and* F_{6l+1} *is a product of two prime different numbers.*

Since Fibonacci numbers and Lucas numbers intervene in our equation, we recall some properties obtained along years:

1.1. *The only perfect square Fibonacci numbers are* F_0, F_1, F_2, F_{12}.

1.2. *g.c.d.*(F_n, F_{n+1})= 1, $\forall \in \mathbb{N}$.

1.3. *The only perfect square Lucas numbers are* L_1, L_3.

1.4. *Between the terms of the Fibonacci sequence and the terms of the Lucas sequence there are the following identities:*
i) $L_n = F_{n-1} + F_{n+1}$;
ii) $L_n^2 - 5F_n^2 = 4(-1)^n$;
iii) $L_n^2 - F_n^2 = 4F_{n-1}F_{n+1}$.

1.5. *If* n *is an even number, then*

$$F_{n-1}F_{n+1} = F_n^2 \pm 1.$$

1.6. *The cycle of the Fibonacci numbers mod* 4 *is*

$$0, 1, 1, 2, 3, 1, (0, 1, 1, 2, 3, 1), ...$$

so the cycle-length of the Fibonacci numbers mod 4 *is* 6.

1.7. F_n *are even numbers if and only if* $n \equiv 0 \pmod 3$.

1.8. *Let* $(F_n)_{n \geq 0}$ *be the Fibonacci sequence. If* F_n *is a prime number, then* n *is a prime number.*

We recall the algorithm of solving the generalized Pell equation:

Proposition 1.1 *Let* d *and* k be integer numbers, $d > 0$, $d \neq h^2$, $\forall h \in \mathbb{N}^*$ *and let be given the generalized Pell equation* $x^2 - dy^2 = k$.
i) *If* $(x_0, y_0) \in \mathbb{N}^* \times \mathbb{N}^*$ *is the minimal solution of the equation* $x^2 - dy^2 = 1$, $\epsilon = x_0 + y_0\sqrt{d}$ *and* (x_i, y_i), $i = 1, ...r$, *are different integer solutions of the equation* $x^2 - dy^2 = k$, *with* $|x_i| \leq \sqrt{|k|\,\epsilon}$ *and* $|y_i| \leq \sqrt{\frac{|k|\epsilon}{d}}$, *then there exists an*

infinity of integer solutions of the given equation and these solutions have the form: $\mu = \pm\mu_i\epsilon^l$ *or* $\mu = \pm\overline{\mu_i}\epsilon^l$, $l\in\mathbb{Z}$, *where* $\mu_i = x_i + y_i\sqrt{d}$, $\overline{\mu_i} = x_i - y_i\sqrt{d}$, $i = 1, ..., r$.
ii)*If the given equation does not have solutions satisfying the above conditions, then it does not have any solutions.*

2 Results

Now we consider the Diophantine equation

$$x^4 - 6x^2y^2 + 5y^4 = 16F_{n-1}F_{n+1}. \quad (1)$$

Proposition 2.1 *All the solutions* $(x, y, n)\in\mathbb{Z}\times\mathbb{Z}\times 3\mathbb{N}$, *of the Diophantine equation* (1) *with* F_{n-1} *a prime number and* $F_{n+1} = p_1p_2$, *where* p_1, p_2 *are different prime natural numbers,are* $(x, y, n) = (\pm L_{6l}, \pm F_{6l}, 6l), l \in \mathbb{N}^*$, *when* $6l-1$ *are prime numbers and* F_{6l+1} *is a product of two prime different numbers.*

Proof. Let $(x, y, n)\in\mathbb{Z} \times \mathbb{Z} \times 3\mathbb{N}$ be a solution of the equation (1).
At the beginning, we note that $x\equiv y \pmod 2$ and $x^2 - 5y^2\leq x^2 - y^2$.

First, we study the situation when $x^2\geq y^2$.

The equation (1) is equivalent with

$$(x^2 - 5y^2)(x^2 - y^2) = 16F_{n-1}F_{n+1}$$

Using the fact that $n \equiv 0 \pmod 3$ and **1.7** we obtain that F_{n-1} and F_{n+1} are odd numbers.
Other remark is that x and y are both even numbers (if x and y are both odd numbers we have $(x^2 - 5y^2)(x^2 - y^2) \equiv 0 \pmod{32}$ but $16F_{n-1}F_{n+1}$ is not divisible by 32 because F_{n-1} and F_{n+1} are odd numbers).
We have the following cases:
Case 1. $x^2 - 5y^2 = r; x^2 - y^2 = \frac{16}{r}F_{n-1}F_{n+1}$, where $r\in\{2, 8\}$.
Case 2. $x^2 - 5y^2 = 4; x^2 - y^2 = 4F_{n-1}F_{n+1}$.
Case 3. $x^2 - 5y^2 = rF_{n-1}; x^2 - y^2 = \frac{16}{r}F_{n+1}$, where $r\in\{2, 8\}$.
Case 4. $x^2-5y^2 = rkF_{n-1}; x^2-y^2 = \frac{16F_{n+1}}{rk}$ or inverse $x^2-5y^2 = \frac{16F_{n+1}}{rk}; x^2-y^2 = rkF_{n-1}$, where $r\in\{2, 8\}$, $k\in\{p_1, p_2\}$.
Case 5. $x^2 - 5y^2 = 4F_{n-1}; x^2 - y^2 = 4F_{n+1}$.
Case 6. $x^2-5y^2 = 4kF_{n-1}; x^2-y^2 = \frac{4F_{n+1}}{k}$, $k \in \{p_1, p_2\}$ or inverse: $x^2-5y^2 = \frac{4F_{n+1}}{k}; x^2 - y^2 = 4kF_{n-1}$, $k\in\{p_1, p_2\}$.
We study then the situation when $x^2<y^2$. There are the following cases:
Case 7. $x^2 - 5y^2 = -rF_{n-1}F_{n+1}; x^2 - y^2 = -\frac{16}{r}$, where $r \in \{2, 8\}$.
Case 8. $x^2 - 5y^2 = -rF_{n+1}; x^2 - y^2 = -\frac{16}{r}F_{n-1}$, where $r \in \{2, 8\}$.
Case 9. $x^2 - 5y^2 = -4F_{n-1}F_{n+1}; x^2 - y^2 = -4$.
Case 10. $x^2 - 5y^2 = -4F_{n+1}; x^2 - y^2 = -4F_{n-1}$.

Case 11. $x^2-5y^2=-\frac{rF_{n+1}}{k}; x^2-y^2=-\frac{16}{r}kF_{n-1}$ or inverse: $x^2-5y^2=-\left\{\frac{16}{r}\right\}kF_{n-1}; x^2-y^2=-\frac{rF_{n+1}}{k}$, where $r\in\{2,8\}$; $k\in\{p_1,p_2\}$.
Case 12. $x^2-5y^2=-\frac{4F_{n+1}}{k}; x^2-y^2=-4kF_{n-1}$ or inverse $x^2-5y^2=-4kF_{n-1}; x^2-y^2=-\frac{4F_{n+1}}{k}$, $k\in\{p_1,p_2\}$.

We analyse step by step all these cases.

Case 1. If $r=2$, we get the system $x^2-5y^2=2; x^2-y^2=8F_{n-1}F_{n+1}$. Since x,y are even numbers, we obtain $x^2-5y^2\equiv 0 \pmod 4$, so the equation $x^2-5y^2=2$ does not have integer solutions.
Similarly with this situation, we obtain that the system does not have integer solutions when $r=8$. Similarly with the Case 1 we obtain that there are no integer solutions in the **Cases 3,4,7,8,11.**

Case 2. $x^2-5y^2=4; x^2-y^2=4F_{n-1}F_{n+1}$.
First we solve the equation $x^2-5y^2=4$. We consider the Pell equation $x^2-5y^2=1$. The minimal solution is $(x_0,y_0)=(9,4)$ and $\epsilon=9+4\sqrt{5}$.
Applying **1.1**, for the equation $x^2-5y^2=4$ we search for solutions $(x,y)\in\mathbb{Z}\times\mathbb{Z}$, with $|x|\leq\sqrt{4\epsilon}$ and $|y|\leq\sqrt{\frac{4\epsilon}{5}}<4$. It results $y\in\{0,\pm1,\pm2,\pm3\}$. But, in our system it is necessary that y be even number, so $y\in\{0,\pm2\}$.
If $y=\pm2$, we obtain $x\notin\mathbb{Z}$.
If $y=0$, we obtain $x=2$. According to Proposition 1.1, we obtain that all integer solutions of the equation $x^2-5y^2=4$ are $(x_l,y_l)\in\mathbb{Z}\times\mathbb{Z}$ such that $x_l+\sqrt{5}y_l=\pm2\epsilon^l$, $l\in\mathbb{N}$. It results:

$$x_l=\pm[(2+\sqrt{5})^{2l}+(2-\sqrt{5})^{2l}],$$
$$y_l=\pm\frac{1}{\sqrt{5}}[(2+\sqrt{5})^{2l}-(2-\sqrt{5})^{2l}],\ l\in\mathbb{N},$$

therefore

$$x_l=\pm L_{6l}, y_l=\pm F_{6l}.$$

We remark (according to **1.4.**(iii)) that all these solutions $(x_l,y_l)=(\pm L_{6l},\pm F_{6l})$ are solutions for the second equation of the considered system ($x^2-y^2=4F_{n-1}F_{n+1}$.)
Therefore, in the Case 2, we obtain solutions of the equation under the hypotheses given in our theorem: $(x_l,y_l,n)=(\pm L_{6l},\pm F_{6l},6l), l\in\mathbb{N}^*$, with $6l-1$ prime numbers and F_{6l+1} a product of two different prime numbers.

Case 5. By substracting the two equations and using the recurrence relation of Fibonacci numbers, we obtain $y^2=F_n$. Applying Proposition 1.1 and the fact that $n\geq1$, it results $n\in\{1,2,12\}$.
$n=1$ is not valid because $1\notin3\mathbb{Z}$.

Similarly $n = 2$ is not valid.
If $n = 12$, we obtain $y \in \{-12, 12\}$, but $x \notin \mathbb{Z}$. So, in this case the equation (1) does not have integer solutions.
Case 6. Since x and y are even numbers we can denote $x = 2x^{'}$, $y = 2y^{'}$, $x^{'}, y^{'} \in \mathbb{Z}$. It is necessary that $x^{'}$ is not congruent with $y^{'}$ (mod 2)(otherwise the equation $x^{'2} - 5y^{'2} = kF_{n-1}$ does not have integer solutions).
We obtain that the system from the case 6. becomes

$$x^{'2} - 5y^{'2} = kF_{n-1}; x^{'2} - y^{'2} = \frac{F_{n+1}}{k}$$

or

$$x^{'2} - 5y^{'2} = \frac{F_{n+1}}{k}; x^{'2} - y^{'2} = kF_{n-1}$$

First we consider the situation: $y \notin 4\mathbb{Z}$. It results $x^{'} \equiv 0$ (mod 2) and $y^{'} \equiv 1$ (mod 2).
We turn back in the equation (1). This is equivalent with

$$x^{'4} - 6x^{'2}y^{'2} + 5y^{'4} = F_{n-1}F_{n+1}. \qquad (2)$$

We consider two subcases.
First subcase: when $n \equiv 3$ (mod 6), using **1.6** we have $F_{n-1} \equiv 1$ (mod 4) and $F_{n+1} \equiv 3$ (mod 4).
We observe that the left side of the equation (2) is $\equiv 1$ (mod 4), but the right side of the same equation is $\equiv 3$ (mod 4). It results that the equation (2)does not have integer solutions, therefore the equation (1)doesn't have integer solutions.
Second subcase: when $n \equiv 0$ (mod 6), using **1.6** we have $F_{n-1} \equiv 1$ (mod 4), $F_{n+1} \equiv 1$ (mod 4), $F_n \equiv 0$ (mod 4). Applying **1.5** we obtain that the equation (2) is equivalent with

$$x^{'4} - 6x^{'2}y^{'2} + 5y^{'4} = F_n^2 \pm 1. \qquad (3)$$

We observe that the left side of the equation (3) is $\equiv$ 5 or 13 (mod16), but the right side of the same equation is $\equiv$ 1 or 15 (mod16). It results that the equation (3) does not have integer solutions, therefore the equation (1) does not have integer solutions.

Now, we consider the situation: $y \in 4\mathbb{Z}$. Since F_{n-1} is a prime number and $F_{n+1} = p_1 p_2$ where p_1, p_2 are prime natural numbers, $p_1 < p_2$, we have the following subcases:
Subcase i): $k = p_1$, and the system

$x^{'2}-5y^{'2}=p_1F_{n-1}; x^{'2}-y^{'2}=p_2$.
We observe immediately that $F_{n-1}<p_2$. We find the natural solutions of the second equation of the system: $x^{'}=\frac{p_2+1}{2}$, $y^{'}=\frac{p_2-1}{2}$. We return in the first equation of the system and we obtain $(p_2-1)(2-p_2)=p_1F_{n-1}-1$. This equality is impossible because the left side is negative and the right side is positive.
Subcase ii): $k=p_2$, and the system $x^{'2}-5y^{'2}=p_1; x^{'2}-y^{'2}=p_2F_{n-1}$, with $p_1<p_2$.
At the first equation of the system we attach the Pell equation $x^{'2}-5y^{'2}=1$. The fundamental solution for this equation is $(x_0^{'},y_0^{'})=(9,4)$ and $\epsilon=9+4\sqrt{5}$.
For the equation $x^{'2}-5y^{'2}=p_1$ we search integer solutions (according Proposition 1.1) $(x^{'},y^{'})$ which satisfy $\left|x^{'}\right|\leq\sqrt{p_1\epsilon}$, $\left|y^{'}\right|\leq\sqrt{\frac{p_1\epsilon}{5}}$. This implies $\left|x^{'}\right|\leq 5\sqrt{p_1}$ and $\left|y^{'}\right|\leq\sqrt{5p_1}$. At the beginning, we search $x^{'},y^{'}$ natural numbers. We suppose that $F_{n-1}<p_2$. From the second equation of the system we obtain $x^{'}-y^{'}=F_{n-1}$, $x^{'}+y^{'}=p_2$, so $x^{'}=\frac{p_2+F_{n-1}}{2}$, $y^{'}=\frac{p_2-F_{n-1}}{2}$.
We obtain:
$\frac{p_2-F_{n-1}}{2}\leq\sqrt{5p_1}$, $\frac{p_2+F_{n-1}}{2}\leq 5\sqrt{p_1}$, therefore $p_1<p_2\leq(5+\sqrt{5})\sqrt{p_1}$. It results $p_1<53$. Since p_1 is an odd prime natural number, we obtain that
$p_1\in\{3,5,7,11,13,17,19,23,29,31,37,41,47\}$. If $p_2<F_{n-1}$, similarly we obtain that $p_1\in\{3,5,7,11,13,17,19,23,29,31,37,41,43,47\}$.
If $p_1\in\{3,7,11,19,23,31,43,47\}$, the equation $x^{'2}-5y^{'2}=p_1$ does not have integer solutions, because $x^{'2}$ cannot be congruent with 3 (mod 4).
If $p_1=13$, we turn back in the equation $x^{'2}-5y^{'2}=13$. Since the last digit of the $5y^{'2}+13$ is 3 or 8, it results that the equation $x^{'2}-5y^{'2}=13$ does not have integer solutions.
If $p_1=17$, we turn back in the equation $x^{'2}-5y^{'2}=17$. Since the last digit of the $5y^{'2}+17$ is 2 or 7, it results that the equation $x^{'2}-5y^{'2}=17$ does not have integer solutions.
Analogously we obtain that the equation $x^{'2}-5y^{'2}=p_1$ does not have integer solutions for $p_1=37$.
If $p_1=5$, we try to find the integer solutions of the equation $x^{'2}-5y^{'2}=5$. Knowing that $\left|y^{'}\right|\leq\sqrt{5p_1}=5$ and $y^{'}$ is even, it results $y^{'}\in\{0,\pm2,\pm4\}$.
If $y^{'}\in\{0,\pm4\}$, it results $x^{'}\notin\mathbb{Z}$.
If $y^{'}=\pm2$, it results $x^{'}=\pm5$. We turn back in the system $x^{'2}-5y^{'2}=5; x^{'2}-y^{'2}=p_2F_{n-1}$ and we obtain $p_2F_{n-1}=21$. But $F_{n-1}\neq 7$, so $p_2=7$, $F_{n-1}=3$. This implies $n=5$, $35=p_1p_2=F_{n+1}=F_6$. This is false. Therefore $(x_1^{'},y_1^{'})=(5,2)$ is not a solution for the equation $x^{'2}-y^{'2}=p_2F_{n-1}$.
But all natural solutions $(x_l^{'},y_l^{'})$ of the equation $x^{'2}-5y^{'2}=5$ are (according to Proposition 1.1) of the form:

$x_l^{'} + y_l^{'}\sqrt{5} = (5 + 2\sqrt{5})(9 + 4\sqrt{5})^l$ or $x_l^{'} + y_l^{'}\sqrt{5} = (5 - 2\sqrt{5})(9 + 4\sqrt{5})^l$, $l \in \mathbb{Z}$. This implies that $x_{l+1}^{'} + y_{l+1}^{'}\sqrt{5} = (x_l^{'} + y_l^{'}\sqrt{5})(9 + 4\sqrt{5})$. So

$$x_{l+1}^{'} = 9x_l^{'} + 20y_l^{'} \tag{4}$$

and

$$y_{l+1}^{'} = 4x_l^{'} + 9y_l^{'}. \tag{5}$$

We obtain that

$$x_{l+1}^{'2} - y_{l+1}^{'2} = 65x_l^{'2} + 319y_l^{'2} + 288x_l^{'}y_l^{'}. \tag{6}$$

Since $(x_1^{'}, y_1^{'}) = (5, 2)$, 5 is not congruent with 0 (mod 3), 2 is not congruent with 0 (mod 3), using the relations (4) and (5) we obtain $x_2^{'}$ is not congruent with 0 (mod 3), $y_2^{'}$ is not congruent with 0 (mod 3),..., $x_l^{'}$ is not congruent with 0 (mod 3), $y_l^{'}$ is not congruent with 0 (mod 3). Using (6) we obtain that $x_{l+1}^{'2} - y_{l+1}^{'2} \equiv 0 \pmod 3$.
If the second equation of the system had a solution, that means $x_{l+1}^{'2} - y_{l+1}^{'2} = p_2 F_{n-1}$, would result $p_2 = 3$ or $F_{n-1} = 3$.
If $p_2 = 3$, it results $F_{n+1} = p_1 p_2 = 15$. This is a contradiction, because there is not $n \in \mathbb{N}$ such that $F_{n+1} = 15$.
If $F_{n-1} = 3$, it results $n = 5$ and $F_{n+1} = F_6 = 8 \neq p_1 p_2$. So, we cannot have $F_{n-1} = 3$.
From the previously proved, it results that the system

$$x^{'2} - 5y^{'2} = 5; x^{'2} - y^{'2} = p_2 F_{n-1}$$

does not have integer solutions.
If $p_1 = 29$, we try to find the integer solutions of the equation $x^{'2} - 5y^{'2} = 29$. Knowing that $\left|y^{'}\right| \leq \sqrt{5p_1} = \sqrt{145}$ and $y^{'}$ is even, it results $y^{'} \in \{0, \pm 2, \pm 4\ \pm 6, \pm 8, \pm 10, \pm 12\}$.
If $y^{'} \in \{0, \pm 4\ \pm 6, \pm 8, \pm 12\}$, it results $x^{'} \notin \mathbb{Z}$.
If $y^{'} = \pm 2$, it results $x^{'} = \pm 7$.
From the system

$$x^{'2} - 5y^{'2} = 29; x^{'2} - y^{'2} = p_2 F_{n-1}$$

we obtain that $p_2 F_{n-1} = 45 = 3^2 \cdot 5$. This is a contradiction with the fact that p_2 and F_{n-1} are prime numbers.
If $y^{'} = \pm 10$, it results $x^{'} = \pm 23$.
From the system $x^{'2} - 5y^{'2} = 29$; $x^{'2} - y^{'2} = p_2 F_{n-1}$

we obtain that $p_2F_{n-1} = 429 = 3\cdot 11\cdot 13$. This is a contradiction with the fact that p_2 and F_{n-1} are prime numbers.
All the natural solutions of the equation $x^{'2} - 5y^{'2} = 29$ are (according to Proposition 1.1) $(x_l^{'}, y_l^{'})$:
$x_l^{'} + y_l^{'}\sqrt{5} = (7+2\sqrt{5})(9+4\sqrt{5})^l$, $l\in\mathbb{Z}$ and $x_l^{'} + y_l^{'}\sqrt{5} = (7-2\sqrt{5})(9+4\sqrt{5})^l$ and $x_l^{'}+y_l^{'}\sqrt{5} = (23+10\sqrt{5})(9+4\sqrt{5})^l$ and $x_l^{'}+y_l^{'}\sqrt{5} = (23-10\sqrt{5})(9+4\sqrt{5})^l$ $l\in\mathbb{Z}$.
Analogously with the case when $p_2 = 5$ we obtain that none of the solutions $(x_l^{'}, y_l^{'})$ of the equation $x^{'2} - 5y^{'2} = 29$ is solution for the equation $x^{'2} - y^{'2} = p_2F_{n-1}$. So, the system $x^{'2} - 5y^{'2} = 29$; $x^{'2} - y^{'2} = p_2F_{n-1}$ does not have integer solutions.
If $p_1 = 41$, we try to find the integer solutions of the equations $x^{'2} - 5y^{'2} = 41$. Knowing that $\left|y^{'}\right| \leq \sqrt{5p_1} = \sqrt{205}$ and $y^{'}$ is even, it results
$y^{'}\in\{0, \pm2, \pm4\ \pm 6, \pm8, \pm10, \pm12, \pm14\}$.
If $y^{'}\in\{0, \pm2\ \pm 6, \pm10, \pm12, \pm14\}$. It results $x^{'}\notin\mathbb{Z}$.
If $y^{'} = \pm4$, it results $x^{'} = \pm11$.
From the system $x^{'2} - 5y^{'2} = 41$; $x^{'2} - y^{'2} = p_2F_{n-1}$ we obtain that $p_2F_{n-1} = 105 = 3\cdot 5\cdot 7$. This is a contradiction with the fact that p_2 and F_{n-1} are prime numbers.
If $y^{'} = \pm8$, it results $x^{'} = \pm19$.
From the system $x^{'2} - 5y^{'2} = 41$; $x^{'2} - y^{'2} = p_2F_{n-1}$ we obtain that $p_2F_{n-1} = 3^3\cdot 11$. This is a contradiction with the fact that p_2 and F_{n-1} are prime numbers.
All the natural solutions of the equation $x^{'2} - 5y^{'2} = 41$ are (according to Proposition 1.1) $(x_l^{'}, y_l^{'})$:
$x_l^{'}+y_l^{'}\sqrt{5} = (11+4\sqrt{5})(9+4\sqrt{5})^l$, $l\in\mathbb{Z}$ and $x_l^{'}+y_l^{'}\sqrt{5} = (11-4\sqrt{5})(9+4\sqrt{5})^l$, $l\in\mathbb{Z}$ and $x_l^{'}+y_l^{'}\sqrt{5} = (19+8\sqrt{5})(9+4\sqrt{5})^l$, $l\in\mathbb{Z}$ and $x_l^{'}+y_l^{'}\sqrt{5} = (19-8\sqrt{5})(9+4\sqrt{5})^l$, $l\in\mathbb{Z}$.
Analogously with the case when $p_2 = 5$ we obtain that none of the solutions $(x_l^{'}, y_l^{'})$ of the equation $x^{'2} - 5y^{'2} = 41$ is solution for the equation $x^{'2} - y^{'2} = p_2F_{n-1}$. So, the system $x^{'2} - 5y^{'2} = 41$; $x^{'2} - y^{'2} = p_2F_{n-1}$ doesn't have integer solutions.
Subcase iii): $k = p_1$, and the system
$x^{'2} - 5y^{'2} = p_2; x^{'2} - y^{'2} = p_1F_{n-1}$, with $p_1 < p_2$
At the begining we consider the situation $F_{n-1} < p_1$. We have:

$$F_{n-1} < p_1 < p_2 \ \Rightarrow$$

$$F_{n-1}^2 < p_1^2 < F_{n+1} \ \Rightarrow$$

$$F_{n-1}(F_{n-1} - 1) < F_n < 2F_{n-1} \ \Rightarrow$$

$$F_{n-1} < 3 = F_4.$$

We obtain $n \in \{1, 2, 3, 4\}$
For $n \in \{1, 2, 3\}$ it results $F_{n-1} \in \{0, 1\}$. This is in contradiction with the fact that F_{n-1} is a prime number.
For $n = 4$, it results $F_{n-1} = 2$ and $F_{n+1} = F_5 = 5$. This is in contradiction with the fact that F_{n+1} is a product of two prime natural numbers.
Now, we consider the situation $F_{n-1} > p_1$.
From the second equation of the system we have natural solutions : $x^{'} = \frac{p_1 + F_{n-1}}{2}$, $y^{'} = \frac{F_{n-1} - p_1}{2}$.
For the first equation from the system we search (according to Proposition 1.1) integer solutions $(x^{'}, y^{'})$ with the properties $\left|x^{'}\right| \leq 5\sqrt{p_2}$ and $\left|y^{'}\right| \leq \sqrt{5p_2}$.
So $\frac{p_1 + F_{n-1}}{2} \leq 5\sqrt{p_2}$, $\frac{F_{n-1} - p_1}{2} \leq \sqrt{5p_2}$.
We obtain:

$$F_{n-1} < \sqrt{5p_2}(\sqrt{5} + 1) \ \Rightarrow$$
$$F_{n-1}^2 < 5(6 + 2\sqrt{5})p_2 < 54p_2 \ \Rightarrow$$
$$F_{n-1}^2 \leq 18p_1p_2 \ \Leftrightarrow F_{n-1}^2 \leq 18F_{n+1} \ \Rightarrow$$
$$F_{n-1}(F_{n-1} - 18) \leq 18F_n \ \Rightarrow$$
$$F_{n-1}(F_{n-1} - 18) \leq 36F_{n-1} \ \Rightarrow$$
$$F_{n-1} < 54.$$

Since F_{n-1} is a prime natural number, it results that $F_{n-1} \in \{3, 5, 13\}$. So $n \in \{5, 6, 8\}$.
If $n = 5$, it results $F_{n+1} = F_6 = 2^3 \neq p_1p_2$.
If $n = 6$, it results $F_{n+1} = F_7 = 13 \neq p_1p_2$.
If $n = 8$, it results $F_{n+1} = F_9 = 2 \cdot 17 = p_1p_2$. Since $p_1 < p_2$ it results $p_1 = 2$ and $p_2 = 17$. This is a contradiction with the fact that F_{n+1} is an odd number.
From the previously proved, we obtain that there are not integer solutions in the case 6.
Case 12. Similarly with the case 6, we obtain that there are not integer solutions in this case.
Case 9. $x^2 - 5y^2 = -4F_{n-1}F_{n+1}; x^2 - y^2 = -4$.
All the integer solutions of the second equation are $(x, y) = (0, 2)$, $(x, y) = (0, -2)$. We go back in the first equation and we obtain $F_{n-1}F_{n+1} = 5$. We observe that there is not any $n \in \mathbf{N}^*$ such that $F_{n-1}F_{n+1} = 5$.
Case 10. $x^2 - 5y^2 = -4F_{n+1}; x^2 - y^2 = -4F_{n-1}$.
By substracting the two equations and using the recurrence relation of Fibonacci numbers, we obtain $y^2 = F_n$. Applying **1.1** and the fact that $n \geq 1$,

and $n\in3\mathbb{N}$, it results $n = 12$. We obtain $y\in\{-12, 12\}$ and $x\notin\mathbb{Z}$. So, there are not integer solutions in this case.

Proposition 2.2 *All the solutions* $(x, y, n)\in\mathbb{Z}\times\mathbb{Z}\times3\mathbb{N}$ *of the Diophantine equation*

$$x^4 - 6x^2y^2 + 5y^4 = 16F_{n-1}F_{n+1},$$

with F_{n+1} *a prime number and* $F_{n-1} = p_1p_2$, *where* p_1, p_2 *are different prime natural numbers,are* $(x, y, n) = (\pm L_{6l}, \pm F_{6l}, 6l), l \in \mathbb{N}^*$, *when* $6l + 1$ *are prime numbers and* F_{6l-1} *is a product of two different prime numbers.*

Proof. It is similarly with the proof of the Proposition 2.1.

Proposition 2.3 *All the solutions* $(x, y, n)\in\mathbb{Z}\times\mathbb{Z}\times3\mathbb{N}$, *of the Diophantine equation*

$$x^4 - 6x^2y^2 + 5y^4 = 16F_{n-1}F_{n+1},$$

with F_{n-1}, F_{n+1} *prime numbers, are* $(x, y, n) = (\pm L_{6l}, \pm F_{6l}, 6l), l \in \mathbb{N}^*$, *when* $6l - 1$ and $6l + 1$ *are prime numbers.*

Proof. It is similarly with the proof of the Proposition 2.1, without the cases **4,6,11,12.**

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