ABOUT ONE PROBLEM OF SYNTHESIS OF OPTIMUM CONTROL BY THERMAL CONDUCTION PROCESS

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The formal procedure of obtaining Bellman equation in the problem of heat conductivity control is stated in [1]. Here we show how the problem of synthesis of optimum control with quadratic criterion of optimum is solved with the help of this equation. For simplification of formulas we used the simplest example which can be easily generalized. It should be noted, that during the solution the nonlinear boundary-value problem for integral-differential equation is derived, which is infinite dimensional analog of well-known Rikkati equation for finite dimensional systems.

1. STATEMENT OF PROBLEM. BELLMAN EQUATION. Let us assume that the control process is described by the function $u(t,x)$ which inside the region of $Q=\{0\le t\le T,\ 0\le x\le 1\}$ satisfies the thermal conduction equation

$$u_t = u_{xx} + q(x)p(t) + f(t,x), \tag{1}$$

and on the boundary Q the homogeneous conditions

$$u(0,x) = 0, \tag{2}$$

$$u_x(t,0) = u_x(t,1) + \alpha u(t,1) = 0,\quad \alpha = const > 0, \tag{3}$$

where f and q are given functions, and control $p(t)$ belongs to $L_2(0,T)$ with open or closed region of values, which we shall term by means of P .

The problem under consideration lies in determining control of such $p[t,u(t,x)]$, that the functional

$$J[p] = \int_0^1 [u(T,x) - \psi(x)]^2 dx + \beta \int_0^T p^2(t)\,dt,\quad \beta = const > 0$$

should have the least possible value . Here T is fixed moment of time, and ψ is given function from $L_2(0,1)$.

With the help of the method, stated in [1] , we can show, that

Bellman functional

$$S[t,u] = \min_{p \in P}\left\{\int_0^1 [u(T,x)-\psi(x)]^2 dx + \beta\int_t^T p^2(t)\,dt\right\}$$

satisfies the equation

$$-\frac{\partial S}{\partial t} = \min_{p(t)\in P}\left\{\beta p^2(t) + p(t)\int_0^1 v(t,x)\,q(x)\,dx - \alpha u(t,1)\,v(t,1) + \int_0^1 [f(t,x)\,v(t,x) - u_x(t,x)\,v_x(t,x)]\,dx\right\} \quad (4)$$

In this case it is supposed, that function $v(t,x)$, which is uniquely determined by Freshe differential $\varphi(u,h)$ of functional S :

$$\varphi(u,h) = \int_0^1 v(t,x)\,h(t,x)\,dx \quad (5)$$

is absolutely continuous on x .

In particular case, when the set of P coincides with the áxle $(-\infty,+\infty)$, we can have

$$p(t) = \frac{1}{2\beta}\int_0^1 v(t,x)\,q(x)\,dx \quad (6)$$

$$\frac{\partial S}{\partial t} = \int_0^1 [u_x v_x - f v]\,dx + \frac{1}{4\beta}\left[\int_0^1 v(t,x)\,q(x)\,dx\right]^2 + \alpha u(t,1)\,v(t,1) \quad (7)$$

from the equation (4).

From the definition of S :

$$S[T,u] = \int_0^1 [u(T,x) - \psi(x)]^2\,dx \quad (8)$$

2. THE CONSTRUCTION OF OPTIMAL CONTROL. We shall find the solution of problem (7)-(8) in the form

$$S[t,u] = \int_0^1\!\!\int_0^1 K(t,x,s)[u(t,x)-\psi(x)][u(t,s)-\psi(s)]\,dx\,ds + \int_0^1 \varphi(t,x)[u(t,x)-\psi(x)]\,dx + \eta(t). \quad (9)$$

Calculating Freshe differential of this functional we find, that

$$\varphi(u,h) = \int_0^1\!\!\int_0^1 [K(t,x,s)+K(t,s,x)][u(t,s)-\psi(s)]\,h(t,x)\,dx\,ds + \int_0^1 \varphi(t,x)\,h(t,x)\,dx$$

and according to formula (5) we have

$$V(t,x)=\int_0^1 [K(t,x,s)+K(t,s,x)][U(t,s)-\psi(s)]\,ds \quad +\varphi(t,x). \tag{10}$$

Substituting (9) and (10) in the equation (7) we **have**

$$\int_0^1\int_0^1 [-K_t(t,x,s)-K_{xx}(t,x,s)-K_{xx}(t,s,x)+\frac{1}{4\beta}K_1(t,x,s)][U(t,x)-\psi(x)]\times$$
$$\times[U(t,s)-\psi(s)]\,dx\,ds+\int_0^1\{-\varphi_t(t,x)-\varphi_{xx}(t,x)-K_2(t,x)-K_3(t,x)\ +$$
$$+\frac{1}{2\beta}\int_0^1\int_0^1[K(t,x,s)+K(t,s,x)]\varphi(t,y)q(y)q(s)\,dy\,ds\}[U(t,x)-\psi(x)]\,dx+$$
$$+\{-\eta_t(t)-\int_0^1\varphi_{xx}(t,x)\psi(x)\,dx-\int_0^1 f(t,x)\varphi(t,x)\,dx+\frac{1}{4\beta}\left(\int_0^1\varphi(t,x)q(x)\,dx\right)^2\}+$$
$$+\{[\varphi_x(t,1)+\alpha\varphi(t,1)]+\int_0^1[K_x(t,1,s)+\alpha K(t,1,s)][U(t,s)-\psi(s)]\,ds\ +$$
$$+\int_0^1[K_x(t,s,1)+\alpha K(t,s,1)][U(t,s)-\psi(s)]\,ds\}\,U(t,1)\ -$$
$$-\{\int_0^1[K_x(t,0,s)+K_x(t,s,0)][U(t,s)-\psi(s)]\,ds+\varphi_x(t,0)\}\,U(t,0)=0$$

where

$$\begin{aligned} K_1(t,x,s)&=\int_0^1\int_0^1[K(t,y,s)+K(t,s,y)][K(t,z,x)+K(t,x,z)]q(y)q(z)\,dy\,dz\\ K_2(t,x)&=\int_0^1[K(t,x,s)+K(t,s,x)]f(t,s)\,ds\\ K_3(t,x)&=\int_0^1[K_{ss}(t,x,s)+K_{ss}(t,s,x)]\psi(s)\,ds \end{aligned} \tag{11}$$

The functional (9) will satisfy the equation (7)for any function $U(t,x)-\psi(x)$, if we demand that $K(t,x,s)$, $\varphi(t,x)$ and $\eta(t)$ satisfy following equations:

$$K_t(t,x,s)+K_{xx}(t,x,s)+K_{xx}(t,s,x)-\frac{1}{4\beta}K_1(t,x,s)=0 \tag{12}$$

$$\left.\begin{aligned} K_x(t,0,s)&=K_x(t,1,s)+\alpha K(t,1,s)=0\\ K_x(t,s,0)&=K_x(t,s,1)+\alpha K(t,s,1)=0 \end{aligned}\right\} \tag{13}$$

$$\varphi_t(t,x)+\varphi_{xx}(t,x)+K_2(t,x)+K_3(t,x)=$$
$$=\frac{1}{2\beta}\int_0^1\int_0^1[K(t,x,s)+K(t,s,x)]\varphi(t,y)q(y)q(s)\,dy\,ds \tag{14}$$

$$\varphi_x(t,0) = \varphi_x(t,1) + \alpha\varphi(t,1) = 0 \tag{15}$$

$$\eta_t(t) + \int_0^1 \varphi_{xx}(t,x)\,\psi(x)\,dx + \int_0^1 f(t,x)\varphi(t,x)\,dx = \frac{1}{4\beta}\left(\int_0^1 \varphi(t,x)\,q(x)\,dx\right)^2 \tag{16}$$

Besides this it follows from (8) and (9), that

$$K(T,x,s) = \delta(s-x),\ \varphi(T,x) = 0,\ \eta(T) = 0. \tag{17}$$

The boundary-value problem (12)-(13) is the infinite dimensional analog of Rikkati equation, which appears during the solution of the problem about the optimum stabilization of system with finite number of degree of freedom.

We find function K in the form

$$K(t,x,s) = \sum_{i,j=1}^{\infty} C_{ij}(t)\,X_i(x)\,X_j(s),\ X_i(x) \equiv \frac{\cos\lambda_i x}{\sqrt{\omega_i}} \tag{18}$$

where λ_i are the eigenvalues of boundary-value problem $X'' + \lambda^2 X = 0$, $X'(0) = 0$, $X'(1) + \alpha X(1) = 0$, which are the positive roots of equation $\lambda\,tg\,\lambda = \alpha$, ω_i^{-1} - the normalization factors.

Substituting the function (18) in equations (12) and (13) we obtain the system of equations with reference to coefficients C_{ij} :

$$\frac{dC_{ij}}{dt} = \lambda_i^2\left[C_{ij} + C_{ji}\right] + \frac{1}{4\beta}\sum_{k,\ell=1}^{\infty}\left[C_{kj} + C_{jk}\right]\left[C_{\ell i} + C_{i\ell}\right]q_k q_\ell \tag{19}$$

and thanks to the conditions (17) we shall have

$$C_{ij}(T) = \delta_{ij},\quad i,j = 1,2,\ldots \tag{20}$$

Here q_i are Fourier coefficients of function $q(x)$.

By analogy the solution of problem (14)-(15) is found in the form of

$$\varphi(t,x) = \sum_{i=1}^{\infty} a_i(t)\,X_i(x). \tag{21}$$

Then coefficients a_i are determined from equations:

$$\left.\begin{aligned} \frac{da_i}{dt} &= \lambda_i^2 a_i + \frac{1}{2\beta}\sum_{k,j=1}^{\infty} q_k q_j\left[C_{ij} + C_{ji}\right]a_k - \sum_{j=1}^{\infty}\left[f_j - \lambda_j^2\psi_j\right]\left[C_{ij} + C_{ji}\right] \\ a_i(T) &= 0,\ i = 1,2,\ldots \end{aligned}\right\} \tag{22}$$

where $f_i(t)$ and ψ_i are Fourier coefficients of functions $f(t,x)$ and $\psi(x)$ correspondingly.

Taking into consideration (21), the equation (16) with the last condition (17) can be rewritten as

$$\eta_t(t)=\sum_{i=1}^{\infty}[\lambda_i^2\psi_i-f_i]a_i+\frac{1}{4\beta}\Big(\sum_{i=1}^{\infty}a_iq_i\Big)^2,\quad \eta(T)=0 \tag{23}$$

So, we obtain a total system of equations for definition of all undetermined values.

From the equation (19) and conditions (20) we can find, that $C_{ij}(t)\equiv 0$ by $i\neq j$ and

$$C_{ii}(t)=\frac{2\beta\lambda_i^2\exp[2\lambda_i^2(t-T)]}{2\beta\lambda_i^2+q_i^2(1-\exp[2\lambda_i^2(t-T)])} \tag{24}$$

That is why we can have

$$a_i(t)=-\int_t^T C_{ii}(\tau)\Big\{\frac{q_i}{\beta}\sum_{k=1}^{\infty}q_ka_k(\tau)-2\gamma_i(\tau)\Big]e^{\lambda_i^2(t-\tau)}d\tau,\ i=1,2,\ldots \tag{25}$$

from (22), where $\gamma_i=f_i-\lambda_i^2\psi_i$.

Multiplying both parts of the i-th equation by q_i and summing them up, we find

$$r(t)=\int_t^T B(t,\tau)r(\tau)\,d\tau+\Gamma(t)$$

where

$$r(t)=\sum_{k=1}^{\infty}q_ka_k(t),\quad B(t,\tau)=-\sum_{i=1}^{\infty}\frac{2q_i^2\lambda_i^2\exp[\lambda_i^2(t+\tau-2T)]}{2\beta\lambda_i^2+q_i^2(1-\exp[2\lambda_i^2(\tau-T)])}$$

$$\Gamma(t)=4\beta\sum_{i=1}^{\infty}\int_t^T\frac{q_i\lambda_i^2[f_i(T)-\lambda_i^2\psi_i]\exp[\lambda_i^2(t+\tau-2T)]}{2\beta\lambda_i^2+q_i^2(1-\exp[2\lambda_i^2(\tau-T)])}\,d\tau.$$

Having determined $r(t)$ and substituted it into (25), we shall have $a_i(t)$, $i=1,2,\ldots$. After that we can easily find $\eta(t)$ from (24). Therefore the functions $K(t,x,\xi)$, $\varphi(t,x)$ and $\eta(t)$ (see $S[t,U]$) are determined uniquely, and by the formula (6) we can obtain optimum control as a functional U. It is not necessary to use the function $\eta(t)$ for the construction of P and therefore we need not solve the problem (23).

Since in every concrete case the problem under study is formally solved completely, then on the basis of obtained formulas we can easily ground the procedure mentioned above, imposing corresponding restrictions on functions, being in its enunciation. But the substantiation of the method of dynamic programming (the obtaining of Bellman equation) demands more profound analysis, similar to that which is proposed in [2] for finite dimensional systems.

In conclusion we must mention, that Bellman's method was used in [3 - 4] for solving analogous control problems. However the authors of these works could not receive optimum control and they had to limit themselves by giving the account of this method.

R e f e r e n c e s

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