

## ABOUT STATIONARITY AND REGULARITY IN VARIATIONAL ANALYSIS

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Dedicated to Boris Mordukhovich on his 60th birthday

**Abstract.** Stationarity and regularity concepts for the three typical for variational analysis classes of objects - real-valued functions, collections of sets, and multifunctions - are investigated. An attempt is made to present a classification scheme for such concepts and to show that properties introduced for objects from different classes can be treated in a similar way. Furthermore, in many cases the corresponding properties appear to be in a sense equivalent. The properties are defined in terms of certain constants which in the case of regularity properties provide also some quantitative characterizations of these properties. The relations between different constants and properties are discussed.

An important feature of the new variational techniques is that they can handle nonsmooth functions, sets and multifunctions equally well  
Borwein and Zhu [8]

### 1. INTRODUCTION

The paper investigates extremality, stationarity and regularity properties of real-valued functions, collections of sets, and multifunctions attempting at developing a unifying scheme for defining and using such properties.

Under different names this type of properties have been explored for centuries. A classical example of a stationarity condition is given by the Fermat theorem on local minima and maxima of differentiable functions. In a sense, any necessary optimality (extremality) conditions define/characterize certain stationarity (singularity/irregularity) properties. The separation theorem also characterizes a kind of extremal (stationary) behavior of convex sets.

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Surjectivity of a linear continuous mapping in the Banach open mapping theorem (and its extension to nonlinear mappings known as Lyusternik-Graves theorem) is an example of a regularity condition. Other examples are provided by numerous constraint qualifications and error bound conditions in optimization problems, qualifying conditions in subdifferential calculus, etc.

Many more properties which can be interpreted as either stationarity or regularity have been introduced (explicitly and in many cases implicitly) and investigated with the development of optimization theory and variational analysis. They are important for optimality conditions, stability of solutions, and numerical methods.

There exist different settings of optimization and variational problems: in terms of single-valued and multivalued mappings and in terms of collections of sets. It is not surprising that investigating stationarity and especially regularity properties of these objects has attracted significant attention. Real-valued functions and collections of sets were examined respectively in [17, 20, 26-29, 32] and [3, 5-7, 14, 20, 25-31, 33, 38, 40-42, 47]. Multifunctions represent the most developed class of objects. A number of useful regularity properties have been introduced and investigated - see [1, 2, 9, 12, 13, 19-21, 23, 27-29, 35-39, 43-46] and the references therein - the most well recognized and widely used being that of *metric regularity*.

In this paper, which continues [29-31], an attempt is made to present a classification scheme for such concepts and to show that, in accordance with the cited above words by Borwein and Zhu, properties introduced for objects from different classes can be treated in a similar way. Furthermore, in many cases the corresponding properties appear to be equivalent.

First of all, stationarity and regularity properties are mutually inverse. For example, the equality  $f'(\bar{x}) = 0$  for a real-valued differentiable at  $\bar{x}$  function  $f$  is a stationarity condition, while the inequality  $f'(\bar{x}) \neq 0$  can be considered as a regularity criterion. Thus, such properties always go in pairs. Given one condition (stationarity or regularity), its negation automatically describes its opposite counterpart.

It seems natural to distinguish between primal space properties and those defined in terms of dual space elements. Metric conditions are primal space properties while their characterizations in terms of normal cones or coderivatives are dual conditions. In some cases primal and dual conditions are equivalent, and dual conditions provide complete characterizations of the corresponding primal space properties. However, there are cases when equivalences do not hold, and one has necessary or sufficient conditions.

Another natural way of classifying stationarity and regularity properties is to distinguish between *basic* ("at a point") and more robust *strict* ("near a point") conditions. In the latter case one can speak about *approximate stationarity* and *uniform regularity*. For instance, dual conditions formulated in terms of usual

Fréchet derivatives or Fréchet subdifferentials/normals belong to the first group, while conditions in terms of strict derivatives or limiting subdifferentials/normals belong to the second one. Metric regularity of multifunctions is a typical example of a primal space uniform regularity property.

The properties can be defined in terms of certain constants which in the case of regularity properties provide also some quantitative characterizations of these properties. It will be demonstrated in the subsequent sections that such constants are convenient when establishing interrelations between the properties.

Obviously not all existing stationarity and regularity properties are discussed in the paper. Only those typical properties have been chosen which better illustrate the classification scheme described above. The content of this paper is not expected to surprise those working in the area of variational analysis. However, the author believes that some relations presented in it can be useful when dealing with specific problems.

The remaining three sections are devoted to our three main objects of interest: real-valued functions, collections of sets, and multifunctions respectively.

In Section 2, we consider stationarity and regularity properties of real-valued functions. The main feature of this class of objects compared to the two others is that, in the nondifferentiable case, one can (and should) distinguish between properties of functions “from below” (from the point of view of minimization) and “from above” (from the point of view of maximization). The terms *inf-stationarity* and *inf-regularity* are used in the paper in the first instance, and *sup-stationarity* and *sup-regularity* in the second one. The “combined” properties are considered as well. A number of stationarity and regularity properties as well as constants characterizing them are introduced. The relations between these constants are summarized in Theorem 2. It can be interesting to note that while two different basic primal space constants are in use, the corresponding strict constants coincide for lower semicontinuous functions on a complete metric space. If, additionally, the space is Asplund, they coincide with the appropriate dual space strict constant. This result (Theorem 2(ix)) improves [32, Theorem 4]. Special attention is given to the differential and convex cases when most of the constants and properties coincide.

In Section 3, collections of sets are considered. The stationarity properties discussed here extend the concept of locally extremal collection introduced in [33] while the relation between the corresponding primal and dual constants formulated in Theorem 4(vi) extends the *extremal principle* [33, 40]. This result improves [29, Theorem 1]. The corresponding regularity properties are discussed as well as their relations with other properties of this kind: *metric inequality (local linear regularity)* [4, 18, 19, 42, 47] and *Jameson's property (G)* [5, 41].

The last Section 4 is devoted to multifunctions with the main emphasis on their regularity properties. The constants characterizing these properties are defined along

the same lines. Metric regularity is treated as an example of a uniform primal space regularity property corresponding to similar properties of real-valued functions and collections of sets. The relations between different constants, including the equality of primal and dual strict constants, are summarized in Theorem 6. Finally, relations are established between the multifunctional regularity/stationarity constants and the corresponding constants defined in the preceding sections for the other two main classes of objects of the current research - real-valued functions and collections of sets.

Mainly standard notations are used throughout the paper.  $B_r(x)$  denotes a closed ball in a metric space with centre at  $x$  and radius  $r$ . A closed unit ball in a normed space is denoted by  $B$ . If  $\Omega$  is a set then  $\text{int } \Omega$  and  $\text{bd } \Omega$  are respectively its interior and boundary. If not explicitly specified otherwise, when considering product spaces we assume that they are equipped with the maximum-type distances or norms:  $d((x_1, y_1), (x_2, y_2)) = \max(d(x_1, x_2), d(y_1, y_2))$ ,  $\|(x, y)\| = \max(\|x\|, \|y\|)$ . Sometimes, in products of normed spaces, the following norm depending on a parameter  $\gamma > 0$  will be used:  $\|(x, y)\|_\gamma = \max(\|x\|, \gamma\|y\|)$ .

## 2. REAL-VALUED FUNCTIONS

### 2.1. Extremality, stationarity, and regularity

The classical criterion characterizing extremum points of real-valued functions is given by the famous *Fermat theorem*.

**Theorem 1 (Fermat).** *If a differentiable function  $f$  has a local minimum or maximum at  $\bar{x}$  then  $f'(\bar{x}) = 0$ .*

This assertion provides a dual ( $f'(\bar{x})$  is an element of the dual space!) *necessary* condition for a local minimum or maximum. It is well known that it actually characterizes a weaker property called *stationarity*.

The concept of stationarity for a real-valued function in the framework of classical analysis can be illustrated by the three examples in Figure 1 which can be found in any textbook on calculus.

For a differentiable function on a normed linear space, the *stationary* behavior near a given point can be characterized in two equivalent ways:

- (P) *Primal characterization:* the increment (and decrement) of the function is infinitely small compared to the increment of the argument.
- (D) *Dual characterization:* the derivative at the point is zero.

If none of the above characterizations holds true then the function is *regular* near the given point.

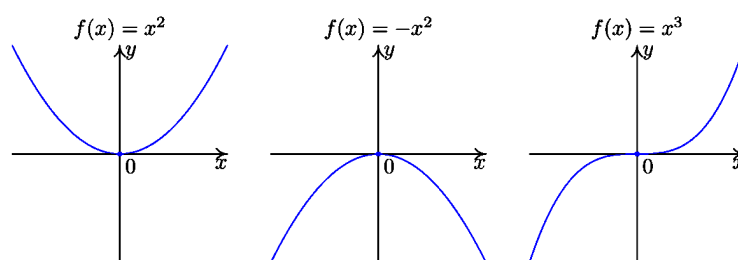


Fig. 1. Stationarity: differentiable case.

## 2.2. Inf-stationarity and inf-regularity

As it is easily seen from the above illustrations, in the differentiable case, the stationarity characterizations do not distinguish between maxima and minima. The nondifferentiable setting is much richer. First of all, stationarity properties of nondifferentiable functions with respect to minimization and maximization are in general essentially different. Besides, these properties can be defined in several different ways.

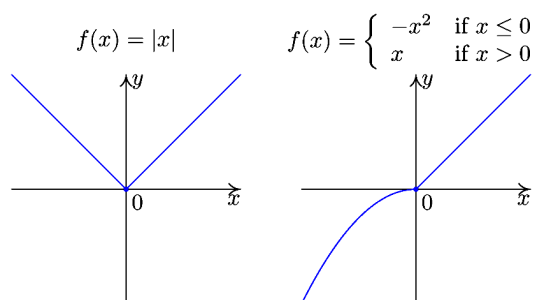


Fig. 2. Inf-stationarity.

The functions presented in Figure 2 clearly possess certain stationarity properties from the point of view of minimization: the decrement of the function is infinitely small (for the first function it is zero) compared to the increment of the argument. Similarly stationarity from the point of view of maximization presumes a similar estimate of the increment of the function. None of the functions in Figure 2 possesses this property.

It is still possible to formulate primal and dual characterizations of stationarity and regularity.

In this section, if not explicitly stated otherwise,  $X$  is a metric space. For all characterizations including dual space objects (subdifferentials) we will assume  $X$  to be a normed linear space.  $f$  is a function on  $X$  with values in the extended real

line  $\mathbb{R}_\infty = \mathbb{R} \cup \{\pm\infty\}$ , finite at  $\bar{x} \in X$ .

We start with *inf-stationarity*, that is stationarity from the point of view of minimization. The following three properties can qualify for generalizing the corresponding primal and dual characterizations (P) and (D).

Inf-stationarity.

(IS1) For any  $\varepsilon > 0$  there exists a  $\rho \in (0, \varepsilon)$  such that

$$(1) \quad f(x) - f(\bar{x}) \geq -\varepsilon\rho, \quad \forall x \in B_\rho(\bar{x}).$$

(IS2) For any  $\varepsilon > 0$  there exists a  $\rho > 0$  such that

$$(2) \quad f(x) - f(\bar{x}) \geq -\varepsilon d(x, \bar{x}), \quad \forall x \in B_\rho(\bar{x}).$$

(ISD) ( $X$  is a normed linear space)  $0 \in \partial f(\bar{x})$ .

In the last property  $\partial f(\bar{x})$  denotes the *Fréchet subdifferential* of  $f$  at  $\bar{x}$ :

$$(3) \quad \partial f(\bar{x}) = \left\{ x^* \in X^* \mid \liminf_{x \rightarrow \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0 \right\}.$$

Clearly, all characterizations of inf-stationarity, formulated above, are satisfied if  $\bar{x}$  is a point of local minimum of  $f$  (see, for example, the first function in Figure 2). They are also satisfied for the second function in Figure 2. Condition (1) (condition (2)) means that  $\bar{x}$  is an  $\varepsilon\rho$ -minimal ( $\varepsilon$ -Ekeland) point of  $f$  on  $B_\rho(\bar{x})$  [21, 22, 36]. In general, (IS1) and (IS2) are not equivalent (see Examples 1 and 2 in [32]). The next proposition shows that (IS1) is weaker than (IS2); when  $X$  is complete, a point satisfying (IS1) can be approximated by points satisfying (IS2).

**Proposition 1.**

(i)  $(IS2) \Rightarrow (IS1)$ .

(ii) Let  $X$  be complete and  $f$  be lower semicontinuous near  $\bar{x}$ . If (IS1) holds true then for any  $\varepsilon > 0$  there exist a  $\rho \in (0, \varepsilon)$  and an  $\hat{x} \in B_\rho(\bar{x})$  such that  $f(\hat{x}) \leq f(\bar{x})$  and

$$f(x) - f(\hat{x}) \geq -\varepsilon d(x, \hat{x}), \quad \forall x \in B_\rho(\hat{x}).$$

(iii) If  $X$  is a normed linear space then  $(IS2) \Leftrightarrow (ISD)$ .

*Proof.* The first and the third assertions follow directly from the definitions. The proof of the second one is a traditional example of application of the *Ekeland variational principle* [15].

If (IS1) holds true, then for any  $\varepsilon > 0$  there exists an  $r \in (0, \varepsilon/2)$  such that

$$f(x) - f(\bar{x}) \geq -\varepsilon r/2, \quad \forall x \in B_r(\bar{x}).$$

Let  $\rho = r/2$ . Then  $\rho < \varepsilon/4 \leq \varepsilon$ . If  $X$  is complete then by the Ekeland variational principle there exists an  $\hat{x} \in B_\rho(\bar{x})$  such that  $f(\hat{x}) \leq f(\bar{x})$  and

$$f(x) - f(\hat{x}) \geq -\varepsilon d(x, \hat{x})$$

for all  $x \in B_r(\bar{x})$ . In particular, the last inequality is valid for all  $x \in B_\rho(\hat{x})$ . ■

Thus, in the nondifferentiable case we have in general two different types of inf-stationarity which primal characterizations are given by (IS1) and (IS2).

If any of these conditions is not satisfied one can speak about the corresponding type of *inf-regularity*.

#### Inf-regularity.

(IR1) There exists an  $\alpha > 0$  and a  $\delta > 0$  such that for any  $\rho \in (0, \delta)$  there is an  $x \in B_\rho(\bar{x})$  satisfying

$$f(x) - f(\bar{x}) < -\alpha\rho.$$

(IR2) There exists an  $\alpha > 0$  such that for any  $\rho > 0$  there is an  $x \in B_\rho(\bar{x})$  satisfying

$$f(x) - f(\bar{x}) < -\alpha d(x, \bar{x}).$$

(IRD) ( $X$  is a normed linear space)  $0 \notin \partial f(\bar{x})$ .

### **2.3. Approximate inf-stationarity and uniform inf-regularity**

The functions not satisfying (IS1) and (IS2) can still possess some features of inf-stationarity near the given point.

For the functions in Figure 3, the point  $\bar{x} = 0$  is definitely far from being inf-stationary. At the same time there are inf-stationary points in any its neighborhood. In such cases it is possible to speak about *approximate inf-stationarity*.

#### Approximate inf-stationarity.

(AIS1) For any  $\varepsilon > 0$  there exists a  $\rho \in (0, \varepsilon)$  and an  $x \in B_\varepsilon(\bar{x})$  such that  $|f(x) - f(\bar{x})| \leq \varepsilon$  and

$$(4) \quad f(u) - f(x) \geq -\varepsilon\rho, \quad \forall u \in B_\rho(x).$$

(AIS2) For any  $\varepsilon > 0$  there exists a  $\rho \in (0, \varepsilon)$  and an  $x \in B_\varepsilon(\bar{x})$  such that  $|f(x) - f(\bar{x})| \leq \varepsilon$  and

$$(5) \quad f(u) - f(x) \geq -\varepsilon d(u, x), \quad \forall u \in B_\rho(x).$$

(AISD) ( $X$  is a normed linear space) For any  $\varepsilon > 0$  there exists an  $x \in B_\varepsilon(\bar{x})$  and an  $x^* \in \partial f(x)$  such that  $|f(x) - f(\bar{x})| \leq \varepsilon$  and  $\|x^*\| \leq \varepsilon$ .

(AISDL) ( $X$  is a normed linear space)  $0 \in \bar{\partial}f(\bar{x})$ .

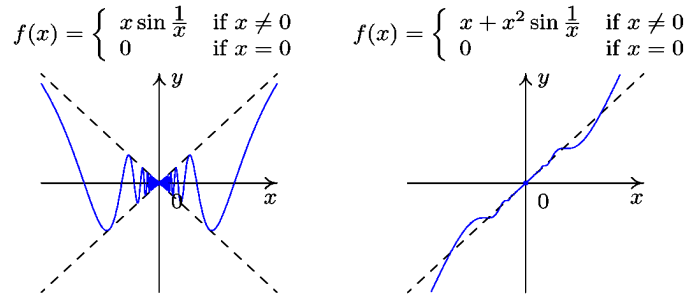


Fig. 3. Approximate inf-stationarity.

In the statement of the last property  $\bar{\partial}f(\bar{x})$  denotes the *limiting subdifferential* of  $f$  at  $\bar{x}$ :

$$(6) \quad \bar{\partial}f(\bar{x}) = \{x^* \in X^* \mid x_k \rightarrow \bar{x}, f(x_k) \rightarrow f(\bar{x}), x_k^* \xrightarrow{w^*} x^*, x_k^* \in \partial f(x_k), k = 1, 2, \dots\},$$

where  $x_k^* \xrightarrow{w^*} x^*$  means that  $x_k^*$  converges to  $x^*$  in the weak\* topology. In contrast to (3), this set can be nonconvex. However, it possesses a certain subdifferential calculus (see [38]). In the convex case, subdifferential (6) coincides with the subdifferential in the sense of convex analysis.

All characterizations of approximate inf-stationarity are satisfied for the functions in Figure 3. Basically approximate inf-stationarity means that in any neighborhood of the given point there is another one at which the corresponding inf-stationarity property “almost” holds.

Once again (AIS1) is obviously weaker than (AIS2). The latter property is referred to in [36] as *stationarity with respect to minimization*.

**Remark.** Notice that the second function in Figure 3 is everywhere differentiable. Moreover,  $f'(0) = 1$  and consequently the function is regular at 0 in the classical sense. Thus, in terms of approximate inf-stationarity, even for differentiable functions, Figure 1 does not present the full list of possibilities. The explanation of this phenomenon is simple: the derivative at 0 of the second function in Figure 3 is not *strict*.



If any of the above conditions is not satisfied one can speak about the corresponding type of *uniform inf-regularity* (a certain property must hold uniformly in a neighborhood of the given point.)

Uniform inf-regularity.

(UIR1) There exists an  $\alpha > 0$  and a  $\delta > 0$  such that for any  $\rho \in (0, \delta)$  and any  $x \in B_\delta(\bar{x})$  with  $|f(x) - f(\bar{x})| \leq \delta$  there is a  $u \in B_\rho(x)$  satisfying

$$f(u) - f(x) < -\alpha\rho.$$

(UIR2) There exists an  $\alpha > 0$  and a  $\delta > 0$  such that for any  $\rho \in (0, \delta)$  and any  $x \in B_\delta(\bar{x})$  with  $|f(x) - f(\bar{x})| \leq \delta$  there is a  $u \in B_\rho(x)$  satisfying

$$f(u) - f(x) < -\alpha d(u, x).$$

(UIRD) ( $X$  is a normed linear space) There exists an  $\alpha > 0$  and a  $\delta > 0$  such that for any  $x \in B_\delta(\bar{x})$  with  $|f(x) - f(\bar{x})| \leq \delta$  and any  $x^* \in \partial f(x)$  it holds  $\|x^*\| > \alpha$ .

(UIRDL) ( $X$  is a normed linear space)  $0 \notin \bar{\partial}f(\bar{x})$ .

**2.4. Constants**

It can be convenient to characterize the inf-stationarity and inf-regularity properties introduced above in terms of certain nonnegative constants:

$$(7) \quad |\theta f|_\rho(\bar{x}) = f(\bar{x}) - \inf_{x \in B_\rho(\bar{x})} f(x) = \sup_{x \in B_\rho(\bar{x})} (f(\bar{x}) - f(x)),$$

$$(8) \quad |\theta f|(\bar{x}) = \liminf_{\rho \downarrow 0} \frac{|\theta f|_\rho(\bar{x})}{\rho},$$

$$(9) \quad |\nabla f|(\bar{x}) = \limsup_{x \rightarrow \bar{x}} \frac{[f(\bar{x}) - f(x)]_+}{d(x, \bar{x})},$$

$$(10) \quad \overline{|\theta f|}(\bar{x}) = \liminf_{\substack{x \xrightarrow{f} \bar{x} \\ \rho \downarrow 0}} \frac{|\theta f|_\rho(x)}{\rho},$$

$$(11) \quad \overline{|\nabla f|}(\bar{x}) = \liminf_{\substack{x \xrightarrow{f} \bar{x} \\ \rho \downarrow 0}} \sup_{u \in B_\rho(x) \setminus \{x\}} \frac{[f(x) - f(u)]_+}{d(u, x)},$$

$$(12) \quad |\partial f|(\bar{x}) = \inf \{ \|x^*\| \mid x^* \in \partial f(\bar{x}) \},$$

$$(13) \quad \widehat{|\partial f|}(\bar{x}) = \liminf_{\delta \downarrow 0} \{ \|x^*\| \mid x^* \in \hat{\partial}_\delta f(\bar{x}) \},$$

$$(14) \quad \overline{|\partial f|}(\bar{x}) = \liminf_{\delta \downarrow 0} \{ \|x^*\| \mid x^* \in \bar{\partial} f(\bar{x}) \}.$$

The following notations and conventions are used in the above formulas:

- In (12)–(14),  $X$  is a normed linear space;
- $[\alpha]_+ = \max(\alpha, 0)$ ;
- $\inf \emptyset = +\infty$ ;
- $x \xrightarrow{f} \bar{x} \Leftrightarrow x \rightarrow \bar{x}$  with  $f(x) \rightarrow f(\bar{x})$ ;
- $\hat{\partial}_\delta f(\bar{x}) = \bigcup \{ \partial f(x) \mid x \in B_\delta(\bar{x}), |f(x) - f(\bar{x})| \leq \delta \}$ .

The last set is called the *strict  $\delta$ -subdifferential* of  $f$  at  $\bar{x}$  (see [24, 27, 28]). Note that the equality  $|\widehat{\partial}f|(\bar{x}) = 0$  does not imply the inclusion  $0 \in \hat{\partial}_\delta f(\bar{x})$  [32, Example 8].

Constant (9) is known as the (strong) *slope* of  $f$  at  $\bar{x}$  [11] (see also [19]). Constant (11) is the *strict slope* of  $f$  at  $\bar{x}$ . Constants (12)–(14) are called respectively the *subdifferential slope*, the *strict subdifferential slope*, and the *limiting subdifferential slope* of  $f$  at  $\bar{x}$  (see [17]).

The equality  $|\theta f|_\rho(\bar{x}) = 0$  for some  $\rho > 0$  (for any  $\rho > 0$ ) is equivalent to  $\bar{x}$  being a point of local (respectively global) minimum of  $f$ . For each of the constants (8)–(13), its equality to zero (being strictly positive) is equivalent to the corresponding inf-stationarity (inf-regularity) characterization:

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| <ul style="list-style-type: none"> <li>• <math> \theta f (\bar{x}) = 0 \Leftrightarrow</math> (IS1);</li> <li>• <math> \nabla f (\bar{x}) = 0 \Leftrightarrow</math> (IS2);</li> <li>• <math> \partial f (\bar{x}) = 0 \Leftrightarrow</math> (ISD);</li> <li>• <math>\overline{ \theta f }(\bar{x}) = 0 \Leftrightarrow</math> (AIS1);</li> <li>• <math>\overline{ \nabla f }(\bar{x}) = 0 \Leftrightarrow</math> (AIS2);</li> <li>• <math> \widehat{\partial}f (\bar{x}) = 0 \Leftrightarrow</math> (AISD);</li> <li>• <math>\overline{ \partial f }(\bar{x}) = 0 \Leftrightarrow</math> (AISDL);</li> </ul> | <ul style="list-style-type: none"> <li>• <math> \theta f (\bar{x}) &gt; 0 \Leftrightarrow</math> (IR1);</li> <li>• <math> \nabla f (\bar{x}) &gt; 0 \Leftrightarrow</math> (IR2);</li> <li>• <math> \partial f (\bar{x}) &gt; 0 \Leftrightarrow</math> (IRD);</li> <li>• <math>\overline{ \theta f }(\bar{x}) &gt; 0 \Leftrightarrow</math> (UIR1);</li> <li>• <math>\overline{ \nabla f }(\bar{x}) &gt; 0 \Leftrightarrow</math> (UIR2);</li> <li>• <math> \widehat{\partial}f (\bar{x}) &gt; 0 \Leftrightarrow</math> (UIRD);</li> <li>• <math>\overline{ \partial f }(\bar{x}) &gt; 0 \Leftrightarrow</math> (UIRDL).</li> </ul> |
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The relationships between the different types of inf-stationarity and inf-regularity are determined by the relations between the corresponding constants. The next theorem summarizes the list of such relations.

**Theorem 2.** *The following assertions hold true:*

- (i)  $|\theta f|(\bar{x}) \leq |\nabla f|(\bar{x})$ ;
- (ii)  $\overline{|\theta f|}(\bar{x}) \leq \liminf_{x \xrightarrow{f} \bar{x}} |\theta f|(x)$ ;
- (iii)  $\overline{|\nabla f|}(\bar{x}) \leq \liminf_{x \xrightarrow{f} \bar{x}} |\nabla f|(x)$ ;
- (iv)  $\overline{|\theta f|}(\bar{x}) \leq \overline{|\nabla f|}(\bar{x})$ ;
- (v) *if  $X$  is complete and  $f$  is lower semicontinuous near  $\bar{x}$ , then  $\overline{|\nabla f|}(\bar{x}) = \overline{|\theta f|}$ .*

*Suppose  $X$  is a normed linear space. Then*

- (vi)  $\widehat{|\partial f|}(\bar{x}) = \liminf_{x \xrightarrow{f} \bar{x}} |\partial f|(x)$ ;
- (vii)  $|\nabla f|(\bar{x}) \leq |\partial f|(\bar{x})$ ;
- (viii)  $\overline{|\nabla f|}(\bar{x}) \leq \widehat{|\partial f|}(\bar{x})$ ;
- (ix) if  $X$  is Asplund and  $f$  is lower semicontinuous near  $\bar{x}$ , then  $\overline{|\nabla f|}(\bar{x}) = \widehat{|\partial f|}(\bar{x})$ ;
- (x) if  $\dim X < \infty$  and  $f$  is lower semicontinuous near  $\bar{x}$ , then  $\widehat{|\partial f|}(\bar{x}) = \overline{|\nabla f|}(\bar{x})$ .

*Proof.* The majority of assertions in the theorem can be found in [32]. The only one which needs proof is the inequality  $\overline{|\nabla f|}(\bar{x}) \geq \widehat{|\partial f|}(\bar{x})$  in assertion (ix).

Let  $X$  be Asplund,  $\overline{|\nabla f|}(\bar{x}) < \alpha$ ,  $\delta > 0$ . Taking into account definition (13), we need to show that there exists an  $\hat{x} \in B_\delta(\bar{x})$  with  $|f(\hat{x}) - f(\bar{x})| \leq \delta$  and an  $x^* \in \partial f(\hat{x})$  such that  $\|x^*\| < \alpha$ .

Chose numbers  $\alpha_1, \alpha_2$ , satisfying  $\overline{|\nabla f|}(\bar{x}) < \alpha_1 < \alpha_2 < \alpha$ . By assertion (v) and definitions (10), (8), and (7), there exists a positive number  $\rho < \min(\alpha_1^{-1}, 1)\delta/2$  and a point  $x_1 \in B_{\delta/2}(\bar{x})$  with  $|f(x_1) - f(\bar{x})| \leq \delta/2$  such that

$$f(u) - f(x_1) > -\rho\alpha_1 \quad \text{for all } u \in B_\rho(x_1).$$

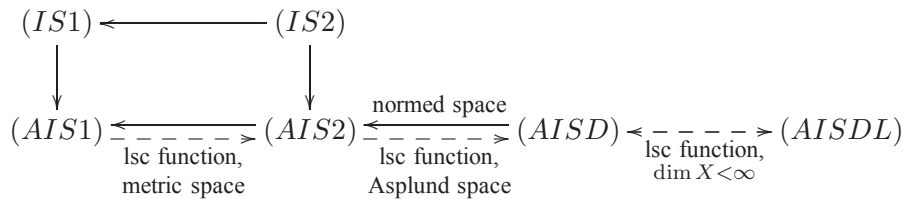
Take  $\rho' = \rho\alpha_1/\alpha_2$ . It follows from the Ekeland variational principle that there exists a point  $x_2 \in B_{\rho'}(x_1)$  such that  $f(x_1) - \rho\alpha_1 < f(x_2) \leq f(x_1)$  and

$$f(u) - f(x_2) + \alpha_2\|u - x_2\| \geq 0 \quad \text{for all } u \in B_\rho(x_1).$$

Since  $x_2$  is an internal point of  $B_\rho(x_1)$  we have  $0 \in \partial(f + f_2)(x_2)$  where  $f_2(u) := \alpha_2\|u - x_2\|$ . Applying the fuzzy sum rule [16] we find a point  $\hat{x} \in B_{\delta/2-\rho}(x_2)$  with  $|f(\hat{x}) - f(x_2)| \leq \delta/2 - \rho\alpha_1$  such that  $\|x^*\| < \alpha$ . Note that  $\|\hat{x} - \bar{x}\| \leq \delta$  and  $|f(\hat{x}) - f(\bar{x})| \leq \delta$ . ■

The inequalities in Theorem 2 can be strict, see [32, Examples 1-4]. Theorem 2(ix) improves [32, Theorem 4].

In accordance with Theorem 2 the relationships between the inf-stationarity concepts can be described by the following diagram:



A similar diagram (with opposite arrows) describes the relationships between the corresponding inf-regularity concepts.

Due to assertion (v) (which is a corollary of Proposition 1), for a lower semi-continuous function  $f$  on a complete metric space conditions (AIS1) and (AIS2) are equivalent. In this case, of course, we also have a single uniform regularity property which is closely related to the *metric regularity* of multifunctions.

If, additionally,  $X$  *Asplund* then, due to (ix), the three approximate inf-stationarity conditions (AIS1), (AIS2), and (AISD) (as well as the corresponding approximate inf-regularity properties) are equivalent. Asplund spaces form a natural class of Banach spaces when working with Fréchet subdifferentials (see [38] for the definition, properties and motivations).

**Corollary 1.1.** *If  $X$  is Asplund and  $f$  is lower semicontinuous near  $\bar{x}$ , then*

- (i) (AIS1)  $\Leftrightarrow$  (AIS2)  $\Leftrightarrow$  (AISD);
- (ii) (UIR1)  $\Leftrightarrow$  (UIR2)  $\Leftrightarrow$  (UIRD).

Conditions (AISD) and (UIRD) can be considered (in the Asplund space setting) as equivalent dual characterizations of the corresponding primal properties.

**2.5. Sup-stationarity and sup-regularity**

Stationarity and regularity properties of nondifferentiable functions from the point of view of maximization (*sup-stationarity* and *sup-regularity*) can be defined in a similar way. They can also be defined in terms of the same constants applied to the function  $-f$ .

- $|\theta(-f)|(\bar{x}) = 0 \Leftrightarrow$  (SS1);      •  $|\theta(-f)|(\bar{x}) > 0 \Leftrightarrow$  (SR1);
- $|\nabla(-f)|(\bar{x}) = 0 \Leftrightarrow$  (SS2);      •  $|\nabla(-f)|(\bar{x}) > 0 \Leftrightarrow$  (SR2);
- $|\partial(-f)|(\bar{x}) = 0 \Leftrightarrow$  (SSD);      •  $|\partial(-f)|(\bar{x}) > 0 \Leftrightarrow$  (SRD);
- $\overline{|\theta(-f)|}(\bar{x}) = 0 \Leftrightarrow$  (ASS1);      •  $\overline{|\theta(-f)|}(\bar{x}) > 0 \Leftrightarrow$  (USR1);
- $\overline{|\nabla(-f)|}(\bar{x}) = 0 \Leftrightarrow$  (ASS2);      •  $\overline{|\nabla(-f)|}(\bar{x}) > 0 \Leftrightarrow$  (USR2);
- $\overline{|\widehat{\partial}(-f)|}(\bar{x}) = 0 \Leftrightarrow$  (ASSD);      •  $\overline{|\widehat{\partial}(-f)|}(\bar{x}) > 0 \Leftrightarrow$  (USRD);
- $\overline{|\partial(-f)|}(\bar{x}) = 0 \Leftrightarrow$  (ASSDL);      •  $\overline{|\partial(-f)|}(\bar{x}) > 0 \Leftrightarrow$  (USRDL).

All functions in Figures 2 and 3 are obviously sup-regular at  $\bar{x}$  (both (SR1) and (SR2) conditions hold true). At the same time the second function in Figure 2 and both functions in Figure 3 are approximately sup-stationary at  $\bar{x}$ .

The relationships between different sup-stationarity (sup-regularity) concepts are similar to those between inf-stationarity (inf-regularity) ones.

The “combined” concepts can also be of interest. It is natural to say that a function is *stationary (regular)* at a point if it is either inf-stationary or sup-stationary (both inf-regular and sup-regular) at this point.

- $\min(|\theta f|(\bar{x}), |\theta(-f)|(\bar{x})) = 0 \Leftrightarrow$  (S1);
- $\min(|\nabla f|(\bar{x}), |\nabla(-f)|(\bar{x})) = 0 \Leftrightarrow$  (S2);
- $\min(|\partial f|(\bar{x}), |\partial(-f)|(\bar{x})) = 0 \Leftrightarrow$  (SD);
- $\min(\overline{|\theta f|}(\bar{x}), \overline{|\theta(-f)|}(\bar{x})) = 0 \Leftrightarrow$  (AS1);
- $\min(\overline{|\nabla f|}(\bar{x}), \overline{|\nabla(-f)|}(\bar{x})) = 0 \Leftrightarrow$  (AS2);
- $\min(\widehat{|\partial f|}(\bar{x}), \widehat{|\partial(-f)|}(\bar{x})) = 0 \Leftrightarrow$  (ASD);
- $\min(|\theta f|(\bar{x}), |\theta(-f)|(\bar{x})) > 0 \Leftrightarrow$  (R1);
- $\min(|\nabla f|(\bar{x}), |\nabla(-f)|(\bar{x})) > 0 \Leftrightarrow$  (R2);
- $\min(|\partial f|(\bar{x}), |\partial(-f)|(\bar{x})) > 0 \Leftrightarrow$  (RD);
- $\min(\overline{|\theta f|}(\bar{x}), \overline{|\theta(-f)|}(\bar{x})) > 0 \Leftrightarrow$  (UR1);
- $\min(\overline{|\nabla f|}(\bar{x}), \overline{|\nabla(-f)|}(\bar{x})) > 0 \Leftrightarrow$  (UR2);
- $\min(\widehat{|\partial f|}(\bar{x}), \widehat{|\partial(-f)|}(\bar{x})) > 0 \Leftrightarrow$  (URD);
- $\min(\overline{|\partial f|}(\bar{x}), \overline{|\partial(-f)|}(\bar{x})) > 0 \Leftrightarrow$  (URDL).

If  $f$  is a continuous function on a complete metric space then

$$(AS1) \Leftrightarrow (AS2) \quad (UR1) \Leftrightarrow (UR2).$$

If, additionally, the space is Asplund then

$$(AS1) \Leftrightarrow (AS2) \Leftrightarrow (ASD) \quad (UR1) \Leftrightarrow (UR2) \Leftrightarrow (URD).$$

### 2.6. Differentiable and convex cases

For differentiable or convex functions most of the stationarity and regularity concepts described above reduce to traditional ones.

In the rest of this section  $X$  is assumed a normed linear space. Recall that  $f$  is called *strictly differentiable* [38, 46] at  $\bar{x}$  (with the derivative  $f'(\bar{x})$ ) if

$$\lim_{x \rightarrow \bar{x}, u \rightarrow \bar{x}} \frac{f(u) - f(x) - \langle f'(\bar{x}), u - x \rangle}{\|u - x\|} = 0.$$

**Proposition 2.** *If  $f$  is Fréchet differentiable at  $\bar{x}$  with the derivative  $f'(\bar{x})$  then*

$$|\theta f|(\bar{x}) = |\nabla f|(\bar{x}) = |\partial f|(\bar{x}) = |\theta(-f)|(\bar{x}) = |\nabla(-f)|(\bar{x}) = |\partial(-f)|(\bar{x}) = \|f'(\bar{x})\|.$$

*If, additionally, the derivative is strict then*

$$\begin{aligned} \overline{|\theta f|}(\bar{x}) &= \overline{|\nabla f|}(\bar{x}) = \widehat{|\partial f|}(\bar{x}) = \overline{|\partial f|}(\bar{x}) \\ &= \overline{|\theta(-f)|}(\bar{x}) = \overline{|\nabla(-f)|}(\bar{x}) = \widehat{|\partial(-f)|}(\bar{x}) = \overline{|\partial(-f)|}(\bar{x}) = \|f'(\bar{x})\|. \end{aligned}$$

**Proposition 3.** *Let  $f$  be convex.*

- (i) If  $|\theta f|_\rho(\bar{x}) > 0$  for some  $\rho > 0$  then  $|\theta f|_\rho(\bar{x}) > 0$  for all  $\rho > 0$ .
- (ii) The function  $\rho \rightarrow |\theta f|_\rho[f](\bar{x})/\rho$  (function  $\rho \rightarrow |\theta f|_\rho[-f](\bar{x})/\rho$ ) is nonincreasing (nondecreasing) on  $\mathbb{R}_+ \setminus \{0\}$ .
- (iii) The following equalities hold true:

$$\overline{|\theta f|}(\bar{x}) = \overline{|\nabla f|}(\bar{x}) = |\theta f|(\bar{x}) = |\nabla f|(\bar{x}) = \sup_{\rho > 0} \frac{|\theta f|_\rho(\bar{x})}{\rho} = \sup_{x \neq \bar{x}} \frac{[f(\bar{x}) - f(x)]_+}{\|x - \bar{x}\|},$$

$$|\theta(-f)|(\bar{x}) = |\nabla(-f)|(\bar{x}) = \inf_{\rho > 0} \frac{|\theta f|_\rho[-f](\bar{x})}{\rho} = \inf_{\rho > 0} \sup_{\|x - \bar{x}\| = \rho} \frac{[f(x) - f(\bar{x})]_+}{\rho}.$$

- (iv)  $|\nabla f|(\bar{x}) \leq |\nabla(-f)|(\bar{x})$ .
- (v)  $\overline{|\nabla f|}(\bar{x}) \leq \overline{|\theta(-f)|}(\bar{x})$ .
- (vi) If  $|\nabla f|(\bar{x}) = |\nabla(-f)|(\bar{x})$  and  $\{x_k\} \subset X$  is a sequence defining  $|\nabla f|(\bar{x})$ , that is  $x_k \rightarrow 0$  and

$$|\nabla f|(\bar{x}) = \lim_{k \rightarrow \infty} \frac{f(\bar{x}) - f(\bar{x} + x_k)}{\|x_k\|}$$

then the limit

$$\lim_{k \rightarrow \infty} \frac{f(\bar{x}) - f(\bar{x} - x_k)}{\|x_k\|}$$

exists and equals  $-|\nabla f|(\bar{x})$ .

- (vii) If  $\dim X < \infty$  and  $f$  is lower semicontinuous near  $\bar{x}$  then  $\overline{|\partial f|}(\bar{x}) = |\widehat{\partial f}|(\bar{x}) = |\partial f|(\bar{x})$ .

Proposition 2 and all assertions in Proposition 3 except the last one are slight reformulations of the corresponding statements from [32]. Assertion (vii) in Proposition 3 follows from the upper semicontinuity of the subdifferential mapping of a convex function.

### 3. COLLECTIONS OF SETS

Starting with the pioneering work by Dubovitskii and Milyutin [14] it is quite natural when dealing with optimality conditions to reformulate optimality in the original optimization problem as a kind of extremal behaviour of a certain collection of sets. Considering collections of sets is a rather general scheme of investigating optimization problems. Any set of “extremality” conditions leads to some optimality conditions for the original problem.

### 3.1. Extremal collections of sets

A typical example of “extremal behaviour” is presented on Figure 4: two convex sets with nonintersecting interiors. In the framework of convex analysis, dual extremality conditions are given by the *separation theorem*.

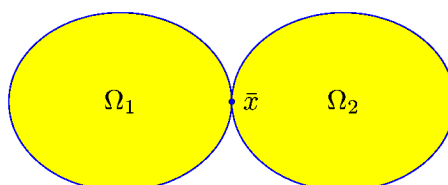


Fig. 4. Extremality of two convex sets.

A pair of sets in Figure 4 can be looked at in a different way: they have a common point and at the same time can be made unintersecting by an arbitrary small translation. Such collections of sets are called *extremal*. This point of view is applicable to nonconvex sets as well. Besides, the sets are not required to have nonempty interiors. See examples in Figure 5. In the last example in Figure 5, the second set consists of a single point  $\bar{x}$ .

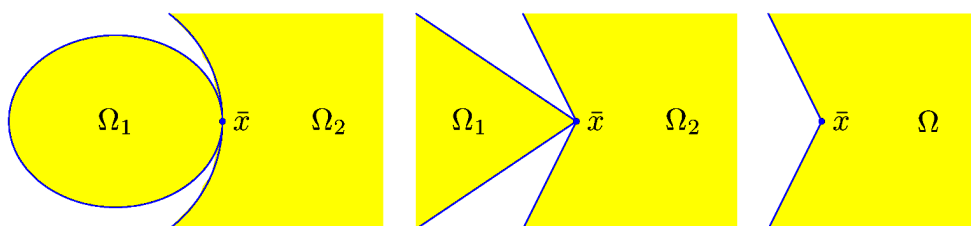


Fig. 5. Extremal collections of sets.

The definition of an *extremal collection* of sets was first introduced in 1980 in [33, 34] (see historical comments in [38]), where a dual characterization of extremality was established. This result can be considered as a generalization of the separation theorem to nonconvex sets and can be used as a tool for proving necessary optimality conditions in nonconvex problems.

For the convex sets in Figure 4 the separation property can be equivalently reformulated in the following way. There exist two *normal* (in the sense of convex analysis) elements  $x_i^* \in N(\bar{x}|\Omega_i)$ ,  $i = 1, 2$ , such that the elements are nonzero:  $\|x_1^*\| + \|x_2^*\| > 0$  while their sum is zero:  $x_1^* + x_2^* = 0$  (Figure 6).

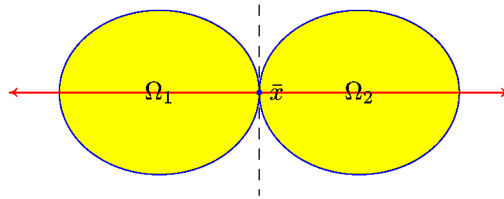


Fig. 6. Separation property.

The same idea can work for nonconvex sets (see Figure 7) if the normal cone in the sense of convex analysis is replaced by its appropriate generalization. This was first done in [33] for spaces admitting Fréchet smooth renorm and then extended in [40] to Asplund spaces. This result is now known as *Extremal principle* (see [38, 46]).

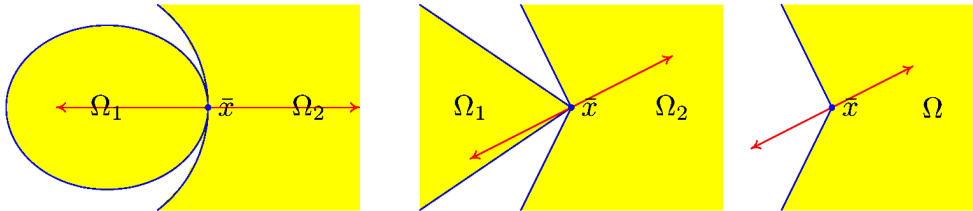


Fig. 7. Nonconvex separation property.

**3.2. Extremal principle**

In this section  $X$  is a normed linear space,  $\Omega_1, \Omega_2, \dots, \Omega_n \subset X$  ( $n > 1$ ),  $\bar{x} \in \bigcap_{i=1}^n \Omega_i$ .

The extremality of the collection of sets  $\Omega_1, \Omega_2, \dots, \Omega_n$  near  $\bar{x}$  can be characterized by the following conditions.

Extremality.

(E)<sub>S</sub> For any  $\varepsilon > 0$  there exist  $a_i \in X, i = 1, 2, \dots, n$ , such that  $\|a_i\| \leq \varepsilon$  and

$$\bigcap_{i=1}^n (\Omega_i - a_i) = \emptyset.$$

(LE)<sub>S</sub> There exists a  $\rho > 0$  such that for any  $\varepsilon > 0$  there are  $a_i \in X, i = 1, 2, \dots, n$ , such that  $\|a_i\| \leq \varepsilon$  and

$$(15) \quad \bigcap_{i=1}^n (\Omega_i - a_i) \cap B_\rho(\bar{x}) = \emptyset.$$



(SP)<sub>S</sub> For any  $\varepsilon > 0$  there exist  $x_i \in \Omega_i \cap B_\varepsilon(\bar{x})$  and  $x_i^* \in N(x_i|\Omega_i)$ ,  $i = 1, 2, \dots, n$ , such that  $\sum_{i=1}^n \|x_i^*\| = 1$  and  $\left\| \sum_{i=1}^n x_i^* \right\| \leq \varepsilon$ .

The subscript “S” in the notations of the above and forthcoming properties in this section means that the properties are defined for collections of sets. Its aim is to avoid confusion with the properties introduced in Sections 2 and 4.

In the last property  $N(x|\Omega)$  denotes the *Fréchet normal cone* to  $\Omega$  at  $x \in \Omega$ :

$$(16) \quad N(x|\Omega) = \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x \rangle}{\|u - x\|} \leq 0 \right\}.$$

Here  $u \xrightarrow{\Omega} x$  means  $u \rightarrow x$  with  $u \in \Omega$ . If  $\Omega$  is convex the set (16) coincides with the normal cone in the sense of convex analysis.

Property (LE)<sub>S</sub> characterizes *local extremality*. Obviously, (E)<sub>S</sub>  $\Rightarrow$  (LE)<sub>S</sub>. On the other hand, (LE)<sub>S</sub> implies property (E)<sub>S</sub> for the collection of  $n + 1$  sets  $\Omega_1, \Omega_2, \dots, \Omega_n, B_\delta(\bar{x})$ .

Property (SP)<sub>S</sub> is a dual condition. It represents a kind of nonconvex *separation property*. Two more versions of this property can be of interest – the basic (SPB)<sub>S</sub> and the limiting (SPL)<sub>S</sub>:

(SPB)<sub>S</sub> There exist  $x_i^* \in N(\bar{x}|\Omega_i)$ ,  $i = 1, 2, \dots, n$ , such that

$$(17) \quad \sum_{i=1}^n \|x_i^*\| > 0 \quad \text{and} \quad \sum_{i=1}^n x_i^* = 0.$$

(SPL)<sub>S</sub> There exist  $x_i^* \in \bar{N}(\bar{x}|\Omega_i)$ ,  $i = 1, 2, \dots, n$ , such that conditions (17) hold true.

Here  $\bar{N}(\bar{x}|\Omega)$  denotes the *limiting normal cone* to  $\Omega$  at  $\bar{x} \in \Omega$ :

$$(18) \quad \bar{N}(\bar{x}|\Omega) = \{x^* \in X^* \mid x_k \rightarrow \bar{x}, x_k^* \xrightarrow{w^*} x^*, x_k^* \in N(x_k|\Omega), k = 1, 2, \dots\},$$

In the convex case, cone (18) also coincides with the normal cone in the sense of convex analysis.

The next assertion is straightforward.

**Proposition 4.**

- (i) (SPB)<sub>S</sub>  $\Rightarrow$  (SP)<sub>S</sub>;
- (ii) If  $\dim X < \infty$  then (SP)<sub>S</sub>  $\Leftrightarrow$  (SPL)<sub>S</sub>.

An advantage of  $(SP)_S$  and  $(SPL)_S$  is that, unlike “fuzzy” condition  $(SP)_S$ , they provide dual criteria “at the point”.

When dealing with fuzzy and limiting conditions like  $(SP)_S$  and  $(SPL)_S$ , the following notation can be convenient ( $\delta \geq 0$ ):

$$(19) \quad \hat{N}_\delta(\bar{x}|\Omega) = \bigcup_{x \in \Omega \cap B_\delta(\bar{x})} N(x|\Omega).$$

This is the *strict  $\delta$ -normal cone* [24, 28] to  $\Omega$  at  $\bar{x} \in \Omega$ . Both cones (18) and (19) can be nonconvex. Using (19), the definition (18) can be rewritten as

$$\bar{N}(\bar{x}|\Omega) = \bigcap_{\delta > 0} \text{cl}^* \hat{N}_\delta(\bar{x}|\Omega),$$

where  $\text{cl}^*$  denotes the sequential weak\* closure in  $X^*$ .

Under certain conditions  $(SP)_S$  is implied by  $(LE)_S$ , and hence provides a dual characterization of local extremality. This is known as Extremal Principle.

**Extremal Principle.**  $(LE)_S \Rightarrow (SP)_S$ .

The following theorem was established in [40, Theorem 3.2] (see also [38, Theorem 2.20]) as a generalization of [33, Theorem 1].

**Theorem 3.** *Let the sets  $\Omega_1, \Omega_2, \dots, \Omega_n$  be locally closed near  $\bar{x}$ . Then the following conditions are equivalent;*

- (i)  *$X$  is Asplund;*
- (ii) *Extremal Principle holds true in  $X$ .*

Thus, in Asplund spaces,  $(SP)_S$  provides a dual necessary condition for local extremality. It has proved to be a useful tool for investigating nonconvex objects far beyond the framework of optimization theory (see the comments in [38]).

Another nonconvex separation property was developed in [7]; see [7] and [30] for the relationships between the two approaches.

Being in general weaker than local extremality, the separation property  $(SP)_S$  can be considered as a dual approximate stationarity condition for a collection of sets near a given point. Similarly to the case of a real-valued function, it is possible to define for a collection of sets some primal space stationarity properties being weaker than local extremality but still implying  $(SP)_S$ .

### 3.3. Stationarity and regularity

A natural way to define for a collection of sets stationarity properties is to use the following conditions.

Stationarity and approximate stationarity.

- (S)<sub>S</sub> For any  $\varepsilon > 0$  there exists a  $\rho \in (0, \varepsilon)$  and  $a_i \in X, i = 1, 2, \dots, n$ , such that  $\|a_i\| \leq \varepsilon\rho$  and (15) holds true.
- (AS)<sub>S</sub> For any  $\varepsilon > 0$  there exists a  $\rho \in (0, \varepsilon)$ ,  $\omega_i \in \Omega_i \cap B_\varepsilon(\bar{x})$ , and  $a_i \in X, i = 1, 2, \dots, n$ , such that  $\|a_i\| \leq \varepsilon\rho$  and

$$\bigcap_{i=1}^n (\Omega_i - \omega_i - a_i) \cap \rho B = \emptyset.$$

**Proposition 5.**  $(LE)_S \Rightarrow (S)_S \Rightarrow (AS)_S$ .

*Proof.* The first implication is obvious. The comparison of (S)<sub>S</sub> and (AS)<sub>S</sub> becomes straightforward if to rewrite (15) in the form

$$\bigcap_{i=1}^n (\Omega_i - \bar{x} - a_i) \cap \rho B = \emptyset. \quad \blacksquare$$

The transition from stationarity to approximate stationarity means that instead of considering each set  $\Omega_i$  near the given point  $\bar{x}$  it is sufficient to find an appropriate point  $\omega_i \in \Omega_i$  close to  $\bar{x}$ , such that the collection of shifted sets  $\Omega_i - \omega_i, i = 1, 2, \dots, n$ , “almost” possesses the stationarity property near 0. Note also that the single common point  $\bar{x}$  in (S)<sub>S</sub> is replaced in (AS)<sub>S</sub> by a collection of points  $\omega_i \in \Omega_i, i = 1, 2, \dots, n$ , each set being considered near its own point.

For the first pair of sets in Figure 8, condition (S)<sub>S</sub> is satisfied while (LE)<sub>S</sub> is not. In the second example in Figure 8 property (AS)<sub>S</sub> holds (consider the points  $\omega_1 \in \Omega_1$  and  $\omega_2 \in \Omega_2$ ) while (S)<sub>S</sub> does not. Note that in the first example basic separation property (SPB)<sub>S</sub> holds true, while separation properties (SP)<sub>S</sub> and (SPL)<sub>S</sub> hold in both examples.

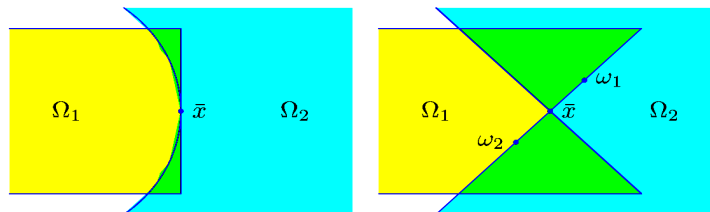


Fig. 8. Stationarity and approximate stationarity.

The negations of primal space stationarity properties (S)<sub>S</sub> and (AS)<sub>S</sub> as well as the dual space properties (SP)<sub>S</sub>, (SPB)<sub>S</sub>, and (SPL)<sub>S</sub> define the corresponding regularity properties for a collection of sets near the given point.

Regularity, uniform regularity, and dual uniform regularity.

(R)<sub>S</sub> There exists an  $\alpha > 0$  and a  $\delta > 0$  such that

$$\bigcap_{i=1}^n (\Omega_i - a_i) \cap B_\rho(\bar{x}) \neq \emptyset$$

for any  $\rho \in (0, \delta)$  and any  $a_i \in X$ ,  $i = 1, 2, \dots, n$ , satisfying  $\|a_i\| \leq \alpha\rho$ .

(UR)<sub>S</sub> There exists an  $\alpha > 0$  and a  $\delta > 0$  such that

$$\bigcap_{i=1}^n (\Omega_i - \omega_i - a_i) \cap \rho B \neq \emptyset$$

for any  $\rho \in (0, \delta)$ ,  $\omega_i \in \Omega_i \cap B_\delta(\bar{x})$ , and  $a_i \in X$ ,  $i = 1, 2, \dots, n$ , satisfying  $\|a_i\| \leq \alpha\rho$ .

(RD)<sub>S</sub> There exists an  $\alpha > 0$  such that

$$(20) \quad \left\| \sum_{i=1}^n x_i^* \right\| \geq \alpha \sum_{i=1}^n \|x_i^*\|$$

for all  $x_i^* \in N(\bar{x}|\Omega_i)$ ,  $i = 1, 2, \dots, n$ .

(URD)<sub>S</sub> There exists an  $\alpha > 0$  and a  $\delta > 0$  such that (20) holds for all  $x_i^* \in \hat{N}_\delta(\bar{x}|\Omega_i)$ ,  $i = 1, 2, \dots, n$ .

(URDL)<sub>S</sub> There exists an  $\alpha > 0$  such that (20) holds for all  $x_i^* \in \bar{N}(\bar{x}|\Omega_i)$ ,  $i = 1, 2, \dots, n$ .

All these regularity properties hold true for the pair of sets on Figure 9.

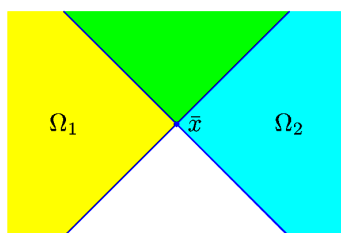


Fig. 9. Regularity.

### 3.4. Constants

Similarly to the case of a real-valued function, it can be convenient to use for describing the defined above extremality, stationarity, and regularity properties of

collections of sets certain nonnegative constants [29-31]:

$$(21) \quad \theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x}) = \sup \left\{ r \geq 0 \mid \bigcap_{i=1}^n (\Omega_i - a_i) \cap B_\rho(\bar{x}) \neq \emptyset, \forall a_i \in B_r \right\},$$

$$(22) \quad \theta[\Omega_1, \dots, \Omega_n](\bar{x}) = \liminf_{\rho \downarrow 0} \frac{\theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x})}{\rho},$$

$$(23) \quad \hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \liminf_{\substack{\omega_i \xrightarrow{\Omega_i} \bar{x} \\ \rho \downarrow 0}} \frac{\theta_\rho[\Omega_1 - \omega_1, \dots, \Omega_n - \omega_n](0)}{\rho},$$

$$(24) \quad \eta[\Omega_1, \dots, \Omega_n](\bar{x}) = \inf \left\{ \left\| \sum_{i=1}^n x_i^* \right\| \mid x_i^* \in N(\bar{x}|\Omega_i), \sum_{i=1}^n \|x_i^*\| = 1 \right\};$$

$$(25) \quad \hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \liminf_{\delta \downarrow 0} \left\{ \left\| \sum_{i=1}^n x_i^* \right\| \mid x_i^* \in \hat{N}_\delta(\bar{x}|\Omega_i), \sum_{i=1}^n \|x_i^*\| = 1 \right\},$$

$$(26) \quad \bar{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \inf \left\{ \left\| \sum_{i=1}^n x_i^* \right\| \mid x_i^* \in \bar{N}(\bar{x}|\Omega_i), \sum_{i=1}^n \|x_i^*\| = 1 \right\}.$$

The notation  $\omega \xrightarrow{\Omega} \bar{x}$  in (23) means that  $\omega \rightarrow \bar{x}$  with  $\omega \in \Omega$ . The last two constants are defined in terms of dual space elements.

The following equivalences are consequences of the definitions.

- $\theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x}) = 0$  for all  $\rho > 0 \iff (E)_S;$
- $\theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x}) = 0$  for some  $\rho > 0 \iff (LE)_S;$
- $\theta[\Omega_1, \dots, \Omega_n](\bar{x}) = 0 \iff (S)_S;$
- $\theta[\Omega_1, \dots, \Omega_n](\bar{x}) > 0 \iff (R)_S;$
- $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) = 0 \iff (AS)_S;$
- $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) > 0 \iff (UR)_S;$
- $\eta[\Omega_1, \dots, \Omega_n](\bar{x}) = 0 \iff (SPB)_S;$
- $\eta[\Omega_1, \dots, \Omega_n](\bar{x}) > 0 \iff (RD)_S;$
- $\hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) = 0 \iff (SP)_S;$
- $\hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) > 0 \iff (URD)_S;$
- $\bar{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) = 0 \iff (SPL)_S;$
- $\bar{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) > 0 \iff (URDL)_S.$

For the regularity properties, the corresponding constant provides a quantitative characterization of this property. It coincides with the exact lower bound of all  $\alpha$  in the inequality defining the property.

The next theorem summarizes the list of relations between the constants.

**Theorem 4.** *Let the sets  $\Omega_1, \Omega_2, \dots, \Omega_n$  be locally closed near  $\bar{x}$ . The following assertions hold true:*

- (i)  $\lim_{\rho \downarrow 0} \theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x}) > 0$  if and only if  $\bar{x} \in \text{int} \cap_{i=1}^n \Omega_i$ ;  
in this case  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \theta[\Omega_1, \dots, \Omega_n](\bar{x}) = \eta[\Omega_1, \dots, \Omega_n](\bar{x}) = \hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \bar{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \infty$ ;
- (ii) if  $\bar{x} \in \text{bd} \bigcap_{i=1}^n \Omega_i$  then  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) \leq 1, \hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) \leq 1$ ;
- (iii) if  $\theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x}) = 0$  for some  $\rho > 0$  then  $\theta[\Omega_1, \dots, \Omega_n](\bar{x}) = 0$ ;
- (iv)  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) \leq \liminf_{\omega_i \rightarrow \bar{x}} \theta[\Omega_1 - \omega_1, \dots, \Omega_n - \omega_n](0) \leq \theta[\Omega_1, \dots, \Omega_n](\bar{x})$ ;
- (v)  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) \leq \hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x})$ ;
- (vi) if  $X$  is Asplund, then  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x})$ ;
- (vii) if  $\dim X < \infty$  then  $\bar{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x})$ .

*Proof.* The majority of assertions in the theorem can be found in [25-31]. The only one which needs proof is the inequality  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) \geq \hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x})$  in assertion (vi).

Let  $X$  be Asplund,  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) < \alpha, \delta > 0$ . Taking into account definition (25), we need to show that there exist  $x_i \in \Omega_i \cap B_\delta(\bar{x}), x_i^* \in N(x_i | \Omega_i), i = 1, 2, \dots, n$ , such that  $\sum_{i=1}^n \|x_i^*\| = 1$  and  $\|\sum_{i=1}^n x_i^*\| < \alpha$ .

Choose numbers  $\alpha_1, \alpha_2$ , satisfying  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) < \alpha_1 < \alpha_2 < \alpha$ , and put  $\gamma = (\alpha_2 + 1)^{-1}$ . By definitions (23) and (21), there exists a positive number  $\rho < \gamma\delta/2$  and points  $\omega_i \in \Omega_i \cap B_{\delta/2}(\bar{x}), a_i \in (\alpha_1\rho)B, i = 1, 2, \dots, n$ , such that

$$\bigcap_{i=1}^n (\Omega_i - \omega_i - a_i) \cap (\rho B) = \emptyset,$$

and consequently

$$f_1(u, v_1, \dots, v_n) := \max_{1 \leq i \leq n} \|v_i - \omega_i - a_i - u\| > 0$$

for all  $u \in \rho B$  and  $v_i \in \Omega_i, i = 1, 2, \dots, n$ . At the same time,  $f_1(0, \omega_1, \dots, \omega_n) = \max_{1 \leq i \leq n} \|a_i\| \leq \alpha_1\rho$ .

Consider the space  $X^{n+1}$  with the norm  $\|\cdot\|_\gamma$  defined by

$$\|(u, v_1, \dots, v_n)\|_\gamma = \max(\|u\|, \gamma \max_{1 \leq i \leq n} \|v_i\|).$$

Then  $X^{n+1}$  is a Banach space (actually it is even Asplund), and we can apply Ekeland variational principle. Take  $\rho' = \rho\alpha_1/\alpha_2$ . It follows that there exist points  $u' \in \rho'B$  and  $\omega'_i \in \Omega_i \cap B_{\rho'/\gamma}(\omega_i)$  such that

$$f_1(u, v_1, \dots, v_n) - f_1(u', \omega'_1, \dots, \omega'_n) + \alpha_2\|(u - u', v_1 - \omega'_1, \dots, v_n - \omega'_n)\|_\gamma > 0$$

for all  $u \in \rho B$  and  $v_i \in \Omega_i, i = 1, 2, \dots, n$ . Note that  $u'$  is an internal point of  $\rho B$ . Hence  $(u', \omega'_1, \dots, \omega'_n)$  is a point of local minimum (on  $X^{n+1}$ ) for the sum  $f_1 + f_2 + f_3$ , where

$$f_2(u, v_1, \dots, v_n) := \alpha_2 \|(u - u', v_1 - \omega'_1, \dots, v_n - \omega'_n)\|_\gamma,$$

$$f_3(v_1, \dots, v_n) := \begin{cases} 0 & \text{if } v_i \in \Omega_i, i = 1, 2, \dots, n, \\ \infty & \text{otherwise.} \end{cases}$$

Thus,  $0 \in \partial(f_1 + f_2 + f_3)(u', \omega'_1, \dots, \omega'_n)$ .

Functions  $f_1$  and  $f_2$  are convex and Lipschitz continuous. We can apply the fuzzy sum rule [16]. Note that  $\max_{1 \leq i \leq n} \|\omega'_i - \omega_i - a_i - u'\| > 0$ . The Fréchet subdifferentials of  $f_1, f_2$ , and  $f_3$  possess the following properties:

(1) if  $(u_1^*, v_{11}^*, \dots, v_{1n}^*) \in \partial f_1(u, v_1, \dots, v_n)$  then

$$(27) \quad u_1^* = - \sum_{i=1}^n v_{1i}, \quad \sum_{i=1}^n \|v_{1i}^*\| = 1$$

for any  $(u, v_1, \dots, v_n)$  near  $(u', \omega'_1, \dots, \omega'_n)$ ;

(2) if  $(u_2^*, v_{21}^*, \dots, v_{2n}^*) \in \partial f_2(u, v_1, \dots, v_n)$  then

$$(28) \quad \|u_2^*\| + \gamma^{-1} \sum_{i=1}^n \|v_{2i}^*\| \leq \alpha_2$$

for any  $(u, v_1, \dots, v_n) \in X^{n+1}$ ;

(3)  $\partial f_3(v_1, \dots, v_n) = \prod_{i=1}^n N(v_i | \Omega_i)$  for any  $v_i \in \Omega_i, i = 1, 2, \dots, n$ .

Note that  $\rho/\gamma < \delta/2$ . Chose an  $\varepsilon \in (0, \gamma)$  such that  $(\alpha_2 + 2)\varepsilon/(\gamma - \varepsilon) \leq \alpha - \alpha_2$ . Applying the fuzzy sum rule we find points  $x_i \in \Omega_i \cap B_{\delta/2 - \rho/\gamma}(\omega'_i), i = 1, 2, \dots, n$ , and elements  $u_1^*, u_2^*, v_{1i}^*, v_{2i}^* \in X^*, i = 1, 2, \dots, n$ , satisfying (27), (28), and  $v_{3i}^* \in N(x_i | \Omega_i), i = 1, 2, \dots, n$ , such that

$$(29) \quad \|u_1^* + u_2^*\| \leq \varepsilon, \quad \sum_{i=1}^n \|v_{1i}^* + v_{2i}^* + v_{3i}^*\| \leq \varepsilon.$$

Then  $\|x_i - \bar{x}\| \leq \|x_i - \omega'_i\| + \|\omega'_i - \omega_i\| + \|\omega_i - \bar{x}\| \leq \delta$ . Denote  $\beta := \sum_{i=1}^n \|v_{2i}^*\|$ . By (28),  $0 \leq \beta \leq \gamma\alpha_2 < 1$ . By the second inequality in (29) and the second equality in (27), we have

$$\sum_{i=1}^n \|v_{3i}^*\| \geq 1 - \beta - \varepsilon \geq \gamma - \varepsilon > 0.$$

The second inequality in (29) implies also  $\|\sum_{i=1}^n (v_{1i}^* + v_{2i}^* + v_{3i}^*)\| \leq \varepsilon$ , and consequently, applying successively the first equality in (27), the first inequality in (29), and inequality (28) and recalling the definition of  $\gamma$ , we obtain

$$\begin{aligned} \left\| \sum_{i=1}^n v_{3i}^* \right\| &\leq \left\| \sum_{i=1}^n v_{1i}^* \right\| + \beta + \varepsilon \\ &\leq \|u_2^*\| + \beta + 2\varepsilon \leq \alpha_2 + (1 - \gamma^{-1})\beta + 2\varepsilon = \alpha_2(1 - \beta) + 2\varepsilon. \end{aligned}$$

Put  $x_i^* = v_{3i}^* / \sum_{i=1}^n \|v_{3i}^*\|$ ,  $i = 1, 2, \dots, n$ . Then obviously  $x_i^* \in N(x_i | \Omega_i)$ ,  $i = 1, 2, \dots, n$ ,  $\sum_{i=1}^n \|x_i^*\| = 1$ , and

$$\left\| \sum_{i=1}^n x_i^* \right\| \leq \frac{\alpha_2(1 - \beta) + 2\varepsilon}{1 - \beta - \varepsilon} = \alpha_2 + \frac{(\alpha_2 + 2)\varepsilon}{1 - \beta - \varepsilon} \leq \alpha_2 + \frac{(\alpha_2 + 2)\varepsilon}{\gamma - \varepsilon} \leq \alpha.$$

The proof is completed. ■

Both inequalities in Theorem 4 (iv) can be strict, see [31, Example 1] for the first inequality and the second example in Figure 8 for the second one.

Due to (ii), if  $\bar{x} \in \text{bd} \bigcap_{i=1}^n \Omega_i$  then constants (23) and (25) are less than or equal to 1. Such an estimate does not hold for constants (21) and (22). Of course,  $\lim_{\rho \downarrow 0} \theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x}) = 0$  due to (i). However, for large  $\rho$ ,  $\theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x})$  can be as large as we wish, see the examples in Figure 8. In the first of these examples,  $\theta[\Omega_1, \Omega_2](\bar{x}) = 0$  since condition (S)<sub>S</sub> is satisfied. The last constant can be large as well; it can even be infinite, see the example in Figure 10 where the sets  $\Omega_1$  and  $\Omega_2$  “strongly overlap”.

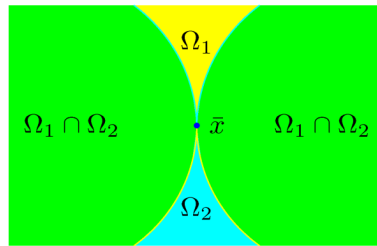
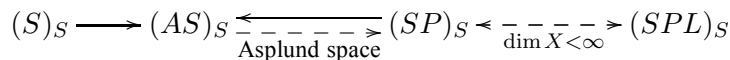


Fig. 10.  $\theta[\Omega_1, \dots, \Omega_n](\bar{x}) = \infty$ .

Theorem 4 (vi) improves [29, Theorem 1].

In accordance with Theorem 4 the relationships between the stationarity concepts for collections of closed sets can be described by the following diagram:





A similar diagram (with opposite arrows) describes relationships between the corresponding regularity concepts.

Due to Proposition 5 the equivalence of  $(AS)_S$  and  $(SP)_S$  is a stronger statement than Extremal Principle. It is called Extended Extremal Principle [28, 31].

**Extended Extremal Principle.**  $(AS)_S \Leftrightarrow (SP)_S$ .

The next theorem extends Theorem 3. It follows from Theorem 3, Proposition 5, and Theorem 4 (vi).

**Theorem 5.** *Let the sets  $\Omega_1, \Omega_2, \dots, \Omega_n$  be locally closed near  $\bar{x}$ . Then the following conditions are equivalent;*

- (i)  $X$  is Asplund;
- (ii) Extremal Principle holds true in  $X$ .
- (iii) Extended Extremal Principle holds true in  $X$ .

### 3.5. Other regularity properties

There exist other important properties which could qualify for being characterizations of a kind of regularity of collections of sets near a given point.

Metric inequality (local linear regularity) [4, 18, 19, 42, 47].

$(MI)_S$  There exists an  $\alpha > 0$  and a  $\delta > 0$  such that

$$d(x, \bigcap_{i=1}^n \Omega_i) \leq \alpha \max_{1 \leq i \leq n} d(x, \Omega_i)$$

for any  $x \in B_\delta(\bar{x})$ .

It is a well-known notion in optimization and approximation theory, playing a key role in establishing linear convergence rate of numerical algorithms. In many articles this property is formulated with the sum replacing maximum in the above inequality. It is not difficult to check that both formulations are equivalent.

The property is satisfied in the example in Figure 9 but fails for the sets in Figure 4. The regularity property  $(R)_S$ , introduced earlier, behaves the same way in both these examples. However, in general the two regularity properties are different and independent. For instance, in all three examples in Figure 5 extremality condition  $(E)_S$  holds true and consequently  $(R)_S$  does not hold, while in the last two examples condition  $(MI)_S$  is satisfied. The same situation can be detected even in the convex case.

**Example 1.** Let  $\Omega_1 = \Omega_2$  be a straight line in  $\mathbb{R}^2$  and  $\bar{x}$  be any point on this line. Then both  $(E)_S$  (and consequently  $(S)_S$ ) and  $(MI)_S$  hold true simultaneously.

The reverse situation is also possible, see the example on Figure 10. Here regularity condition  $(R)_S$  holds true while condition  $(MI)_S$  does not.

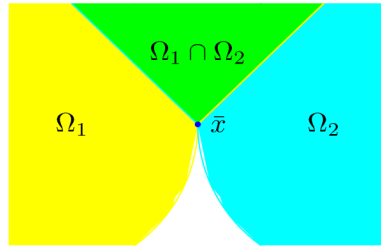


Fig. 11.  $(R)_S$  holds while  $(MI)_S$  does not.

The next property is obviously stronger than  $(MI)_S$  since it requires the metric inequality to hold not only for the original collection of sets but also for all small translations of the sets, with an estimate being uniform.

Uniform metric inequality [30, 31].

$(UMI)_S$  There exists an  $\alpha > 0$  and a  $\delta > 0$  such that

$$d\left(x, \bigcap_{i=1}^n (\Omega_i - x_i)\right) \leq \alpha \max_{1 \leq i \leq n} d(x, (\Omega_i - x_i))$$

for any  $x \in B_\delta(\bar{x})$  and  $x_i \in \delta B, i = 1, 2, \dots, n$ .

Note that  $(UMI)_S$  does not hold in Example 1.

The next proposition recaptures [30, Theorem 1]. It presents an equivalent representation of the uniform regularity constant  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x})$ , which yields immediately the equivalence of uniform metric inequality  $(UMI)_S$  and uniform regularity property  $(UR)_S$ .

**Proposition 6.** 
$$\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \liminf_{\substack{x \rightarrow \bar{x}, x_i \rightarrow 0 \\ x \notin \bigcap_{i=1}^n (\Omega_i - x_i)}} \frac{\max_{1 \leq i \leq n} d(x, \Omega_i - x_i)}{d\left(x, \bigcap_{i=1}^n (\Omega_i - x_i)\right)}.$$

**Corollary 6.1.**  $(UR)_S \Leftrightarrow (UMI)_S \Rightarrow (MI)_S$ .

Dual space regularity conditions  $(URD)_S$  and  $(URDL)_S$  are actually certain regularity conditions imposed on collections of strict  $\delta$ -normal cones and limiting

normal cones respectively. In general, regularity conditions for collections of cones in a dual space, when applied to normal cones, generate certain dual space regularity conditions for collections of sets in the primal space. An important example is provided by *Jameson's property (G)* (see [5, 41]). Applied to  $\delta$ -normal and limiting normal cones it produces the following two regularity properties.

Regularity based on Jameson's property (G).

(G)<sub>S</sub> There exists an  $\alpha > 0$  and a  $\delta > 0$  such that for any  $x^* \in \sum_{i=1}^n \hat{N}_\delta(\bar{x}|\Omega_i)$  there

are  $x_i^* \in \hat{N}_\delta(\bar{x}|\Omega_i)$ ,  $i = 1, 2, \dots, n$ , satisfying  $\sum_{i=1}^n x_i^* = x^*$  and

$$(30) \quad \alpha \sum_{i=1}^n \|x_i^*\| \leq \|x^*\|.$$

(GL)<sub>S</sub> There exists an  $\alpha > 0$  such that for any  $x^* \in \sum_{i=1}^n \bar{N}(\bar{x}|\Omega_i)$  there are  $x_i^* \in$

$\bar{N}(\bar{x}|\Omega_i)$ ,  $i = 1, 2, \dots, n$ , such that  $\sum_{i=1}^n x_i^* = x^*$ , and (30) holds true.

Note that (G)<sub>S</sub> (as well as its limiting version (GL)<sub>S</sub>) is a rather weak regularity condition. It is satisfied in all examples considered in this paper. In the convex setting, it is used as a complement to the *strong conical hull intersection property* when characterizing (MI)<sub>S</sub> [41, 47]. See [3] for the discussion of the role played by (G)<sub>S</sub> in variational analysis and some historical comments.

The next proposition provides upper estimates for the dual uniform regularity constants (25) and (26) in terms of the data involved in the definitions of properties (G)<sub>S</sub> and (GL)<sub>S</sub>. it follows immediately from the definitions.

**Proposition 7.** *Let the sets  $\Omega_1, \Omega_2, \dots, \Omega_n$  be locally closed near  $\bar{x}$ . Then*

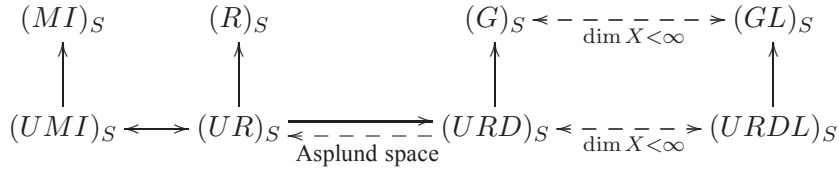
$$(i) \quad \hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) \leq \lim_{\delta \downarrow 0} \inf_{0 \neq x^* \in \sum_{i=1}^n \hat{N}_\delta(\bar{x}|\Omega_i)} \sup_{\substack{x_i^* \in \hat{N}_\delta(\bar{x}|\Omega_i) \\ \sum_{i=1}^n x_i^* = x^*}} \frac{\|x^*\|}{\sum_{i=1}^n \|x_i^*\|};$$

$$(ii) \quad \bar{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) \leq \inf_{0 \neq x^* \in \sum_{i=1}^n \bar{N}(\bar{x}|\Omega_i)} \sup_{\substack{x_i^* \in \bar{N}(\bar{x}|\Omega_i) \\ \sum_{i=1}^n x_i^* = x^*}} \frac{\|x^*\|}{\sum_{i=1}^n \|x_i^*\|}.$$

The constants in the right-hand sides of the inequalities in Proposition 7 characterize properties (G)<sub>S</sub> and (GL)<sub>S</sub>. We will denote them  $\hat{\eta}_G[\Omega_1, \dots, \Omega_n](\bar{x})$  and  $\bar{\eta}_G[\Omega_1, \dots, \Omega_n](\bar{x})$  respectively.

**Corollary 7.1.**  $(URD)_S \Rightarrow (G)_S, \quad (URDL)_S \Rightarrow (GL)_S.$

In accordance with Theorem 4 and Corollaries 6.1 and 7.1 the relationships between the regularity concepts for collections of sets can be described by the following diagram:



A similar diagram (with opposite arrows) describes the relationships between the corresponding stationarity concepts.

Note that conditions  $(R)_S$ ,  $(MI)_S$ , and  $(G)_S$  are independent.

**3.6. Convex case**

For convex sets the concepts of extremality, local extremality, stationarity and approximate stationarity coincide; regularity and uniform regularity coincide too. This follows from the next proposition established in [30].

**Proposition 8.** *Let  $\Omega_1, \Omega_2, \dots, \Omega_n$  be convex.*

- (i) *If  $\theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x}) > 0$  for some  $\rho > 0$  then  $\theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x}) > 0$  for all  $\rho > 0$ .*
- (ii) *The function  $\rho \rightarrow \theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x})/\rho$  is nonincreasing on  $\mathbb{R}_+ \setminus \{0\}$ .*
- (iii)  $\theta[\Omega_1, \dots, \Omega_n](\bar{x}) = \sup_{\rho > 0} \theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x})/\rho$ .
- (iv)  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \theta[\Omega_1, \dots, \Omega_n](\bar{x})$ .
- (v) *If  $\text{int } \Omega_i \neq \emptyset, i = 1, 2, \dots, n - 1$ , then  $\theta[\Omega_1, \dots, \Omega_n](\bar{x}) = 0$  if and only if  $\bigcap_{i=1}^{n-1} \text{int } \Omega_i \cap \Omega_n = \emptyset$ .*
- (vi)  $\eta[\Omega_1, \dots, \Omega_n](\bar{x}) = \hat{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \bar{\eta}[\Omega_1, \dots, \Omega_n](\bar{x})$ .
- (vii)  $\hat{\eta}_G[\Omega_1, \dots, \Omega_n](\bar{x}) = \bar{\eta}_G[\Omega_1, \dots, \Omega_n](\bar{x})$   
 $= \inf_{0 \neq x^* \in \sum_{i=1}^n N(\bar{x}|\Omega_i)} \sup \left\{ \|x^*\| / \sum_{i=1}^n \|x_i^*\| \mid x_i^* \in N(\bar{x}|\Omega_i), \sum_{i=1}^n x_i^* = x^* \right\}$ .

Note that  $(MI)_S$  and  $(G)_S$  ( and its limiting version  $(GL)_S$ ) can still be weaker than  $(R)_S$  – see Example 1 above.

4. MULTIFUNCTIONS

Multifunctions (set-valued mappings) represent another typical and very convenient setting for dealing with optimization/variational problems, with their regularity

being the key to different stability issues, subdifferential calculus, constraint qualifications, etc., see [1, 8, 10, 19-21, 36, 38, 46].

In this section, along the lines of Section 2, we discuss some regularity and stationarity concepts of multifunctions closely related to the corresponding properties of real-valued functions and collections of sets investigated in the preceding sections.

Consider a multifunction  $F : X \rightrightarrows Y$  between metric spaces and a point  $(\bar{x}, \bar{y}) \in \text{gph } F = \{(x, y) \in X \times Y \mid y \in F(x)\}$ . If not explicitly stated otherwise, we assume that  $X \times Y$  is a metric space with the maximum type distance:  $d((x_1, y_1), (x_2, y_2)) = \max(d(x_1, x_2), d(y_1, y_2))$ . When formulating dual space characterizations, we will assume  $X$  and  $Y$  to be normed linear spaces.

**4.1. Regularity**

The next three properties represent analogs of inf- and sup-regularity properties discussed in Section 2.

Regularity.

(Cov)<sub>M</sub> There exists an  $\alpha > 0$  and a  $\delta > 0$  such that for any  $\rho \in (0, \delta)$

$$(31) \quad B_{\alpha\rho}(\bar{y}) \subset F(B_\rho(\bar{x})).$$

(SeR)<sub>M</sub> There exists an  $\alpha > 0$  and a  $\delta > 0$  such that for any  $y \in B_\delta(\bar{y})$

$$\alpha d(\bar{x}, F^{-1}(y)) \leq d(y, \bar{y}).$$

(RD)<sub>M</sub> ( $X$  and  $Y$  are normed linear spaces)  $D^*F^{-1}(\bar{y}, \bar{x})(0) = \{0\}$ .

In the last property  $D^*F^{-1}(\bar{y}, \bar{x}) : X^* \rightrightarrows Y^*$  denotes the *Fréchet coderivative* of  $F^{-1}$  at  $(\bar{y}, \bar{x})$ . Since  $F$  and  $F^{-1}$  share the same graph in  $X \times Y$  the coderivative mapping can be defined by

$$(32) \quad D^*F^{-1}(\bar{y}, \bar{x})(x^*) = \{y^* \in Y^* \mid (x^*, -y^*) \in N((\bar{x}, \bar{y})|\text{gph } F)\}.$$

The subscript “M” in the notations of the above and forthcoming properties in this section means that the properties are defined for multifunctions.

Property (Cov)<sub>M</sub> can be interpreted as  $\alpha$ -covering of  $F$  at  $(\bar{x}, \bar{y})$ . (RD)<sub>M</sub> is a dual space regularity property.

Property (SeR)<sub>M</sub> is a weakened “at a point” version of the famous *metric regularity* property (see property (MR)<sub>M</sub> below) when the point  $x = \bar{x}$  (as well as the corresponding to it point  $\bar{y} \in F(\bar{x})$ ) is fixed in (34). Another “at a point” version of (MR)<sub>M</sub> corresponds to fixing  $y = \bar{y}$  in (34). This very useful property was felicitously coined by Dontchev and Rockafellar [13] as *subregularity*. To distinguish from subregularity we are going to call property (SeR)<sub>M</sub> *semiregularity*. A

multifunction being subregular at a point is equivalent to its inverse being *calm*. Similarly property  $(\text{SeR})_M$  means that  $F^{-1}$  is *Lipschitz lower semicontinuous* (with rank  $\alpha$ ) [21] at  $(\bar{y}, \bar{x})$ . It will be shown in Theorem 6 (i) that properties  $(\text{Cov})_M$  and  $(\text{SeR})_M$  are equivalent.

Corollary 9.1 below shows that property  $(\text{Cov})_M$  generalizes inf- and sup-regularity properties (IR1) and (SR1) of real-valued functions, while property  $(\text{RD})_M$  generalizes properties (IRD) and (SRD). Property  $(\text{SeR})_M$  can be considered as an analog of properties (IR2) and (SR2). At the same time the realization of property  $(\text{SeR})_M$  for the cases of the epigraphical and hypographical multifunctions leads in general to stronger inf- and sup-regularity properties. In the general setting of metric spaces a complete analog of (IR2) and (SR2) does not exist. The latter two properties depend heavily on the linear and order structure in the image space.

When  $F$  is single-valued near  $\bar{x}$  all three properties  $(\text{Cov})_M$ ,  $(\text{SeR})_M$ , and  $(\text{RD})_M$  are in general stronger than the corresponding “combined” regularity properties discussed in Section 2.

As in the preceding sections, the next step is to define the uniform analogs of regularity properties  $(\text{Cov})_M$ ,  $(\text{SeR})_M$ , and  $(\text{RD})_M$ . It can be done along the same lines. As a result one obtains the following three uniform regularity properties. All of them are very well known and widely used in variational analysis, see e.g. [21, 38, 46].

Uniform regularity.

$(\text{UCov})_M$  There exists an  $\alpha > 0$  and a  $\delta > 0$  such that for any  $\rho \in (0, \delta)$  and any  $(x, y) \in \text{gph } F \cap B_\delta(\bar{x}, \bar{y})$

$$(33) \quad B_{\alpha\rho}(y) \subset F(B_\rho(x)).$$

$(\text{MR})_M$  There exists an  $\alpha > 0$  and a  $\delta > 0$  such that for any  $x \in B_\delta(\bar{x})$ ,  $y \in B_\delta(\bar{y})$

$$(34) \quad \alpha d(x, F^{-1}(y)) \leq d(y, F(x)).$$

$(\text{URD})_M$  ( $X$  and  $Y$  are normed linear spaces) There exists an  $\alpha > 0$  and a  $\delta > 0$  such that  $\alpha \|y^*\| \leq \|x^*\|$  for all  $(x^*, y^*) \in \hat{N}_\delta((\bar{x}, \bar{y}) | \text{gph } F)$ .

$(\text{URDL})_M$  ( $X$  and  $Y$  are normed linear spaces)  $\bar{D}_M^* F^{-1}(\bar{y}, \bar{x})(0) = \{0\}$ .

The set  $\hat{N}_\delta((\bar{x}, \bar{y}) | \text{gph } F)$  in the uniform dual regularity condition  $(\text{URD})_M$  denotes the strict  $\delta$ -normal cone to the graph of  $F$  (see definition (19)), while  $\bar{D}_M^* F^{-1}(\bar{y}, \bar{x})$  in the limiting uniform dual regularity condition  $(\text{URDL})_M$  is the *mixed (limiting) coderivative* [38] of  $F^{-1}$  at  $(\bar{y}, \bar{x})$ :

$$\begin{aligned} \bar{D}_M^* F^{-1}(\bar{y}, \bar{x})(x^*) = \{y^* \in X^* \mid (x_k, y_k) \xrightarrow{\text{gph } F} (\bar{x}, \bar{y}), x_k^* \rightarrow x^*, y_k^* \xrightarrow{w^*} y^*, \\ y_k^* \in D^* F^{-1}(y_k, x_k)(x_k^*), k = 1, 2, \dots\}, \end{aligned}$$

Note that the above definition requires “mixed” convergence of the components in the sequence  $(x_k^*, y_k^*)$ : norm convergence of  $x_k^*$  and  $w^*$ -convergence of  $y_k^*$ . Of course, this difference can be of importance only in infinite dimensional spaces.

The uniform covering property  $(\text{UCov})_M$  is also known as *local covering*, *openness at a linear rate*, or *linear openness* (see [38]), while  $(\text{MR})_M$  represents *local metric regularity* – one of the central concepts of variational analysis (see [19, 21, 38, 46]).

Conditions  $(\text{UCov})_M$ ,  $(\text{MR})_M$ , and  $(\text{URD})_M$  obviously strengthen the corresponding “nonuniform” conditions  $(\text{Cov})_M$ ,  $(\text{SeR})_M$ , and  $(\text{RD})_M$ .

#### 4.2. Constants

As in the preceding sections, it can be convenient to characterize the above regularity concepts in terms of certain nonnegative constants:

$$(35) \quad \theta_\rho[F](\bar{x}, \bar{y}) = \sup\{r \geq 0 \mid B_r(\bar{y}) \subset F(B_\rho(\bar{x}))\},$$

$$(36) \quad \theta[F](\bar{x}, \bar{y}) = \liminf_{\rho \downarrow 0} \frac{\theta_\rho[F](\bar{x}, \bar{y})}{\rho},$$

$$(37) \quad \vartheta[F](\bar{x}, \bar{y}) = \liminf_{y \rightarrow \bar{y}} \frac{d(y, \bar{y})}{d(\bar{x}, F^{-1}(y))},$$

$$(38) \quad \hat{\theta}[F](\bar{x}, \bar{y}) = \liminf_{\substack{(x,y) \xrightarrow{\text{gph } F} (\bar{x}, \bar{y}) \\ \rho \downarrow 0}} \frac{\theta_\rho[F](x, y)}{\rho},$$

$$(39) \quad \hat{\vartheta}[F](\bar{x}, \bar{y}) = \liminf_{\substack{x \rightarrow \bar{x}, y \rightarrow \bar{y} \\ y \notin F(x)}} \frac{d(y, F(x))}{d(x, F^{-1}(y))},$$

$$(40) \quad \eta[F](\bar{x}, \bar{y}) = \inf \{ \|x^*\| \mid (x^*, y^*) \in N((\bar{x}, \bar{y}) \mid \text{gph } F), \|y^*\| = 1 \},$$

$$(41) \quad \hat{\eta}[F](\bar{x}, \bar{y}) = \liminf_{\delta \downarrow 0} \left\{ \|x^*\| \mid (x^*, y^*) \in \hat{N}_\delta((\bar{x}, \bar{y}) \mid \text{gph } F), \|y^*\| = 1 \right\},$$

$$(42) \quad \bar{\eta}[F](\bar{x}, \bar{y}) = \inf \{ \|x^*\| \mid (x^*, y^*) \in \bar{N}((\bar{x}, \bar{y}) \mid \text{gph } F), \|y^*\| = 1 \}.$$

The following equivalences are obvious.

- $\theta[F](\bar{x}, \bar{y}) > 0 \Leftrightarrow (\text{Cov})_M$ ;      •  $\eta[F](\bar{x}, \bar{y}) > 0 \Leftrightarrow (\text{RD})_M$ ;
- $\vartheta[F](\bar{x}, \bar{y}) > 0 \Leftrightarrow (\text{SeR})_M$ ;      •  $\hat{\eta}[F](\bar{x}, \bar{y}) > 0 \Leftrightarrow (\text{URD})_M$ ;
- $\hat{\theta}[F](\bar{x}, \bar{y}) > 0 \Leftrightarrow (\text{UCov})_M$ ;      •  $\bar{\eta}[F](\bar{x}, \bar{y}) > 0 \Leftrightarrow (\text{URDL})_M$ ;
- $\hat{\vartheta}[F](\bar{x}, \bar{y}) > 0 \Leftrightarrow (\text{MR})_M$ ;

The above constants provide quantitative characterizations of the corresponding regularity properties. Wherever the property is defined by an inequality, the constant coincides with the exact lower bound of all  $\alpha$  in this inequality.

For any of the constants, its equality to zero can be interpreted as a kind of stationary/singular/irregular behavior of the multifunction. The exact definitions can be easily obtained from the ones of the corresponding regularity properties.

The next theorem contains the list of relations between the regularity constants.

**Theorem 6.** *The following assertions hold true:*

- (i)  $\theta[F](\bar{x}, \bar{y}) = \vartheta[F](\bar{x}, \bar{y})$ ;
- (ii)  $\hat{\theta}[F](\bar{x}, \bar{y}) \leq \theta[F](\bar{x}, \bar{y})$ ;  $\hat{\vartheta}[F](\bar{x}, \bar{y}) \leq \vartheta[F](\bar{x}, \bar{y})$ ;
- (iii)  $\hat{\theta}[F](\bar{x}, \bar{y}) = \hat{\vartheta}[F](\bar{x}, \bar{y})$ .

*Suppose  $X$  and  $Y$  are normed linear spaces. Then*

- (iv)  $\vartheta[F](\bar{x}, \bar{y}) \leq \eta[F](\bar{x}, \bar{y})$ ;
- (v)  $\hat{\vartheta}[F](\bar{x}, \bar{y}) \leq \eta[F](\bar{x}, \bar{y})$ ;
- (vi)  $\hat{\theta}[F](\bar{x}, \bar{y}) \leq \hat{\eta}[F](\bar{x}, \bar{y})$ ;
- (vii) *if  $X$  and  $Y$  are Asplund spaces and  $\text{gph } F$  is locally closed near  $(\bar{x}, \bar{y})$  then  $\hat{\eta}[F](\bar{x}, \bar{y}) = \hat{\theta}[F](\bar{x}, \bar{y})$ ;*
- (viii) *if  $\dim X + \dim Y < \infty$  and  $\text{gph } F$  is locally closed near  $(\bar{x}, \bar{y})$  then  $\bar{\eta}[F](\bar{x}, \bar{y}) = \hat{\eta}[F](\bar{x}, \bar{y})$ .*

*Proof.* (i). Let  $0 < \alpha < \theta[F](\bar{x}, \bar{y})$ . By (36) there exists a  $\delta > 0$  such that for any  $\rho \in (0, \delta)$  the inequality  $\theta_\rho[F](\bar{x}, \bar{y}) > \alpha\rho$  holds. By (35) this implies (31). Chose a  $\delta' \in (0, \alpha\delta)$ . For any  $y \in B_{\delta'}(\bar{y})$  take  $\rho = d(y, \bar{y})/\alpha$ . Then  $\rho < \delta$  and  $y \in B_{\alpha\rho}(\bar{y})$ . It follows from (31) that there exists an  $x \in F^{-1}(y) \cap B_\rho(\bar{x})$ , and consequently

$$\alpha d(\bar{x}, F^{-1}(y)) \leq \alpha d(x, \bar{x}) \leq \alpha\rho = d(y, \bar{y}).$$

By (37) this implies  $\vartheta[F](\bar{x}, \bar{y}) \geq \alpha$ , and consequently  $\vartheta[F](\bar{x}, \bar{y}) \geq \theta[F](\bar{x}, \bar{y})$ .

To prove the opposite inequality chose an  $\alpha$  satisfying  $0 < \alpha < \vartheta[F](\bar{x}, \bar{y})$ . By (37) there exists a  $\delta > 0$  such that for any  $y \in B_\delta(\bar{y}) \setminus \{\bar{y}\}$  one has

$$\alpha d(\bar{x}, F^{-1}(y)) < d(y, \bar{y}).$$

Denote  $\delta' = \delta/\alpha$  and take any  $\rho \in (0, \delta')$  and  $y \in B_{\alpha\rho}(\bar{y})$ ,  $y \neq \bar{y}$ . Then  $y \in B_\delta(\bar{y})$ , and it follows from the last inequality that there exists an  $x \in F^{-1}(y)$  such that

$$d(x, \bar{x}) \leq d(y, \bar{y})/\alpha \leq \rho.$$

The same conclusion holds trivially for  $y = \bar{y}$ : take  $x = \bar{x}$ . By (35) and (36) this implies,  $\theta[F](\bar{x}, \bar{y}) \geq \alpha$ , and consequently  $\theta[F](\bar{x}, \bar{y}) \geq \vartheta[F](\bar{x}, \bar{y})$ .

(ii). The inequalities follow directly from the definitions.



(iii). The proof is similar to that of (i). Let  $0 < \alpha < \hat{\theta}[F](\bar{x}, \bar{y})$ . By (38) there exists a  $\delta > 0$  such that for any  $\rho \in (0, \delta)$  and any  $(x, v) \in \text{gph } F \cap B_\delta(\bar{x}, \bar{y})$  the inequality  $\theta_\rho[F](x, v) > \alpha\rho$  holds, and consequently  $B_{\alpha\rho}(v) \subset F(B_\rho(x))$ . Chose positive numbers  $\delta_1 < \delta$ ,  $\delta_2 < \min(\alpha\delta, \delta - \delta_1)$ , an  $x \in B_{\delta_1}(\bar{x})$ , a  $y \in B_{\delta_1}(\bar{y})$ , and a  $v \in F(x) \cap B_{\delta_2}(y)$ . Take  $\rho = d(y, v)/\alpha$ . Then  $\rho < \delta$ ,  $(x, v) \in B_\delta(\bar{x}, \bar{y})$ , and  $y \in B_{\alpha\rho}(v)$ . It follows that there exists a  $u \in F^{-1}(y) \cap B_\rho(x)$ , and consequently

$$\alpha d(x, F^{-1}(y)) \leq \alpha d(x, u) \leq \alpha\rho = d(y, v).$$

Taking the infimum over  $v$  in the right-hand side of the above inequality we obtain

$$\alpha d(x, F^{-1}(y)) \leq d(y, F(x) \cap B_{\delta_2}(y)) = d(y, F(x)).$$

By (39) this implies  $\hat{\vartheta}[F](\bar{x}, \bar{y}) \geq \alpha$ , and consequently  $\hat{\vartheta}[F](\bar{x}, \bar{y}) \geq \hat{\theta}[F](\bar{x}, \bar{y})$ .

To prove the opposite inequality chose an  $\alpha$  satisfying  $0 < \alpha < \hat{\vartheta}[F](\bar{x}, \bar{y})$ . By (39) there exists a  $\delta > 0$  such that for any  $x \in B_\delta(\bar{x})$  and  $y \in B_\delta(\bar{y})$  with  $y \notin F(x)$  one has

$$(43) \quad \alpha d(x, F^{-1}(y)) < d(y, F(x)).$$

Denote  $\delta' = \min(\alpha^{-1}, 1)\delta/2$  and take any  $\rho \in (0, \delta')$ ,  $(x, v) \in \text{gph } F \cap B_{\delta'}(\bar{x}, \bar{y})$ , and  $y \in B_{\alpha\rho}(v)$  with  $y \notin F(x)$ . Then  $x \in B_\delta(\bar{x})$ ,  $y \in B_\delta(\bar{y})$ , and it follows from (43) that there exists a  $u \in F^{-1}(y)$  such that

$$d(u, x) \leq d(y, v)/\alpha \leq \rho.$$

The same conclusion holds trivially for  $y \in F(x)$ : take  $u = x$ . Thus  $B_{\alpha\rho}(v) \subset F(B_\rho(x))$ . By (35) and (38) this implies,  $\hat{\theta}[F](\bar{x}, \bar{y}) \geq \alpha$ , and consequently  $\hat{\theta}[F](\bar{x}, \bar{y}) \geq \hat{\vartheta}[F](\bar{x}, \bar{y})$ .

(iv). Let  $X$  and  $Y$  be normed linear spaces,  $0 < \alpha < \vartheta[F](\bar{x}, \bar{y})$ , and  $(x^*, y^*) \in N((\bar{x}, \bar{y}) | \text{gph } F)$ ,  $\|y^*\| = 1$ . Chose a sequence  $z_k \in Y$ ,  $k = 1, 2, \dots$ , such that  $\|z_k\| = 1$  and  $\langle y^*, z_k \rangle \rightarrow 1$  as  $k \rightarrow \infty$ , and set  $y_k = \bar{y} + z_k/k$ . By (37) there exists a sequence  $x_k \in F^{-1}(y_k)$ ,  $k = 1, 2, \dots$ , satisfying  $\alpha\|x_k - \bar{x}\| \leq \|y_k - \bar{y}\|$  for all sufficiently large  $k$ . Then for large  $k$ , we have  $0 < \|(x_k, y_k) - (\bar{x}, \bar{y})\| \leq \beta/k$ , where  $\beta = \max(\alpha^{-1}, 1)$ , and consequently

$$\limsup_{k \rightarrow \infty} k(\langle y^*, y_k - \bar{y} \rangle + \langle x^*, x_k - \bar{x} \rangle) \leq 0.$$

The last inequality yields

$$\|x^*\| \geq \left( \liminf_{k \rightarrow \infty} k\|x_k - \bar{x}\| \right)^{-1} \geq \left( \alpha^{-1} \lim_{k \rightarrow \infty} k\|y_k - \bar{y}\| \right)^{-1} = \alpha.$$

This obviously implies (iv).

(v). The inequality follows directly from the definitions.

(vi). The proof is similar to that of (iv). Let  $X$  and  $Y$  be normed linear spaces and  $0 < \alpha < \hat{\theta}[F](\bar{x}, \bar{y})$ . By (38) and (35), there exists a  $\delta > 0$  such that  $B_{\alpha\rho}(y) \subset F(B_\rho(x))$  for any  $\rho \in (0, \delta)$  and any  $(x, y) \in \text{gph } F \cap B_\delta(\bar{x}, \bar{y})$ . Let  $(x^*, y^*) \in N((x, y) | \text{gph } F)$ ,  $\|y^*\| = 1$ . Chose a sequence  $z_k \in Y$ ,  $k = 1, 2, \dots$ , such that  $\|z_k\| = 1$  and  $\langle y^*, z_k \rangle \rightarrow 1$  as  $k \rightarrow \infty$ , and set  $y_k = y + z_k/k$ ,  $\rho_k = 1/(k\alpha)$ . Then for sufficiently large  $k$  we have  $\rho_k < \delta$ ,  $y_k \in B_{\alpha\rho_k}(y)$ , and consequently there exists an  $x_k \in F^{-1}(y_k) \cap B_{\rho_k}(x)$ . Thus,  $(x_k, y_k) \in \text{gph } F$  and

$$\begin{aligned} \alpha\|x_k - x\| &\leq \rho_k = \|y_k - y\|, \\ 0 < \|(x_k, y_k) - (x, y)\| &\leq \beta/k, \end{aligned}$$

where  $\beta = \max(\alpha^{-1}, 1)$ , and consequently

$$\begin{aligned} \limsup_{k \rightarrow \infty} k(\langle y^*, y_k - y \rangle + \langle x^*, x_k - x \rangle) &\leq 0, \\ \|x^*\| &\geq \left( \liminf_{k \rightarrow \infty} k\|x_k - x\| \right)^{-1} \geq \left( \alpha^{-1} \lim_{k \rightarrow \infty} k\|y_k - y\| \right)^{-1} = \alpha. \end{aligned}$$

The above inequality obviously implies (vi).

(vii). Let  $X$  and  $Y$  be Asplund spaces,  $\text{gph } F$  be locally closed near  $(\bar{x}, \bar{y})$ ,  $\delta > 0$ , and  $\alpha > \hat{\theta}[F](\bar{x}, \bar{y})$ . Taking into account (vi) only the opposite inequality needs to be proved. Chose a  $\gamma \in (0, \alpha^{-1})$  and an  $\alpha_1 \in (\hat{\theta}[F](\bar{x}, \bar{y}), \alpha)$ . Then there exists a positive number  $\rho < \min(\gamma^{-1}, 1)\delta/2$  and points  $x \in B_{\delta/2}(\bar{x})$ ,  $y \in F(x) \cap B_{\delta/2}(\bar{y})$  and  $w \in B_{\alpha_1\rho}(y)$  such that  $F^{-1}(w) \cap B_\rho(x) = \emptyset$ . In other words,  $\|v - w\| > 0$  for all  $(u, v) \in \text{gph } F$  with  $u \in B_\rho(x)$ . At the same time,  $\|y - w\| \leq \alpha_1\rho$ . It is the right time now to apply Ekeland variational principle.

Note that  $\hat{\eta}[F](\bar{x}, \bar{y})$  does not change if the norm on  $X \times Y$  is replaced by an equivalent one. So we have some freedom with the choice of an appropriate norm. Define a norm on  $X \times Y$  depending on the  $\gamma$ :  $\|(u, v)\|_\gamma = \max(\|u\|, \gamma\|v\|)$ . Chose an  $\alpha_2 \in (\alpha_1, \alpha)$  and set  $\rho' = \rho\alpha_1/\alpha_2$ . Then there exists a point  $(x', y') \in \text{gph } F$  such that  $\|(x', y') - (x, y)\|_\gamma \leq \rho'$ ,  $\|y' - w\| \leq \|y - w\|$ , and

$$\|v - w\| + \alpha_2\|(u, v) - (x', y')\|_\gamma \geq \|y' - w\|$$

for all  $(u, v) \in \text{gph } F$  near  $(x', y')$ . Thus,  $(x', y')$  is a point of local minimum for the sum of three functions on  $X \times Y$  given by  $f_1(u, v) = \|v - w\|$ ,  $f_2(u, v) = \alpha_2\|(u, v) - (x', y')\|_\gamma$ , and  $f_3(u, v) = 0$  if  $(u, v) \in \text{gph } F$ ,  $f_3(u, v) = \infty$  otherwise (the indicator function of  $\text{gph } F$ ), and consequently  $0 \in \partial(f_1 + f_2 + f_3)(x', y')$ .

Note that  $f_1$  and  $f_2$  are convex and Lipschitz continuous, and  $\|y' - w\| > 0$  since  $x' \in B_\rho(x)$ . The Fréchet subdifferentials of  $f_1$ ,  $f_2$ , and  $f_3$  possess the following properties:

- (1) if  $(u^*, v^*) \in \partial f_1(u, v)$  then  $u^* = 0$  and  $\|v^*\| = 1$  for any  $(u, v)$  near  $(x', y')$ ;
- (2) if  $(u^*, v^*) \in \partial f_2(u, v)$  then  $\|u^*\| + \gamma^{-1}\|v^*\| \leq \alpha_2$  for any  $(u, v) \in X \times Y$ ;
- (3)  $\partial f_3(u, v) = N((u, v) | \text{gph } F)$  for any  $(u, v) \in \text{gph } F$ .

Chose an  $\varepsilon \in \left(0, \min \left\{ \frac{1-\alpha_2\gamma}{\gamma+1}, \frac{\alpha-\alpha_2}{\alpha+1} \right\}\right)$ . Applying the fuzzy sum rule [16] we conclude that there exists a point  $(\hat{x}, \hat{y}) \in \text{gph } F$  satisfying  $\|\hat{x} - x'\| \leq \delta/2 - \rho$ ,  $\|\hat{y} - y'\| \leq \delta/2 - \gamma^{-1}\rho$ , a number  $\beta \in [0, \alpha_2]$ , and an element  $(x^*, y^*) \in N((\hat{x}, \hat{y}) | \text{gph } F)$  such that  $\|x^*\| \leq \beta + \varepsilon$  and  $\|y^*\| \geq 1 - \gamma(\alpha_2 - \beta) - \varepsilon > 0$ . Then  $\|\hat{x} - \bar{x}\| \leq \delta$ ,  $\|\hat{y} - \bar{y}\| \leq \delta$ , and

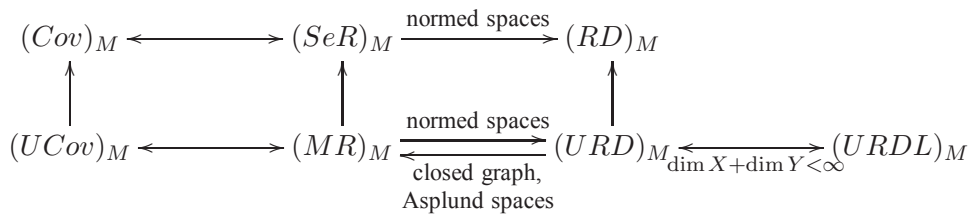
$$\frac{\|x^*\|}{\|y^*\|} \leq \frac{\beta + \varepsilon}{1 - \gamma(\alpha_2 - \beta) - \varepsilon}.$$

Since  $1 - \alpha_2\gamma - (\gamma + 1)\varepsilon > 0$  the right-hand side of the above inequality is an increasing function of  $\beta$ , and consequently attains its maximum on  $[0, \alpha_2]$  at  $\beta = \alpha_2$ . Thus,

$$\frac{\|x^*\|}{\|y^*\|} \leq \frac{\alpha_2 + \varepsilon}{1 - \varepsilon} < \alpha.$$

It follows from (41) that  $\hat{\eta}[F](\bar{x}, \bar{y}) \leq \alpha$ , and consequently  $\hat{\eta}[F](\bar{x}, \bar{y}) \leq \hat{\theta}[F](\bar{x}, \bar{y})$ . (viii) follows from the definitions. ■

In accordance with Theorem 6 the relationships between the regularity concepts can be described by the following diagram:



Unlike regularity properties (IR1) and (IR2) (as well as (SR1) and (SR2)) of real-valued functions, their set-valued analogs  $(Cov)_M$  and  $(SeR)_M$  are equivalent. Condition  $(RD)_M$  can be strictly weaker than  $(Cov)_M$  even in finite dimensions - see Examples 2 and 3 below. The relationships between the uniform regularity properties are well known and can be found for instance in [38, Theorems 1.52, 1.54, 4.1, 4.5], see also [23, 37].

### 4.3. Epigraphical and hypographical multifunctions

Consider again an extended real-valued function  $f : X \rightarrow \mathbb{R}_\infty$ , finite at  $\bar{x} \in X$ , and its epigraph

$$\text{epi } f = \{(x, y) \in X \times \mathbb{R} : f(x) \leq y\}.$$

The epigraphical multifunction  $\text{epi } f(\cdot) : X \rightrightarrows \mathbb{R}$  can be defined by the relation

$$y \in \text{epi } f(x) \iff (x, y) \in \text{epi } f.$$

In this subsection we are going to compare regularity properties of this multifunction with the corresponding properties of  $f$  considered in Section 2. The next proposition gives the relations between the corresponding constants.

**Proposition 9.**

- (i)  $\theta_\rho[\text{epi } f](\bar{x}, f(\bar{x})) = |\theta f|_\rho(\bar{x})$ ,  $\rho > 0$ ;
- (ii)  $\theta[\text{epi } f](\bar{x}, f(\bar{x})) = |\theta f|(\bar{x})$ ;
- (iii)  $\vartheta[\text{epi } f](\bar{x}, f(\bar{x})) \leq |\nabla f|(\bar{x})$ ;
- (iv)  $\hat{\theta}[\text{epi } f](\bar{x}, f(\bar{x})) = \overline{|\theta f|}(\bar{x})$ ;
- (v)  $\hat{\vartheta}[\text{epi } f](\bar{x}, f(\bar{x})) \leq \overline{|\nabla f|}(\bar{x})$ ;
- (vi) if  $X$  is complete and  $f$  is lower semicontinuous near  $\bar{x}$ , then  $\hat{\vartheta}[\text{epi } f](\bar{x}, f(\bar{x})) = \overline{|\nabla f|}(\bar{x})$ .

Suppose  $X$  is a normed linear space. Then

- (vii)  $\eta[\text{epi } f](\bar{x}, f(\bar{x})) = |\partial f|(\bar{x})$ ;
- (viii)  $\hat{\eta}[\text{epi } f](\bar{x}, f(\bar{x})) = \widehat{|\partial f|}(\bar{x})$ ;
- (iv) if  $\dim X < \infty$  and  $f$  is lower semicontinuous near  $\bar{x}$  then  $\bar{\eta}[\text{epi } f](\bar{x}, f(\bar{x})) = \overline{|\partial f|}(\bar{x})$ .

*Proof.* (i)-(iii). Let  $f(x) \leq y$ . Inclusion  $B_r(y) \subset \text{epi } f(B_\rho(x))$  for some  $\rho > 0$  and  $r > 0$  is obviously equivalent to the existence of an  $u \in B_\rho(x)$  such that  $f(u) \leq y - r$ . This implies the inequality

$$y - \inf_{u \in B_\rho(x)} f(u) \geq r.$$

On the other hand, the above condition implies that for any  $r' \in (0, r)$  there exists an  $u \in B_\rho(x)$  such that  $f(u) - y \leq -r'$ . These observations and definition (35) prove that

$$(44) \quad \theta_\rho[\text{epi } f](x, y) = y - \inf_{u \in B_\rho(x)} f(u).$$

Putting in (44)  $x = \bar{x}$  and  $y = f(\bar{x})$  and taking into account definition (7) we arrive at assertion (i). Assertion (ii) follows from (i) and definitions (8) and (36). Inequality (iii) is a consequence of (ii), assertion (i) in Theorem 2 and assertion (i) in Theorem 6.

(iv). Let  $\delta > 0$ . It follows from (44) that

$$(45) \quad \inf_{y \in \text{epi } f(x) \cap B_\delta(f(\bar{x}))} \theta_\rho[\text{epi } f](x, y) = \max(f(x), f(\bar{x}) - \delta) - \inf_{u \in B_\rho(x)} f(u).$$

In particular, if  $|f(x) - f(\bar{x})| \leq \delta$  we have

$$\inf_{y \in \text{epi } f(x) \cap B_\delta(f(\bar{x}))} \theta_\rho[\text{epi } f](x, y) = f(x) - \inf_{u \in B_\rho(x)} f(u) = |\theta f|_\rho(x),$$

and definitions (13) and (38) imply the inequality  $\hat{\theta}[\text{epi } f](\bar{x}, f(\bar{x})) \leq \overline{|\theta f|}(\bar{x})$ .

To prove the opposite inequality, assume that  $0 < \alpha < \overline{|\theta f|}(\bar{x})$ . By (13) and (7), there exists a  $\delta > 0$  such that for any  $x \in B_\delta(\bar{x})$  with  $|f(x) - f(\bar{x})| \leq \delta$  and any  $\rho \in (0, \delta)$  there is an  $u \in B_\rho(x)$  satisfying

$$(46) \quad f(x) - f(u) > \alpha\rho.$$

Put  $\delta' = \delta/(\alpha + 1)$  and take any  $(x, y) \in B_{\delta'}(\bar{x}, f(\bar{x}))$  and  $\rho \in (0, \delta')$ . Thus,  $f(x) \leq y < f(\bar{x}) + \delta$ . If  $f(x) \geq f(\bar{x}) - \delta$  then there is an  $u \in B_\rho(x)$  satisfying (46). If  $f(x) < f(\bar{x}) - \delta$  then  $(f(\bar{x}) - \delta') - f(x) > \delta - \delta' = \alpha\delta' \geq \alpha\rho$ . Consequently, in any case, we have

$$\max(f(x), f(\bar{x}) - \delta') - \inf_{u \in B_\rho(x)} f(u) > \alpha\rho,$$

and it follows from the representation (45) that  $\theta_\rho[\text{epi } f](x, y) > \alpha\rho$ . By (38), the last inequality implies the estimate  $\hat{\theta}[\text{epi } f](\bar{x}, f(\bar{x})) \geq \alpha$ , and consequently the required inequality  $\hat{\theta}[\text{epi } f](\bar{x}, f(\bar{x})) \geq \overline{|\theta f|}(\bar{x})$ .

Assertions (v) and (vi) follow from (iv) due to assertions (iv) and (v) in Theorem 2 and Theorem 6 (iii).

(vii)-(viii). Let  $X$  be a normed linear space. Conditions  $f(x) \leq y$ ,  $|y^*| = 1$ , and  $(x^*, y^*) \in N((x, y) | \text{epi } f)$  obviously imply  $f(x) = y$ ,  $y^* = -1$ , and  $(x^*, -1) \in N((x, f(x)) | \text{epi } f)$ . The last inclusion is equivalent (see [28, 38]) to  $x^* \in \partial f(\bar{x})$ . Assertions (vii) and (viii) follow from definitions (10), (12), (40), and (41).

Assertion (ix) follows from (viii) and Theorems 2 (x) and 6 (viii). ■

Proposition 9 implies certain relationships between regularity properties of  $\text{epi } f(\cdot)$  and the corresponding inf-regularity properties of  $f$ .

**Corollary 9.1.** *Let  $Y = \mathbb{R}$ ,  $\text{gph } F = \text{epi } f$ , and  $\bar{y} = f(\bar{x})$ . Then*

- (i)  $(Cov)_M \Leftrightarrow (IR1), \quad (SeR)_M \Rightarrow (IR2);$
- (ii)  $(UCov)_M \Leftrightarrow (UIR1), \quad (MR)_M \Rightarrow (UIR2);$
- (iii) *if  $X$  is complete and  $f$  is lower semicontinuous near  $\bar{x}$ , then*  
 $(MR)_M \Leftrightarrow (UIR2);$

- (iv) if  $X$  is a normed linear space then  
 $(RD)_M \Leftrightarrow (IRD), \quad (URD)_M \Leftrightarrow (UIRD);$
- (v) if  $\dim X < \infty$  and  $f$  is lower semicontinuous near  $\bar{x}$  then  
 $(URDL)_M \Leftrightarrow (UIRDL).$

Proposition 9 implies also similar relationships between regularity properties of the hypographical multifunction  $\text{hyp } f(\cdot)$  (with the graph  $\text{hyp } f = \{(x, y) \in X \times \mathbb{R} : f(x) \geq y\}$ ) and the corresponding sup-regularity properties of  $f$ .

It can also be of interest to compare regularity properties of  $f$  considered as a special case of multifunction with the corresponding “combined” regularity properties of  $f$  discussed in Section 2. The next proposition shows that the “multifunctional” properties are in general stronger than their “scalar” counterparts from Section 2.

**Proposition 10.**

- (i)  $\theta_\rho[f](\bar{x}, f(\bar{x})) \leq \min(|\theta f|_\rho(\bar{x}), |\theta(-f)|_\rho(\bar{x}));$
- (ii)  $\theta[f](\bar{x}, f(\bar{x})) \leq \min(|\theta f|(\bar{x}), |\theta(-f)|(\bar{x}));$
- (iii)  $\vartheta[f](\bar{x}, f(\bar{x})) \leq \min(|\nabla f|(\bar{x}), |\nabla(-f)|(\bar{x}));$
- (iv)  $\hat{\theta}[\text{epi } f](\bar{x}, f(\bar{x})) \leq \min(\overline{|\theta f|}(\bar{x}), \overline{|\theta(-f)|}(\bar{x}));$
- (v)  $\hat{\vartheta}[\text{epi } f](\bar{x}, f(\bar{x})) \leq \min(\overline{|\vartheta f|}(\bar{x}), \overline{|\vartheta(-f)|}(\bar{x})).$

Suppose  $X$  is a normed linear space. Then

- (vi)  $\eta[f](\bar{x}, f(\bar{x})) \leq \min(|\partial f|(\bar{x}), |\partial(-f)|(\bar{x}));$
- (vii)  $\hat{\eta}[\text{epi } f](\bar{x}, f(\bar{x})) \leq \min(\widehat{|\partial f|}(\bar{x}), \widehat{|\partial(-f)|}(\bar{x}));$
- (viii)  $\bar{\eta}[\text{epi } f](\bar{x}, f(\bar{x})) \leq \min(\overline{|\partial f|}(\bar{x}), \overline{|\partial(-f)|}(\bar{x})).$

The inequalities in Propositions 9 and 10 as well as the corresponding implications in Corollary 9.1 can be strict. The next two examples illustrate inequality (iii) in Propositions 9 and 10.

**Example 2.** Consider a sequence of positive numbers  $\alpha_n = 1/2^{2^n}, n = 0, 1, \dots$ . Obviously,  $\alpha_n = \alpha_{n-1}^2, \alpha_n \rightarrow 0,$  and  $\alpha_n/\alpha_{n-1} \rightarrow 0$  as  $n \rightarrow \infty$ . Using this sequence define a piecewise constant lower semicontinuous real-valued function (see Figure 12)

$$f(x) = \begin{cases} -1/2, & \text{if } x \leq -1/2, \\ -\alpha_n, & \text{if } -\alpha_{n-1} < x \leq -\alpha_n, \ n = 1, 2, \dots, \\ 0, & \text{if } x = 0, \\ \alpha_n, & \text{if } \alpha_n < x \leq \alpha_{n-1}, \ n = 1, 2, \dots, \\ 1/2, & \text{if } x > 1/2. \end{cases}$$

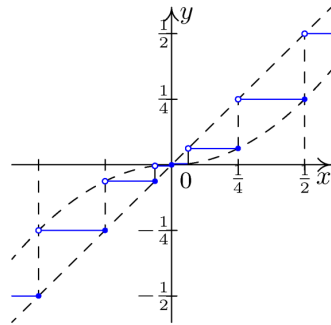


Fig. 12. Example 2.

For this function, inequality (iii) in Proposition 9 is strict. Indeed, it is easy to check that  $|\nabla f|(0) = \lim_{n \rightarrow \infty} (-f(-\alpha_n))/\alpha_n = 1$ . At the same time for the sequence  $y_n = -2\alpha_n$ ,  $n = 1, 2, \dots$ , one obviously has  $-\alpha_{n-1} \leq y_n < -\alpha_n$ ,  $(\text{epi } f(\cdot))^{-1}(y_n) = (-\infty, -\alpha_{n-1}]$ , and consequently  $d(y_n, 0)/d(0, (\text{epi } f(\cdot))^{-1}(y_n)) = 2\alpha_n/\alpha_{n-1} \rightarrow 0$  as  $n \rightarrow \infty$ . It follows that  $\vartheta[\text{epi } f](0, 0) = 0 < |\nabla f|(0)$ . Regularity condition (IR2) holds true while condition  $(\text{SeR})_M$  does not hold for  $\text{epi } f(\cdot)$  at  $(0, 0)$ .

Note that  $\partial f(0) = \emptyset$ , and consequently  $|\partial f|(0) = \infty$ . All uniform inf-regularity constants equal zero.

Similarly,  $|\nabla(-f)|(0) = 1$  while  $\vartheta[\text{hyp } f](0, 0) = 0$ .

Since both (IR2) and (SR2) hold true for  $f$  at 0, condition (R2) holds as well. At the same time condition  $(\text{SeR})_M$  does not hold for  $f$  at  $(0, 0)$ : if  $-\alpha_{n-1} < y < -\alpha_n$  or  $\alpha_n < y < \alpha_{n-1}$  then  $f^{-1}(y) = \emptyset$ , and consequently  $\vartheta[f](0, 0) = 0 < \min(|\nabla f|(0), |\nabla(-f)|(0))$ .

Note also that  $|\theta f|_\rho(0) > 0$  and  $|\theta(-f)|_\rho(0) > 0$  for any  $\rho > 0$  while  $\theta_\rho[f](0, 0) = 0$ .

The above example can be modified in such a way that the function becomes Lipschitz continuous while all the main conclusions remain true.

**Example 3.** Let  $\beta_n = (\alpha_{n-1} + \alpha_n)/2$ ,  $n = 1, 2, \dots$ , where the sequence  $\{\alpha_n\}$  is defined in Example 2. Obviously,  $\beta_n \rightarrow 0$ , and  $\alpha_n/\beta_n = 2\alpha_{n-1}/(1 + \alpha_{n-1}) \rightarrow 0$  as  $n \rightarrow \infty$ . Define a piecewise linear real-valued function (see Figure 13)

$$f(x) = \begin{cases} -1/2, & \text{if } x \leq -1/2, \\ 2x + \alpha_{n-1}, & \text{if } -\alpha_{n-1} < x \leq -\beta_n, \quad n = 1, 2, \dots, \\ -\alpha_n, & \text{if } -\beta_n < x \leq -\alpha_n, \quad n = 1, 2, \dots, \\ 0, & \text{if } x = 0, \\ \alpha_n, & \text{if } \alpha_n \leq x < \beta_n, \quad n = 1, 2, \dots, \\ 2x - \alpha_{n-1}, & \text{if } \beta_n \leq x < \alpha_{n-1}, \quad n = 1, 2, \dots, \\ 1/2, & \text{if } x \geq 1/2. \end{cases}$$

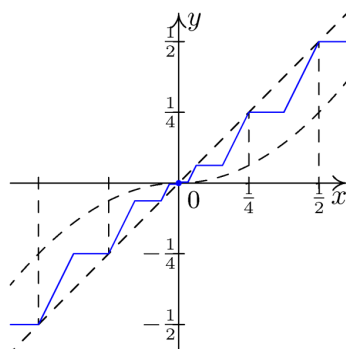


Fig. 13. Example 3.

Most of the arguments used in Example 2 are applicable to this function as well. The main difference is that now  $f^{-1}(y) \neq \emptyset$  if  $|y| \leq 1/2$ . However, it is still not difficult to construct a sequence of numbers  $y_n \rightarrow 0$  such that  $y_n/d(0, f^{-1}(y_n)) \rightarrow 0$  as  $n \rightarrow \infty$ .

#### 4.4. Multifunctions and collections of sets

A multifunction is a single object. Nevertheless there exists a close relationship between regularity properties of multifunctions and the corresponding properties of collections of sets considered in Section 3.

A multifunction  $F : X \rightrightarrows Y$  with  $(\bar{x}, \bar{y}) \in \text{gph } F$  remains our main object of interest. In this subsection we are assuming that  $X$  and  $Y$  are normed linear spaces. We are going to establish relationships between regularity properties of  $F$  and those of the following pair of sets in the product space  $X \times Y$ :

$$(47) \quad \Omega_1 = \text{gph } F, \quad \Omega_2 = X \times \{\bar{y}\}$$

Obviously  $(\bar{x}, \bar{y}) \in \Omega_1 \cap \Omega_2$ .

Consider first another multifunction  $\Phi : X \rightarrow X \times Y$  given by

$$(48) \quad \Phi(x) = \{(u, y) \in X \times Y \mid y \in F(x + u/2)\}.$$

Thus,  $(x, u, y) \in \text{gph } \Phi \Leftrightarrow (x + u/2, y) \in \text{gph } F$ . In particular,  $(\bar{x}, 0, \bar{y}) \in \text{gph } \Phi$ . Some properties of  $\Phi$  needed for computing its regularity constants are provided by the next proposition.

**Proposition 11.** *Let multifunction  $\Phi : X \rightarrow X \times Y$  be given by (48),  $(x, u, y) \in \text{gph } \Phi$ . Then*

$$(i) \quad B_r(u, y) \subset \Phi(B_\rho(x)) \Leftrightarrow B_r(y) \subset \bigcap_{u' \in B_r(u)} F(B_\rho(x + u'/2)) \quad (\rho > 0 \ r \geq 0);$$

$$(ii) \quad (x^*, u^*, y^*) \in N((x, u, y) \mid \text{gph } \Phi) \Leftrightarrow [x^* = 2u^* \text{ and } (x^*, y^*) \in N((x + u/2, y) \mid \text{gph } F)].$$



*Proof.* (i) Due to definition (48), condition  $B_r(u, y) \subset \Phi(B_\rho(x))$  is equivalent to inclusion  $B_r(y) \subset F(B_\rho(x) + u/2)$  being valid for any  $u' \in B_r(u)$ . The assertion follows immediately.

(ii) Condition  $(x^*, u^*, y^*) \in N((x, u, y) | \text{gph } \Phi)$  by definition means that

$$(49) \quad \limsup_{\substack{(x', u', y') \rightarrow (x, u, y) \\ y' \in F(x' + u'/2)}} \frac{\langle x^*, x' - x \rangle + \langle u^*, u' - u \rangle + \langle y^*, y' - y \rangle}{\|(x' - x, u' - u, y' - y)\|} \leq 0.$$

Take any  $w \in X$  with  $\|w\| = 1$  and set  $x_t = x + tw$ ,  $u_t = u - 2tw$ . Then  $x_t \rightarrow x$ ,  $u_t \rightarrow u$  as  $t \rightarrow 0$ ,  $y \in F(x_t + u_t/2)$ , and  $\|(x_t - x, u_t - u, y - y)\| = 2t$ . It follows from (49) that  $\langle x^* - 2u^*, w \rangle \leq 0$ , and consequently  $x^* = 2u^*$ . Thus, condition (49) takes the form

$$(50) \quad \limsup_{\substack{(x', u', y') \rightarrow (x, u, y) \\ y' \in F(x' + u'/2)}} \frac{\langle x^*, x' - x + (u' - u)/2 \rangle + \langle y^*, y' - y \rangle}{\|(x' - x, u' - u, y' - y)\|} \leq 0.$$

Now for any  $x' \in X$  set  $x'' = (x + x' - u/2)/2$  and  $u' = u/2 + x' - x$ . Then  $x'' + u'/2 = x'$ ,  $\|x'' - x\| = \|u' - u\|/2 \leq \|u' - u\| = \|x' - x - u/2\|$ , and  $x'' \rightarrow x$ ,  $u' \rightarrow u$  as  $x' \rightarrow x + u/2$ . It follows from (50) (with  $x'$  replaced by  $x''$ ) that

$$(51) \quad \limsup_{\substack{(x', y') \rightarrow (x + u/2, y) \\ y' \in F(x')}} \frac{\langle x^*, x' - x - u/2 \rangle + \langle y^*, y' - y \rangle}{\|(x' - x - u/2, y' - y)\|} \leq 0,$$

that is  $(x^*, y^*) \in N((x + u/2, y) | \text{gph } F)$ .

Conversely, for any  $x', u' \in X$  set  $x'' = x' + u'/2$ . Then  $\|x'' - x - u/2\| \leq \|x' - x\| + \|u' - u\|/2 \leq (3/2) \max(\|x' - x\|, \|u' - u\|)$ ,  $\|(x'' - x - u/2, y' - y)\| \leq (3/2)\|(x' - x, u' - u, y' - y)\|$ , and  $x'' \rightarrow x + u/2$  as  $x' \rightarrow x$  and  $u' \rightarrow u$ . Hence (51) implies (50). ■

The next corollary provides expressions for some regularity constants of  $\Phi$ . It follows from Proposition 11 and definitions of the constants.

**Corollary 11.1.** *Let multifunction  $\Phi : X \rightarrow X \times Y$  be given by (48). Then*

- (i)  $\theta_\rho[\Phi](\bar{x}, 0, \bar{y}) = \sup \left\{ r \geq 0 \mid B_r(\bar{y}) \subset \bigcap_{u \in rB} F(B_\rho(\bar{x} + u/2)) \right\}$  ( $\rho > 0$ );
- (ii)  $\eta[\Phi](\bar{x}, 0, \bar{y}) = \inf \{ \|x^*\| \mid (x^*, y^*) \in N((\bar{x}, \bar{y}) | \text{gph } F), \|x^*\|/2 + \|y^*\| = 1 \}$ ;
- (iii)  $\hat{\eta}[\Phi](\bar{x}, 0, \bar{y}) = \liminf_{\delta \downarrow 0} \{ \|x^*\| \mid (x^*, y^*) \in \hat{N}_\delta((\bar{x}, \bar{y}) | \text{gph } F), \|x^*\|/2 + \|y^*\| = 1 \}$ ;

(iv) If  $\dim X + \dim Y < \infty$  then

$$\bar{\eta}[\Phi](\bar{x}, 0, \bar{y}) = \inf\{\|x^*\| \mid (x^*, y^*) \in \bar{N}((\bar{x}, \bar{y}) \mid \text{gph } F), \|x^*\|/2 + \|y^*\| = 1\}.$$

*Proof.* (i) follows from Proposition 11 (i) and definition (35).

(ii) and (iii) follow from Proposition 11 (ii) and definitions (40) and (41) respectively, taking into account that  $\|(u^*, y^*)\| = \|u^*\| + \|y^*\| = \|x^*\|/2 + \|y^*\|$ .

(iv) follows from (iii), definitions (26) and (42), and Theorem 6 (viii). ■

The next proposition provides some relations between the regularity constants of  $\Phi$  and  $F$ .

**Proposition 12.** Let multifunction  $\Phi : X \rightarrow X \times Y$  be given by (48). Then

- (i)  $\min(\theta_{\rho/2}[F](\bar{x}, \bar{y}), \rho) \leq \theta_\rho[\Phi](\bar{x}, 0, \bar{y}) \leq \theta_\rho[F](\bar{x}, \bar{y})$  ( $\rho > 0$ );
- (ii)  $\min(\theta[F](\bar{x}, \bar{y})/2, 1) \leq \theta[\Phi](\bar{x}, 0, \bar{y}) \leq \theta[F](\bar{x}, \bar{y})$ ;
- (iii)  $\min(\hat{\theta}[F](\bar{x}, \bar{y})/2, 1) \leq \hat{\theta}[\Phi](\bar{x}, 0, \bar{y}) \leq \hat{\theta}[F](\bar{x}, \bar{y})$ ;
- (iv)  $\eta[\Phi](\bar{x}, 0, \bar{y}) = \eta[F](\bar{x}, \bar{y}) / (\eta[F](\bar{x}, \bar{y})/2 + 1)$ ;
- (v)  $\hat{\eta}[\Phi](\bar{x}, 0, \bar{y}) = \hat{\eta}[F](\bar{x}, \bar{y}) / (\hat{\eta}[F](\bar{x}, \bar{y})/2 + 1)$ ;
- (vi) If  $\dim X + \dim Y < \infty$  then  $\bar{\eta}[\Phi](\bar{x}, 0, \bar{y}) = \bar{\eta}[F](\bar{x}, \bar{y}) / (\bar{\eta}[F](\bar{x}, \bar{y})/2 + 1)$ .

*Proof.* (i) Let  $0 < r < \theta_\rho[\Phi](\bar{x}, 0, \bar{y})$ . It follows from Corollary 11.1 (i) that  $B_r(\bar{y}) \subset F(B_\rho(\bar{x}))$ , and consequently  $r \leq \theta_\rho[F](\bar{x}, \bar{y})$ . Hence  $\theta_\rho[\Phi](\bar{x}, 0, \bar{y}) \leq \theta_\rho[F](\bar{x}, \bar{y})$ .

Let  $0 < r < \min(\theta_{\rho/2}[F](\bar{x}, \bar{y}), \rho)$ . Then  $B_r(\bar{y}) \subset F(B_{\rho/2}(\bar{x})) \subset F(B_\rho(\bar{x}) + u)$  for any  $u \in (r/2)B$ . It follows from Corollary 11.1 (i) that  $r \leq \theta_\rho[\Phi](\bar{x}, 0, \bar{y})$ . Hence  $\min(\theta_{\rho/2}[F](\bar{x}, \bar{y}), \rho) \leq \theta_\rho[\Phi](\bar{x}, 0, \bar{y})$ .

(ii) follows from (i).

(iii) If  $(x, u, y) \in \text{gph } \Phi$  then  $(x + u/2, 0, y) \in \text{gph } \Phi$  and condition  $B_r(u, y) \subset \Phi(B_\rho(x))$  is equivalent to  $B_r(0, y) \subset \Phi(B_\rho(x + u/2))$ . This implies that  $\theta_\rho[\Phi](x, u, y) = \theta_\rho[\Phi](x + u/2, 0, y)$ . The assertion follows now from (i).

(iv)-(vi) follow from Corollary 11.1 (ii)-(iv). It is sufficient to notice that for any cone  $K \in (X \times Y)^*$  one has

$$\begin{aligned} & \inf\{\|x^*\| \mid (x^*, y^*) \in K, \|x^*\|/2 + \|y^*\| = 1\} \\ &= \inf\left\{\frac{\|x^*\|}{\|x^*\|/2 + 1} \mid (x^*, y^*) \in K, \|y^*\| = 1\right\} \\ &= \frac{\inf\{\|x^*\| \mid (x^*, y^*) \in K, \|y^*\| = 1\}}{\inf\{\|x^*\| \mid (x^*, y^*) \in K, \|y^*\| = 1\}/2 + 1}. \end{aligned}$$

■

The next proposition specifies constants (21)-(26) for the pair of sets (47) in terms of the corresponding constants of  $\Phi$ .

**Proposition 13.** *Let sets  $\Omega_1$  and  $\Omega_2$  be given by (47). Then*

- (i)  $\theta_\rho[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \min(\theta_\rho[\Phi](\bar{x}, 0, \bar{y})/2, \rho)$  ( $\rho > 0$ );
- (ii)  $\theta[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \min(\theta[\Phi](\bar{x}, 0, \bar{y})/2, 1)$ ;
- (iii)  $\hat{\theta}[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \min(\hat{\theta}[\Phi](\bar{x}, 0, \bar{y})/2, 1)$ ;
- (iv)  $\eta[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \min(\eta[\Phi](\bar{x}, 0, \bar{y})/2, 1)$ ;
- (v)  $\hat{\eta}[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \min(\hat{\eta}[\Phi](\bar{x}, 0, \bar{y})/2, 1)$ ;
- (vi) *If  $\dim X + \dim Y < \infty$  then  $\bar{\eta}[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \min(\bar{\eta}[\Phi](\bar{x}, 0, \bar{y})/2, 1)$ .*

*Proof.* (i) Let  $r \geq 0$ . Take any  $(u_1, v_1), (u_2, v_2) \in X \times Y$  satisfying

$$(52) \quad \|(u_1, v_1)\| \leq r, \|(u_2, v_2)\| \leq r,$$

$$(53) \quad [\Omega_1 - (u_1, v_1)] \cap [\Omega_2 - (u_2, v_2)] \cap B_\rho(\bar{x}, \bar{y}) \neq \emptyset.$$

Note that  $\Omega_2 - (u_2, v_2) = X \times \{\bar{y} - v_2\}$ . Condition (53) means that  $\|v_2\| \leq \rho$  and there exists an  $u' \in B_\rho(\bar{x})$  such that  $(u' + u_1, \bar{y} + v_1 - v_2) \in \text{gph } F$ . Hence, condition (53) holding for any  $(u_1, v_1), (u_2, v_2) \in X \times Y$  satisfying (52) is equivalent to two conditions: 1)  $r \leq \rho$ , and 2)  $B_{2r}(\bar{y}) \subset F(B_\rho(\bar{x}) + u)$  for any  $u \in rB$ . The assertion follows from Corollary 11.1 (i).

(ii) follows from (i).

(iii) Let  $\omega_1 = (x, y) \in \Omega_1, \omega_2 \in \Omega_2, \rho > 0$ . Then  $\Omega_2 - \omega_2 = X \times \{0\}, \Omega_2 + \omega_1 - \omega_2 = X \times \{y\}$ , and consequently (taking into account (i))

$$\theta_\rho[\Omega_1 - \omega_1, \Omega_2 - \omega_2](0) = \theta_\rho[\text{gph } F, X \times \{y\}](x, y) = \min(\theta_\rho[\Phi](x, 0, y)/2, \rho).$$

The assertion follows from definitions (23) and (39).

(iv)-(v) Taking into account that  $N((x, \bar{y}) | \Omega_2) = \{0\} \times Y^*$  for any  $x \in X$ , we have

$$(54) \quad \eta[\Omega_1, \Omega_2](x, y) = \inf \{ \|x^*\| + \|y^* + v^*\| \mid (x^*, y^*) \in N((x, y) | \text{gph } F), \\ v^* \in Y^*, \|x^*\| + \|y^*\| + \|v^*\| = 1 \},$$

$$(55) \quad \hat{\eta}[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \lim_{\delta \downarrow 0} \inf_{(x, y) \in \text{gph } F \cap B_\delta(\bar{x}, \bar{y})} \eta[\Omega_1, \Omega_2](x, y).$$

Since there are no restrictions on  $v^*$ , the set of points  $x^*, y^*, v^*$ , participating in definition (54), is nonempty. Obviously  $\eta[\Omega_1, \Omega_2](x, y) \leq 1$ . If  $y^* = 0$  in (54) then  $\|x^*\| + \|y^* + v^*\| = 1$ . Hence

$$(56) \quad \eta[\Omega_1, \Omega_2](x, y) = 1 \quad \text{if } y^* = 0 \text{ for all } (x^*, y^*) \in N((x, y) | \text{gph } F).$$

Otherwise, we can limit ourselves to considering only triples  $(x^*, y^*, v^*)$  with  $y^* \neq 0$  when evaluating the infimum in (54), and  $\eta[\Omega_1, \Omega_2](x, y)$  can be represented in the following way:

$$\eta[\Omega_1, \Omega_2](x, y) = \inf_{\substack{(x^*, y^*) \in N((x, y) | \text{gph } F) \\ y^* \neq 0, \|x^*\| + \|y^*\| \leq 1}} (\|x^*\| + \inf_{\|v^*\| = 1 - (\|x^*\| + \|y^*\|)} \|y^* + v^*\|).$$

By the triangle inequality, the internal infimum in the last formula is achieved when

$$v^* = -y^*[1 - (\|x^*\| + \|y^*\|)]/\|y^*\|.$$

In this case,

$$\|y^* + v^*\| = \left| \|y^*\| - \|v^*\| \right| = \left| \|x^*\| + 2\|y^*\| - 1 \right|,$$

and consequently

$$\eta[\Omega_1, \Omega_2](x, y) = \inf_{\substack{(x^*, y^*) \in N((x, y) | \text{gph } F) \\ y^* \neq 0, \|x^*\| + \|y^*\| \leq 1}} (\|x^*\| + \left| \|x^*\| + 2\|y^*\| - 1 \right|).$$

The last formula can be rewritten as

$$\eta[\Omega_1, \Omega_2](x, y) = \inf_{\substack{(x^*, y^*) \in N((x, y) | \text{gph } F) \\ y^* \neq 0}} \inf_{0 \leq t \leq (\|x^*\| + \|y^*\|)^{-1}} (t\|x^*\| + |t\|x^*\| + 2t\|y^*\| - 1|).$$

Since  $\|x^*\| \leq \|x^*\| + 2\|y^*\|$ , the infimum over  $t$  in the above formula is attained at  $t = (\|x^*\| + 2\|y^*\|)^{-1}$ , and consequently

$$(57) \quad \begin{aligned} & \eta[\Omega_1, \Omega_2](x, y) \\ &= \inf_{\substack{(x^*, y^*) \in N((x, y) | \text{gph } F) \\ y^* \neq 0}} \|x^*\| / (\|x^*\| + 2\|y^*\|) \\ &= \inf \{ \|x^*\| \mid (x^*, y^*) \in N((\bar{x}, \bar{y}) | \text{gph } F), \|x^*\| + 2\|y^*\| = 1 \}. \end{aligned}$$

Putting in (57)  $(x, y) = (\bar{x}, \bar{y})$  and comparing the formula with Corollary 11.1 (ii) we come to assertion (iv). Combining (55), (56), and (57) and comparing the outcome with Corollary 11.1 (iii) we arrive at (v).

(vi) follows from (v) and Theorems 4 (vii) and 6 (viii). ■

Taking into account Proposition 12, the first three assertions in Proposition 13 strengthen the corresponding estimates in [30, Theorem 2] (see Theorem 7 below).

Combining Propositions 12 and 13 we arrive at the next theorem providing relations between regularity constants of multifunction  $F$  and the corresponding pair of sets (47).

**Theorem 7.** *Let sets  $\Omega_1$  and  $\Omega_2$  be given by (47). Then*

- (i)  $\theta_\rho[\Omega_1, \Omega_2](\bar{x}, \bar{y}) \leq \min(\theta_\rho[F](\bar{x}, \bar{y})/2, \rho) \leq \theta_{2\rho}[\Omega_1, \Omega_2](\bar{x}, \bar{y})$  ( $\rho > 0$ );
- (ii)  $\theta[\Omega_1, \Omega_2](\bar{x}, \bar{y}) \leq \min(\theta[F](\bar{x}, \bar{y})/2, 1) \leq 2\theta[\Omega_1, \Omega_2](\bar{x}, \bar{y})$ ;
- (iii)  $\hat{\theta}[\Omega_1, \Omega_2](\bar{x}, \bar{y}) \leq \min(\hat{\theta}[F](\bar{x}, \bar{y})/2, 1) \leq 2\hat{\theta}[\Omega_1, \Omega_2](\bar{x}, \bar{y})$ ;
- (iv)  $\eta[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \eta[F](\bar{x}, \bar{y}) / (\hat{\eta}[F](\bar{x}, \bar{y}) + 2)$ ;
- (v)  $\hat{\eta}[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \hat{\eta}[F](\bar{x}, \bar{y}) / (\hat{\eta}[F](\bar{x}, \bar{y}) + 2)$ ;
- (vi) *If  $\dim X + \dim Y < \infty$  then  $\bar{\eta}[\Omega_1, \Omega_2](\bar{x}, \bar{y}) = \bar{\eta}[F](\bar{x}, \bar{y}) / (\bar{\eta}[F](\bar{x}, \bar{y}) + 2)$ .*

The above theorem implies equivalence of the corresponding regularity properties of  $F$  and the pair of sets (47).

**Corollary 7.1.** *The equivalences below refer to multifunction  $F$  and the pair of sets (47) at  $(\bar{x}, \bar{y})$ .*

$$\begin{aligned} (Cov)_M &\Leftrightarrow (R)_S & (UCov)_M &\Leftrightarrow (UR)_S; \\ (RD)_M &\Leftrightarrow (RD)_S & (URD)_M &\Leftrightarrow (URD)_S. \end{aligned}$$

*If  $\dim X + \dim Y < \infty$  then  $(URDL)_M \Leftrightarrow (URDL)_S$ .*

Thus, regularity properties of a multifunction are equivalent to the corresponding properties of a certain collection of sets. The equivalence of the two sets of properties is in fact deeper. Given a collection of sets, it is possible to construct a multifunction such that its regularity properties are equivalent to the corresponding properties of the collection of sets.

Let  $\Omega_1, \Omega_2, \dots, \Omega_n \subset X$  ( $n > 1$ ) be a collection of sets in a normed linear space  $X$  and  $\bar{x} \in \bigcap_{i=1}^n \Omega_i$ .

Consider a multifunction  $F : X \rightrightarrows X^n$  given by

$$(58) \quad F(x) = (\Omega_1 - x) \times (\Omega_2 - x) \times \dots \times (\Omega_n - x).$$

Obviously  $(\bar{x}, 0) \in \text{gph } F$ .

**Theorem 8.** *Let multifunction  $F : X \rightrightarrows X^n$  be given by (58). Then*

- (i)  $\theta_\rho[\Omega_1, \dots, \Omega_n](\bar{x}) = \theta_\rho[F](\bar{x}, 0)$  ( $\rho > 0$ );
- (ii)  $\theta[\Omega_1, \dots, \Omega_n](\bar{x}) = \theta[F](\bar{x}, 0)$ ;

- (iii)  $\hat{\theta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \hat{\theta}[F](\bar{x}, 0)$ ;
- (iv)  $\eta[\Omega_1, \dots, \Omega_n](\bar{x}) = \eta[F](\bar{x}, 0)$ ;
- (v)  $\hat{\wedge}[\Omega_1, \dots, \Omega_n](\bar{x}) = \hat{\wedge}[F](\bar{x}, 0)$ ;
- (vi) If  $\dim X < \infty$  then  $\bar{\eta}[\Omega_1, \dots, \Omega_n](\bar{x}) = \bar{\eta}[F](\bar{x}, 0)$ .

*Proof.* All assertions follow easily from the definitions of the corresponding constants. The first three were established in [30, Theorem ]. Below we prove (v).

Let  $(x, y) \in \text{gph } F$  and  $(x^*, y^*) \in N((x, y) | \text{gph } F)$ . The first inclusion means that  $y = (\omega_1 - x, \omega_2 - x, \dots, \omega_n - x)$  for some  $\omega_i \in \Omega_i$ ,  $i = 1, 2, \dots, n$ , while the second one can be expressed as

$$(59) \quad \limsup_{\substack{u \rightarrow x, \\ \substack{\Omega_i \\ i=1,2,\dots,n} \\ v_i \rightarrow \omega_i}} \frac{\langle x^*, u - x \rangle + \sum_{i=1}^n \langle y_i^*, (v_i - u) - (\omega_i - x) \rangle}{\max(\|u - x\|, \max_{1 \leq i \leq n} \|(v_i - u) - (\omega_i - x)\|)} \leq 0,$$

where  $y^* = (y_1^*, y_2^*, \dots, y_n^*)$ . Fixing  $u = x$  and, for any  $i = 1, 2, \dots, n$ ,  $v_j = \omega_j$  when  $j \neq i$ , we obtain from (59):

$$(60) \quad y_i^* \in N(\omega_i | \Omega_i).$$

Similarly, fixing in (59)  $v_i = \omega_i$  for all  $i = 1, 2, \dots, n$ , leads to the equality

$$(61) \quad x^* = y_1^* + y_2^* + \dots + y_n^*.$$

On the other hand, inclusions (60) (for  $i = 1, 2, \dots, n$ ) and equality (61) obviously imply (59). The assertion follows from definitions (25) and (41).  $\blacksquare$

Multifunction (58) was used for a similar purpose in Ioffe [19].

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