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ABOUT STABILITY OF EQUILIBRIUM SHAPES

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Abstract. We discuss the stability of "critical" or "equilibrium" shapes of a shape-dependent energy functional. We analyze a problem arising when looking at the positivity of the second derivative in order to prove that a critical shape is an optimal shape. Indeed, often when positivity -or coercivity- holds, it does for a weaker norm than the norm for which the functional is twice differentiable and local optimality cannot be *a priori* deduced. We solve this problem for a particular but significant example. We prove "weak-coercivity" of the second derivative uniformly in a "strong" neighborhood of the equilibrium shape.

Résumé. Nous nous intéressons à la stabilité des formes critiques ou d'"équilibre" d'une énergie dépendant de la forme. Dans le but de montrer qu'une forme critique est une forme optimale, nous étudions la positivité de la dérivée seconde. En effet, quand elle a lieu, la coercivité n'est vraie que dans une norme plus faible que celle pour laquelle l'énergie est différentiable: l'optimalité locale ne peut donc pas en être déduite *a priori.* Nous résolvons cette difficulté dans un cas particulier mais néanmoins significatif. Nous établissons de la "coercivité faible" de la dérivée seconde uniformément dans un voisinage "fort" de la forme d'équilibre.

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1. Introduction

We consider here the question of stability of equilibrium shapes which can be stated as follows. Let $\Omega \mapsto E(\Omega)$ be a real valued functional defined on a family $\mathcal{O}$ of subsets Ω of $\mathbb{R}^n$. Let Ω_0 be an equilibrium shape for $E(.)$, that is a shape at which the first derivative of $E(.)$ on $\mathcal{O}$ vanishes (we also say "critical shape", see below for a precise definition). By stability, we mean that $E(\Omega_0)$ is a strict local extremum, say a minimum, that is

$$E(\Omega_0) < E(\Omega) \tag{1.1}$$

for all Ω close enough to Ω_0 and in $\mathcal{O}$. If $E(.)$ represents the total energy in some shape equilibrium problem, this definition coincides with the classical notion of stability.

One of the difficulties is to understand properly the meaning of "being close to Ω_0" and therefore to choose the right topology on $\mathcal{O}$. One of the classical techniques is then to compute the second derivative of $E(.)$ at Ω_0 and to prove that it is strictly positive. However, in many applications, one is led to a situation where the second derivative of $E(.)$ at Ω_0 is coercive (*i.e.* strictly positive) for a certain norm which turns out to be

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weaker than the norm for which differentiability and Taylor formula hold. Consequently, the existence of a local minimum does not follow, even for the stronger topology. In order to show what could happen, let us consider an elementary example of such a situation (not taken from shape optimization). Let $L^2(0,1)$ and $H_0^1(0,1)$ be equipped with their usual norm and recall that $||\cdot||_{L^2} \leq ||\cdot||_{H_0^1}$. Consider the functional defined as

$$E(u) = \|u\|^2_{L^2(0,1)} - \|u\|^4_{H_0^1(0,1)}.$$

One can check that E is twice differentiable in H_0^1 and that

$$E'(0) \equiv 0,\ \forall h \in H_0^1(0,1),\ E''(0).(h,h) = 2\|h\|^2_{L^2(0,1)}.$$

Therefore, $E''(0)$ is coercive for the weaker norm L^2. This yields some "weak stability": indeed, there is a local minimum in each direction $u_{=}0 \in H_0^1$ since, for $t \in \mathbb{R}$, $E(tu_0) = t^2(||u_0||_{L^2} - t^2||u_0||^4_{H_0^1})$. However, there is no local minimum for E, even for the strong topology, since there is no $r > 0$ such that

$$\|u\|_{H_0^1(0,1)} < r \implies E(u) > E(0) = 0 \ i.e.\ \|u\|^2_{L^2(0,1)} > \|u\|^4_{H_0^1(0,1)},$$

as one can always construct a sequence in $H_0^1(0,1)$ such that

$$\|u_n\|_{H_0^1(0,1)} = r/2, \|u_n\|_{L^2(0,1)} \to 0 \text{ when } n \to +\infty.$$

Our goal is to precisely analyze this difficulty in a particular, but significant situation coming from shape optimization. Here, the second derivative of $E(.)$ will exist in a $\mathcal{C}^{2,\alpha}$-norm around Ω_0, but coercivity will only hold with respect to the $H^{1/2}(\partial\Omega_0)$-norm. This situation is typical in shapes problems. Here we choose $E(.)$ to be the energy associated with the classical Dirichlet problem and the measure of the admissible domains is supposed to be given. This model problem arises in many examples: let us for instance mention the case where $E(.)$ is the total energy in a problem of equilibrium shapes for liquid metals confined in a electro-magnetic field (see *e.g.* [1, 12, 13, 15]) We will restrict ourself to a simple two-dimensional model for which stability was already investigated in [3, 7–9, 14, 19]. The critical shapes we consider are assumed to be regular. Stability of a critical shape Ω_0 will mean that $E(\Omega_0)$ is a strict minimum for $E(\Omega)$ among the admissible domains Ω in some $\mathcal{C}^{2,\alpha}$-neighborhood of Ω_0 with the same measure as Ω_0.

We prove here for this problem that stability does occur when $H^{1/2}$-positivity of the second derivative holds on the tangent subspace of constraints. The main idea is to compute the second derivative not only at the equilibrium shape Ω_0 but around Ω_0 in the $\mathcal{C}^{2,\alpha}$-sense and to prove a uniform $H^{1/2}(\partial\Omega_0)$-coercivity in a $\mathcal{C}^{2,\alpha}$-neighborhood of Ω_0. This yields at least the existence of a local minimum in the $\mathcal{C}^{2,\alpha}$-topology. This technique, while developed here only in a specific case, may actually be used in many other situations like for the "exterior shaping problem" where the Dirichlet problem is set in the exterior of the shapes, or also when the functional depends on the perimeter and for more general functional depending on the state and on the gradient of the state (see [4]). We think that the estimate of the variation of the second derivative is by itself interesting and might lead to other applications. Note that the above question of choice of topologies was, in particular, raised in [7].

2. The problem and the results

Let k be a given real-valued function with compact support in $\mathbb{R}^2$ and belonging to the $\mathcal{C}^{0,\alpha}(\mathbb{R}^2)$ Hölder space ($\alpha \in (0,1)$) and let $S >$ meas(support(k)) be a given constant where meas(.) stands for the Lebesgue measure. We first define the family $\mathcal{O}$ of admissible shapes to be the family of open bounded subsets Ω of $\mathbb{R}^2$ with $\mathcal{C}^{2,\alpha}$-boundary such that

$$\text{meas}(\Omega) = S, \tag{2.1}$$

$$\text{support}(k) \subset \Omega. \tag{2.2}$$

We consider the shaping function E from $\mathcal{O}$ into $\mathbb{R}$ defined by

$$E(\Omega) = -\int_\Omega |\nabla\, u_\Omega|^2,$$

where $|.|$ denotes here the Euclidian norm and the function u_Ω is the solution of the Dirichlet problem

$$\begin{cases} -\Delta u_\Omega = k & \text{in } \Omega, \\ u_\Omega = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Note that $E(\Omega)$ is, up to a positive constant, the "energy" associated with the Dirichlet problem (2.3) which is

$$\frac{1}{2}\int_\Omega |\nabla\, u_\Omega|^2 - \int_\Omega k\, u_\Omega = -\frac{1}{2}\int_\Omega |\nabla\, u_\Omega|^2 = -\int_\Omega k\, u_\Omega,$$

the above equalities following easily from multiplying (2.3) by u_Ω. Note also that (see *e.g.* Sect. 4), with the above regularity assumptions

$$\forall \Omega \in \mathcal{O}, u_\Omega \in \mathcal{C}^{2,\alpha}(\Omega).$$

We now consider Ω_0 a critical point of E under the constraint (2.1) that is to say an open set Ω_0 in $\mathcal{O}$ where the derivative of $E(\Omega) + \Lambda$ meas(Ω) with respect to Ω vanishes for some $\Lambda \in \mathbb{R}$. As we will verify later (see Sect. 5), this means - and we will assume it throughout the paper:

$$|\nabla u_{\Omega_0}|^2 \quad = \quad \Lambda \text{ on } \partial\Omega_0. \tag{2.4}$$

The constant Λ is the Lagrange multiplier corresponding to the constraint meas$(\Omega) = S$. We will recall below (see beginning of Sect. 3) the definition of the spaces $\mathcal{C}^{2,\alpha}$ and of their norms.

We want to find sufficient conditions for the stability of Ω_0 using second derivatives of the augmented functional $J(\Omega) = E(\Omega) + \Lambda$ meas(Ω). Let us first recall some facts about these derivatives.

Let us denote

$$\mathcal{V}(\eta) := \{\Theta \in \mathcal{C}^{2,\alpha}\ (\mathbb{R}^2, \mathbb{R}^2); \|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha} < \eta\}. \tag{2.5}$$

Note that, for η small enough, any $\Theta \in \mathcal{V}(\eta)$ is a diffeomorphism. For all $\Theta \in \mathcal{V}(\eta)$, we set $\bar{J}(\Theta) := J(\Theta(\Omega_0))$. Then the second (classical) derivative of $\bar{J}$ exists and one can show (see *e.g.* [2, 6, 16–18, 20, 21] or also Sect. 5 here) that, since Ω_0 is a critical shape, the second derivative at $\Theta = \mathrm{Id}(=$ Identity) has a specific structure, namely

$$\forall \xi \in \mathcal{C}^{2,\alpha}(\mathbb{R}^2, \mathbb{R}^2),\ \bar{J}''(\mathrm{Id})(\xi, \xi) = B(\xi \cdot \boldsymbol{n}_{|\partial\Omega_0}, \xi \cdot \boldsymbol{n}_{|\partial\Omega_0}) \tag{2.6}$$

where B is a continuous bilinear form on $\mathcal{C}^{1,\alpha}(\partial\Omega_0, \mathbb{R})$, $\boldsymbol{n}$ is the unit exterior normal vector field to $\partial\Omega_0$ and $\cdot_{|\partial\Omega_0}$ denotes the restriction of a function to $\partial\Omega_0$. As we will see in Section 5, in the situation we consider here, the explicit expression of B is given as follows: set $m = \xi \cdot n_{|\partial\Omega_0}$, then

$$B(m, m) = 2\,\Lambda \int_{\partial\Omega_0} C_0(m)\, m + \mathcal{C}\, m^2, \tag{2.7}$$

where $\mathcal{C}$ denotes the curvature of $\partial\Omega_0$ and C_0 denotes the so-called "capacity" or "Steklov-Poincaré" operator on $\partial\Omega_0$. We refer *e.g.* to [5] or to Section 5 for a precise definition and properties of this operator but we can already mention that:

$$\int_{\partial\Omega_0} C_0(m)\, m = \int_{\Omega_0} |\nabla M|^2$$

where M is the harmonic extension of m to Ω_0. Consequently, the first part of the integral is always strictly positive (except if $M = 0$ which happens only if m is constant on each boundary of the connected components of Ω_0).

A *necessary condition for stability* is that the second derivative of $\bar{J}$ be positive on the subspace which is tangent to the constraint, namely the subspace of functions m as above such that $\int_{\partial\Omega_0} m = 0$. Obviously, it is the case if, for instance, Ω_0 is convex; then B is even coercive in the space $H^{1/2}(\partial\Omega_0)$ if $\Lambda > 0$. More general situations are described in [4, 11]. See also more comments on this at the end of this paper.

The question we address here is *the converse*. For this, one has to assume, as usual, that the second derivative is *strictly positive* in some sense. The natural space of coercivity here is $H^{1/2}(\partial\Omega_0)$. The question is then the following: assume that there exists $c > 0$ such that

$$\forall m \in \mathcal{C}^{2,\alpha}(\partial\Omega_0, \mathbb{R}) \ \ with \ \int_{\partial\Omega_0} m = 0, \ \ B(m,m) \geq c\,||m||^2_{H^{1/2}(\partial\Omega_0)}. \tag{2.8}$$

Then, is $E(\Omega_0)$ a strict local minimum with the constraint (2.1), local at least in the $\mathcal{C}^{2,\alpha}$-topology?

We prove here that the answer to this question is positive:

Theorem 2.1 (*Existence of a local strict minimum*)**.** *Assume $\partial\Omega_0$ is of class $\mathcal{C}^{4,\alpha}$ and (2.8) holds for some $c > 0$. Then there exists $\eta > 0$ such that for all Θ in $\mathcal{V}(\eta)$ with* $\mathrm{meas}\big(\Theta(\Omega_0)\big) = \mathrm{meas}(\Omega_0)$ *and different from the identity*

$$E\big(\Theta(\Omega_0)\big) > E(\Omega_0).$$

The main point in the proof of this result will be contained in Theorem 2.5. In order to state it, we need to introduce a few notations and to recall some more or less known facts on small regular perturbations of Ω_0 and of $\partial\Omega_0$. Since we are in dimension two, $\partial\Omega_0$ is a union of q disjoint regular Jordan curves. For simplicity, we will write the proof when $q = 1$ (*i.e.* Ω_0 is simply connected). The changes needed in the general case are obvious (see the remark at the end of Sect. 2.2 or also [4]).

Let γ denote a function in $\mathcal{C}^{k,\alpha}([0, L], \mathbb{R}^2)$, with $k \geq 2$, whose image is $\partial\Omega_0$, that is

$$\begin{cases} \gamma([0, L]) = \partial\Omega_0, \gamma(0) = \gamma(L), \\ \gamma \ \text{ is one-to-one from } [0, L) \ \text{ into } \partial\Omega_0, \\ \forall s \in [0, L], ||\gamma'(s)|| = 1. \end{cases}$$

Here the parameter s is the length parameter and L is the total length of $\partial\Omega_0$. We denote by $\boldsymbol{n}$ the unit exterior normal derivative to $\partial\Omega_0$. The orientation is chosen so that so that $\boldsymbol{n}(\gamma(s)) = R_{-\pi/2}(\gamma'(s))$ where $R_{-\pi/2}$ is the rotation of angle $-\pi/2$ in $\mathbb{R}^2$ or also $\boldsymbol{n}(\gamma(s)) = (\gamma_2'(s), -\gamma_1'(s))$ where $\gamma = (\gamma_1, \gamma_2)$. We will often write simply $\boldsymbol{n}(s) = \boldsymbol{n}(\gamma(s))$.

For $\tau > 0$, we denote by $\mathcal{T}_\tau$ the tubular neighborhood of $\partial\Omega_0$ with radius τ, that is

$$\mathcal{T}_\tau = \{x \in \mathbb{R}^2; \mathrm{distance}(x, \partial\Omega_0) < \tau\}.$$

Lemma 2.2 (Normal representation of small perturbations of $\partial\Omega_0$)**.** *Assume $\partial\Omega_0$ is of class $C^{2,\alpha}$. Then, there exists $\tau_1 > 0$ such that the mapping*

$$(s, \tau) \in [0, L) \times (-\tau_1, \tau_1) \to \gamma(s) + \tau\,\boldsymbol{n}(s) \in \mathcal{T}_{\tau_1}$$

is one-to-one. Moreover, there exists $\eta_1 > 0$ such that for all $\Theta \in \mathcal{V}(\eta_1)$, there exists a unique $d_\Theta \in \mathcal{C}^{1,\alpha}([0, L], (-\tau_1, \tau_1))$ with $d_\Theta(0) = d_\Theta(L)$ and such that $s \to \gamma(s) + d_\Theta(s)\ \boldsymbol{n}(s)$ is one-to-one from $[0, L)$ into $\Theta(\partial\Omega_0)$. If $\partial\Omega_0$ is of class $\mathcal{C}^{3,\alpha}$, then d_Θ is of class $\mathcal{C}^{2,\alpha}$ and we have

$$\|d_\Theta\|_{2,\alpha} \leq C\,\|\Theta - \mathrm{Id}\|_{2,\alpha}. \tag{2.9}$$

Here our goal is to minimize the functional $E(.)$ on open subsets with a prescribed measure. Therefore we want to go from Ω_0 to $\Theta(\Omega_0)$ for any $\Theta \in \mathcal{V}(\eta_1)$ with $\text{meas}(\Theta(\Omega_0)) = \text{meas}(\Omega_0)$, by a regular path $t \in [0,1] \mapsto \Omega(t)$ where $\text{meas}(\Omega(t)) = \text{meas}(\Omega_0)$. This can be done through *normal deformations* obtained from the *flow of a divergence-free vector field* as stated in the next proposition.

We first need to extend the vector-field $\boldsymbol{n}$ to the tubular neighborhood $\mathcal{T}_{\tau_1}$ of $\partial\Omega_0$. We do it as follows: if $x \in \mathcal{T}_{\tau_1}$, there exists (s,τ) unique in $[0,L) \times (-\tau_1,\tau_1)$ such that $x = \gamma(s) + \tau\boldsymbol{n}(s)$; then we define $\boldsymbol{n}(x)$ as $\boldsymbol{n}(\gamma(s))$. Thus for all $(s,\tau) \in [0,L) \times (-\tau_1,\tau_1)$,

$$\boldsymbol{n}(\gamma(s) + \tau\,\boldsymbol{n}(s)) = \boldsymbol{n}(s). \tag{2.10}$$

Proposition 2.3 (Area-preserving normal deformations of Ω_0)**.** *We assume $\partial\Omega_0$ is of class $\mathcal{C}^{4,\alpha}$. There exist $\eta_2 > 0, \tau_2 > 0$ such that, for all $\Theta \in \mathcal{V}(\eta_2)$ with* $\text{meas}(\Theta(\Omega_0)) = \text{meas}(\Omega_0)$*, there exists a divergence-free vector field $\boldsymbol{X}_\Theta \in \mathcal{C}^{2,\alpha}(\mathbb{R}^2,\mathbb{R}^2)$ and $m_\Theta \in \mathcal{C}^{2,\alpha}(\mathcal{T}_{\tau_2},\mathbb{R})$ such that $\boldsymbol{X}_\Theta = m_\Theta\,\boldsymbol{n}$ on $\mathcal{T}_{\tau_2}$ and the flow Φ_Θ of $\boldsymbol{X}_\Theta$ maps $\partial\Omega_0$ onto $\Theta(\partial\Omega_0)$ at time $t = 1$. Moreover, we have for all $t \in [0,1]$,*

$$\|\Phi_\Theta(t) - \text{Id}_{\mathbb{R}^2}\|_{2,\alpha} \le C\,\|\Theta - \text{Id}_{\mathbb{R}^2}\|_{2,\alpha}. \tag{2.11}$$

Recall that the flow Φ_Θ of the vector field $\boldsymbol{X}_\Theta$ is the solution of

$$\begin{cases} \partial_t\Phi_\Theta(t,x) = \boldsymbol{X}_\Theta(\Phi_\Theta(t,x)) \\ \Phi_\Theta(0,x) = x. \end{cases}$$

Since $\text{div}\boldsymbol{X}_\Theta = 0$, we have $\det\,\Phi_\Theta(t,x) \equiv 1$, so that, in particular, $\text{meas}(\Phi_\Theta(t,\Omega_0)) = \text{meas}(\Omega_0)$. We will use this area-preserving path $t \to \Phi_\Theta(t,\Omega_0)$ going from Ω_0 to $\Phi_\Theta(1,\Omega_0) = \Theta(\Omega_0)$ and control the variation of $E(\cdot)$ along this path by studying the second derivative with respect to t of

$$e_\Theta(t) := E(\Phi_\Theta(t,\Omega_0)).$$

Lemma 2.4. *Assume Ω_0 has a $\mathcal{C}^{4,\alpha}$-boundary. Then, for all $\Theta \in \mathcal{V}(\eta_2)$, $t \mapsto e_\Theta(t)$ is twice differentiable on* $[0,1]$.

A proof of this lemma can be found in [7] (see also [6,16,20,21]). The main arguments are also given here while computing $e''_\Theta(t)$ (see Sect. 5).

We can now state our key-result.

Theorem 2.5. *Assume $\partial\Omega_0$ is of class $\mathcal{C}^{4,\alpha}$. Then there exist $\eta_0 > 0$ and a function $\omega :]0,\eta_0] \mapsto \mathbb{R}$ with $\lim_{r\downarrow 0}\omega(r) = 0$, such that for all $\eta \in (0,\eta_0]$ and all $\Theta \in \mathcal{V}(\eta)$ with* $\text{meas}(\Theta(\Omega_0)) = \text{meas}(\Omega_0)$*, we have for all $t \in [0,1]$,*

$$|e''_\Theta(t) - e''_\Theta(0)| \le\; \omega(\eta)\;\|m_\Theta\|^2_{H^{1/2}(\partial\Omega_0)}. \tag{2.12}$$

It is easy to guess how Theorem 1 can be deduced from this theorem by application of Taylor formula (see the end of this paper). The main point is that around the equilibrium shape Ω_0, if the positivity condition (2.8) holds, then the second derivative will actually be $H^{1/2}(\partial\Omega_0)$-coercive *uniformly in a $\mathcal{C}^{2,\alpha}$-neighborhood* of Ω_0.

The above property depends on the nature of the various terms which come out in the expression of the second derivative. Obviously, it can be shown to be valid for many other similar functionals like those already mentioned above (see [4,11]).

With respect to the regularity assumptions, the hypothesis that the critical point should be of class $\mathcal{C}^{4,\alpha}$ is not a restriction since Henrot and the second author have shown in [13] that if a regular Jordan curve with a $\mathcal{C}^2$ boundary is a critical point for this functional, then it is in fact analytic.

3. Proofs of geometrical results

Let us recall the classical definitions of Hölder spaces and norms. Let D be an open subset of $\mathbb{R}^p, p \geq 1$ and let $q \geq 1$. Then the $\mathcal{C}^2(\bar{D}, \mathbb{R}^q)$-norm is defined as

$$\|u\|_2 = \|u\|_{L^\infty(D,\mathbb{R}^q)} + \sup_{1\leq i\leq n} \|\partial_i u\|_{L^\infty(D,\mathbb{R}^q)} + \sup_{1\leq i,j\leq n} \|\partial^2_{i,j} u\|_{L^\infty(D,\mathbb{R}^q)}.$$

For $\alpha \in (0,1)$, the Hölder space $\mathcal{C}^{2,\alpha}(\bar{D}, \mathbb{R}^q)$ is the subspace of $\mathcal{C}^2$-functions such that

$$\max_{1\leq i,j\leq n} \sup_{\substack{(x,y)\in D^2,\\ x\neq y}} \frac{\|\partial^2_{i,j} u(x) - \partial^2_{i,j} u(y)\|}{\|x-y\|^\alpha} < +\infty,$$

and the $\mathcal{C}^{2,\alpha}$-norm is defined as

$$\|u\|_{2,\alpha} = \|u\|_2 + \max_{1\leq i,j\leq n} \sup_{\substack{(x,y)\in D^2,\\ x\neq y}} \frac{\|\partial^2_{i,j} u(x) - \partial^2_{i,j} u(y)\|}{D\|x-y\|^\alpha}\cdot$$

It gives to $\mathcal{C}^{2,\alpha}(\bar{D}, \mathbb{R}^q)$ a Banach space structure. A similar definition may be given for $\mathcal{C}^{k,\alpha}(\bar{D}, \mathbb{R}^q), k \geq 0$ by replacing the second derivatives by the kth derivatives.

3.1. **Proof of Lemma 2.2.**

This is rather classical but we give here an elementary proof whose ingredients will be partly used later in this paper.

Assume first that $\partial\Omega_0$ is of class $\mathcal{C}^2$. Let us consider the mapping T from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}^2$ defined by $T(s,\tau) = \gamma(s) + \tau\, \boldsymbol{n}(s)$ which is of class $\mathcal{C}^1$ and L-periodic in s. The derivative of T at $(s_0, 0)$ is given by

$$T'(s_0, 0) = [\gamma'(s_0) \;\; \boldsymbol{n}(s_0)]$$

so that $\det T'(s_0, 0) = -1$. By the inverse mapping theorem, T is a local $\mathcal{C}^1$-diffeomorphism.

Let us show that, if τ_1 is small enough, T is also a global bijection from $[0, L) \times (-\tau_1, \tau_1)$ into $\mathcal{T}_{\tau_1}$.

Let $\tau(s_0), \eta(s_0) > 0$ be such that T is a diffeomorphism from $(s_0 - \eta(s_0), s_0 + \eta(s_0)) \times (-\tau(s_0), \tau(s_0))$ onto a neighborhood $\omega(s_0)$ of $\gamma(s_0)$. Up to still reducing $\tau(s_0)$, one can also assume that

$$\omega(s_0) \cap \partial\Omega_0 = T\big((s_0 - \eta(s_0), s_0 + \eta(s_0)) \times \{0\}\big). \tag{3.1}$$

Let $r(s_0)$ be such that $B(\gamma(s_0), r(s_0)) \subset \omega(s_0)$. One can find a finite covering of $\partial\Omega_0$ by the union of the $\{\partial\Omega_0 \cap B(\gamma(s_i), \frac{1}{2}\, r(s_i)), s_i \in [0, L), i = 1...p\}$. Next we set:

$$r_1 := \min\{r(s_i), i = 1...p\}, \tau_1 := \min\{r_1/5, \tau(s_i), i = 1...p\}.$$

Then the announced global bijection property holds. Indeed, assume

$$\gamma(s) + \tau\, \boldsymbol{n}(s) = \gamma(\hat{s}) + \hat{\tau}\, \boldsymbol{n}(\hat{s}), \text{ with } \tau, \hat{\tau} \in (-\tau_1, \tau_1), s, \hat{s} \in [0, L). \tag{3.2}$$

Then

$$\|\gamma(s) - \gamma(\hat{s})\| \leq \tau + \hat{\tau} \leq 2\, \tau_1 < \frac{1}{2}\, r_1.$$

Therefore, $\gamma(s)$ and $\gamma(\hat{s})$ belong to the same ball $B(\gamma(s_i), r(s_i))$. Then, by (3.1), there exist

$$\sigma, \hat{\sigma} \in (-\eta(s_i) + s_i, s_i + \eta(s_i)), \sigma = s \bmod (L), \hat{\sigma} = \hat{s} \bmod (L),$$

such that

$$\gamma(s) = \gamma(\sigma), \gamma(\hat{s}) = \gamma(\hat{\sigma}).$$

Then, (3.2) can be rewritten

$$\gamma(\sigma) + \tau\, \boldsymbol{n}(\sigma) = \gamma(\hat{\sigma}) + \hat{\tau}\, \boldsymbol{n}(\hat{\sigma}), \text{ with } \tau, \hat{\tau} \in (-\tau_1, \tau_1), \sigma, \hat{\sigma} \in (-\eta(s_i) + s_i, s_i + \eta(s_i)).$$

By the local bijection property of T, $\sigma = \hat{\sigma}, \tau = \hat{\tau}$. Since now γ is a bijection from $[0, L)$ onto $\partial\Omega_0$, $s, \hat{s} \in [0, L)$ and $\gamma(s) = \gamma(\hat{s})$ imply finally that $s = \hat{s}$.

For the regularity, we know that $(\tilde{s}(x), \tilde{\tau}(x)) = T^{-1}(x)$ is of class $\mathcal{C}^1$ on $\mathcal{T}_{\tau_1}$. We then easily check that $\nabla\tilde{\tau}(x) = \boldsymbol{n}(\tilde{s}(x))$ (see 3.4 below). Therefore, $\tilde{\tau}'$ is of class $\mathcal{C}^1$ and $\tilde{\tau}$ is of class $\mathcal{C}^2$ (note that $\tilde{\tau}$ is the distance function to $\partial\Omega_0$ and it is classical that it has the regularity of $\partial\Omega_0$, see *e.g.* [10]). If γ is of class $\mathcal{C}^{k,\alpha}$, then T and T^{-1} are of class $\mathcal{C}^{k-1,\alpha}$. So is $\tilde{\tau}'$ so that $\tilde{\tau}$ is of class $\mathcal{C}^{k,\alpha}$.

Let us now consider $\Theta \in \mathcal{V}(\eta_1), \eta_1 > 0$; recall that Θ is then of class $\mathcal{C}^{2,\alpha}$ and that γ is assumed to be of class $\mathcal{C}^{k,\alpha}$ with $k = 2$ or 3. The mapping $Y(s) = \Theta(\gamma(s))$ is of class $\mathcal{C}^{2,\alpha}$ and, if η_1 is small enough so that $Y(s) \in \mathcal{T}_{\tau_1}$ for all s, with the previous notations and $\psi(s) := \tilde{s}(Y(s))$, one can write

$$Y(s) = \gamma(\psi(s)) + \tilde{\tau}(Y(s))\boldsymbol{n}(\psi(s)). \tag{3.3}$$

For the regularity, ψ is of class $\mathcal{C}^{k-1,\alpha}$ and $\tilde{\tau}(Y)$ of class $\mathcal{C}^{2,\alpha}$. Let us check that ψ is invertible by proving that its derivative does not vanish. It is given by

$$\psi'(s) = \tilde{s}'(Y(s))Y'(s).$$

We can deduce the expression of $\tilde{s}'$ by inverting

$$T'(s,\tau) = [\gamma'(s) + \tau\boldsymbol{n}'(s) \;\; \boldsymbol{n}(s)] \tag{3.4}$$

and we easily obtain

$$\nabla\tilde{s}(x) = -\gamma'(\tilde{s}(x))/\det T'(x). \tag{3.5}$$

Since T is a diffeomorphism, $\det T'$ does not vanish so that ψ' vanishes if and only if $0 = \gamma'(s) \cdot Y'(s)$. It is not the case if η_1 is small enough since

$$|\gamma'(s) \cdot Y'(s) - 1| = |\gamma'(s) \cdot (Y'(s) - \gamma'(s))\| \leq \|Y'(s) - \gamma'(s)\| \leq C\, \|\Theta - \mathrm{Id}\|_{\mathcal{C}^1}$$

where we used $Y' = D\Theta\, \gamma'$ for the last inequality. Therefore ψ is invertible and is a bijection from $[0, L)$ into $[\psi(0), \psi(0) + L)$. Plugging $\psi^{-1}(s)$ in place of s into (3.3) leads to

$$Y(\psi^{-1}(s)) = \gamma(s) + d_\Theta(s)\boldsymbol{n}(s) \tag{3.6}$$

where we set $d_\Theta(s) = \tilde{\tau}(Y(\psi^{-1}(s)))$. We check that d_Θ is of class $\mathcal{C}^{k-1,\alpha}$.

We finish the proof of Lemma 2.2 by obtaining the estimate on d_Θ when $k = 3$. We will use the following technical lemma whose proof is left to the reader:

Lemma 3.1. *Let $p, q \geq 1$, $A : \mathbb{R}^p \to \mathbb{R}^p$ of class $\mathcal{C}^{2,\alpha}$, $\beta : \mathbb{R}^q \to \mathbb{R}^p$ of class $\mathcal{C}^{2,\alpha}$ and $B : \mathbb{R}^p \to \mathbb{R}^p$ of class $\mathcal{C}^{3,\alpha}$. Then, there exists $C_1 = C_1(\|\beta\|_{\mathcal{C}^{2,\alpha}})$ and $C_2 = C_2(\|B\|_{\mathcal{C}^{3,\alpha}}, \|A\|_{\mathcal{C}^{2,\alpha}})$ such that*

$$\|A \circ \beta\|_{\mathcal{C}^{2,\alpha}} \leq C_1 \|A\|_{\mathcal{C}^{2,\alpha}}, \tag{3.7}$$

$$\|B \circ A - B\|_{\mathcal{C}^{2,\alpha}} \leq C_2 \|A - \mathrm{Id}\|_{\mathcal{C}^{2,\alpha}}. \tag{3.8}$$

Remark. Note that C_2 involves the third derivative of B.

We first apply twice the above lemma to bound the $\mathcal{C}^{2,\alpha}$-norm of $\psi = \tilde{s}(\Theta(\gamma))$ by a constant depending on the data and on η_1 (we use here that $\tilde{s}$ is of class $\mathcal{C}^{2,\alpha}$ and therefore that γ is of class $\mathcal{C}^{3,\alpha}$). We deduce that the $\mathcal{C}^{2,\alpha}$-norm of ψ^{-1} is also bounded by a similar constant.

Next we apply the above lemma with $A = d_\Theta(\psi)$ and $\beta = \psi^{-1}$ to get

$$\|d_\Theta\|_{\mathcal{C}^{2,\alpha}} \leq \|d_\Theta(\psi)\|_{\mathcal{C}^{2,\alpha}}.$$

Then we use

$$d_\Theta(\psi(s)) = \tilde{\tau} \circ \Theta \circ \gamma(s) = \tilde{\tau} \circ \Theta \circ \gamma(s) - \tilde{\tau} \circ \gamma(s) = [\tilde{\tau} \circ \Theta - \tilde{\tau}] \circ \gamma,$$

and we apply again the first part of the above lemma with $A = \tilde{\tau} \circ \Theta - \tilde{\tau}$ and $\beta = \gamma$ to obtain

$$\|d_\Theta \circ \psi\|_{\mathcal{C}^{2,\alpha}} \leq C \|\tilde{\tau} \circ \Theta - \tilde{\tau}\|_{\mathcal{C}^{2,\alpha}}.$$

Finally we use the second part of the above lemma to conclude (recall that by previous remarks, here $\tilde{\tau}$ is of class $\mathcal{C}^{3,\alpha}$).

3.2. Proof of Proposition 2.3

We will construct the function m_Θ from $\mathbb{R}^2$ into $\mathbb{R}$ so that $X_\Theta = m_\Theta \boldsymbol{n}$ be divergence-free. Here, $\boldsymbol{n}$ denotes the extension of the unit normal to $\mathcal{T}_{\tau_1}$ as defined in (2.10). We will use the local coordinates (s, τ) introduced in Lemma 1 to first define m in a neighborhood of $\partial\Omega_0$. Let us compute $\mathrm{div}(\boldsymbol{n})$ by differentiating

$$x = \gamma(\tilde{s}(x)) + \tilde{\tau}(x)\boldsymbol{n}(x)$$

with respect to x:

$$\mathrm{Id} = D\gamma(\tilde{s}(x))D\tilde{s}(x) + \boldsymbol{n}(x)D\tilde{\tau}(x) + \tilde{\tau}(x)D\boldsymbol{n}(x).$$

We use the expressions of the derivatives of $\tilde{s}, \tilde{\tau}$ obtained above ($\nabla\tilde{s}$ is given in (3.5) and $\nabla\tilde{\tau}(x) = \boldsymbol{n}(x)$) and we take the trace of the latter equality to get

$$2 = -(\mathrm{det}T'(x))^{-1} + 1 + \tilde{\tau}(x)\mathrm{div}(\boldsymbol{n})(x).$$

By using also (see 3.4),

$$\det T'(s, \tau) = \det(\gamma', \boldsymbol{n})(s) + \tau\det(\boldsymbol{n}', \boldsymbol{n})(s) = -1 + \tau\det(\boldsymbol{n}', \boldsymbol{n})(s),$$

we deduce the expression of $\mathrm{div}(\boldsymbol{n})$:

$$\mathrm{div}(\boldsymbol{n})(s, \tau) = \tau^{-1}(1 + (\mathrm{det}T')^{-1}) = \frac{\det(\boldsymbol{n}', \boldsymbol{n})(s)}{\tau\det(\boldsymbol{n}', \boldsymbol{n})(s) - 1}. \tag{3.9}$$

From now on, we introduce the notation $a(s) := \det(\boldsymbol{n}, \boldsymbol{n}')(s)$. Note that $a(s)$ is exactly the curvature at $\gamma(s)$ of the curve $\partial\Omega_0$ seen from inside. It is of class $\mathcal{C}^{2,\alpha}$ since $\partial\Omega_0$ is assumed to be of class $\mathcal{C}^{4,\alpha}$. An important remark on (3.9) is that

$$\forall (s,\tau) \in \mathcal{T}_{\tau_1},\ 1+\tau a(s) > 0,$$

since $\det T' \neq 0$ on $\mathcal{T}_{\tau_1}$. We fix $\tau_2 \in (0,\tau_1)$ (depending only on $\partial\Omega_0$ and τ_1) such that

$$\forall (s,\tau) \in \bar{\mathcal{T}}_{\tau_2},\ 1+\tau a(s) \geq c_{\tau_2} > 0. \tag{3.10}$$

For vector fields of the form $\boldsymbol{X} = m(s,\tau)\boldsymbol{n}$, we have

$$\mathrm{div}\boldsymbol{X} = (\nabla m, \boldsymbol{n}) + m\mathrm{div}\boldsymbol{n} = \partial_\tau m(s,\tau) + \frac{a(s)m(s,\tau)}{1+\tau a(s)}, \tag{3.11}$$

so that $\boldsymbol{X} = m\boldsymbol{n}$ will be divergence-free if it satisfies:

$$\big(1+\tau a(s)\big)\partial_\tau m(s,\tau) + a(s)m(s,\tau) = 0. \tag{3.12}$$

This leads to

$$m(s,\tau) = \frac{f(s)}{1+\tau a(s)} \tag{3.13}$$

where f is to be determined as follows: let $\Theta \in \mathcal{V}(\eta_2)$ with $\eta_2 = \min\{\eta_1, \tau_2\}$ (where η_1, τ_2 are defined in Lemma 1 and 3.10); then f should be such that the following boundary conditions be satisfied

$$\Phi(0,\partial\Omega_0) = \partial\Omega_0, \Phi(1,\partial\Omega_0) = \Theta(\partial\Omega_0) \tag{3.14}$$

where $\Phi(t,.)$ is the flow of $\boldsymbol{X}$, that is the solution of

$$\begin{cases} \partial_t\Phi(t,x) &= \boldsymbol{X}(\Phi(t,x)),\ t>0, \\ \Phi(0,x) &= x. \end{cases}$$

Since $\boldsymbol{X} = m\boldsymbol{n}$, the trajectories of $\Phi(t,x)$ starting at $x = \gamma(s) \in \partial\Omega_0$ are parallel to $\boldsymbol{n}$, at least for small t, so that

$$\Phi(t,\gamma(s)) = \gamma(s) + \tau(s,t)\boldsymbol{n}(s). \tag{3.15}$$

The above system reduces to the scalar ordinary differential equation in $\tau(.,.)$

$$\partial_t\tau(s,t) = m(s,\tau(t,s)), \tag{3.16}$$

and the boundary conditions (3.14) become

$$\tau(s,0) = 0, \tau(s,1) = d_\Theta(s), \tag{3.17}$$

where d_Θ is defined in Lemma 1. According to (3.13), the equation (3.16) is equivalent to the existence of a function $C(s)$ such that

$$\frac{1}{2}a(s)\tau(s,t)^2 + \tau(s,t) = f(s)t + C(s). \tag{3.18}$$

Now f, C have to satisfy (3.17): the condition $\tau(s,0)=0$ leads to $C(s)=0$ and the other one to

$$f(s) = -\big[\frac{1}{2}a(s)d_\Theta(s)^2 + d_\Theta(s)\big]. \tag{3.19}$$

It remains to prove that, with this choice of f, the equation (3.18) does have a solution $\tau(s,t)$ for $t\in[0,1]$. It is indeed the case since it is a quadratic equation in $\tau(s,t)$ with discriminant

$$\Delta(t) = 1 + \big[a(s)^2 d_\Theta(s)^2 + 2a(s)d_\Theta(s)\big]t.$$

This quantity is linear in t, nonnegative at $t=0$ as well as at $t=1$ since:

$$\Delta(1) = \big[a(s)d_\Theta(s)+1\big]^2.$$

Recall that, by (3.10) and $\eta_2\le\eta_1$, $1+a(s)d_\Theta(s)\ge c_{\tau_2}$. We deduce the existence on $[0,1]$ of a solution to (3.18) given by

$$\begin{cases} \tau(s,t) = \big[-1+\sqrt{1+t(a(s)^2d_\Theta(s)^2+2a(s)d_\Theta(s))}\big]/a(s), \text{if } \ a(s)\neq 0\\ \tau(s,t) = td_\Theta(s), \text{if } \ a(s)=0. \end{cases} \tag{3.20}$$

At this step, for all $\Theta\ \in\mathcal{V}(\eta_2)$, we have constructed a divergence-free vector-field $\boldsymbol{X}=m\boldsymbol{n}$ in the neighborhood $\mathcal{T}_{\tau_1}$ of $\partial\Omega_0$ where m, given by the formulas (3.13, 3.19) is of class $\mathcal{C}^{2,\alpha}$. We will now use it to define the divergence-free vector-field $\boldsymbol{X}_\Theta$ on the whole space $\mathbb{R}^2$ (see also the remark at the end of this paragraph).

We denote by ζ a $C_0^\infty(\mathbb{R}^2,\mathbb{R})$-function, identically equal to 1 on $\mathcal{T}_{\tau_2}$ and with compact support in $\mathcal{T}_{\tau_1}$. Recall that $\Phi([0,1]\times\partial\Omega_0)\subset\mathcal{T}_{\tau_2}$. Since $\mathrm{div}(\boldsymbol{X})=0$ on $\mathcal{T}_{\tau_1}$, there exists $\xi\in\mathcal{C}^{3,\alpha}$ locally so that $\boldsymbol{X}=(\xi_y,-\xi_x)$. But, as proved *e.g.* in [13], this is valid globally on $\mathcal{T}_{\tau_1}$ since

$$0=\int_{\partial\Omega_0}(\boldsymbol{X}_\Theta,\boldsymbol{n}) = \int_{\partial\Omega_0} m = \int_0^L m(s,0)\,\mathrm{d}s, \tag{3.21}$$

as we check below. Then we set

$$\boldsymbol{X}_\Theta := ((\xi\zeta)_y, -(\xi\zeta)_x).$$

Obviously $\boldsymbol{X}_\Theta$ coincides with $\boldsymbol{X}=m\boldsymbol{n}$ on $\mathcal{T}_{\tau_2}$ and, by construction of m, satisfies all the conclusions of Proposition 2.3.

To obtain the final estimate on Φ_Θ, recall first that, at least on $\mathcal{T}(\eta_2)$

$$\boldsymbol{X}_\Theta = \frac{f(\tilde{s})}{1+\tilde{\tau}\,a(\tilde{s})}\boldsymbol{n}(\tilde{s}),\ f=-\frac{1}{2}ad_\Theta^2 - d_\Theta.$$

By using Lemma 3.1 and the estimate of Lemma 2.2, we deduce

$$\|\boldsymbol{X}_\Theta\|_{\mathcal{C}^{2,\alpha}} \le C\,\|d_\Theta\|_{\mathcal{C}^{2,\alpha}} \le C\,\|\Theta-\mathrm{Id}\|_{\mathcal{C}^{2,\alpha}}.$$

Again by Lemma 3.1

$$\|\boldsymbol{X}_\Theta(\Phi_\Theta(t))\|_{\mathcal{C}^{2,\alpha}} \le C\big(\|\Phi_\Theta(t)\|_{\mathcal{C}^{2,\alpha}}\big)\|\boldsymbol{X}_\Theta\|_{\mathcal{C}^{2,\alpha}}.$$

We then deduce the announced estimate on Φ_Θ from the latter estimates and the fact that Φ_Θ is the flow of $\boldsymbol{X}_\Theta$.

It remains to check (3.21). This is coming from the assumption that $\text{meas}(\Theta(\Omega_0)) = \text{meas}(\Omega_0)$. Indeed, this implies

$$0 = \int_0^L \{ \big(\gamma(s) + d_\Theta(s)\boldsymbol{n}(s)\big) \wedge \big(\gamma(s) + d_\Theta(s)\boldsymbol{n}(s)\big)' - \gamma(s) \wedge \gamma(s)' \}\, \mathrm{d}s$$

which gives (we drop the s-dependence)

$$0 = \int_0^L d_\Theta(n \ \wedge \gamma' + \gamma \wedge n') + d'_\Theta \gamma \wedge n + d^2_\Theta n \wedge n'$$

or after integrating the term in d' by parts and using the notation $a(s)$ introduced before:

$$0 = \int_0^L 2d_\Theta + d^2_\Theta a(s).$$

But, according to (3.13, 3.19), this is exactly (3.21).

Remark. About the extension of $\boldsymbol{X}$ outside $\mathcal{T}_{\tau_1}$: the fact that $\text{div}\boldsymbol{X} = 0$ on $\mathcal{T}_{\tau_1}$ implies that, for **any** extension of $\boldsymbol{X}$ to $\mathbb{R}^2$, $t \to \text{meas}\big(\Phi(t, \Omega_0)\big)$ is linear in t since the second derivative vanishes. As $\Phi(1)$ is a diffeomorphism and $\Phi(1, \partial\Omega_0) = \Theta(\partial\Omega_0)$, then $\Phi(1, \Omega_0) = \Theta(\Omega_0)$. Since $\text{meas}\big(\Theta(\Omega_0)\big) = \text{meas}(\Omega_0)$, then $t \to \text{meas}\big(\Phi(t, \Omega_0)\big)$ is constant. Therefore, **any** extension of $\boldsymbol{X}$ would be convenient for our purpose. This remark is actually to be used in the general case when $\partial\Omega_0$ is a finite number of Jordan's curves and $\mathcal{T}_{\tau_1}$ a neighborhood of $\partial\Omega_0$. Then, if Ω_0 is for instance not connected, we would not be able to extend $\boldsymbol{X}$ as a free-divergence vector field on $\mathbb{R}^2$.

3.3. Some more estimates

With the notations of the previous section, we have:

Proposition 3.2. *Under the assumption of Proposition 2.3, there exists a constant C depending only on the data such that for all $\Theta \in \mathcal{V}(\eta_2)$ and all $t \in [0,1]$*

$$\|m_\Theta(\Phi_\Theta(t)) - m_\Theta\|_{L^2(\partial\Omega_0)} \le C\|m_\Theta\|_{L^2(\partial\Omega_0)}\|\Theta - \text{Id}\|_{2,\alpha}. \tag{3.22}$$

$$\|m_\Theta(\Phi_\Theta(t)) - m_\Theta\|_{H^{1/2}(\partial\Omega_0)} \le C\|m_\Theta\|_{H^{1/2}(\partial\Omega_0)}\|\Theta - \text{Id}\|_{2,\alpha}. \tag{3.23}$$

Proof. Recall (see 3.13, 3.20) that, in terms of local coordinates and with the previous notations

$$m_\Theta(\Phi_\Theta(t, \gamma(s))) - m_\Theta(\gamma(s)) = m(s, \tau(s,t)) - m(s,0) = f(s)[(1 + \tau(s,t)a(s))^{-1} - 1], \tag{3.24}$$

$$f(s) = m(s,0) = m_\Theta(\gamma(s)) = -\big[\frac{1}{2}a(s)d_\Theta(s)^2 + d_\Theta(s)\big]. \tag{3.25}$$

As a consequence

$$\|m_\Theta(\Phi_\Theta(t,.)) - m_\Theta(.)\|_{L^2(\partial\Omega_0)} \le \|m_\Theta(.)\|_{L^2(\partial\Omega_0)} \sup_{s\in[0,L]} |(1 + \tau(s,t)a(s))^{-1} - 1)|.$$

We know from (3.10) that $1 + \tau(s,t)a(s)$ is bounded from below by c_{τ_2} on $\{(s,t) \in [0,L] \times [0,1]\}$. Therefore, since $(1 + \tau a)^{-1} - 1 = -\tau a/(1 + \tau a)$

$$\sup_{s\in[0,L]} |(1 + \tau(s,t)a(s))^{-1} - 1)| \le c_{\tau_2}^{-1}\|a\|_\infty \sup_{(s,t)\in[0,L]\times[0,1]} |\tau(s,t)|.$$

Using (3.20), we bound τ from above by

$$|\tau(s,t)| \leq |a(s)|d_\Theta(s)^2 + 2|d_\Theta(s)| \leq C\|d_\Theta\|_\infty \leq C\|\Theta - \mathrm{Id}\|_{2,\alpha},$$

the latter inequality coming from (2.9). This proves the first estimate of Proposition 3.2. For the second one, we use the following lemma:

Lemma 3.3. *Let $v \in H^{1/2}(\partial\Omega_0), w \in \mathcal{C}^1(\partial\Omega_0)$. Then, $v\,w$ belongs to $H^{1/2}(\partial\Omega_0)$ and*

$$\|v\,w\|_{H^{1/2}(\partial\Omega_0)} \leq C\|v\|_{H^{1/2}(\partial\Omega_0)}\|w\|_{\mathcal{C}^1(\partial\Omega_0)} \tag{3.26}$$

for some constant C depending only on Ω_0.

We postpone the proof of this lemma and continue the proof of the proposition. From this lemma and the expression (3.24), we obtain:

$$\|m_\Theta(\Phi_\Theta(t,.)) - m_\Theta(.)\|_{H^{1/2}(\partial\Omega_0)} \leq \|m_\Theta(.)\|_{H^{1/2}(\partial\Omega_0)}\|(1+\tau(.,t)a(.))^{-1} - 1)\|_{\mathcal{C}^1([0,L])}.$$

By differentiation with respect to s, we see that

$$\|\partial_s(1+\tau(.,t)a(.))^{-1}\|_{L^\infty([0,L])} \leq c_{\tau_2}^2\|\partial_s(a(.)\tau(.,t))\|_\infty.$$

Using the expression of τ in (3.20), we obtain the existence of C such that

$$\|\partial_s(a(.)\tau(.,t))\|_\infty \leq C\|d_\Theta\|_{\mathcal{C}^1} \leq C\|\Theta - \mathrm{Id}\|_{2,\alpha}.$$

This finishes the proof of the proposition.

Proof of Lemma 3.3. Let V be an harmonic extension of v to Ω_0 and W a $\mathcal{C}^1$-extension of w to Ω_0 so that, for some constant C depending only on Ω_0,

$$\|V\|_{H^1(\Omega_0)} \leq C\|v\|_{H^{1/2}(\partial\Omega_0)}, \|W\|_{\mathcal{C}^1(\Omega_0)} \leq C\|w\|_{\mathcal{C}^1(\partial\Omega_0)}.$$

We then have

$$\|v\,w\|_{H^{1/2}(\partial\Omega_0)} \leq C\,\|V\,W\|_{H^1(\Omega_0)}.$$

We now use

$$\|V\,W\|_{L^2(\Omega_0)} \leq \|V\|_{L^2(\Omega_0)}\|W\|_{L^\infty(\Omega_0)},$$

$$\|\nabla V\,W\|_{L^2(\Omega_0)} \leq \|\nabla V\|_{L^2(\Omega_0)}\|W\|_{L^\infty(\Omega_0)},\ \|V\,\nabla W\|_{L^2(\Omega_0)} \leq \|V\|_{L^2(\Omega_0)}\|\nabla W\|_{L^\infty(\Omega_0)}$$

and the estimate (3.26) follows.

We finish this section by stating some more geometric estimates.

Proposition 3.4. *There is a constant $C > 0$ such that for all $\Theta \in \mathcal{V}(\eta_2)$ and all $t \in [0,1]$:*

1. $\|D\Phi_\Theta(t) - \mathrm{Id}_{\mathbb{R}^2}\|_{L^\infty} + \|D^2\Phi_\Theta(t)\|_{L^\infty} \leq C\,\|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}$
2. $\|D\Phi_\Theta(t)^{-1} - \mathrm{Id}_{\mathbb{R}^2}\|_{L^\infty} + \|D\big[D\Phi_\Theta(t)^{-1}\big]\|_{L^\infty} \leq C\,\|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}$
3. *if $\boldsymbol{n}_t$ denotes the unit normal field to Ω_t,*

$$\|\boldsymbol{n}_t(\Phi_\Theta(t)) - \boldsymbol{n}\|_{\mathcal{C}^1(\partial\Omega_0)} \leq C\,\|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha} \tag{3.27}$$

4. *if* $J(t) = \det D\Phi_\Theta(t)\, |{}^t D\Phi_\Theta(t)^{-1}\boldsymbol{n}|$ *then*

$$\|J(t) - 1\|_{L^\infty(\partial\Omega_0)} \leq C\, \|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}. \tag{3.28}$$

Proof of Proposition 3.4. Note first that, thanks to the estimate (2.11), it is sufficient to bound each of the four expressions in the Proposition by $C\, \|\Phi_\Theta(t) - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}$.

The first estimate comes from the definition of the $\mathcal{C}^{2,\alpha}$-norm applied to $\Phi_\Theta(t) - \mathrm{Id}_{\mathbb{R}^2}$.

For the second one, use that for all $x \in \mathbb{R}^2$

$$D\Phi_\Theta(t)^{-1}(x) = \mathrm{Id}_{\mathbb{R}^2} + \sum_{n=1}^{\infty} [\mathrm{Id}_{\mathbb{R}^2} - D\Phi_\Theta(t)(x)]^n. \tag{3.29}$$

It follows that (up to still reducing η_2)

$$\|D\Phi_\Theta(t)^{-1}(x) - \mathrm{Id}_{\mathbb{R}^2}\| \leq \frac{\|D\Phi_\Theta(t)(x) - \mathrm{Id}_{\mathbb{R}^2}\|}{1 - \|D\Phi_\Theta(t)(x) - \mathrm{Id}_{\mathbb{R}^2}\|} \leq C \|D\Phi_\Theta(t)(x) - \mathrm{Id}_{\mathbb{R}^2}\|.$$

For the estimate on the second derivative, we differentiate (3.29) and bound norms from above similarly.

For the third estimate, let us denote

$$z(s) := \partial_s[\Phi_\Theta(t, \gamma(s))] = D\Phi_\Theta(t, \gamma(s))\gamma'(s).$$

Since $\boldsymbol{n}(s)$ and $\boldsymbol{n}_t\big(\Phi_\Theta(t, \gamma(s))\big)$ are respectively deduced from $\gamma'(s)$ and $z(s)/|z(s)|$ by a rotation of angle $-\pi/2$, we have

$$\|\boldsymbol{n}_t(\Phi_\Theta(t)) - \boldsymbol{n}\|_{\mathcal{C}^1(\partial\Omega_0)} = \|\frac{z}{|z|} - \gamma'\|_{\mathcal{C}^1(\partial\Omega_t)}.$$

We will estimate the right-hand side. We have

$$||z(s)| - 1| = ||z(s)| - |\gamma'(s)|| \leq |z(s) - \gamma'(s)| \leq \|D\Phi_\Theta(t, \gamma(s)) - \mathrm{Id}\| \leq \|\Phi_\Theta(t) - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}.$$

If η_2 is chosen small enough – and we assume it here – this implies

$$||z(s)| - 1| \leq C\, \|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha} \leq 1/2, |z(s)| \geq 1/2, ||z(s)|^{-1} - 1| \leq C\, \|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}$$

$$\Big|\frac{z(s)}{|z(s)|} - \gamma'(s)\Big| \leq |z(s)|\big||z(s)|^{-1} - 1\big| + |z(s) - \gamma'(s)| \leq C\, \|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha},$$

whence the L^∞-estimate of $z|z|^{-1} - \gamma'$. For the L^∞-estimate of its derivative, note first that

$$z'(s) = D^2\Phi_\Theta(t, \gamma(s))(\gamma'(s), \gamma'(s)) + D\Phi_\Theta(t, \gamma(s))\gamma''(s)$$

so that

$$|z'(s) - \gamma''(s)| \leq \|D^2\Phi_\Theta(t)\|_\infty + \|D\Phi_\Theta(t) - \mathrm{Id}\||\gamma''(s)| \leq C\, \|\Phi_\Theta - \mathrm{Id}\|_{2,\alpha}.$$

Now, we use

$$\partial_s(z(s)|z(s)|^{-1}) = z'(s)|z(s)|^{-1} - z(s)|z(s)|^{-3}\langle z(s), z'(s)\rangle.$$

We treat this expression as before, using in particular the estimate of $||z(s)|^{-1} - 1|$, the only new term being $\langle z(s), z'(s)\rangle$ which we estimate as follows:

$$|\langle z, z'\rangle| = |\langle z, z'\rangle - \langle \gamma', \gamma''\rangle| \leq |\langle z - \gamma', z'\rangle + \langle \gamma', z' - \gamma''\rangle|$$

and, as shown before, we obtain a bound from above by $C\,\|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}$.

Finally, the variation of $J(t)$ in (3.28) is then easily estimated.

4. Properties of u_{Ω_t}

Here again we fix $\Theta \in \mathcal{V}(\eta_2)$. We consider the solution u_{Ω_t} of the following Dirichlet problem on the moving domain $\Omega_t = \Phi_\Theta(t, \Omega_0)$

$$\begin{cases} -\Delta u_{\Omega_t} = k & \text{in } \Omega_t, \\ u_{\Omega_t} = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{4.1}$$

We will often write $u(t)$ for u_{Ω_t}. We will now make estimates on the "transported" solution $\tilde{u}(t)$ defined on the fixed domain Ω_0 by

$$\tilde{u}(t) = u_{\Omega_t} \circ \Phi_\Theta(t) \;\; (= u(t) \circ \Phi_\Theta(t)).$$

Then $\tilde{u}(t)$ is solution of a new problem on the fixed domain, namely

$$-L(\Theta, t)\,\tilde{u}(t) = k \circ \Phi_\Theta(t) \;\text{ on }\; \Omega_0, \tilde{u}(t) = 0 \;\text{ on }\; \partial\Omega_0, \tag{4.2}$$

where $L(\Theta, t)$ is the differential operator explicitly given by:

$$\begin{aligned} L(\Theta, t) &= a_{i,j}(t) D_{i,j} + b_i(t) D_i & (4.3)\\ &= \big[\partial_1 \Psi_t^i \partial_1 \Psi_t^j + \partial_2 \Psi_t^i \partial_2 \Psi_t^j\big] D_{i,j} + \big[\partial^2_{1,1} \Psi_t^i + \partial^2_{2,2} \Psi_t^i\big] D_i' & (4.4) \end{aligned}$$

where $\Psi_t = \Phi_\Theta(t)^{-1}$. We then have the following main estimate.

Proposition 4.1. *There exists a function $\omega : [0,1] \to \mathbb{R}$ with $\omega(0^+) = \lim_{r\downarrow 0} \omega(r) = 0$ ("modulus of continuity") such that:*

$$\sup_{t\in[0,1]} \|\tilde{u}(t) - u_{\Omega_0}\|_{\mathcal{C}^2(\bar{\Omega}_0)} \leq \omega\big(\|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}\big). \tag{4.5}$$

Remark. The modulus of continuity ω depends on the regularity of the right-hand side k in (4.1). If we assume k of class $\mathcal{C}^2$, then one can choose $\omega(\eta) = C\,\eta$ and even have the $\mathcal{C}^{2,\alpha}$ norm instead of the $\mathcal{C}^2$ norm. More comments and indications of the proof of this remark may be found after the proof of the proposition.

Proof of Proposition 4.1. The main idea is here to use Schauder's estimates in Hölder spaces for the solution of the Dirichlet problem governed by an uniformly elliptic operator with Hölder continuous coefficients. Indeed, we have:

Lemma 4.2. *Assume η_2 is small enough. Then, there is a constant U depending only on the data and on the $\mathcal{C}^{2,\alpha}$-norm of Θ such that for all $t \in [0,1]$*

$$\|\tilde{u}(t)\|_{2,\alpha,\Omega_0} \leq U. \tag{4.6}$$

Proof of Lemma 4.2. Let us first remark that the operators $L(\Theta, t)$ are uniformly elliptic. Indeed, the part of second order of $L(\Theta, t)$ can be written matricially as

$$A(\Theta, t) = \begin{pmatrix} \partial_1 \Psi_t^1 & \partial_2 \Psi_t^1 \\ \partial_1 \Psi_t^2 & \partial_2 \Psi_t^2 \end{pmatrix} \begin{pmatrix} \partial_1 \Psi_t^1 & \partial_1 \Psi_t^2 \\ \partial_2 \Psi_t^1 & \partial_2 \Psi_t^2 \end{pmatrix} = D\Psi_t \, {}^t D\Psi_t.$$

This proves that $A(\Theta, t)$ is a symmetric nonnegative matrix. Since $\det D\Psi_t = 1 \neq 0$, it is positive definite. The smallest eigenvalue being a continuous function of t, it is uniformly bounded from below on the compact interval $[0,1]$. Actually the bound depends only on the $\mathcal{C}^{2,\alpha}$-norm of Θ if η_2 is small enough since

$$\begin{aligned} \|D\Psi_t \, {}^t D\Psi_t - \mathrm{Id}\| &\leq \|D\Psi_t \, {}^t D\Psi_t - D\Psi_t\| + \|D\Psi_t - \mathrm{Id}\| \\ &\leq \|\Psi_t - \mathrm{Id}\|_{2,\alpha} \leq C \, \|\Theta - \mathrm{Id}\|_{2,\alpha} \leq C \, \eta_2. \end{aligned}$$

Now, since $\Psi_t = \Phi_\Theta(t)^{-1}$ is of class $\mathcal{C}^{2,\alpha}$, the coefficients of $L(\Theta, t)$ are $\mathcal{C}^{0,\alpha}$ with a norm depending only on the $\mathcal{C}^{2,\alpha}$-norm of $\Phi_\Theta(t)^{-1}$, that is also on the $\mathcal{C}^{2,\alpha}$-norm of Θ. From classical Schauder's estimates applied to the equation (4.2) (see for example [10]), there is constant C depending only on the data and on the $\mathcal{C}^{2,\alpha}$ -norm of Θ such that

$$\|\tilde{u}(t)\|_{2,\alpha,\Omega_0} \leq C \, \|k \circ \Phi_\Theta(t)\|_{0,\alpha,\Omega_0}.$$

Since $k \in \mathcal{C}^{0,\alpha}$ and $\Phi_\Theta(t) \in \mathcal{C}^{2,\alpha}$, then $k \circ \Phi_\Theta(t) \in \mathcal{C}^{0,\alpha}$ and the Lemma 4.2 is proved.

We can now finish the proof of Proposition 4.1. We define ω as

$$\forall \eta \in]0, \eta_2[, \;\; \omega(\eta) := \sup_{\Theta \in \mathcal{V}(\eta), t \in [0,1]} \|\tilde{u}(t) - u_{\Omega_0}\|_{\mathcal{C}^2(\bar{\Omega}_0)}.$$

Lemma 4.2 guarantees that this quantity is well defined, and we just need to prove that $\omega(0^+) = 0$. This follows from the compact embedding of $\mathcal{C}^{2,\alpha}(\Omega_0)$ into $\mathcal{C}^2(\bar{\Omega}_0)$. Indeed, if $\omega(0^+) \neq 0$, there are $a > 0$ and sequences $t_n \in [0,1], \Theta_n \in \mathcal{V}(\eta_2), \Omega_n = \Phi_{\Theta_n}(t_n, \Omega_0), \tilde{u}_n = u_{\Omega_n} \circ \Phi_{\Theta_n}(t_n)$ such that

$$\|\Theta_n - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha} \leq 1/n, \tag{4.7}$$

$$\|\tilde{u}_n - u_{\Omega_0}\|_{\mathcal{C}^2(\bar{\Omega}_0)} \geq a > 0. \tag{4.8}$$

The sequence $\tilde{u}_n$ is bounded in $\mathcal{C}^{2,\alpha}(\bar{\Omega}_0)$ and by compactness, there is a subsequence converging in $\mathcal{C}^2(\bar{\Omega}_0)$ to some $u_{lim} \in \mathcal{C}^2(\bar{\Omega}_0)$ and a subsequence of t_n converging to some $t_{lim} \in [0,1]$. We then remark that

$$L(\Theta_n, t_n)\tilde{u}_n - \Delta u_{\mathrm{lim}} = \Big[L(\Theta_n, t_n) - \Delta\Big]\tilde{u}_n + \Delta\Big[\tilde{u}_n - u_{\mathrm{lim}}\Big].$$

Passing to the limit as $n \to \infty$ shows that u_{lim} is solution of the equation

$$\begin{aligned} \Delta u &= k \quad \text{in } \Omega_0, \\ u &= 0 \quad \text{on } \partial\Omega_0. \end{aligned}$$

This is in contradiction with (4.8) as the Dirichlet problem has a unique solution in $\mathcal{C}^2(\bar{\Omega}_0)$ namely u_{Ω_0}. This proves that $\omega(0^+) = 0$.

Remark. As a consequence of Proposition 4.1, $\tilde{u}(t)$ is continuous into $\mathcal{C}^2(\bar{\Omega}_0)$ at $t_0 = 0$. Since $t_0 = 0$ is not particular for problem (4.1), the same is true for $t \to u(t) \circ \Phi_\Theta(t) \circ \Phi_\Theta(t_0)^{-1}$ at $t = t_0 \in [0,1]$ and, by composition:

$$t \in [0,1] \to \tilde{u}(t) \in \mathcal{C}^2(\bar{\Omega}_0) \;\text{ is continuous.} \tag{4.9}$$

Remark. Another approach to estimate a norm of the difference $\tilde{u}(t) - u_{\Omega_0}$ would be the following. For simplicity, we set $u_0 := u_{\Omega_0}$ and $\tilde{k}(t) := k \circ \Phi_\Theta(t)$. We have from (4.2) and from (4.1) with $t = 0$

$$-\Delta(\tilde{u}(t) - u_0) = \tilde{k}(t) - k + [L(\Theta, t) - \Delta](\tilde{u}(t)). \tag{4.10}$$

Since $\tilde{u}(t) - u_0 = 0$ on $\partial\Omega_0$, we can apply the Schauder's estimates again to (4.10)

$$\|\tilde{u}(t) - u_0\|_{2,\alpha,\Omega_0} \leq C \, \|\tilde{k}(t) - k\|_{0,\alpha,\Omega_0} + \|[L(\Theta, t) - \Delta](\tilde{u}(t))\|_{0,\alpha,\Omega_0}.$$

Thanks to the $\mathcal{C}^{2,\alpha}$ estimate of Lemma 4.2, the $\mathcal{C}^{0,\alpha}$ norm of the second term of the right-hand side is easily estimated by $\|\Phi_\Theta(t) - \mathrm{Id}\|_{2,\alpha}$. But, even if $\tilde{k}(t) - k$ is indeed in $\mathcal{C}^{0,\alpha}$, its $\mathcal{C}^{0,\alpha}$-norm cannot be estimated in terms of $\|\Phi_\Theta(t) - \mathrm{Id}\|_{2,\alpha}$.

What one can at least say is that:

$$\|\tilde{k}(t) - k\|_\infty \leq \|k\|_{0,\alpha} \, \|\Phi_\Theta(t) - \mathrm{Id}\|_\infty^\alpha.$$

By interpolation, for $\epsilon \in (0, \alpha)$, we get an estimate of $\|\tilde{k}(t) - k\|_{0,\epsilon}$ with a modulus of continuity in $\eta^{\alpha-\epsilon}$. This yields (4.1) and even a $\mathcal{C}^{2,\epsilon}$-estimate with an explicit modulus of continuity. One can also check that, if k is of class $\mathcal{C}^2$, then, the $\mathcal{C}^{0,\alpha}$-norm (and even the Lipschitz-norm) of $\tilde{k}(t) - k$ can be estimated by $C \, \|\Phi_\Theta(t) - \mathrm{Id}\|_{2,\alpha}$. Then one obtains a $\mathcal{C}^{2,\alpha}$-estimate of $\tilde{u}(t) - u_{\Omega_0}$.

5. Proof of the theorems

Again, we fix Θ a diffeomorphism in $\mathcal{V}(\eta_2)$ such that $\mathrm{meas}\big(\Theta(\Omega_0)\big) = \mathrm{meas}(\Omega_0)$. As in the previous sections, we consider the function m_Θ, the vector field $\boldsymbol{X}_\Theta = m_\Theta \, \boldsymbol{n}$, Φ_Θ, the domains $\Omega_t = \Phi_\Theta(t, \Omega_0)$, the solutions $u(t) = u_{\Omega_t}$, $\tilde{u}(t) = u(t) \circ \Phi_\Theta(t)$. As announced in the introduction, we study the second derivatives of $t \in [0, t] \to E(\Omega_t) + \Lambda \, \mathrm{meas}(\Omega_t)$. Since, $\forall t \in [0, 1]$, $\mathrm{meas}(\Omega_t) = \mathrm{meas}(\Omega_0)$, it coincides with the second derivative of $t \to E(\Omega_t)$. We denote

$$e_\Theta(t) := E(\Omega_t) = -\frac{\mathrm{d}}{\mathrm{d}t} \|u(t, .)\|^2_{H_0^1(\Omega_t)}.$$

To compute the second derivative, we use the following classical lemma [6, 16, 21].

Lemma 5.1. *Let $H : [0, 1] \times \mathbb{R}^2 \mapsto \mathbb{R}$ be such that $H, \partial_t H, \nabla H \in \mathcal{C}([0, 1]; L^1(\mathbb{R}^2))$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} H(t, x) \mathrm{d}x = \int_{\Omega_t} \big[\partial_t H(t, x) + \mathrm{div}\{H(t, x) \boldsymbol{X}_\Theta(x)\}\big] \mathrm{d}x. \tag{5.1}$$

Remark. As usual in shape differentiation, this lemma may be extended to functions defined only on Ω_t. If for instance (this will be enough for us here),

$$H(t) \circ \Phi_\Theta(t) \in \mathcal{C}^1([0, 1]; \mathcal{C}^0(\bar{\Omega}_0)) \cap \mathcal{C}^0([0, 1]; \mathcal{C}^1(\bar{\Omega}_0)), \tag{5.2}$$

then we may define $\partial_t H(t) \in \mathcal{C}^0(\bar{\Omega}_t)$ in such a way that (5.1) remains valid. This may be done through a given linear continuous extension operator $\mathcal{P}$ from $\mathcal{C}^k(\bar{\Omega}_0)$ into $\mathcal{C}^k(\mathbb{R}^2)$, $k = 0, 1, 2$. We set

$$\bar{H}(t) := \mathcal{P}\big(H(t) \circ \Phi_\Theta(t)\big) \circ \Phi_\Theta^{-1}(t), \tag{5.3}$$

and, by definition $\partial_t H(t) := \partial_t \bar{H}(t)$ on $\bar{\Omega}_t$ (note that $H(t)$ and $\bar{H}(t)$ also coincide on $\bar{\Omega}_t$). Applying Lemma 5.1 as stated above to $\bar{H}$ yields the formula (5.1) for H.

Here, $\tilde{u}(t) = u(t) \circ \Phi_\Theta(t)$ is continuous into $\mathcal{C}^2(\bar{\Omega}_0)$ (see (4.9)) so that its extension $\bar{u}$ is well-defined. As we will see below, its derivative $\partial_t u (= \partial_t \bar{u})$ is well-defined and

$$\partial_t u \circ \Phi_\Theta(t) \in \mathcal{C}^0(0,1; \mathcal{C}^1(\bar{\Omega}_0)). \tag{5.4}$$

We will use this in the next computations and we will often write u for $\bar{u}$.

By differentiating formally (4.1) with respect to t, we see that $\partial_t u(t)$ should be solution of

$$\begin{cases} -\Delta(\partial_t u(t)) = 0 & \text{in } \Omega_t, \\ \partial_t u(t, \Phi_\Theta(t)) + \langle \nabla u(t, \Phi_\Theta(t)), \boldsymbol{X}_\Theta(\Phi_\Theta(t))\rangle = 0 & \text{on } \partial\Omega_0 \\ \text{or also } \partial_t u(t) + \langle \nabla u(t), \boldsymbol{X}_\Theta \rangle = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{5.5}$$

Since $\nabla u(t) \in \mathcal{C}^{1,\alpha}(\partial\Omega_t)$, classical regularity results from Schauder theory (see *e.g.* [10]) ensure the existence of a $\mathcal{C}^{1,\alpha}$-solution to this equation with a $\mathcal{C}^{1,\alpha}$-norm uniformly bounded for $t \in [0,1]$. Moreover, its composition with $\Phi_\Theta(t)$ is continuous with values into $\mathcal{C}^1(\bar{\Omega}_0)$ since it is the case of $t \to \nabla u(t) \circ \Phi_\Theta(t)$. We easily deduce that this solution is not hing but $\partial_t u$ and the formal computation is justified as well as (5.4).

Derivation of the magnetic energy

We apply Lemma 5.1 to $H = |\nabla u|^2 = |\nabla \bar{u}|^2$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,.)\|^2_{H^1_0(\Omega_t)} = 2\int_{\Omega_t} \langle \nabla u(t,.), \nabla \partial_t u(t,.)\rangle \mathrm{d}x + \int_{\Omega_t} \operatorname{div}\big[|\nabla u(t,.)|^2 \boldsymbol{X}_\Theta\big]\mathrm{d}x.$$

The first term of this sum vanishes since by Green's formula, we have:

$$\int_{\Omega_t} \langle \nabla u(t,.), \nabla \partial_t u(t,.)\rangle \mathrm{d}x = -\int_{\Omega_t} u(t,.)\Delta_x \partial_t u(t,.)\mathrm{d}x + \int_{\partial\Omega_t} u(t,.)\langle \nabla \partial_t u(t,.), \boldsymbol{n}_t\rangle \mathrm{d}\sigma.$$

This is equal to zero since $u(t) = 0$ on $\partial\Omega_t$ and $\Delta\partial_t u = 0$ on Ω_t; (recall that we denote by $\boldsymbol{n}_t$ the unit normal derivative to Ω_t). So we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t,.)\|^2_{H^1_0(\Omega_t)} = \int_{\Omega_t} \operatorname{div}\big[|\nabla u(t,.)|^2 \boldsymbol{X}_\Theta\big]\mathrm{d}x. \tag{5.6}$$

As a first consequence, using Green's formula, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}_{t=0}[\Lambda \operatorname{meas}(\Omega_t) + E(\Omega_t)] = \int_{\partial\Omega_0} [\Lambda - |\nabla u_{\Omega_0}|^2]\langle \boldsymbol{X}_\Theta, \boldsymbol{n}\rangle \mathrm{d}\sigma.$$

Since Ω_0 is a critical shape, we see on this formula (valid for any vector field in place of $\boldsymbol{X}_\Theta$) that $\Lambda = |\nabla u_{\Omega_0}|^2$ on $\partial\Omega_0$ as announced in (2.4).

Next, applying Lemma 5.1 (at least formally, see below) to the expression (5.6) leads to the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|u(t,.)\|^2_{H^1_0(\Omega_t)} = \int_{\Omega_t} \operatorname{div}\big[\partial_t |\nabla u(t,.)|^2 \boldsymbol{X}_\Theta\big]\mathrm{d}x + \int_{\Omega_t} \operatorname{div}\{\operatorname{div}\big[|\nabla u(t,.)|^2 \boldsymbol{X}_\Theta\big]\boldsymbol{X}_\Theta\}\mathrm{d}x. \tag{5.7}$$

As such, these expressions may not be defined, but we integrate by parts and we set

$$\begin{aligned} e''_\Theta(t) &= -2B(t) - A(t), \\ B(t) &= \int_{\partial\Omega_t} \langle \nabla \partial_t u(t), \nabla u(t)\rangle \langle \boldsymbol{X}_\Theta, \boldsymbol{n}_t\rangle \mathrm{d}\sigma, \\ A(t) &= \int_{\partial\Omega_t} \langle \nabla |\nabla u(t)|^2, \boldsymbol{X}_\Theta\rangle \langle \boldsymbol{X}_\Theta, \boldsymbol{n}_t\rangle \mathrm{d}\sigma, \end{aligned} \tag{5.8}$$

where we used that div $\boldsymbol{X}_\Theta = 0$ to simplify the expression of $A(t)$. To justify this computation, we may apply Lemma 5.1 to $H_n(t) = \mathrm{div}\big[|\nabla U_n(t)|^2 \boldsymbol{X}_\Theta\big]$ where U_n are regularized approximations by convolution in $\mathbb{R}^N$ of $\bar u$. Because of the regularity of $\bar u$ at the boundary, the convergence under the integrals holds uniformly,

The study of $A(t)$

We first change variable in the integral on the moving boundary to get an integral on the fixed boundary. The Jacobian we need is estimated in (3.28) of Proposition 3.4. We use also that

$$\nabla u(t) \circ \ \Phi_\Theta(t) = ^t \big[D\Phi_\Theta(t)\big]^{-1} \nabla \tilde u(t)$$

and the similar formula with $|\nabla u(t)|^2$ in place of $u(t)$.

We will simply write Φ_t for $\Phi_\Theta(t)$, $\boldsymbol{X}$ for $\boldsymbol{X}_\Theta$, m for m_Θ, $\tilde{\boldsymbol{X}}(t)$ for $\boldsymbol{X}_\Theta \circ \Phi_\Theta(t)$, $\tilde m(t)$ for $m_\Theta \circ \Phi_\Theta(t)$ and $\tilde{\boldsymbol{n}}_t(t)$ for $\boldsymbol{n}_t \circ \Phi_\Theta(t)$. Note that by (2.10), (3.15), $\boldsymbol{n} \circ \Phi_\Theta(t) = \boldsymbol{n}$ on $\mathcal{T}(\eta_2)$. We obtain

$$\begin{aligned} A(t) &= \int_{\partial\Omega_t} \langle \nabla |\nabla u(t)|^2, \boldsymbol{X}\rangle \langle \boldsymbol{X}, \boldsymbol{n}_t\rangle \mathrm{d}\sigma, \\ &= \int_{\partial\Omega_0} \langle {}^tD\Phi_t^{-1} \nabla |{}^tD\Phi_t^{-1}\nabla \tilde u(t)|^2, \tilde{\boldsymbol{X}}(t)\rangle \langle \tilde{\boldsymbol{X}}(t), \tilde{\boldsymbol{n}}_t(t)\rangle J(t)\,\mathrm{d}\sigma, \\ &= \int_{\partial\Omega_0} \tilde m(t)^2 \langle {}^tD\Phi_t^{-1} \nabla |{}^tD\Phi_t^{-1}\nabla \tilde u(t)|^2, \boldsymbol{n}\rangle \langle \boldsymbol{n}, \tilde{\boldsymbol{n}}_t(t)\rangle J(t)\,\mathrm{d}\sigma. \end{aligned}$$

The goal is now to estimate $A(t) - A(0)$. We set

$$\begin{cases} a_0(t) := \tilde m(t), \;\; a_1(t) := \langle {}^tD\Phi_t^{-1} \nabla |{}^tD\Phi_t^{-1}\nabla \tilde u(t)|^2, \boldsymbol{n}\rangle \\ a_2(t) := \langle \boldsymbol{n}, \tilde{\boldsymbol{n}}_t(t)\rangle, \;\; a_3(t) := J(t). \end{cases}$$

We denote by C any constant depending only on the $\mathcal{C}^{2,\alpha}$-norm of Θ and we set $\eta := \|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}$. We have the following estimates for i in $\{2,3\}$ and for all t in $[0,1]$

$$\|a_i(t)\|_{L^\infty(\partial\Omega_0)} \le C, \;\; \|a_i(t) - a_i(0)\|_{L^\infty(\partial\Omega_0)} \le C\,\eta. \tag{5.9}$$

This is coming from (3.27) in Proposition 3.4. For $i = 1$, we will prove below the following for all t in $[0,1]$

$$\|a_1(t)\|_{L^\infty(\partial\Omega_0)} \le C, \qquad \|a_1(t) - a_1(0)\|_{L^\infty(\partial\Omega_0)} \le C\,\omega(\eta), \tag{5.10}$$

where ω is the modulus of continuity appearing in Proposition 4.1. Next, we also have for all t in $[0,1]$

$$\|a_0(t)\|_{L^2(\partial\Omega_0)} \le C\|m\|_{L^2(\partial\Omega_0)}, \qquad \|a_0(t) - a_0(0)\|_{L^2(\partial\Omega_0)} \le C\,\eta\,\|m\|_{L^2(\partial\Omega_0)}. \tag{5.11}$$

This is the content of the first part of Proposition 3.2. Now, we write

$$A(t) - A(0) = \int_{\partial\Omega_0} [a_0(t)^2 - a_0(0)^2] \prod_{i=1,2,3} a_i(t) + \int_{\partial\Omega_0} a_0(t)^2 [\prod_{i=1,2,3} a_i(t) - \prod_{i=1,2,3} a_i(0)]. \tag{5.12}$$

The second integral is bounded by $C\,\|m\|^2_{L^2(\partial\Omega_0)}\|\prod_{i=1,2,3} a_i(t) - \prod_{i=1,2,3} a_i(0)\|_\infty$. We easily check that

$$\|\prod_{i=1,2,3} a_i(t) - \prod_{i=1,2,3} a_i(0)\|_\infty \leq C \sum_{i=1,2,3} \|a_i(t) - a_i(0)\|_\infty \leq C\,(\omega(\eta) + \eta),$$

the last inequality coming from (5.9, 5.10). Next we will assume for simplicity that $\omega(\eta) \geq C\,\eta$.

The first integral in (5.12) is bounded above by

$$\|\prod_{i=1,2,3} a_i(t)\|_\infty \|a_0(t) - a_0(0)\|_{L^2(\partial\Omega_0)} \|a_0(t) + a_0(0)\|_{L^2(\partial\Omega_0)} \leq C\,\eta\,\|m\|^2_{L^2(\partial\Omega_0)},$$

where we used (5.11) for the last inequality. Finally, for all t in $[0,1]$, we get

$$|A(t) - A(0)| \leq C\,\omega(\eta)\|m\|^2_{L^2(\partial\Omega_0)}. \tag{5.13}$$

It remains to prove (5.10). The L^∞-bound on $a_1(t)$ is obvious from the $\mathcal{C}^2$-estimates of Lemma 4.2 on $\tilde{u}(t)$. For the L^∞-estimate on the difference, inserting $\nabla|{}^tD\Phi_t^{-1}\nabla\tilde{u}(t)|^2$, we obtain a first bound by

$$C\,\|{}^tD\Phi_t^{-1} - \mathrm{Id}\|_\infty + C\,\|\nabla|{}^tD\Phi_t^{-1}\nabla\tilde{u}(t)|^2 - \nabla|\nabla u(0)|^2\|_\infty.$$

Using the $\mathcal{C}^2$-estimates of Proposition 4.1 for the second term, we prove that this is bounded by $C\,\omega(\eta)$.

The study of $B(t)$

Recall that

$$B(t) = \int_{\partial\Omega_t} \langle\nabla\partial_t u(t), \nabla u(t)\rangle\langle \boldsymbol{X}_\Theta, \boldsymbol{n}_t\rangle \mathrm{d}\sigma.$$

As $u(t)$ is constant along $\partial\Omega_t$, its gradient is normal and therefore

$$\nabla u(t) = \langle\nabla u(t), \boldsymbol{n}_t\rangle\boldsymbol{n}_t.$$

If we denote $\partial_{n_t} u := \langle\nabla u(t), \boldsymbol{n}_t\rangle$, we deduce that

$$\langle\nabla\partial_t u(t), \nabla u(t)\rangle = \partial_{n_t} u\,\langle\nabla\partial_t u(t), \boldsymbol{n}_t\rangle. \tag{5.14}$$

Since also $\boldsymbol{X}_\Theta = m\,\boldsymbol{n}$, the boundary condition for $\partial_t u(t)$ may be rewritten (see (5.5))

$$\partial_t u(t) + m\,\partial_{n_t} u\langle\boldsymbol{n}_t, \boldsymbol{n}\rangle = 0 \;\text{ on }\; \partial\Omega_t. \tag{5.15}$$

Let us now introduce the Steklov-Poincaré operator C_t on Ω_t defined from $H^{1/2}(\partial\Omega_t)$ into $H^{-1/2}(\partial\Omega_t)$ as follows: if $z \in H^{1/2}(\partial\Omega_t)$, we consider the harmonic extension Z of z to Ω_t and we define $C_t(z) := \langle\nabla Z, \boldsymbol{n}_t\rangle$. For the properties of this operator, one can refer for example to [5]. A simple computation shows that,

$$\int_{\partial\Omega_t} z\,C_t(z) = \int_{\Omega_t} |\nabla Z|^2,$$

this, at least for regular enough functions z (in general, the first integral is to be replaced by $\langle z, C_t(z)\rangle_{H^{1/2}\times H^{-1/2}}$). Then, if we set $z(t) := -m\,\partial_{n_t} u\langle\boldsymbol{n}_t, \boldsymbol{n}\rangle$, we have by (5.5, 5.14, 5.15)

$$\langle\nabla\partial_t u(t), \boldsymbol{n}_t\rangle = C_t(z(t)),\;\; \langle\nabla\partial_t u(t), \nabla u(t)\rangle = \partial_{n_t} u\,C_t(z(t)).$$

This gives a new expression for $B(t)$:

$$B(t) = -\int_{\partial\Omega_t} z(t) C_t(z(t)) = -\int_{\Omega_t} |\nabla Z(t)|^2,$$

where $Z(t)$ is the harmonic extension of $z(t)$ to Ω_t. We also denote $\tilde{Z}(t) := Z(t) \circ \Phi_t$, $\tilde{z}(t) := z(t) \circ \Phi_t$. We then have to estimate

$$B(t) - B(0) = \int_{\Omega_0} |\nabla Z(0)|^2 - \int_{\Omega_0} |{}^t D\Phi_t^{-1} \nabla \tilde{Z}(t)|^2,$$

(recall that the Jacobian $\det D\Phi_t$ is here equal to 1). We denote again by C any constant depending only on the $\mathcal{C}^{2,\alpha}$-norm of Θ and we set again $\eta := \|\Theta - \mathrm{Id}_{\mathbb{R}^2}\|_{2,\alpha}$.

Lemma 5.2. *We have the following main estimates: for all $t \in [0,1]$*

$$\|\tilde{z}(t)\|_{H^{1/2}(\partial\Omega_0)} \leq C\,\|m\|_{H^{1/2}(\partial\Omega_0)},\ \ \|\tilde{z}(t) - z(0)\|_{H^{1/2}(\partial\Omega_0)} \leq C\,\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)}$$
$$\|\tilde{Z}(t)\|_{H^1(\Omega_0)} \leq C\,\|m\|_{H^{1/2}(\partial\Omega_0)},\ \ \|\tilde{Z}(t) - Z(0)\|_{H^1(\Omega_0)} \leq C\,\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)}.$$

Assuming this lemma, we obtain

$$\begin{aligned}
|B(t) - B(0)| &\leq \|{}^t D\Phi_t^{-1}\nabla\tilde{Z}(t) + \nabla Z(0)\|_{L^2} \|{}^t D\Phi_t^{-1}\nabla\tilde{Z}(t) - \nabla Z(0)\|_{L^2} \\
&\leq C\,\|m\|_{H^{1/2}(\partial\Omega_0)} \big[\|{}^t D\Phi_t^{-1}\nabla\tilde{Z}(t) -{}^t D\Phi_t^{-1}\nabla Z(0)\|_{L^2} + \|{}^t D\Phi_t^{-1}\nabla Z(0) - \nabla Z(0)\|_{L^2}\big] \\
&\leq C\,\|m\|_{H^{1/2}(\partial\Omega_0)} \big[\|\nabla\tilde{Z}(t) - \nabla Z(0)\|_{L^2} + C\,\eta\|\nabla Z(0)\|_{L^2}\big] \\
&\leq C\,\|m\|_{H^{1/2}(\partial\Omega_0)} [\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)} + \eta\|m\|_{H^{1/2}(\partial\Omega_0)}].
\end{aligned}$$

This together with (5.8, 5.13) finishes the proof of Theorem 2.5. It only remains to prove Lemma 5.2.

Proof of Lemma 5.2. Recall that

$$z(t) := -m\,\partial_{n_t} u \langle \boldsymbol{n}_t, \boldsymbol{n}\rangle,\ \ \tilde{z}(t) := z(t) \circ \Phi_t.$$

From Lemma 3.3, we have

$$\|\tilde{z}(t)\|_{H^{1/2}(\partial\Omega_0)} \leq \|m\|_{H^{1/2}(\partial\Omega_0)} \|(\partial_{n_t} u \circ \Phi_t)\langle \tilde{\boldsymbol{n}}_t, \boldsymbol{n}\rangle\|_{\mathcal{C}^1(\partial\Omega_0)}.$$

From Propositions 3.4 and 4.1, we easily deduce the first estimate on $\tilde{z}$ in Lemma 5.2. For the difference, we write

$$\tilde{z}(t) - z(0) = \tilde{z}(t) + m\langle \nabla u_0, \boldsymbol{n}\rangle = m\,[\langle \nabla u_0, \boldsymbol{n}\rangle - (\partial_{n_t} u \circ \Phi_t)\langle \tilde{\boldsymbol{n}}_t, \boldsymbol{n}\rangle].$$

Again, by Lemma 3.3, we have to estimate the $\mathcal{C}^1$-norm of $\langle \nabla u_0, \boldsymbol{n}\rangle - (\partial_{n_t} u \circ \Phi_t)\langle \tilde{\boldsymbol{n}}_t, \boldsymbol{n}\rangle$ which is bounded by the sum

$$\|\langle \nabla u_0, \boldsymbol{n}\rangle(1 - \langle \tilde{\boldsymbol{n}}_t, \boldsymbol{n}\rangle)\|_{\mathcal{C}^1} + \|\langle \tilde{\boldsymbol{n}}_t, \boldsymbol{n}\rangle[\langle \nabla u_0, \boldsymbol{n}\rangle - \partial_{n_t} u \circ \Phi_t]\|_{\mathcal{C}^1}.$$

The first term is estimated as expected by $C\,\eta$ thanks to (3.27). The second one depends on the $\mathcal{C}^1$-norm of $\langle \nabla u_0, \boldsymbol{n}\rangle - \partial_{n_t} u \circ \Phi_t = \langle \nabla u_0, \boldsymbol{n}\rangle - \langle {}^t D\Phi_t^{-1}\nabla\tilde{u}(t), \tilde{n}(t)\rangle$. We use Propositions 4.1 and 3.4 to estimate it by $\omega(\eta)$ and the part concerning $\tilde{z}$ in the Lemma is complete.

Since $Z(0)$ is harmonic on Ω_0 with value $z(0)$ at the boundary, we have

$$\|Z(0)\|_{H^1(\Omega_0)} \leq C\,\|z(0)\|_{H^{1/2}(\partial\Omega_0)} \leq C\,\|m\|_{H^{1/2}(\partial\Omega_0)}.$$

The estimate on the H^1-norm of $\tilde{Z}(t) - Z(0)$ starts with the equation

$$L(\Theta,t)(\tilde{Z}(t)) = 0 \ \text{ on } \ \Omega_0,$$

which we rewrite

$$-\Delta(\tilde{Z}(t) - Z(0)) = [L(\Theta,t) - \Delta](\tilde{Z}(t)) \ \text{ on } \ \Omega_0.$$

This implies that

$$\|\tilde{Z}(t) - Z(0)\|_{H^1(\Omega_0)} \leq C\,[\|\tilde{z}(t) - z(0)\|_{H^{1/2}(\partial\Omega_0)} + \|[L(\Theta,t) - \Delta](\tilde{Z}(t))\|_{H^{-1}(\Omega_0)}. \tag{5.16}$$

The first term has just been estimated by $C\,\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)}$. As we check below, for all $w \in H^1(\Omega_0)$, we have

$$\|[L(\Theta,t) - \Delta](w)\|_{H^{-1}(\Omega_0)} \leq C\,\eta\,\|w\|_{H^1(\Omega_0)}. \tag{5.17}$$

We now decompose

$$[L(\Theta,t) - \Delta](\tilde{Z}(t)) = [L(\Theta,t) - \Delta][\tilde{Z}(t) - Z(0)] + [L(\Theta,t) - \Delta](Z(0)),$$

to obtain

$$\|[L(\Theta,t) - \Delta](\tilde{Z}(t))\|_{H^{-1}(\Omega_0)} \leq C\,\eta\|\tilde{Z}(t) - Z(0)\|_{H^1(\Omega_0)} + C\,\eta\,\|m\|_{H^{1/2}(\partial\Omega_0)}. \tag{5.18}$$

Together with (5.16), this implies

$$\|\tilde{Z}(t) - Z(0)\|_{H^1(\Omega_0)}(1 - C\,\eta) \leq C\,\omega(\eta)\|m\|_{H^{1/2}(\partial\Omega_0)}$$

from which we deduce the last estimate of Lemma 5.2.

It remains to prove (5.17). We do it by duality as follows. Let $\psi \in \mathcal{C}_0^\infty$, then

$$\int_{\Omega_0} [L(\Theta,t) - \Delta]w\ \psi = \int_{\Omega_0} \big[\sum_{i,j}(a_{i,j}(t) - a_{i,j}(0))D_{i,j}w + \sum_i (b_i(t) - b_i(0))D_i w\big]\ \psi,$$

where $a_{i,j}, b_i$ are the coefficients of $L(\Theta,t)$ (see 4.3). Now

$$\begin{aligned}
\Big|\int_{\Omega_0} (a_{i,j}(t) - a_{i,j}(0))D_{i,j}w\ \psi\Big| &= \Big|\int_{\Omega_0} [D_j(a_{i,j}(t) - a_{i,j}(0))\,\psi + (a_{i,j}(t) - a_{i,j}(0))D_j\psi]D_i w\Big|,\\
&\leq C\,\|a_{i,j}(t) - a_{i,j}(0)\|_{\mathcal{C}^1}\|\psi\|_{H_0^1}\|w\|_{H^1},\\
&\leq C\,\eta\,\|\psi\|_{H_0^1}\|w\|_{H^1},\\
\Big|\int_{\Omega_0} (b_i(t) - b_i(0))D_i w\ \psi\Big| &\leq C\,\|b_i(t) - b_i(0)\|_{L^\infty}\|\psi\|_{L^2}\|w\|_{H^1},\\
&\leq C\,\eta\,\|\psi\|_{L^2}\|w\|_{H^1}.
\end{aligned}$$

The estimate (5.17) follows.

Proof of Theorem 2.1. Let $\Theta \in \mathcal{V}(\eta)$ where η is small enough so that Theorem 2.5 applies. We write Taylor formula at order 2 for $t \to e_\Theta(t) + \Lambda\,\mathrm{meas}(\Omega_t)$ which is of class $\mathcal{C}^2$. As $\mathrm{meas}(\Omega_t)$ is constant and Ω_0 is assumed to be a critical point for the constrained functional, $e'_\Theta(0) = 0$ and the Taylor's formula writes

$$e_\Theta(1) = e_\Theta(0) + \int_0^1 (1-t) e''_\Theta(t) \mathrm{d}t.$$

We have

$$e''_\Theta(0) = -2B(0) - A(0) = \int_{\partial\Omega_0} 2z(0)\, C_0(z(0)) - m^2 \langle \nabla |\nabla u_0|^2, \boldsymbol{n}\rangle.$$

Here $z(0) = -m\,\langle \nabla u_0, \boldsymbol{n}\rangle$ and $|\langle \nabla u_0, \boldsymbol{n}\rangle|^2 = |\nabla u_0|^2 = \Lambda$. On the other hand (see below),

$$\langle \nabla |\nabla u_0|^2, \boldsymbol{n}\rangle = -2\,\Lambda\,\mathcal{C} \tag{5.19}$$

where $\mathcal{C}$ is the curvature of $\partial\Omega_0$ seen from inside. Therefore

$$e''_\Theta(0) = 2\,\Lambda \int_{\partial\Omega_0} mC_0(m) + \mathcal{C}\, m^2, \tag{5.20}$$

which is the expression we announced in (2.7). Now, m is of class $\mathcal{C}^{2,\alpha}$ and satisfies (see (3.21)) $\int_{\partial\Omega_0} m = 0$. By assumption (2.8),

$$e''_\Theta(0) \geq c\, \|m\|^2_{H^{1/2}(\partial\Omega_0)}. \tag{5.21}$$

Recall that this occurs when Ω_0 is convex for example. But by Theorem 2.5, we have for all $t \in [0,1]$,

$$e''_\Theta(t) \geq e''_\Theta(0) - C\,\omega(\eta)\, \|m\|^2_{H^{1/2}(\partial\Omega_0)}.$$

Therefore, there exists $\eta_0 > 0$ such that, for all $\Theta \in \mathcal{V}(\eta_0)$ and $\forall t \in [0,1]$,

$$e''_\Theta(t) \geq C\, \|m\|^2_{H^{1/2}(\partial\Omega_0)},$$

and

$$e_\Theta(1) - e_\Theta(0) \geq C\, \|m\|^2_{H^{1/2}(\partial\Omega_0)},$$

which is strictly positive if $m \neq 0$ whence the theorem.

Let us finally check (5.19) by the following elementary local computation (inspired from [7]) where we assume that Ω_0 is locally above the graph of the function $f : (-\epsilon, +\epsilon) \to \mathbb{R}$ with $f(0) = f'(0) = 0$. For simplicity, we write u instead of u_0 that is: $u_0 = u(x,y)$. The function u is such that:

$$\forall x \in (-\epsilon, +\epsilon),\ u(x, f(x)) = 0, \langle \nabla u_0, \boldsymbol{n}\rangle = -u_y(x, f(x)).$$

By differentiation, we have

$$\begin{aligned} 0 &= u_x(x, f(x)) + u_y(x, f(x)) f'(x), \\ 0 &= u_{xx}(x, f(x)) + 2u_{xy}(x, f(x)) f'(x) + u_{yy}(x, f(x)) f'(x)^2 + u_y(x, f(x)) f''(x), \end{aligned}$$

which gives at $x = 0:\ 0 = u_x(0,0),\ 0 = u_{xx}(0,0) + u_y(0,0)f''(0)$. We also have

$$\langle \nabla |\nabla u_0|^2, \boldsymbol{n}\rangle = -\partial_y\{(u_x)^2 + (u_y)^2\} = -2\,(u_x u_{xy} + u_y u_{yy}).$$

Recall now that the right-hand side k in (2.4) is compactly supported in Ω_0. By regularity, we have $u_{xx}+u_{yy} = 0$ on the boundary. Therefore, at $x = 0$, we obtain

$$\langle \nabla |\nabla u_0|^2, \boldsymbol{n}\rangle = 2\,u_y(0,0)u_{xx}(0,0) = -2\,(u_y(0,0))^2\, f''(0),$$

and $(u_y(0,0))^2 = |\nabla u_0|^2 = \Lambda$. The formula (5.19) follows since $f''(0) = \mathcal{C}$.

Remark. If we do not assume k to be compactly supported in Ω_0, then we have to use $u_{yy} = -u_{xx} - k$ instead so that

$$\langle \nabla |\nabla u_0|^2, \boldsymbol{n}\rangle = -2\,u_y[-u_{xx} - k] = -2\,(u_y(0,0))^2\, f''(0) + 2\,k\,u_y = -2\,\Lambda\,\mathcal{C} - 2\,k\,\langle \nabla u_0, \boldsymbol{n}\rangle.$$

We know that $|\langle \nabla u_0, \boldsymbol{n}\rangle|$ is constant on $\partial\Omega_0$ and equal to $\Lambda^{1/2}$. Assume $\partial\Omega_0$ is connected. Then, by regularity, $\langle \nabla u_0, \boldsymbol{n}\rangle$ itself is constant and equal to $\epsilon\Lambda^{1/2}$ with $\epsilon = +1$ or -1. The sign is determined by the relation

$$\int_{\Omega_0} k = \int_{\Omega_0} -\Delta u_0 = \int_{\partial\Omega_0} -\langle \nabla u_0, \boldsymbol{n}\rangle = -\epsilon\,\Lambda^{1/2}\,\mathrm{length}(\partial\Omega_0),$$

which shows that $\epsilon = -\,\mathrm{sign}(\int_{\Omega_0} k)$. Finally, the second derivative becomes

$$e''_\Theta(0) = 2\,\Lambda \int_{\partial\Omega_0} mC_0(m) +\, m^2\left[\mathcal{C} -\, k\,\Lambda^{-1/2}\mathrm{sign}(\int_{\Omega_0} k)\right]. \tag{5.22}$$

6. About the coercivity of $e''_\Theta(0)$

If k is compactly supported in Ω_0, then expression (5.20) is valid. Note that then the stability depends only on the geometry of $\partial\Omega_0$. If moreover $\partial\Omega_0$ is convex, then the coercivity (5.21) holds. Obviously, this extends to curves $\mathcal{C}^2$-close to convex curves.

If k is identically equal to a positive constant, then any disk of radius $R = \sqrt{S/\pi}$ is a critical shape. We have $k\,S = \Lambda^{1/2}\,2\pi R$ and by (5.22)

$$e''_\Theta(0) = 2\,\Lambda \int_{\partial\Omega_0} mC_0(m) - \frac{1}{R}\,m^2 = 2\,\Lambda\langle [C_0 - \frac{1}{R}\mathrm{Id}](m), m\rangle_{H^{-1/2}\times H^{1/2}}.$$

We easily check that this vanishes for $m = \sin\,\theta$ and $m = \cos\,\theta$ so that $e''_\Theta(0)$ does not satisfy (2.8) or (5.21). This obviously corresponds to the fact that the disk remains a critical shape when moved by translation. One can check that $e''_\Theta(0)$ is however positive on the set of $m's$ orthogonal to the linear space spanned by $\{1, \cos, \sin\}$ and this is also a consequence of Theorem 2.5 (see *e.g.* [11, 14]).

One may also compute what happens for radial functions k. Then the disk centered at the origin and of radius $R = \sqrt{S/\pi}$ is a critical shape and (we assume for instance $\int_{\Omega_0} k \geq 0$)

$$e''_\Theta(0) = 2\,\Lambda \int_{\partial\Omega_0} mC_0(m) + m^2[\frac{1}{R} - \frac{2\pi R k(R)}{\int_{\Omega_0} k}].$$

We easily check that

$$\int_{\partial\Omega_0} mC_0(m) \geq \frac{1}{R}\int_{\partial\Omega_0} m^2.$$

Therefore, for η positif and small

$$e''_{\Theta}(0) \geq 2\,\Lambda\{\eta\,\|m\|^2_{H^{1/2}} + \int_{\partial\Omega_0} \frac{2\,m^2}{R}[1 - \frac{\eta}{2} - \frac{k(R)\pi R^2}{\int_{\Omega_0} k}]\,\},$$

so that $e''_{\Theta}(0)$ is coercive if $k(R)\langle\frac{\int_{\Omega_0} k}{\pi R^2}\cdot$

More detailed studies of such quadratic forms may be found in [11] where more general functional involving the perimeter of the shapes (*i.e.* surface tension) are considered. Moreover, the case of the less stable "exterior" problem (or "exterior shaping problem") is also treated where the Dirichlet problem is set in the exterior of the shapes. The positivity is then more difficult to study. An extension of the results of this paper to the case with surface tension can be found in [4] as well as N-dimensional situations.

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