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Article

About Analytical Approximate Solutions of the Van der Pol Equation in the Complex Domain

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Abstract: In the article, the existence of solutions for the Van der Pol differential equation is proved, and the approximate structure of such solutions in the analyticity domain is obtained. In the proof, the majorant method was applied not to the right side of the differential equation, as per usual, but to the solution to the nonlinear differential equation under consideration. Results of the numerical study are presented.

Keywords: nonlinear differential equation of the second order; movable singular point; analytical approximate solution; analyticity domain

MSC: 34G20; 35A05

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1. Introduction

Some problems of the theory of nonlinear oscillations lead to the necessity to solve the nonlinear Van der Pol differential equation [1–6]. The given equation describes both self-oscillations and the process of their establishment [1]. The energy for self-oscillations is supplied with small-amplitude oscillations [3–6]. For instance, this equation arises in the study of chains containing vacuum tubes. The Van der Pol equation describes the universal mechanism of the appearance of self-oscillations through the Andronov–Hopf bifurcation and proves the possibility of both quasi-harmonic and relaxation oscillations [7,8]. Relaxation oscillations reduce to the study of solutions of the corresponding differential system of two first-order equations with a small parameter at the derivative [9]. This equation is a mathematical model (with a number of simplifying assumptions) of a triode tube study of solutions of the corresponding differential system of two first-order equations with a small parameter at the derivative [9]. This equation is a mathematical model (with a number of simplifying assumptions) of a triode tube oscillator in the case of a cubic characteristic of the lamp. In studying the Van der Pol equation, qualitative and asymptotic theories of ordinary differential equations are widely used [7–11]. In recent decades, the Van der Pol equation has been used in modeling the vibrational movements of a human limb, in a model of exciting and inhibiting neural interactions, in modeling a bipedal musculoskeletal system, in plasma oscillations, and in neural networks when processing and transmitting information, etc. [2,12].

When studying the Van der Pol equation by methods of the qualitative theory of differential equations, the expression for solutions themselves remains behind the scenes and is often ignored [7]. The application of the asymptotic theory of differential equations faces a serious question: what should we do with moving singular points? Thus far, there is no answer to this question. As all the numerical methods for solving differential equations do not work effectively in the complex domain, we apply an analytical approximate method, when the structure of the solution is expressed as some terms of power series. Since the Van der Pol equation is nonlinear, the existence region for solutions is divided into two parts: analyticity regions and the neighborhoods of moving singular points. In this paper, we propose a modification of the classical majorant method [13] for the domain of analyticity, which was effectively implemented to study a number of classes of nonlinear differential equations [14–18]. The author's analytical method includes solving six problems that are formulated and solved for some nonlinear differential equations in the papers [14–17]:

(1) Proof of the theorem of existence and uniqueness in the domain of analyticity and building the analytical approximate solution;
(2) Proof of the theorem of existence and uniqueness in the neighborhood of a movable singular point and building the analytical approximate solution;
(3) Influence of perturbation of a movable singular point on the structure of the analytical approximate solution in the neighborhood of a movable singular point;
(4) Obtaining exact criteria for the existence of moving singular points (necessary, necessary and sufficient conditions);
(5) Influence of perturbation of the initial data on the structure of the analytical approximate solution in the domain of analyticity;
(6) On the exact boundaries of the application area of the analytical approximate solution in the neighborhood of the approximate value of the movable singular point.

In this paper, we are focused on solving the first problem only.

The development of the proposed author's method for differential equations of the second and third orders can be found in [19–21].

Let us consider the initial problem for the Van der Pol equation

$$\frac{d^2w}{dz^2} = -a(w^2-1)\frac{dw}{dz} - w, \tag{1}$$

$$w(z_0) = w_0, \;\; w'(z_0) = w_1, \tag{2}$$

where $a = const$ is a real parameter. We formulate and prove the existence and uniqueness theorem for solutions of problems (1)–(2) in the analyticity domain.

2. Main Result for the Case $|a| \geq 1$

Theorem 1. *Solution to the initial problem (1)–(2), where $|a| = const \geq 1$ is an analytic function*

$$w(z) = \sum_{n=0}^{\infty} C_n (z-z_0)^n \tag{3}$$

in the domain

$$|z-z_0| < \rho, \tag{4}$$

where

$$\rho = \frac{1}{|\,a\,|\,M(M+2)}, \qquad M = max\{|w_0|,\ |w_1|\}. \tag{5}$$

Proof. We will apply the modified majorant method. Assuming that the solution to problem (1)–(2) is an analytic function (3), we will show the uniqueness of the coefficients of the series (3) and obtain a formula for calculating the domain in which the series (3) is convergent. Substituting (3) into Equation (1), we obtain

$$\sum_{n=2}^{\infty} C_n\, n(n-1)(z-z_0)^{n-2} = -a \sum_{n=0}^{\infty} C_n^{***}\,(z-z_0)^n - \sum_{n=0}^{\infty} C_n (z-z_0)^n, \tag{6}$$

where

$$(w^2-1)\, w' = \sum_{n=0}^{\infty} C_n^{***} (z-z_0)^n;$$

$$(w^2 - 1) = \sum_{n=0}^{\infty} C_n^{**}(z - z_0)^n; \;\; w^2 = \sum_{n=0}^{\infty} C_n^{*}(z - z_0)^n; \;\; C_n^* = \sum_{i=0}^{n} C_i \, C_{n-i}; \tag{7}$$

$$C_0^{**} = C_0^* - 1; \;\; C_n^{**} = C_n^* \;\; \forall n = 1, 2, \ldots; \;\; C_n^{***} = \sum_{i=0}^{n} C_i^{**}(n + 1 - i) \, C_{n+1-i};$$

from (6), we derive the recurrence relation for calculating the coefficients:

$$C_n \, n(n-1) = -a \, C_{n-2}^{***} - C_{n-2}. \tag{8}$$

The relation (8) guarantees the uniqueness of the coefficients C_n. The initial condition (2) determines the values of the coefficients C_0 and C_1. From (8), we find expressions

$$C_2 = -\frac{1}{2}(aC_0^2 C_1 - aC_1 + C_0);$$

$$C_3 = -\frac{1}{6}(a(-aC_0^4 C_1 + 2aC_0^2 C_1 - C_0^3 - aC_1 + C_0 + 2C_0 C_1^2) + C_1). \tag{9}$$

Based on the structure of the coefficients C_n $(n > 2)$, we check the estimation hypothesis for the coefficients C_n:

$$|C_n| \le \frac{|\, a \,|^{n-1} \, M^{n-1}}{n(n-1)}(M + 2)^n, \tag{10}$$

where $M = max\{|w_0|, \; |w_1|\}$ and satisfies the conditions of the theorem. Indeed, from (8), it follows

$$|C_{n+1}| = \left| \, \frac{1}{n(n+1)}(C_{n-1}^{***} - C_{n-1}) \, \right|$$

$$= \left| \, \frac{1}{n(n+1)}\left(-a \sum_{i=0}^{n-1} C_i^{**}(n-i)C_{n-i} - C_{n-1}\right) \, \right|$$

$$\le \frac{1}{n(n+1)} \left| \, -a \sum_{i=0}^{n-1} C_i^{*}(n-i)C_{n-i} - C_{n-1} \, \right|$$

$$\le \frac{1}{n(n+1)} \left| \, -a\Big(\sum_{i=0}^{n-1} \Big(\sum_{j=0}^{i} \frac{|\, a \,|^{j-1} \, M^{j-1}}{j^*(j-1)^*}(M+2)^j \, \frac{|\, a \,|^{i-j-1} \, M^{i-j-1}(M+2)^{i-j}}{(i-j)^*(i-j-1)^*} \Big) \right.$$

$$\left. \times \frac{a^{n-i-2} M^{n-i-2}}{(n-i-1)^*}(M+2)^{n-i} \Big) + \frac{a^{n-2} M^{n-2}}{(n-1)(n-2)}(M+2)^{n-1} \right|.$$

Taking into account relations

$$j^* = \begin{cases} 1, & j = 0, \\ j, & j = 1, 2, \ldots, \end{cases} \qquad (j-1)^* = \begin{cases} 1, & j = 1, \\ j-1, & j = 0, 2, \ldots, \end{cases}$$

$$(i-j)^* = \begin{cases} 1, & i = j, \\ i-j, & i \ne j, \end{cases} \qquad (i-j-1)^* = \begin{cases} 1, & i-j = 1, \\ j-1, & i-j \ne 1, \end{cases}$$

$$(n-j-1)^* = \begin{cases} 1, & i = n-1, \\ n-i-1, & i \ne n-1. \end{cases}$$

After some transformations, we obtain

$$|C_{n+1}| \le \frac{|\, a \,|^{n-1} \, M^{n-2}(M+2)^{n-1}}{n(n+1)} \left| \, \frac{n}{n-1}(M+2) + \frac{1}{(n-1)(n-2)} \, \right|$$

$$\leq \frac{|\, a \,|^{n} \; M^{n}}{(n+1)n}(M+2)^{n+1}.$$

Consider the series

$$\sum_{n=0}^{\infty} v_n (z-z_0)^n, \tag{11}$$

where

$$v_n = \frac{|\, a \,|^{n-1} \; M^{n-1}}{n(n-1)}(M+2)^{n}.$$

It is a majorant for the series (3). The series (11) is convergent in the domain

$$|z-z_0| < \rho, \quad \rho = \frac{1}{|\, a \,| \; M(M+2)}, \tag{12}$$

therefore, the series (3) is also convergent in domain (12). □

The Theorem 1 allows us to construct an analytic approximate solution in domain (12):

$$w_N(z) = \sum_{n=0}^{N} C_n (z-z_0)^n. \tag{13}$$

Theorem 2. *For an analytical approximate solution (13) of the initial problem (1)–(2), provided $|\, a \,| = const \geq 1$, in domain (12), the error estimate is valid*

$$\Delta w_N(z) \leq \frac{|\, a \,|^{N} \; M^{N} (M+2)^{N+1} \,|\, z-z_0 \,|^{N+1}}{(N+1)(N+2)(1-|\, a \,| \; M(M+2) \,|\, z-z_0 \,|)}, \tag{14}$$

where M is determined by the Formula (5).

Proof. By definition

$$\Delta w_N(z) = |\, w(z) - w_N(z) \,| = \left| \sum_{n=N+1}^{\infty} C_n (z-z_0)^n \right|$$

$$\leq \left| \sum_{n=N+1}^{\infty} \frac{|\, a \,|^{n-1} \; M^{n-1}}{n(n+1)} (M+2)^n (z-z_0)^n \right| \leq \frac{|\, a \,|^{N} \; M^{N} (M+2)^{N+1} \,|\, z-z_0 \,|^{N+1}}{(N+1)(N+2)(1-|\, a \,| \; M(M+2) \,|\, z-z_0 \,|)}.$$

□

3. Main Result for the Case $|a| \leq 1$

In the case $|a| \leq 1$, we prove the following existence and uniqueness theorem for the solution to Equation (1) in the analytic domain.

Theorem 3. *The solution to problem (1)–(2), where $|a| \leq 1$, is an analytic function (3) in the domain*

$$|\, z-z_0 \,| < \rho, \quad where \quad \rho = \frac{1}{M(M+1)}. \tag{15}$$

Proof. Similarly, as in Theorem 1, we substitute (3) into Equation (1) and so obtain the relation (6) with the notation (7). The relation (6) leads to the recurrence relation (8), which establishes the uniqueness of the coefficients C_n. Given the conditions of the theorem for $|a| \leq 1$, we prove the estimate for the coefficients C_n:

$$|\, C_n \,| \leq \frac{|\, a \,| \;\; M^{n-1}(M+1)^n + M}{n(n-1)}, \tag{16}$$

where $M = max\{| w_0 |, | w_1 |\}$. From (6)–(7), for $n > 2$, it follows

$$| C_{n+1} |= \frac{1}{n(n+1)} | C^{***}_{n-1} - C_{n-1} |\leq \frac{1}{n(n+1)} | -a \sum_{i=0}^{n-1} (\sum_{j=0}^{i} C_j C_{i-j})(n-i) C_{n-i} - C_{n-1} |$$

$$\leq \frac{1}{n(n+1)} | -a \Big(\sum_{i=0}^{n-1} (\sum_{j=0}^{i} \frac{| a | M^{j-1}(M+1)^j + M}{j^*(j-1)^*} \times \frac{| a | M^{i-j-1}(M+1)^{i-j} + M}{(i-j)^*(i-j-1)^*}) \times \frac{aM^{n-i-1}(M+1)^{n-i-1} + M}{(n-i-1)^*} \Big) + \frac{aM^{n-2}(M+1)^{n-1} + M}{(n-1)(n-2)} |,$$

where

$$C^*_n = \sum_{i=0}^{n} C_i\, C_{n-i}; C^{**}_0 = C^*_0 - 1;\ \ C^{**}_n = C^*_n\ \ \forall n = 1, 2, ...;\ C^{***}_n = \sum_{i=0}^{n} C^{**}_i (n+1-i)\, C_{n+1-i},$$

$$(n-i-1)^* = \begin{cases} 1, & i = n-1, \\ n-i-1, & i \neq n-1. \end{cases}$$

After transformation, the last inequality yields

$$| C_{n+1} |\leq \frac{1}{n(n+1)} | -a \frac{M^{n-3}(M+1)^{n-1} n}{n-1} + \frac{M^{n-2} | a | (M+1)^{n-1} + M}{(n-1)(n-2)} |$$

$$\leq \frac{| a | M^n (M+1)^{n+1} + M}{n(n+1)} = B_n.$$

Considering the next series

$$\sum_{n=0}^{\infty} B_n (z - z_0)^n,$$

converging in the domain

$$| z - z_0 |< \rho,\ \text{ where }\ \rho = \frac{1}{M(M+1)}$$

as majorants for the series (3), we obtain the proof for convergence of the series (3) in domain (15). □

Remark 1. *In the case $|a| = 1$, the estimate (16) takes the form*

$$| C_n |\leq \frac{M}{n(n-1)}$$

and domain (15) becomes $|z - z_0| < 1$.

The proved Theorem 3 allows us to obtain an a priori error estimate for the analytical approximate solution (13) in the case when $|a| \leq 1$.

Theorem 4. *For an analytical approximate solution (13) of the initial problem (1)–(2), provided $|a| \leq 1$ in domain (15), the following estimate is true*

$$\Delta w_N(z) \leq \frac{| a | M^N (M+1)^{N+1} | z - z_0 |^{N+1}}{N(N+1)(1 - M(M+1) | z - z_0 |)} + \frac{M | z - z_0 |^{N+1}}{N(N+1)(1 - | z - z_0 |)}, \tag{17}$$

where $M = max\{|w_0|, |w_1|\}$.

Proof. Based on the classical approach, taking into account the estimate (14), we have

$$\Delta w_N(z) = |w(z) - w_N(z)| = |\sum_{n=N+1}^{\infty} C_n (z - z_0)^n|$$

$$\leq |\sum_{n=N+1}^{\infty} \frac{(aM^{n-1}(M+1)^n + M)}{n(n-1)} (z - z_0)^n \,|$$

$$\leq \frac{|\, a \,| \;\; M^N (M+1)^{N+1} \,|\, z - z_0 \,|^{N+1}}{N(N+1)(1 - M(M+1) \,|\, z - z_0 \,|)} + \frac{M \,|\, z - z_0 \,|^{N+1}}{N(N+1)(1 - \,|\, z - z_0 \,|)}.$$

An expression of an a priori error estimate is obtained for the domain

$$|\, z - z_0 \,| < \rho,$$

where $\rho = \frac{1}{M(M+1)}$. □

Remark 2. *In the case $|a| = 1$, the expression of the a priori estimate takes the form*

$$\Delta w_N(z) \leq \frac{M \;\, |\, z - z_0 \,|^{N+1}}{N(N+1)(1 - |z - z_0|)}$$

in the domain $|z - z_0| < 1$.

Remark 3. *The a priori estimate in Theorem 4 is more accurate than the estimate in Theorem 2, and the domain given in Theorem 4 is wider than the domain given in Theorem 2.*

4. Discussion of the Results

In this section, we provide calculations for three parameter values: $a = 2 > 1$, $a = 1$ and $a = 0.01 < 1$. Numerical characteristics of the solutions for each case are presented. The theoretical results obtained are also valid for the real interval. Example 1 presents the calculation result for the real domain, while examples 2 and 3 present calculations in the complex domain.

Example 1. *Let us consider the initial problem (1)–(2), $a = 2$, where*

$$w_0 = 0, \quad w_1 = 1/2. \tag{18}$$

We will look for an approximate solution (13), where N = 10. Substituting (13) into Equation (1) and taking into account the initial conditions (2), (18), we find an approximate solution in the form

$$\begin{aligned} w = {} & \frac{3943}{483840} z^{10} - \frac{589}{40320} z^9 - \frac{11}{240} z^8 - \frac{41}{560} z^7 \\ & - \frac{37}{480} z^6 - \frac{3}{80} z^5 + \frac{1}{16} z^4 + \frac{1}{4} z^3 + \frac{1}{2} z^2 + \frac{1}{2} z. \end{aligned} \tag{19}$$

We estimate the error of the solution (19). According to (13), we find the value $M = 1/2$. The convergence radius given by (15) is equal to

$$|z - z_0| < 0.4\,.$$

Consider $z_1 = 1/3$ belonging to the region obtained above. The calculation results are shown in Table 1.

Here, $w_9(z_1)$ is an approximate solution, Δw_{10} is an a priori error estimate, Δ_1 is an a posteriori error estimate. The a posteriori error estimate is the desired value of the error estimate, which allows us to determine the structure of the analytical approximate solution (the value of N in

Formula (13)). For an approximate solution (13) with accuracy $\varepsilon = 10^{-5}$ (Theorem 2), $N = 34$ is necessary. The summands from 11th to 34th in total do not exceed 10^{-5}. Therefore, $w_9(z)$ in the resulting domain has an accuracy of 10^{-5}.

Table 1. Numerical characteristics of example 1.

z_1	$w_{10}(z_1)$	Δw_{10}	Δ_1
1/3	0.231952	0.00611764	10^{-5}

Example 2. *For the approximate solution $w_{10}(z)$ (see (13)), taking into account the initial data*

$$w_0 = 0, \quad w_1 = 1 + i. \tag{20}$$

and the parameter value $|a| = 1$, we have

$$\begin{aligned} w = -(1.00936 - 1.02409i) \cdot 10^{-5}z^{10} + (6.84529 + 3.97563i) \cdot 10^{-6}z^9 \\ +(1.00027 - 1.02223i) \cdot 10^{-4}z^8 - (2.11626 + 1.94485) \cdot 10^{-4}z^7 \\ -(5.69272 - 6.52594i) \cdot 10^{-4}z^6 + (8.35417 + 8.3075i) \cdot 10^{-3}z^5 \\ +(8.33375 - 24.9996i) \cdot 10^{-4}z^4 - 0.16665(1+i)z^3 + 0.005(1+i)z^2 + (1+i)z. \end{aligned} \tag{21}$$

According to (12), we obtain the domain

$$|z - z_0| < \frac{1}{\sqrt{2}(1+\sqrt{2})}.$$

We consider $z_1 = 1/4$. The calculations for example 2 are presented in Table 2.

Here, $w_{10}(z_1)$ is the value of analytical approximate solution, $\Delta w_{10}(z_1)$ is an a priori error estimate, and Δ_1 is an a posteriori error estimate. According to Theorem 2, we must take $N = 37$ for $\Delta_1 = 10^{-5}$. The summands from the 11th to the 37th in the structure $w_{37}(z)$ do not exceed $\varepsilon = 10^{-5}$. Therefore, the obtained analytical approximate solution $w_{10}(z)$ in a given domain has accuracy $\varepsilon = 10^{-5}$.

Table 2. Numerical characteristics of example 2.

z_1	$w_{10}(z_1)$	Δw_{10}	Δ_1
1/4	0.24772 + 0.247707 i	0.00769043	10^{-5}

Example 3. *For the initial conditions*

$$w_0 = 0, \quad w_1 = i \tag{22}$$

and $a = 10^{-2}$, we look for an approximate solution

$$\begin{aligned} w = iz + 0.005iz^2 - 0.16665iz^3 + 4.16667 \cdot 10^{-8}iz^4 + 0.0083425iz^5 - 2.63806 \cdot 10^{-4}iz^6 \\ -2.01388 \cdot 10^{-4}iz^7 + 4.95973 \cdot 10^{-5}iz^8 + 2.80602 \cdot 10^{-6}iz^9 - 5.0754 \cdot 10^{-6}iz^{10} \end{aligned} \tag{23}$$

in the domain $\mid z - z_0 \mid < 1/2$. For the value $z_1 = 1/3$, the calculations are presented in Table 3.

Here, $w_{10}(z_1)$ is the value of the approximate solution, $\Delta w_{10}(z_1)$ is an a priori error estimate (Theorem 4), Δ_1 is an a posteriori error estimate. According to Theorem 4, we must take $N = 17$ for $\Delta_1 = 10^{-7}$. The summands from the 11th to the 17th in the structure $w_{17}(z)$ do not exceed $\varepsilon = 10^{-7}$. Therefore, the obtained analytical approximate solution has accuracy $\varepsilon = 10^{-7}$.

Table 3. Numerical characteristics of example 3.

z_1	$w_{10}(z_1)$	Δw_{10}	Δ_1
1/3	$0.327751i$	$3.22998 \cdot 10^{-6}$	10^{-7}

5. Conclusions

The existence theorem for solutions of the Van der Pol differential equation is proved, and the structure of its analytical approximate solution in the analyticity domain is obtained. In the proof of the theorem, a modified majorant method is used, which allows one to obtain a constructive existence theorem: the fact of the existence of an analytic solution, a formula for calculating the analytic domain, the structure of an approximate solution, and a priori estimates for an approximate solution. Theoretical results are illustrated by numerical study.

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