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About the influence of pre-stress upon adiabatic perturbations of the Earth

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Summary. In this paper we examine the influence of the state of stress in the equilibrium configuration of the Earth (i.e. the pre-stress) upon its adiabatic perturbations. The equations governing these perturbations to the first order (Woodhouse & Dahlen; Dahlen) are re-derived using a Lagrangian approach. Different expressions of the sesquilinear form associated to the elastic-gravitational operator are given. One of these provides a way to extend to hydrostatically pre-stressed solids the criterion of local stability given by Friedman & Schutz for uniformly rotating fluids. Then the propagation in the Earth of seismic wavefronts is considered. It is shown that the nature of these different wavefronts is entirely determined by the quadratic coefficients of the development of the specific internal energy variation, corresponding to isentropic evolution, with respect to the Lagrangian finite deformation tensor. Expressions for the velocities of the various waves are given as functions of incidence angle and pre-stress for orthotropic elastic material. In the particular case where the elastic parameters depend only on one coordinate of a curvilinear system and the axis of orthotropy of the material coincides with the corresponding natural base vector, the elastodynamic equations are reduced to a simple system for a displacement stress vector, using surface operators. In particular for spherical geometry, equations are obtained which generalize to orthotropic pre-stress those given by Alterman *et al.* and Takeuchi & Saito.

Key words: anisotropy, gravito-elastodynamics, normal modes, perturbation, pre-stress

1 Introduction

Since at least Love (1911) it has been common to establish the first-order equations governing the adiabatic perturbations of the Earth by considering the Eulerian perturbations of the equilibrium configuration (see, e.g. Pekeris & Jarosh 1958; Dahlen 1972). Sections 2 and 3 review how these equations may be derived using a Lagrangian approach. Equation

(36) of the sesquilinear form associated with the elastic-gravitational operator may be used as a starting point for the mathematical definition and study of this operator. Equations (38) and (39) are perhaps less well known. In the hydrostatic case they allow the energy to be expressed as a quadratic form of the Eulerian perturbations of the various physical quantities, and thus local stability to be considered in a similar way to that used for perfect fluids (Friedman & Schutz 1978b).

It seems of some importance to introduce the tensor c^{ijkl} which arises naturally in the development of the specific internal energy variation. Section 4 shows indeed that this tensor determines the nature of the different seismic wavefronts.

Following Alterman, Jarosh & Pekeris (1959) and Takeuchi & Saito (1972), we show in Sections 5, 6 and 7 how the elastic-gravitational equations can be separated in the case of lateral invariance of the mechanical properties of the material. The hypothesis of orthotropy made by Takeuchi & Saito (1972) in spherical geometry is extended here to the pre-stress for more general stratifications. It is the broadest hypothesis which may be adopted in a global approach. Indeed the assumption of lateral invariance implies that the different elastic tensors are invariant under parallel displacements over the constitutive surfaces of the stratification. Thus, considering closed surfaces in a global approach, we are led to assume that these different tensors are at each point invariant under rotation around the direction normal to the stratification.

2 Preliminaries

Let us consider a deformable body which is in equilibrium in a Euclidean space E in which we introduce a convenient curvilinear coordinate system (x^i) with metric tensor g^{ij}. A deformation of this body from this reference configuration is defined at each instant by a map f_t:

$$\forall \ a \in V, a \to f_t(a) = x(a, t) = a + u(a, t) \in V_t,$$

where $u(a, t)$ is the Lagrangian displacement of the material point a, defined in the volume V, from its initial position to the corresponding one at t in the deformed volume V_t.

Let us denote F the linear tangent application of f_t at a, Du the Euclidian derivative mapping of u at (a, t), ϵ the Lagrangian finite deformation tensor and e the Eulerian one. F is an isomorphism of E, which is naturally defined from the tangent space $T_a(E)$ at point a on to the tangent space at the deformed point $x(a, t)$, $T_x(E)$; ϵ and e may be considered as bilinear symmetrical forms on E (ϵ is naturally defined on $T_a(E)$ and e on $T_x(E)$). In the local frame $(e_i)_{i=1,3}$ at point a, these different tensors are expressed by:

$$Du(e_q) = D_q u^p e_p, \quad F^p{}_q = g^p{}_q + D_q u^p,$$

$$\epsilon_{pq} = D_p u_q + D_q u_p + D_p u^k \, D_q u_k, \quad e_{ij} = (F^{-1})^p{}_i (F^{-1})^q{}_j \, \epsilon_{pq}. \tag{1}$$

At each time the different expressions of virtual work result in an evaluation of (differential) linear forms on the field space of u. The virtual strain work is usually expressed as follows (see, e.g. Bamberger 1981; Malvern 1969; Truesdell 1972):

$$\delta W_S = \int_{V_t} T^{ij} \delta e_{ij} \, dVt = \int_V \sigma^{pq} \delta \epsilon_{pq} \, dV, \tag{2}$$

where:

$$\delta e_{ij} = D_i \delta u_j + D_j \delta u_i, \quad \delta \epsilon_{pq} = F^i{}_p F^i{}_q \, \delta e_{ij}.$$

This defines the Cauchy stress tensor T and the (second) Piola–Kirchhoff stress tensor σ, linked to each other by:

$$\sigma^{pq} = (1+\theta)\,\frac{\partial a^p}{\partial x^i}\,\frac{\partial a^q}{\partial x^j}\,T^{ij} = (1+\theta)\,\{(F^{-1}\times F^{-1})\,(T)\}^{pq} \tag{3}$$

where $\times$ denotes the tensorial product and $\theta = \det(F) - 1$ the volumetric dilatation which may be expressed up to the second order in Du as:

$$\theta = D_i u^i + \{(D_i u^i)^2 - D_i u^j D_j u^i\}/2. \tag{4}$$

The Cauchy stress tensor is indeed directly related to the state of stress at each given point of the Euclidean space. The actual density force acting at the point considered on the element of surface of unit normal n is precisely $T(n)$. The state of stress in the equilibrium configuration is represented by T_0 ($=\sigma_0$). This initial state of stress is called the pre-stress when considering the stress evolution during the perturbation of the equilibrium. In order to describe the perturbation of the different physical quantities, two approaches are usually adopted.

The first one corresponds to a Lagrangian description which consists of evaluating the perturbation from a given point in the reference configuration by following the considered material particle. One way to evaluate this Lagrangian perturbation is to follow the evolution in the frame strained by the field u (i.e. in considering co-moving coordinates, see Taub 1969 and Friedman & Schutz 1978a). In the case of the Cauchy stress tensor, one obtains in the local frame at reference point a:

$$\delta_l' T^{pq} = \{F^{-1}\times F^{-1}(T)\}^{pq} - T_0^{pq}.$$

Taking into account (3) and (4), this yields to the first order in Du:

$$\delta_l' T^{pq} = (\sigma - \sigma_0)^{pq} - \mathrm{div}(u)\sigma^{pq}. \tag{5}$$

A more common way simply considers the evolution by following the particle in the embedding Euclidean space:

$$\delta_l T = T_{x_t} - T_0.$$

In the local frame at initial point a, this yields, taking into account (3):

$$\delta_l T^{pq} = \{F\times F(\sigma)\}^{pq}/(1+\theta) - T_0^{pq}.$$

Thus one deduces from (1) and (4) that to the first order in Du:

$$\delta_l T^{pq} = (\sigma - \sigma_0)^{pq} - \sigma^{pq}\,\mathrm{div}(u) + \sigma^{pk} D_k u^q + \sigma^{qk} D_k u^p. \tag{6}$$

The second point of view is the Eulerian one which consists of considering the evolution of the physical quantities in the local frame at a given point of the space. In the hypothesis of small displacement, that is to the first order in u, the Eulerian perturbation may be related to the Lagrangian ones as follows:

$$\delta_e T^{pq} = \delta_l T^{pq} - u^k D_k T^{pq}, \tag{7a}$$

and (Taub 1969; Friedman & Schutz 1978a):

$$\delta_e T^{pq} = \delta_l' T^{pq} - L_u T^{pq}, \tag{7b}$$

where L_u is the Lie derivative with respect to u (see, e.g. Choquet-Bruhat *et al.* 1982; Doubrovine *et al.* 1982):

$$L_u T^{pq} = u^k D_k T^{pq} - T^{pk} D_k u^q - T^{qk} D_k u^p,\; -L_u g^{pq} = g^{pk} D_k u^q + g^{qk} D_k u^p.$$

The mechanical equation of motion (or virtual works principle, i.e. the stationarity of the total energy with respect to field u, at each instant) is usually written in Lagrangian form as follows (see, e.g. Bamberger 1981; Malvern 1969; Marsden & Hughes 1978; Truesdell 1972):

$$D_p \{(1 \times F)(\sigma)\}^{pj} + \rho\Psi^j\{x(a, t)\} = 0 \quad \text{in } V, \tag{8a}$$

$$F^j{}_q \sigma^{qp} n_p = (l + \chi)\,\Phi^j \qquad \text{on } \partial V \text{ as well as on each interface,} \tag{8b}$$

where $\{(1 \times F)\}(\sigma)^{pj} = \sigma^{pq} F^j{}_q$ is the (first or Piola–Lagrange) Piola–Kirchhoff tensor, ρ the density in the reference configuration, Ψ the body force density, Φ the surface density of force acting on the actual deformed boundary and χ the surface dilatation of this boundary with respect to the initial configuration with outward unit normal n.

Equation (8a) has the disadvantage of being hybrid; this is due to the fact that the (first) Piola–Kirchhoff tensor works naturally on the product of the spaces tangent at a and at x: $T_a(E) \times T_x(E)$. In the coordinate point of view this means that the index j in (8) refers to $T_x(E)$ and the index p to $T_a(E)$. In order to obtain a purely Lagrangian formulation (i.e. on $T_a(E)$ exclusively) one may multiply (8a) by $(F^{-1})^l{}_j$. Indeed summing up j indices leads to:

$$D_p \sigma^{pl} + \sigma^{pq} (F^{-1} D_p F)^l{}_q + \rho F^{-1} (\Psi)^l = 0,$$

which to the first order in Du, $D^2 u$ and with the help of (1) yields:

$$D_p \sigma^{pl} + \sigma^{pq} D_p D_q u^l + \rho\{\Psi - Du(\Psi)\}^l = 0.$$

Taking into account the equilibrium of the initial configuration, that is:

$$D_p\, \sigma_0^{pl} + \rho\Psi_0^l = 0 \;\text{ in } V\,(\sigma_0(n) = \Phi_0 \text{ on } \partial V), \tag{9}$$

one finally deduces:

$$D_p\{(\sigma - \sigma_0)^{pl} + \sigma_0^{pq} D_q u^l\} + (\sigma - \sigma_0)^{pq} D_p D_q u^l$$
$$+ \rho\{(\Psi - \Psi_0) - Du(\Psi - \Psi_0)\}^l = 0 \quad \text{in } V, \tag{10}$$

which constitutes the Lagrangian formulation of the equation governing to the first order the perturbation inside the body.

Taking into account (3) and the expression of the outward unit vector n_t normal to the deformed boundary as a function of its analogue n in the reference configuration:

$$n_t = \frac{1 + \theta}{1 + \chi} (F^{-1})^*(n),$$

where the asterisk denotes the adjoint operator with respect to the Euclidean structure, it is clear that equation (8b) is equivalent to the Eulerian formulation:

$$T(n_t) = \Phi. \tag{11}$$

Anyway, denoting by a dot the Euclidean scalar product in E, one obtains to the first order in Du:

$$\chi = \operatorname{div}(u) - Du(n) \cdot n \tag{12a}$$

$$n_t = n - Du^*(n) + \{Du(n) \cdot n\} n \tag{12b}$$

and from (3), (4) and (8b) or (11):

$$T(n_t) = \{1 - \operatorname{div}(u) + Du(n) \cdot n\}\, \sigma(n) + Du\{\sigma(n)\} = \Phi. \tag{12c}$$

3 The different forms of the equations governing to the first-order adiabatic perturbations of the Earth

Let us suppose that the Earth is in equilibrium under uniform rotation with instantaneous rotation vector Ω about its centre of mass G. As usual, we take this rotating space, centred at G, as the Lagrangian reference space. The fact that this reference configuration does not correspond to a physical state, because of the existence of tidal forces, needs to be pointed out. However, such an abstract configuration has the advantage of representing a mean thermodynamical state of the Earth which should not differ much from the real one.

3.1 CONSTITUTIVE EQUATIONS AND INTERNAL ENERGY

Adopting a purely elastic behaviour for the solid parts of the Earth V_S and a perfectly fluid rheology for the external core V_F leads, to the first order in Du for any adiabatic perturbations of the equilibrium, to the following expressions in the local frame attached to the reference point a (see Appendix 1):

$$(\sigma - \sigma_0)^{ij} = c^{ijkl} D_k u_l \qquad \text{in } V_\mathrm{S}, \tag{13a}$$

$$\delta_l T^{ij} = (T - T_0)^{ij} = -g^{ij}\delta_l p = p_0 \gamma \operatorname{div}(u) g^{ij} \qquad \text{in } V_\mathrm{F}, \tag{13b}$$

where $\gamma\{= \rho/p_0(\partial p/\partial \rho)_R\}$ is the adiabatic index of the fluid and c^{ijkl}/ρ are the quadratic coefficients in the development of the specific internal energy variation, corresponding to an isentropic evolution, with respect to the finite deformation tensor. Thus the tensor c^{ijkl} has the following symmetries:

$$c^{ijkl} = c^{klij} = c^{jikl} = c^{ijlk}. \tag{14}$$

It may be of interest to note that whereas an Eulerian description seems well suited to the fluid case, it is the second Piola–Kirchhoff tensor which naturally appears from the thermodynamical principles in the case of a solid (see Bamberger 1981; Malvern 1969). But, as a matter of fact, the constitutive equation of a perfect fluid may be regarded as a particular form of that of a solid. Indeed, from (13a), (5), (6) and (7) it is deduced to the first order in Du:

$$\delta_l' T^{ij} = c^{ijkl} D_k u_l - \sigma_0^{ij} \operatorname{div}(u) \tag{15a}$$

$$\delta_l T^{ij} = d^{ijkl} D_k u_l \tag{15b}$$

$$\delta_e T^{ij} = d^{ijkl} D_k u_l - u^k D_k \sigma_0^{ij} \tag{15c}$$

with:

$$d^{ijkl} = c^{ijkl} - \sigma_0^{ij} g^{kl} + \sigma_0^{ik} g^{jl} + \sigma_0^{jk} g^{il}. \tag{16}$$

It is then clear, by comparison with (13b), that in the case of a perfect fluid:

$$\sigma_0^{ij} = -p_0 g^{ij}, \quad c^{ijkl} = p_0(\gamma - 1) g^{ij} g^{kl} + p_0 (g^{ik} g^{jl} + g^{jk} g^{il}).$$

The tensor d^{ijkl} contains the symmetries (14) in the case of an hydrostatic pre-stress. This tensor has been adopted by Takeuchi & Saito (1972) in order to represent elasticity. The tensor d^{ijkl}, however, no longer has these symmetries in the case of anisotropic pre-stress. Dahlen (1972) and Woodhouse & Dahlen (1978) have adopted as reference tensor the orthogonal projection of d^{ijkl} (with respect to the usual scalar product: $t \cdot t' = t^{ijkl} t'_{ijkl}$) on to the space of tensors characterized by the symmetries (14).

As the evolution is supposed adiabatic, the variation δI of the internal energy is nothing else but the opposite of the work of the interior forces. That is, the strain work minus the mutual work W_M due to slipping at interfaces or to non-local actions as gravitational effects (the variation of gravitational potential energy may be also considered separately); thus (see Appendix 1):

$$\delta I = \int_V \delta_l i \quad dm - W_M \tag{17a}$$

where $\delta_l i$ is the Lagrangian perturbation of the specific internal energy which is expressed up to the second order in Du as:

$$\rho\delta_l i = \sigma_0^{ij} D_i u_j + \frac{1}{2}(c^{ijkl} + \sigma_0^{ik} g^{jl}) D_i u_j D_k u_l. \tag{17b}$$

If we consider a free evolution in the neighbourhood of the reference state of the Earth, considered as a closed system, the forces reduce to inertia and gravitation:

$$\delta_l \Psi = \Psi(x) - \Psi_0(a) = g'(x) - g_0'(a) - 2\Omega \times \partial_t u - \partial_{tt}^2 u \tag{18}$$

where g' is the gravity field:

$$g'(x) - g_0'(a) = g(x) - g_0(a) - \Omega \times (\Omega \times u),$$

$$g(x) - g_0(a) = G \int_V \left\{ \frac{x' - x}{|x' - x|^3} - \frac{a' - a}{|a' - a|^3} \right\} dm', \quad dm = \rho dV,$$

and G the universal gravitational constant.

In the hypothesis that the domain V_t remains contractible (i.e. without hole), we deduce to the first order in u:

$$g - g_0 = G \int_V \left\{ \frac{u' - u}{|a' - a|^3} - 3(a' - a) \frac{(a' - a) \cdot (u' - u)}{|a' - a|^5} \right\} dm', \tag{19a}$$

where u' represents $u(a')$.

This yields the usual expression:

$$g - g_0 = \text{grad}\,(\psi + u \cdot g_0) - Du^*(g_0) = Dg_0(u) + \delta_e g = Dg_0(u) + \text{grad}\,\psi, \tag{19b}$$

where:

$$\psi = -G \int_V \frac{(a' - a) \cdot u'}{|a' - a|^3} dm' = -G \int_V \frac{\text{div}(\rho' u')}{|a' - a|} dV' - G \int_{\Sigma_\rho} \frac{[\rho'] u' \cdot n'}{|a' - a|} d\Sigma'.$$

Here Σ_ρ consists of all the surfaces of discontinuity of ρ and $[\rho]$ denotes the jump of ρ through Σ_ρ in the direction of n. ψ is the potential of mass redistribution which verifies (in the sense of distribution over E):

$$\Delta\psi = 4\pi G\, \text{div}(\rho u). \tag{20}$$

Thus the traces of $(\text{grad}\,\psi - 4\pi G\rho u) \cdot n$ on both sides of each interface are identical, as well as that of ψ.

The variation of gravitational potential energy is:

$$\delta P = -\frac{G}{2} \int_{V \times V} \left\{ \frac{1}{|x' - x|} - \frac{1}{|a' - a|} \right\} dm dm',$$

and a second-order development yields:

$$\delta P = -\int_V u \cdot g_0 \, dm + \frac{G}{4} \int_{V\times V} \left\{ \frac{|u' - u|^2}{|a' - a|^3} - 3 \frac{\{(a' - a) \cdot (u' - u)\}^2}{|a' - a|^5} \right\} dm dm'$$

$$= -\int_V u \cdot \frac{g + g_0}{2} \, dm. \tag{21}$$

3.2 LAGRANGIAN AND EULERIAN FORMULATIONS OF ELASTODYNAMIC EQUATIONS OF THE EARTH

Making use of (13a), equation (10) may be rewritten to the first order in Du, D^2u:

$$\partial^2_{tt} u + 2\Omega \times \partial_t u + A(u) = 0 \quad \text{in } V, \tag{22}$$

where the operator A is defined by:

$$A(u)^j = -\frac{1}{\rho} D_i \tau^{ij} - \{g - g_0 - \Omega \times (\Omega \times u)\}^j \tag{23a}$$

with:

$$\tau^{ij} = (c^{ijkl} + \sigma_0^{ik} g^{jl}) D_k u_l$$

$$D_i \sigma_0^{ij} + \rho\{g_0 - \Omega \times (\Omega \times a)\}^j = 0, \tag{23b}$$

and where $g - g_0$ is given by (19), to the first order in u.

These equations correspond to those given by Woodhouse & Dahlen (1978) and constitute the Lagrangian formulation of the elastodynamic equations inside the Earth.

Taking into account expressions (15c, 16) of the Eulerian perturbation of the Cauchy stress tensor, we may deduce from (23):

$$A(u)^j = -\frac{1}{\rho} \{D_i \delta_e T^{ij} - g_0'^j \operatorname{div}(\rho u) + \rho(\operatorname{grad} \psi)^j\}, \tag{24a}$$

that is:

$$A(u)^j = -\frac{1}{\rho} \delta_e (D_i T^{ij} + \rho g_0'^j). \tag{24b}$$

This shows that equation (22) may be obtained to the first order in u, Du, D^2u at each inner point of the reference configuration V, as the Eulerian perturbation of the equilibrium equation. This approach, which has been adopted by Takeuchi & Saito (1972) in the hydrostatic case and by Dahlen (1972) in the general case, leads to:

$$\rho \partial^2_{tt} u^j + 2\rho(\Omega \times \partial_t u)^j - D_i(d^{ijkl} D_k u_l - u^k D_k \sigma_0^{ij}) + \operatorname{div}(\rho u) g_0'^j$$
$$- \rho(\operatorname{grad} \psi)^j = 0 \text{ in } V, \tag{25}$$

where the expression of d^{ijkl} is given in (16), and constitutes the Eulerian form of the elastodynamic equations inside the Earth. It may be of some interest to note that this Eulerian interpretation of the elastodynamic equations in the reference configuration fails at boundaries, unless they are assumed undeformable.

3.3 BOUNDARY CONDITIONS

The usual hypothesis that the domain V_t remains contractible implies that at each inner interface, assumed to be closed and regular, the different parts V_{t+}, V_{t-} remain in contact. This yields, to the first order, $u \cdot n$ continuous across the reference interface:

$$[u] \cdot n = u_+ \cdot n - u_- \cdot n = 0, \tag{26}$$

where n is the unit normal in the reference configuration, oriented in the direction of the jump (i.e. from V_- toward V_+).

Taking into account (13a) it is deduced from boundary conditions (9), (12c) that to the first order in u, Du, (Dahlen 1972):

$$[T(n_t)] = [\delta_l\{T(n)\}] = D\{\sigma_0(n)\}[u],$$

and thus:

$$[(\sigma - \sigma_0)(n) + Du\{\sigma_0(n)\} - \{\mathrm{div}(u) - Du(n) \cdot n\}\sigma_0(n)] = D\{\sigma_0(n)\}[u]. \tag{27}$$

Apart from eventual source areas, the solid interfaces are usually considered as welded. Therefore, the boundary conditions on these interfaces are:

$$[u] = 0, \quad [(\sigma - \sigma_0)(n) + Du\{\sigma_0(n)\} - \{\mathrm{div}(u) - Du(n) \cdot n\}\sigma_0(n)] = 0. \tag{28}$$

Let us consider $\tau^*(n)$, (see 23b):

$$\tau^*(n) = (\sigma - \sigma_0)(n) + Du\,\{\sigma_0(n)\}, \tag{29}$$

and $\tau'^*(n)$ defined by:

$$\tau'^*(n) = (\sigma - \sigma_0)(n) + Du\{\sigma_0(n)\} + Du^*\{\sigma_0(n)\} - \mathrm{div}(u)\sigma_0(n). \tag{30}$$

From (12) and (13a) and to the first order in Du, it is clear that:

$$\tau^*(n) = (1 + \chi)T(n_t) - \sigma_0(n). \tag{31}$$

Considering expressions (15b) and (16) one deduces that $\tau'^*(n)$ coincides with $(\delta_l T)(n)$ if the pre-stress is hydrostatic. In the general case, however:

$$\tau'^*(n) = (\delta_l T)(n) - \sigma_0\{Du^*(n)\} + Du^*\{\sigma_0(n)\},$$

that is:

$$\tau'^*(n)^j = \tau'^{ij} n_i, \tag{32a}$$

with:

$$\tau'^{ij} = \tau^{ij} + \sigma_0^{il} g^{jk} D_k u_l - \mathrm{div}(u)\,\sigma_0^{ij}. \tag{32b}$$

In terms of these vectors, (28) may be rewritten:

$$[u] = 0, \quad [\tau^*(n)] = 0 \quad \text{or} \quad [\tau'^*(n)] = \{\sigma_0(n) \cdot [Du(n)]\}\, n - \{n \cdot [Du(n)]\}\, \sigma_0(n). \tag{33}$$

(Note that the latter term is null if n is an eigendirection of σ_0.)

At the external boundary ∂V, neglecting atmospheric pressure, it is easily deduced from (9) and (12c) that:

$$\sigma_0(n) = \tau^*(n) = \tau'^*(n) = 0, \tag{34}$$

where n represents the unit outward normal to ∂V.

At an inner interface Σ where a fluid is involved:

$$\sigma_0(n) = -p_0 n.$$

From (26), (27), (29), (30) it is deduced in Appendix 2 that the boundary conditions may take the following forms (equivalent to the one given by Woodhouse & Dahlen 1978):

$$[u]\cdot n = 0,\ [\tau^*(n)] = -n\,\mathrm{div}_\Sigma(p_0[u]) - p_0 W[u]$$
$$= -n\,\mathrm{div}_\Sigma(p_0[u]) + p_0[Du^*(n) - \{Du(n)\cdot n\}n], \tag{35a}$$

$$[u]\cdot n = 0,\ [\tau'^*(n)] = -([u]\cdot \mathrm{grad}_\Sigma\, p_0)n. \tag{35b}$$

Here div_Σ and grad_Σ are the usual surface operators, with:

$$\mathrm{div}\, u = \mathrm{div}_\Sigma\{u - (u\cdot n)n\} + Du(n)\cdot n + (c_1 + c_2)u\cdot n,$$

$$\mathrm{grad}\, p_0 = \mathrm{grad}_\Sigma\, p_0 + (\mathrm{grad}\, p_0\cdot n)n,$$

where c_1 and c_2 are the principal curvatures of Σ at the considered point, i.e. the eigenvalues of the Weingarten operator W which is the self-adjoint operator defined on the space tangent at Σ, $T_a(\Sigma)$, by:

$$\forall\, v\in T_a(\Sigma)\ W(v) = D_v(n)\in T_a(\Sigma).$$

Note that when pre-stress is hydrostatic on both sides of the interface and ρ is discontinuous across it, the equilibrium of the reference configuration yields easily that the right side of (35b) vanishes.

3.4 SESQUILINEAR FORM ASSOCIATED TO OPERATOR A AND LOCAL STABILITY

From these boundary conditions it may be deduced that A is a symmetric unbounded operator in $L^2_C(V, dm)$. Indeed, from Appendix 3:

$$(A(u)|v) = \int_V (c^{ijkl} + \sigma_0^{ik} g^{jl}) D_i u_j D_k \bar{v}_l + \rho\{(\Omega\cdot u)(\Omega\cdot\bar{v}) - \Omega^2 u\cdot\bar{v}\}\ dV$$
$$+ \frac{G}{2}\int_{V\times V}\left\{\frac{(u'-u)\cdot(\bar{v}'-\bar{v})}{|a'-a|^3} - 3\,\frac{\{(a'-a)\cdot(u'-u)\}\{(a'-a)\cdot(\bar{v}'-\bar{v})\}}{|a'-a|^5}\right\} dm\,dm'$$
$$+ \int_\Sigma p_0 [\![[u]\cdot \mathrm{grad}_\Sigma(\bar{v}\cdot n) + [\bar{v}]\cdot\mathrm{grad}_\Sigma(u\cdot n)$$
$$- [W\{u - (u\cdot n)n\}\cdot\{\bar{v} - (\bar{v}\cdot n)n\}]]\!]\, d\Sigma, \tag{36}$$

where $(|)$ denotes the inner product of $L^2_C(V, dm)$:

$$(\,|\,) = \int_V \cdot\ \bar{}\ dm,$$

Σ consists of all interfaces where a fluid is involved and $\bar{}$ denotes complex conjugation. The integration over $V\times V$ involved in the second term of (36) is due to the non-local character of mass redistribution effects. In order to avoid this term, the potential of mass redistribution ψ is usually introduced via (19b) (Pekeris & Jarosh 1958). However ψ must then be constrained by equation (20). Adopting a variational approach, one may (Backus & Gilbert 1967; Johnson & Smylie 1977; Woodhouse & Dahlen 1978) use the method of Lagrange multipliers in order to derive an unconstrained stationary problem related to the operator A.

The integration over Σ corresponds to the mutual work caused by slipping at interfaces Σ. More precisely, considering a real evolution u, expressions (17) and (21) yield:

$$\delta I = \int_V (\sigma_0^{ij} D_i u_j - \rho g_0 \cdot u)\, dV + \frac{G}{4} \int_{V \times V} \left\{ \frac{|u' - u|^2}{|a' - a|^3} - 3\, \frac{\{(a' - a)\cdot(u' - u)\}^2}{|a' - a|^5} \right\} dm\, dm'$$

$$+ \frac{1}{2} \int_V (c^{ijkl} + \sigma_0^{ik} g^{jl}) D_i u_j D_k u_l\, dV + \int_0^t dt \int_\Sigma [(1 + \chi_t)\, T(n_t) \cdot \partial_t u]\, d\Sigma,$$

with the unit normal n to Σ oriented in accordance with the jump.

From relation (31) and Appendix 3, we deduce:

$$\int_0^t dt \int_\Sigma [(1 + \chi_t)\, T(n_t) \cdot \partial_t u]\, d\Sigma = \int_0^t dt \int_\Sigma [\{\sigma_0(n) + \tau^*(n)\} \cdot \partial_t u]\, d\Sigma$$

$$= \int_\Sigma p_0\, [\![-[u] \cdot n + [u] \cdot \mathrm{grad}_\Sigma (u \cdot n) - [W\{u - (u \cdot n) n\} \cdot \{u - (u \cdot n) n\}]/2]\!]\, d\Sigma,$$

and finally, taking into account the equilibrium of the reference state (23b):

$$\delta I = \frac{1}{2} \left\{ \{A(u) | u\} + \int_V \{ |\Omega \times (a + u)|^2 - |\Omega \times a|^2 \}\, dm \right\}. \tag{37}$$

But other expressions of $(A(u)|v)$ may be obtained for which the surface integral term is very simple or cancels. Indeed, as derived in Appendix 4:

$$(A(u)|v) = \int_V [\![(c^{ijkl} - \sigma_N g^{ij} g^{kl} + \sigma_0^{ik} g^{jl} + \sigma_N g^{il} g^{jk}) D_i u_j D_k \bar{v}_l + S\{ -\rho D g_0' u \cdot \bar{v}$$

$$+ \{D\bar{v}(u) - u\, \mathrm{div}(\bar{v})\} \cdot \mathrm{grad}\, \sigma_N \}]\!]\, dV - \int_E \frac{\mathrm{grad}\, \psi(u) \cdot \mathrm{grad}\, \psi(\bar{v})}{4\pi G}\, dV$$

$$- \int_\Sigma S\{ (u \cdot n) [\bar{v}] \cdot \mathrm{grad}_\Sigma\, p_0 \}\, d\Sigma, \tag{38a}$$

and also:

$$(A(u)|v) = \int_V [\![(c^{ijkl} - \sigma_N g^{ij} g^{kl} + \sigma_0^{ik} g^{jl} + \sigma_N g^{il} g^{jk}) D_i u_j D_k \bar{v}_l + S\{ 2\, \rho g_0' . u\, \mathrm{div}(\bar{v})$$

$$+ (g_0' \cdot u)(\bar{v} \cdot \mathrm{grad}\, \rho) + (\mathrm{grad}\, \sigma_N + \rho g_0')(D\bar{v}(u) - u\, \mathrm{div}(\bar{v})) \}]\!]\, dV$$

$$- \int_E \frac{\mathrm{grad}\, \psi(u) \cdot \mathrm{grad}\, \psi(\bar{v})}{4\pi G}\, dV + \int_\Sigma S\{ (\bar{v} \cdot n) [u \cdot (\rho g_0' - \mathrm{grad}_\Sigma\, p_0)] \}\, d\Sigma$$

$$+ \int_{\Sigma'} S\{ [\rho]\, (g_0' \cdot u)(\bar{v} \cdot n) \}\, d\Sigma, \tag{38b}$$

where Σ is comprised as in (36) of all interfaces where a fluid is involved and Σ' of all welded interfaces. $S\{\ \}$ denotes the symmetric part:

$S\{a(u, v)\} = \{a(u, v) + \overline{a(v, u)}\}/2,$

and here σ_N is any regular scalar function the value of which is $-p_0$ in V_F and 0 on ∂V. Under the common assumption that the interfaces belong to a regular family of surfaces we may choose $\sigma_N = \sigma_0(n) \cdot n$, where n is here the unit normal to the surface of the family at the considered point.

In the hydrostatic case with the choice $\sigma_N = -p_0$, (38a) yields:

$$(A(u)\,|\,v) = \int_V d^{ijkl} D_i u_j D_k \bar{v}_l + \rho S\{u \cdot g_0' \operatorname{div}(\bar{v}) - \bar{v} \cdot \operatorname{grad}(u \cdot g_0')\} \; dV$$

$$-\int_E \frac{\operatorname{grad} \psi(u) \cdot \operatorname{grad} \psi(\bar{v})}{4\pi G} dV - \int_\Sigma S\{(u \cdot n)[\bar{v}] \cdot \operatorname{grad}_\Sigma p_0\} d\Sigma. \tag{39a}$$

Also, with the use of (15c) it is easy to see that (38b) reduces to:

$$\{A(u)|v\} = \int_V \left\{(\delta_e T_u)_{ij} (d^{-1})^{ijkl} (\delta_e T_{\bar{v}})_{kl} + \frac{u \cdot g_0' \bar{v} \cdot g_0'}{|g_0'|^2} \{\operatorname{grad} \rho - \rho^2 (d^{-1})_i^{\,i\,k}{}_k g_0'\} \cdot g_0' \right\} dV$$

$$-\int_E \frac{(\delta_e g_u)(\delta_e g_{\bar{v}})}{4\pi G} dV + \int_{\Sigma_\rho} [\rho]\,(g_0' \cdot n)(u \cdot n)(\bar{v} \cdot n)\, d\Sigma. \tag{39b}$$

The latter expression provides a way to extend the criterion of local stability (Friedman & Schutz 1978b) to hydrostatically pre-stressed solids. Indeed relation (15b) shows that it is possible to select the Cauchy stress tensor T and the specific entropy h as state variables in order to describe to the first order the elastic evolution around an hydrostatic configuration. Then, supposing the evolution is isentropic ($\delta_l h = 0$) leads to:

$$\delta_l \rho = -\rho \operatorname{div}(u) = -\rho tr\{d^{-1}(\delta_l T)\},$$

so that, for an arbitrary evolution:

$$\delta_l \rho = -\rho tr(d^{-1} \delta_l T) + \left(\frac{\partial \rho}{\partial h}\right)_T \delta_l h.$$

Applying this relation to Eulerian perturbations, we deduce:

$$\delta_e \rho = -\operatorname{div}(\rho u) = -\rho tr(d^{-1} \delta_e T) + \left(\frac{\partial \rho}{\partial h}\right)_T \delta_e h,$$

and finally, with (15c) in the case of an isentropic evolution:

$$\delta_e h = -\left(\frac{\partial h}{\partial \rho}\right)_T \{\operatorname{grad} \rho - \rho^2 (d^{-1})_i^{\,i\,k}{}_k g_0'\} \cdot u = -\operatorname{grad} h \cdot u.$$

This shows that:

$$\operatorname{grad} h = \left(\frac{\partial h}{\partial \rho}\right)_T \{\operatorname{grad} \rho - \rho^2 (d^{-1})_i^{\,i\,k}{}_k g_0'\} /\!/ g_0' \; (\operatorname{grad} \rho /\!/ g_0'),$$

and therefore that the second term of the right side of (39b) is proportional to the product of Eulerian perturbations of specific entropy.

If we consider now an evolution with viscosity, say Newtonian, instead of (22) the real displacement verifies:

$$\partial_{tt}^2 u + 2\Omega \times \partial_t u + A(u) = -B(\partial_t u) \tag{40}$$

where B is a positive self-adjoint operator. $(B(\partial_t u) \mid \partial_t u)$ represents the power of viscous irreversibility, i.e. the rate of entropy increase.

Let us now introduce, after Friedman & Schutz (1978b), the canonical energy $E_{C,R}$ (canonical with respect to the symplectic structure related to equation (22)):

$$E_{C,R} = \{(A(u) \mid u) + (\partial_t u \mid \partial_t u)\}/2.$$

Then, formally, it is straightforward from (40) that:

$$\partial_t E_{C,R} = -(B(\partial_t u) \mid \partial_t u) \leqslant 0. \tag{41}$$

So that, in order to ensure stability, it must necessarily be assumed that A is positive in the space of displacements orthogonal to uniform translations. That is the space of kinematically admissible displacements satisfying the constraint that the reference space is centred at G: $\int u dm = 0$. (The fact that in the presence of other celestial bodies, the trajectory of G is not uniform contributes simply to tidal forces in the reference space, see Wahr (1981).) Indeed, let us suppose that A is not positive in this space. Then it would be possible to find a kinematically admissible displacement u_0 such that: $(A(u_0) \mid u_0) < 0$. Adopting this displacement with $(\partial_t u)_0 = 0$ as initial conditions of the evolution problem (40), it would follow by integration of (41):

$$-(A(u) \mid u) \geqslant -(A(u_0) \mid u_0) > 0.$$

But (40) yields also:

$$-(A(u) \mid u) = (\partial^2_{tt} u \mid u) + 2\Omega. \int_V u \times \partial_t u \, dm + (B(\partial_t u) \mid u).$$

Thus it would be impossible that u converges and that $\partial_t u$ and $\partial_{tt} u$ vanish for the topology of $H^1(V_S \cup V_F)$. Indeed, since the form $(B.\mid.)$ is continuous for this topology in the case of Newtonian viscosity, $-(A(u) \mid u)$ would tend to zero. This clearly contradicts asymptotic stability.

Consider now local conditions and thus displacements with support in the vicinity of a given point. In the case of a fluid, the independence of $\delta_e p$ and $\delta_e h$ leads us to assume (see expression (39b) for $(A(u) \mid u)$) that:

$$g_0' \cdot \operatorname{grad} \rho \geqslant \rho^2 (d^{-1})^{i\ k}_{\ i\ k} g_0'^2. \tag{42}$$

This is the generalized Schwarzschild criterion for uniformly rotating fluids (Friedman & Schutz 1978b). In the case of a solid, $\delta_e T$ and $\delta_e h$ may no longer be fixed independently in the neighbourhood of a point. However, for a negative value of $g_0' \cdot (\operatorname{grad} \rho - \rho^2 (d^{-1})^{i\ k}_{\ i\ k} g_0')$ instability arises if the tensor d^{ijkl} is sufficiently close to the direction of $g^{ij} g^{kl}$. For instance, in the isotropic case, given a value of the bulk modulus $(\kappa = \lambda + 2\mu/3)$ and a negative value of:

$$\operatorname{grad} \rho - \rho^2 (d^{-1})^{i\ k}_{\ i\ k} \, g_0' \cdot g_0' = \{\operatorname{grad} \rho^2 g_0'/\kappa\} \cdot g_0',$$

there is a critical value of the rigidity (μ) under which local instability appears.

4 Pre-stress and seismic wavefronts

The influence of pre-stress upon plane wave propagation in homogeneous media has been already investigated in geophysical literature (see, e.g. Nikitin & Chesnokov 1981). Here we will examine a related problem: how the pre-stress affects seismic wavefront propagation, and thus travel-time analysis. Following Hadamard (1903) (see also Bamberger 1981;

Truesdell 1972; Vlaar 1968) let us consider an acceleration wavefront, that is a travelling surface S_t across which the relative acceleration admits a jump: $s = [\partial_{tt}u]$. Then it may be deduced (Hadamard lemma) that the second-order spatial derivative of u also admits a discontinuity across the front: $[D_iD_ju] = n_in_j{}^s/U^2$, where n is the unit normal vector to the surface S_t representing the wavefront with normal velocity U in the reference configuration.

In order to obtain the eigenvalue problem which governs the polarization and the velocity of these fronts at a given point of the reference configuration, the jump operator has to be applied to equations (23) or (25). Taking into account the regularity of u, Du and $g - g_0$, this yields at a point where the different elastic parameters are regular:

$$M^i{}_js^j = \rho U^2 s^i, \text{ with } M^{ij} = c^{kilj}\, n_kn_l + g^{ij}\sigma_0^{kl}\, n_kn_l = d^{kilj}n_kn_l. \tag{43}$$

Thus it is clear that the polarization of the different acceleration waves corresponds to the different eigendirections of the symmetrical tensor $c^{kilj}n_kn_l$. Therefore it is in fact c^{ijkl} which determines the nature of the seismic fronts in the Earth. If c^{ijkl} is isotropic only *P*- and *S*-wavefronts exist and the influence of pre-stress anisotropy manifests itself only in the eigenvalues of (43), i.e. is confined to the dependence of the velocity of the wavefronts on their incidence. The importance of this anisotropy is thus related to the ratio between the usual Lamé parameters and the pre-stress deviator. Considering the maximum order of magnitude generally adopted for the latter in the lithosphere, a major effect is not to be expected (probably less than 1 per cent in velocity anisotropy).

In the case where c^{ijkl} is anisotropic, the velocities corresponding to the three kinds of fronts are affected by both c^{ijkl} and σ_0^{ij}. More precisely, let us assume that the anisotropy of c^{ijkl} is sufficiently weak to permit a (mathematical) perturbation approach (see Backus 1970), to the eigenvalue problem (43). Denoting λ' and μ' the Lamé coefficients of the orthogonal projection of c^{ijkl} on to the isotropic tensors space:

$$c^{ijkl} = \lambda' g^{ij}g^{kl} + \mu'(g^{ik}g^{jl} + g^{jk}g^{il}) + \delta^{ijkl},$$

it follows, in the case of quasi-compressional wavefronts:

$$\begin{aligned}\rho U_{qP}^2 &= \lambda' + 2\mu' + \sigma_0^{ij}n_in_j + \delta^{ijkl}n_in_jn_kn_l \\ &= \sigma_0^{ij}n_in_j + c^{ijkl}n_in_jn_kn_l \;\; (= d^{ijkl}n_in_jn_kn_l).\end{aligned} \tag{44}$$

Thus only a weak inference upon the deviatoric part of the state of stress may be expected from the knowledge of quasi-compressional wavefront velocity as a function of incidence.

The problem of determining 'inherent' anisotropy of the material, i.e. that which is related to the inner structure of material, independently of the state of stress, has been investigated by Nikitin & Chesnokov (1981). They have proposed an evaluation of the change of the quadratic adiabatic elastic parameters c^{ijkl} related to the deviatoric part of the pre-stress.

5 The orthotropic material case

Let us now consider in more detail the case where at each solid point the material is orthotropic (or transversely isotropic), that is its mechanical properties are invariant under rotation around a given axis. This results in supposing that c^{ijkl} and σ_0^{ij} are both orthotropic with respect to the same axis (the hypothesis for σ_0^{ij} is not essential in this section). Moreover let us suppose that it is possible to choose, at least locally, a curvilinear coordinate system such that at each point the vector of the natural basis e_1, associated to the first coordinate, corresponds to this axis of orthotropy and is orthogonal to the other two vectors of the basis which will be referred to by a Greek index: $(e_\alpha)\,\alpha = 2, 3$.

Let η_{ij} be the linearized deformation tensor: $\eta_{ij} = (D_i u_j + D_j u_i)/2$. For any rotation of axis e_1, η has five linearly independant invariants of second order. It is usual to choose:

$$(\eta^2{}_2 + \eta^3{}_3)^2,\quad (\eta^1{}_1)^2,\quad \eta^1{}_1(\eta^2{}_2 + \eta^3{}_3),\quad \eta^1{}_2\eta^2{}_1 + \eta^1{}_3\eta^3{}_1,\quad \eta^2{}_3\eta^3{}_2 - \eta^2{}_2\eta^3{}_3.$$

Thus supposing c^{ijkl} orthotropic yields:

$$c^{ijkl}\eta_{ij}\eta_{kl} = A'(\eta^2{}_2 + \eta^3{}_3)^2 + C'(\eta^1{}_1)^2 + 2F'\,\eta^1{}_1(\eta^2{}_2 + \eta^3{}_3) + 4L'(\eta^1{}_2\eta^2{}_1 + \eta^1{}_3\eta^3{}_1) + 4N'(\eta^2{}_3\eta^3{}_2 - \eta^2{}_2\eta^3{}_3). \tag{45}$$

Considering an adiabatic evolution and taking into account (13a) we deduce from (45) to the first order in Du:

$$\sigma^{\alpha\beta} = \sigma_0^{\alpha\beta} + \{(A' - 2N')\,\eta^\gamma{}_\gamma + F'\,\eta^1{}_1\}\, g^{\alpha\beta} + 2N'\,\eta^{\alpha\beta},$$
$$\sigma^{\alpha 1} = \sigma_0^{\alpha 1} + 2L'\eta^{\alpha 1},\quad \sigma^{11} = \sigma_0^{11} + F'\,\eta^\gamma{}_\gamma g^{11} + C'\eta^{11}. \tag{46}$$

Substituting these expressions into (23b) leads to:

$$\tau^{\alpha\beta} = \{(A' - 2N')D_\gamma u^\gamma + F'D_1 u^1\}\, g^{\alpha\beta} + N'g^{\beta\gamma}D_\gamma u^\alpha + (N' + \sigma_T)g^{\alpha\gamma}D_\gamma u^\beta,$$
$$\tau^{\alpha 1} = (L' + \sigma_T)g^{\alpha\beta}D_\beta u^1 + L'g^{11}D_1 u^\alpha,\quad \tau^{1\alpha} = L'g^{\alpha\beta}D_\beta u^1 + (L' + \sigma_N)g^{11}D_1 u^\alpha,$$
$$\tau^{11} = g^{11}\,\{F'D_\gamma u^\gamma + (C' + \sigma_N)D_1 u^1\}, \tag{47}$$

where $\sigma_N = \sigma_0{}^1{}_1$, $\sigma_T = \sigma_0{}^2{}_2 = \sigma_0{}^3{}_3$.

Thus it may be deduced that the eigenvalue problem (43) takes the form:

$$M^i{}_j s^j = \rho U^2 s^i, \quad \text{with } M^i{}_j = P^i{}_j + (\sigma_0^{kl} n_k n_l)\, g^i{}_j \text{ and} \tag{48}$$

$$P = \begin{vmatrix} C'n^1 n_1 + L'(n^2 n_2 + n^3 n_3) & (F' + L')n^1 n_2 & (F' + L')n^1 n_3 \\ (F' + L')n^2 n_1 & L'n^1 n_1 + A'n^2 n_2 + N'n^3 n_3 & (A' - N')n^2 n_3 \\ (F' + L')n^3 n_1 & (A' - N')n^3 n_2 & L'n^1 n_1 + N'n^2 n_2 + A'n^3 n_3 \end{vmatrix}$$

The eigenvalues of P are:

$$N'(n^2 n_2 + n^3 n_3) + L'n^1 n_1,\quad \{A'(n^2 n_2 + n^3 n_3) + C'n^1 n_1 + L' \pm \Delta\}/2,$$

with:

$$\Delta^2 = \{A'(n^2 n_2 + n^3 n_3) + C'n^1 n_1 - L'\}^2 + 4n^1 n_1 (n^2 n_2 + n^3 n_3)\,\{(F' + L')^2 - (A' - L')(C' - L')\}.$$

The expression of the different acceleration wave velocities follows:

$$\rho U_{S_1}^2 = N \sin^2\theta + L\cos^2\theta,$$

$$\rho U^2 \left\{ \begin{matrix} P \\ S_2 \end{matrix} = \frac{1}{2}\,\{A\sin^2\theta + C\cos^2\theta + L - (\sigma_N - \sigma_T)\sin^2\theta\} \pm \Delta \right\}, \tag{49}$$

where θ is here the angle between the unit normal to the front n and e_1, and:

$$\Delta^2 = \{A \sin^2 \theta + C \cos^2 \theta - L + (\sigma_N - \sigma_T) \sin^2 \theta\}^2 + \sin^2 (2\theta) \; [(F + L)^2 - \{(A - L + (\sigma_N - \sigma_T)\} (C - L)],$$

$$A = A' + \sigma_T = \rho U^2_{qPT}, \;\; C = C' + \sigma_N = \rho U^2_{qPN},$$

$$F = F' - \sigma_N, \;\; L = L' + \sigma_N = \rho U^2_{qSN}, \;\; N = N' + \sigma_T = \rho U^2_{qS_1 T}. \tag{50a}$$

These latter coefficients permit the definition of an orthotropic tensor b^{ijkl} containing the symmetries (14) by a relation analogous to (45). Thus:

$$b^{ijkl} \eta_{ij} \bar{\eta}'_{kl} = c^{ijkl} \eta_{ij} \bar{\eta}'_{kl} + \sigma_N (2\eta^i{}_j \bar{\eta}'^j{}_i - \eta^i{}_i \bar{\eta}'^j{}_j) + (\sigma_T - \sigma_N)(2\eta^\alpha{}_\beta \bar{\eta}'^\beta{}_\alpha - \eta^\alpha{}_\alpha \bar{\eta}'^\beta{}_\beta). \tag{50b}$$

Note that in the hydrostatic case this tensor is equal to d^{ijkl} and then corresponds to Takeuchi & Saito's (1972) choice.

The consideration of the case where:

$$(F' + L')^2 = (A' - L')(C' - L'),$$

and where:

$$\rho U^2_{qS_1} = N \sin^2 \theta + L \cos^2 \theta, \;\; \rho U^2_{qS_2} = \{L - (\sigma_N - \sigma_T)\} \sin^2 \theta + L \cos^2 \theta,$$

$$\rho U^2_{qP} = A \sin^2 \theta + C \cos^2 \theta,$$

also illustrates the weak resolving power of the knowledge of velocity anisotropy upon the deviatoric part of the stress. (The approximative approach (44) yields the same result for qP fronts when: $A' + C' = 4L' + 2F'$.)

6 Equations of the 'displacement stress vector'

Let us now suppose that the tensors σ_0^{ij} and c^{ijkl} are invariant under any parallel displacement over surfaces defined, at least locally, by $x^1 = c^t$. This means that the medium is generally stratified and that A, C, F, L, N, σ_N and σ_T are functions only of the first coordinate. We will make the same assumption for ρ.

In order to reduce the equation of elastodynamic and avoid derivatives of the different elastic parameters, it is natural from (23) to introduce the vector (29):

$$t = \tau^*(e_1) = (\sigma - \sigma_0)(e_1) + Du\{\sigma_0(e_1)\}, \quad (t^j = c_1{}^{jkl} D_k u_l + \sigma_N D_1 u^j). \tag{51}$$

Unfortunately this vector does not allow boundary conditions at the solid–fluid interfaces (see 35a) as simply as that of the vector $(\delta_l T)(e_1)$ (see 15b, 16, 25) used by Alterman *et al.* (1959) and by Takeuchi & Saito (1972) in hydrostatic situations. In order to generalize the choice adopted by these authors while keeping simple boundary conditions and avoiding derivatives of the deviator $(\sigma_N - \sigma_T)$ of the pre-stress, we also have to introduce the vector (see 30, 32, 35b):

$$s = \tau'^*(e_1) = (\delta_l T)(e_1) - \sigma_0\{Du^*(e_1)\} + Du^*\{\sigma_0(e_1)\},$$

$$(s^j = t^j + \sigma_N g^{jk} D_k u_1 - \sigma_{01}{}^j D_k u^k = b_1{}^{jkl} D_k u_l). \tag{52}$$

From (47) it is straightforward that:

$$D_1 u' = (\tau_1{}^1 - F' D_\alpha u^\alpha)/(C' + \sigma_N), \quad D_1 u^\beta = (\tau_1{}^\beta - L' g^{\alpha\beta} D_\alpha u_1)/(L' + \sigma_N). \tag{53}$$

Taking into account (50) and (52), this yields:

$$D_1 u^1 = (\tau_1'^1 - F D_\alpha u^\alpha)/C, \quad D_1 u^\beta = -g^{\alpha\beta} D_\alpha u_1 + \tau_1'^\beta / L. \tag{54}$$

Let ∇ denote the Levi–Civita covariant derivation (see Choquet-Bruhat, Dewitt-Morette & Dilliard-Bleick 1982; Doubrovine, Novikov & Fomenko 1982) over surfaces defined locally by $x^1 = c^t$ and Γ^i_{jk} the Christoffel symbols related to the Euclidean derivation D for the system of curvilinear coordinates (x^i). Since it is assumed that the first coordinate is orthogonal to the other two, the symbols related to ∇ are equal to those corresponding to D. So that, for instance:

$$D_\alpha u^\beta = \nabla_\alpha u^\beta + \Gamma^\beta_{\alpha 1} u^1, \quad D_\alpha \tau^{\beta\gamma} = \nabla_\alpha \tau^{\beta\gamma} + \Gamma^\beta_{\alpha 1} \tau^{1\gamma} + \Gamma^\gamma_{\alpha 1} \tau^{\beta 1},$$

and it may be deduced (Appendix 5) from (23) and (47):

$$\begin{aligned}
g^{11} D_1(\tau_1{}^\beta) = {} & \rho \delta_t \Psi^\beta + \left(\frac{F'^2}{C' + \sigma_N} + N' - A'\right) g^{\beta\alpha} \nabla_\alpha \nabla_\gamma u^\gamma - \frac{F'}{C' + \sigma_N} g^{\beta\alpha} \nabla_\alpha \tau_1{}^1 \\
& + g^{11} N' (\Gamma^\beta_{1\gamma} \Gamma^\gamma_{1\delta} - \Gamma^\alpha_{1\alpha} \Gamma^\beta_{1\delta}) u^\delta - (N' + \sigma_T) g^{\alpha\gamma} \nabla_\alpha \nabla_\gamma u^\beta \\
& - g^{11}\left(\Gamma^\alpha_{1\alpha} t^\beta + \frac{L'}{L' + \sigma_N} \Gamma^\beta_{1\alpha} t^\alpha\right) - \left(\sigma_T + \frac{L' \sigma_N}{L' + \sigma_N}\right) \Gamma^\beta_{1\alpha} g^{\alpha\gamma} (\nabla_\gamma u^1 + \Gamma^1_{\gamma i} u^i) \\
& - \nabla_\alpha \left(u^1 \left\{(2N' + \sigma_T) g^{\beta\gamma} \Gamma^\alpha_{\gamma 1} + \left(A' - 2N' - \frac{F'^2}{C' + \sigma_N}\right) g^{\beta\alpha} \Gamma^\gamma_{1\gamma}\right\}\right).
\end{aligned} \tag{55a}$$

$$\begin{aligned}
D_1(\tau_1{}^1) = {} & \rho \delta_t \Psi_1 - \frac{L'}{L' + \sigma_N} (\nabla_\alpha t^\alpha - \Gamma^1_{1\alpha} t^\alpha) + \Gamma^\alpha_{1\alpha} \left(\frac{F'}{C' + \sigma_N} - 1\right) t^1 \\
& + (2N' + \sigma_T) \Gamma^\gamma_{1\beta} (\nabla_\gamma u^\beta + \Gamma^\beta_{1\gamma} u^1) + \Gamma^\alpha_{\alpha 1}\left(A' - 2N' - \frac{F'^2}{C' + \sigma_N}\right) (\nabla_\gamma u^\gamma + \Gamma^\gamma_{1\gamma} u^1) \\
& - \left(\frac{L' \sigma_N}{L' + \sigma_N} + \sigma_T\right) g_{11} \{ g^{\alpha\beta} \nabla_\alpha \nabla_\beta u^1 + g^{\alpha\beta} \nabla_\alpha (\Gamma^1_{\beta\gamma} u^\gamma + \Gamma^1_{1\beta} u^1) \\
& + g^{\alpha\beta} \Gamma^1_{1\alpha} (\nabla_\beta u^1 + \Gamma^1_{\beta i} u^i) \}.
\end{aligned} \tag{55b}$$

where (18): $\delta_t \Psi = \partial^2_{tt} u + 2\Omega \times \partial_t u - D g_0'(u) - \operatorname{grad} \psi$.

Let us now suppose that the surfaces $x^1 = c^t$ are parallel (or may be considered parallel, for at least a limited range of x^1). This results in $\Gamma^1_{\alpha 1} = 0$, or equivalently $\Gamma^\alpha_{11} = 0$. The first lines of coordinate are straight lines, g_{11} is a function only of the first coordinate and possibly by means of a change of coordinate we may suppose that $g_{11} = 1$. Let us then introduce the scalar field u_N, t_N, s_N and the tangent vector fields u_T, t_T, s_T such that:

$$u = u_N e_1 + u_T, \quad t = t_N e_1 + t_T, \quad s = s_N e_1 + s_T.$$

Let W, Δ_T, div_T, grad_T denote respectively the Weingaten operator, the Beltrami operator, the divergence and the gradient over surfaces $x^1 = c^t$:

$$\{W(u_T)\}^\beta = \Gamma^\beta_{1\alpha} u^\alpha, \quad (\Delta_T u_T)^\beta = g^{\alpha\gamma} \nabla_\alpha \nabla_\gamma u^\beta, \quad \Delta_T \phi = g^{\alpha\gamma} \nabla_\alpha \nabla_\gamma \phi,$$

$$\operatorname{div}_T(u_T) = \nabla_\alpha u^\alpha, \quad (\operatorname{grad}_T \phi)^\beta = g^{\beta\alpha} \nabla_\alpha \phi. \tag{56}$$

The equilibrium of the reference configuration (9) yields:

$$g_0' = -|g_0'| e_1, \ D_1 \sigma_N + \operatorname{tr}(W)(\sigma_N - \sigma_T) - \rho |g_0'| = 0, \tag{57}$$

where tr denotes the trace operator.

Furthermore, introducing $\chi = D_1\psi - 4\pi G\rho u_N$, and taking into account:

$$\Delta\psi = \Delta_T\psi + \mathrm{tr}(W)D_1\psi + D_{11}\psi,$$

we deduce from (20):

$$D_1\chi = 4\pi G\rho\ \mathrm{div}_T u_T - \Delta_T\psi - \mathrm{tr}(W)\chi$$

$$D_1\psi = \chi + 4\pi G\rho u_N \tag{58}$$

ψ and χ continuous at each interface ($\rho = 0$ outside V).

Therefore, with the help of:

$$\mathrm{div}(g_0') = -D_1|g_0'| - \mathrm{tr}(W)|g_0'| = -4\pi G\rho + 2|\Omega|^2, \tag{59}$$

it follows that:

$$\delta_l\Psi_N = (\partial_{tt}^2 u + 2\Omega\times\partial_t u)_N - \mathrm{tr}(W)|g_0'|u_N - 2|\Omega|^2 u_N - \chi$$

$$\delta_l\Psi_T = (\partial_{tt}^2 u + 2\Omega\times\partial_t u)_T + |g_0'|W(u_T) - \mathrm{grad}_T\psi. \tag{60}$$

Then, taking into account (56), (60), the hypothesis: $\Gamma^1_{1\alpha} = 0$ and the relations:

$$W^2 - \mathrm{tr}(W)W + \det(W) = 0,$$

$$\nabla_\alpha(g^{\beta\gamma}\Gamma^\alpha_{1\gamma}) = \nabla_\gamma(g^{\beta\gamma}\Gamma^\alpha_{1\alpha}) = \{\mathrm{grad}_T(\mathrm{tr}(W)\}^\beta,$$

$$(\nabla_\alpha\Gamma^\beta_{1\gamma} = \partial_\alpha\Gamma^\beta_{1\gamma} + \Gamma^\beta_{\alpha\delta}\Gamma^\delta_{1\gamma} - \Gamma^\delta_{\alpha\gamma}\Gamma^\beta_{1\delta} = \partial_1\Gamma^\beta_{\alpha\gamma}), \tag{61}$$

equations (53), (55) may be rewritten:

$$D_1 u_N = (-F'\{\mathrm{tr}(W)u_N + \mathrm{div}_T u_T\} + t_N)/(C' + \sigma_N),$$

$$D_1 u_T = (L'\{-\mathrm{grad}_T u_N + W(u_T)\} + t_T)/(L' + \sigma_N)$$

$$\begin{aligned}
D_1 t_N = {} & \Bigg\{(2N' + \sigma_T)\mathrm{tr}(W^2) - \left(\frac{F'^2}{C' + \sigma_N} + 2N' - A'\right)\mathrm{tr}(W)^2 \\
& - \left(\frac{L'\sigma_N}{L' + \sigma_N} + \sigma_T\right)\Delta_T + \rho\{\partial_{tt}^2 - |g_0'|\,\mathrm{tr}(W) - 2|\Omega|^2\}\Bigg\}u_N + 2\rho\{\Omega\times\partial_t u\}_N \\
& + \Bigg\{\left(\frac{L'\sigma_N}{L' + \sigma_N} + 2\sigma_T + 2N'\right)\mathrm{div}_T W - \left(\frac{F'^2}{C' + \sigma_N} + 2N' - A'\right)\mathrm{tr}(W)\,\mathrm{div}_T \\
& - (2N' + \sigma_T)\,\mathrm{grad}_T(\mathrm{tr}W)\cdot\Bigg\}u_T - \left(1 - \frac{F'}{C' + \sigma_N}\right)\mathrm{tr}(W)t_N - \frac{L'}{L' + \sigma_N}\mathrm{div}_T t_T - \rho\chi,
\end{aligned}$$

$$\begin{aligned}
D_1 t_T = {} & \left(\frac{F'^2}{C' + \sigma_N} + 2N' - A'\right)\mathrm{grad}_T(\mathrm{tr}(W)u_N) - \left(\frac{L'\sigma_N}{L' + \sigma_N} + 2\sigma_T + 2N'\right)W\,\mathrm{grad}_T u_N \\
& - (2N' + \sigma_T)u_N\,\mathrm{grad}_T\mathrm{tr}(W) + \Bigg\{\left(\frac{F'^2}{C' + \sigma_N} + N' - A'\right)\mathrm{grad}_T\mathrm{div}_T \\
& - (N' + \sigma_T)\Delta_T - N'\det(W) + \left(\frac{L'\sigma_N}{L' + \sigma_N} + \sigma_T\right)W^2 + \rho(\partial_{tt}^2 + |g_0'|W)\Bigg\}u_T \\
& + 2\rho(\Omega\times\partial_t u)_T - \frac{F'}{C' + \sigma_N}\mathrm{grad}_T t_N - \left\{\mathrm{tr}(W) + \frac{L'}{L' + \sigma_N}W\right\}t_T - \rho\,\mathrm{grad}_T\psi.
\end{aligned} \tag{62}$$

Furthermore, from (52), (54), (57) and (61) we may deduce:

$$D_1 s_N = D_1 t_N + \{(\sigma_N - \sigma_T)\mathrm{tr}(W) - \rho\,|g_0'|\}\{\mathrm{div}_T u_T + \mathrm{tr}(W)u_N\}$$
$$+ \sigma_N\{\Delta_T u_N - u_T \cdot \mathrm{grad}_T \mathrm{tr}(W) - \mathrm{div}_T s_T/L\} + \sigma_N \mathrm{tr}(W^2)u_N$$
$$- \sigma_N\,\mathrm{tr}(W)(s_N/C - F\{\mathrm{div}_T u_T + \mathrm{tr}(W)\,u_N\}/C),$$

$$D_1 s_T = D_1 t_T + \{\rho\,|g_0'| - (\sigma_N - \sigma_T)\mathrm{tr}(W)\}(\mathrm{grad}_T u_N - Wu_T)$$
$$- \sigma_N(F\,\mathrm{grad}_T\{\mathrm{tr}(W)\,u_N + \mathrm{div}_T u_T\}/C + \mathrm{grad}_T s_N/C + W(s_T)/L).$$

Substituting (62) into these latter relations and taking into account (50), (52), (54) and (58) we finally deduce:

$$D_1 u_N = (-F\{\mathrm{tr}(W)\,u_N + \mathrm{div}_T u_T\} + s_N)/C,\quad D_1 u_T = -\mathrm{grad}_T u_N + Wu_T + s_T/L,$$

$$D_1 s_N = (2N\,\mathrm{tr}(W^2) - (F^2/C + 2N - A)\mathrm{tr}(W)^2 + (\sigma_N - \sigma_T)\{\Delta_T + \mathrm{tr}(W^2)\}$$
$$+ \rho\{\partial_{tt}^2 - 2\mathrm{tr}(W)|g_0'| - 2|\Omega|^2\})\,u_N + 2\rho(\Omega \times \partial_t u)_N + (2N\,\mathrm{div}_T W$$
$$- (F^2/C + 2N - A)\mathrm{tr}(W)\,\mathrm{div}_T - \rho\,|g_0'|\,\mathrm{div}_T - (2N + \sigma_N - \sigma_T)\,\mathrm{grad}_T\{\mathrm{tr}(W)\}\cdot\,)\,u_T$$
$$- \mathrm{tr}(W)\,(1 - F/C)s_N - \mathrm{div}_T s_T - \rho\chi,$$

$$D_1 s_T = (F^2/C + 2N - A)\,\mathrm{grad}_T\{\mathrm{tr}(W)u_N\} + (\rho\,|g_0'| - 2NW)\,\mathrm{grad}_T u_N$$
$$- 2Nu_N\,\mathrm{grad}_T\,\mathrm{tr}(W) + \{(F^2/C + N - A)\,\mathrm{grad}_T \mathrm{div}_T - N\{\Delta_T + \det(W)\}$$
$$+ \rho\partial_{tt}^2\}u_T + 2\rho(\Omega \times \partial_t u)_T - (F/C)\,\mathrm{grad}_T s_N - \{\mathrm{tr}(W) + W\}s_T - \rho\,\mathrm{grad}_T\psi,$$

$$D_1\psi = \chi + 4\pi G\rho u_N,\quad D_1\chi = 4\pi G\rho\,\mathrm{div}_T u_T - \Delta_T\psi - \mathrm{tr}(W)\chi. \tag{63a}$$

In the fluid areas (15), (50), (52):

$$A = C = F = p_0\gamma,\ L = N = 0,\ \sigma_N = \sigma_T = -p_0,\ s = p_0\gamma\,\mathrm{div}(u)e_1 = -(\delta_l p)e_1.$$

Thus equations (63a) reduce to:

$$s_T = 0,\ D_1 u_N = -\mathrm{tr}(W)u_N - \mathrm{div}_T u_T + s_N/(p_0\gamma),$$
$$D_1 s_N = \rho(\{\partial_{tt}^2 - 2|g_0'|\,\mathrm{tr}(W) - 2|\Omega|^2\}u_N - 2(\Omega \times \partial_t u)_N - |g_0'|\,\mathrm{div}_T u_T - \chi),$$
$$\partial_{tt}^2 u_T + 2(\Omega \times \partial_t u)_T = \mathrm{grad}_T\{s_N/\rho + \psi - |g_0'|u_N\}.$$

This result may be also directly derived from the usual equation:

$$\mathrm{grad}\,\{p_0\gamma\,\mathrm{div}(u)/\rho + \psi + u\cdot g_0'\} - \mathrm{div}(u)\,\{g_0' + p_0\gamma\,\mathrm{grad}(1/\rho)\} = \partial_{tt}^2 u + 2\Omega \times \partial_t u,$$

corresponding to (25) for a perfect fluid.

From (52), (26), (33), (34), (35b) and (58), it is straightforward that the boundary conditions are:

s_N, s_T, ψ, χ continuous at each interface ($\rho = 0$ outside V if necessary),

u_N continuous at each inner interface,

u_T continuous at each solid–solid interface. (63b)

As far as the variational approach is concerned under these hypotheses of orthotropy and lateral invariance over parallel stratification, expression (38a) for $(A(u)|v)$ becomes (see

Appendix 6):

$$(A(u)\mid v)=\int_V b^{ijkl}D_iu_jD_k\bar{v}_l+\rho S\{u\,.\,g_0'\operatorname{div}(\bar{v})-\bar{v}\,.\operatorname{grad}(u\,.\,g_0')\}$$
$$+(\sigma_N-\sigma_T)(\operatorname{tr}(W^2)u_N\bar{v}_N-\operatorname{grad}_Tu_N\cdot\operatorname{grad}_T\bar{v}_N-S\{u_N\bar{v}_T\,.\operatorname{grad}_T\operatorname{tr}(W)\})\,dV$$
$$-\int_E\frac{\operatorname{grad}\psi(u)\cdot\operatorname{grad}\psi(\bar{v})}{4\pi G}\,dV \tag{64a}$$

and thus:

$$(A(u)\mid u)=\int_V [\![\,|u_N|^2(N\{2\operatorname{tr}(W)^2-\operatorname{tr}(W)^2\}-2\rho\,|\Omega|^2-2\rho\,|g_0'|\operatorname{tr}(W)$$
$$+(\sigma_N-\sigma_T)\operatorname{tr}(W^2))-(\sigma_N-\sigma_T)|\operatorname{grad}_Tu_N|^2-N|\operatorname{div}_Tu_T|^2$$
$$-Nu_T\,.\{\Delta_T+\operatorname{grad}_T\operatorname{div}_T+\det(W)\}\bar{u}_T$$
$$+(A-N-F^2/C)|\operatorname{tr}(W)u_N+\operatorname{div}_Tu_T|^2+|s_N|^2/C$$
$$+|s_T|^2/L-R\{(\rho\,|g_0'|+2N\operatorname{tr}(W))u_N\operatorname{div}_T\bar{u}_T+(4NW-\rho\,|g_0'|)\bar{u}_T\cdot\operatorname{grad}_Tu_N$$
$$+(\sigma_N-\sigma_T+4N)u_N\bar{u}_T\,.\operatorname{grad}_T\operatorname{tr}(W)-\rho\chi\bar{u}_N-\rho\bar{u}_T\cdot\operatorname{grad}_T\psi\}]\!]\,dV. \tag{64b}$$

Neglecting rotation, it is well known that in the case of plane and spherical stratification the solution of (63) may be decoupled as *SH*- and *PSV*-waves. More precisely, let us define the subspaces E_{SH} and E_{PSV} (mutually orthogonal in $L^2_C(V,dm)$) by:

$E_{SH}=\{u:u_1=0,\ \operatorname{div}(u)=0\}$, $E_{PSV}=\{u:\operatorname{Curl}(u)_1=0\}$.

It follows then from (63) that E_{SH} and E_{PSV} reduce A in the case of parallel stratification only when the two curvatures of x^1 surfaces are equal at each point, i.e. only in the case of plane and spherical stratification (see Jobert 1976). The case of cylindrical stratification is also interesting to consider. There *PSV*- and *SH*-waves propagating in a direction perpendicular to the cylindrical axis are decoupled whatever the stratification (assumed here to be parallel). (For such *SH*-waves it is clear indeed that: $u_T\cdot\operatorname{grad}_T\{\operatorname{tr}(W)\}=0$ and $\operatorname{div}_T(Wu_T)=0$.) However the subspaces E_{SH} and E_{PSV} do not reduce A in this case, and so the corresponding waves are generally coupled.

7 Pre-stress and spherically symmetrical earth models

Adopting a spherically symmetrical earth configuration in order to draw an average global model of the Earth, we have to assume that the material properties are invariant under any rotation about the centre of mass. This implies that the material will be considered as orthotropic with the radial direction as the axis of symmetry at each point.

In the case of spherical geometry the Weingarten operator is simply $W=I/r$, where r denotes the radial distance of the considered point. Therefore, neglecting rotation, it follows from (63a) that:

$$D_ru_N=\{-F(2u_N/r+\operatorname{div}_Tu_T)+s_N\}/C,\quad D_ru_T=-\operatorname{grad}_Tu_N+u_T/r+s_T/L,$$
$$D_rs_N=\{4(A-N-F^2/C)/r^2+(\sigma_N-\sigma_T)(\Delta_T+2/r^2)+\rho(\partial^2_{tt}-4|g_0|/r)\}u_N$$
$$+\{2(A-N-F^2/C)/r-\rho\,|g_0|\}\operatorname{div}_Tu_T-2(1-F/C)s_N/r-\operatorname{div}_Ts_T-\rho\chi,$$
$$D_rs_T=\{2(F^2/C+N-A)/r+\rho\,|g_0|\}\operatorname{grad}_Tu_N+\{(F^2/C+N-A)\operatorname{grad}_T\operatorname{div}_T$$
$$-N(\Delta_T+1/r^2)+\rho\partial^2_{tt}\}u_T-(F/C)\operatorname{grad}_Ts_N-3s_T/r-\rho\operatorname{grad}_T\psi,$$
$$D_r\psi=\chi+4\pi G\rho u_N,\quad D_r\chi=4\pi G\rho\operatorname{div}_Tu_T-\Delta_T\psi-2\chi/r. \tag{65}$$

We may thus deduce the equations governing the radial dependence of the free modes of the model (i.e. the eigenfunctions of the operator A). Indeed these free oscillations may be expressed as (see Alterman *et al.* 1959):

$$\left.\begin{matrix}u\\s\end{matrix}\right\} = y^{lm}_{\{{7 \atop 8}}(r)\,\mathrm{Curl}\,(Y^m_l\, re_r) \quad \text{(toroidal oscillation of the subspace } E_{SH}),$$

$$\left.\begin{matrix}u\\s\end{matrix}\right\} = y^{lm}_{\{{1 \atop 2}}(r)\,Y^m_l\, e_r + y^{lm}_{\{{3 \atop 4}}(r)\,\mathrm{grad}_T(Y^m_l\, r) \quad \text{(spheroidal oscillation of the subspace } E_{PSV}). \tag{66}$$

Here the Y^m_l ($|m| \leqslant l$) denote the usual orthonormal spherical harmonic functions:

$$Y^m_l(\theta, \phi) := (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}\, P^m_l(\cos\theta)\exp(im\,\phi),$$

P^m_l the associated Legendre functions, θ and ϕ colatitudinal and longitudinal angles.

For a toroidal oscillation with eigenfrequency $\omega/2\pi$, taking into account:

$$\Delta_T(Y^m_l) = -l(l+1)Y^m_l/r^2, \quad \Delta_T\{\mathrm{Curl}\,(Y^m_l\, re_r)\} = \{1 - l(l+1)\}\,\mathrm{Curl}\,(Y^m_l re_r)/r^2,$$

it is straightforward from (64) that:

$$\frac{dy_7}{dr} = y_7/r + y_8/L, \quad \frac{dy_8}{dr} = \{(l-1)(l+2)N/r^2 - \rho\omega^2\}\, y_7 - 3y_8/r, \tag{67a}$$

with $y_8 = 0$ (63b) at each boundary of the solid part.

For spheroidal oscillation with eigenfrequency $\omega/2\pi$, adopting:

$$\psi = y^{lm}_5(r)\,Y^m_l, \quad \chi = y^{lm}_6(r)\,Y^m_l,$$

and taking into account:

$$\Delta_T(Y^m_l) = -l(l+1)Y^m_l/r^2, \quad \Delta_T\{\mathrm{grad}_T(Y^m_l r)\} = \{1 - l(l+1)\}\,\mathrm{grad}_T(Y^m_l r)/r^2$$

it follows easily from (64) that:

$$\frac{dy_1}{dr} = \{-2Fy_1/r + y_2 + l(l+1)Fy_3/r\}/C, \quad \frac{dy_3}{dr} = -(y_1 - y_3)/r + y_4/L,$$

$$\begin{aligned}\frac{dy_2}{dr} = {} & \{4(A - N - F^2/C)/r^2 + (l-1)(l+2)(\sigma_T - \sigma_N)/r - \rho(4|g_0|/r + \omega^2)\}y_1\\ & - 2(1 - F/C)\,y_2/r - l(l+1)\{2(A - N - F^2/C)/r - \rho|g_0|\}y_3/r\\ & + l(l+1)\,y_4/r - \rho y_6,\end{aligned}$$

$$\begin{aligned}\frac{dy_4}{dr} = {} & -\{2(A - N - F^2/C)/r - \rho|g_0|\}y_1/r - (F/C)\,y_2/r\\ & - (\{l(l+1)(F^2/C - A) + 2N\}/r^2 + \rho\omega^2)\,y_3 - 3y_4/r - \rho y_5/r,\end{aligned}$$

$$\frac{dy_5}{dr} = 4\pi G\rho y_1 + y_6, \quad \frac{dy_6}{dr} = -4\pi G\rho l(l+1)y_3/r + l(l+1)\,y_5/r^2 - 2y_6/r. \tag{67b}$$

From (63b) it may be deduced that the boundary conditions take the form:

$y_2 = y_4 = y_6 + (l+1)y_5/b = 0$, at the free surface $r = b$,

$y_1, y_2, y_3, y_4, y_5, y_6$ continuous at each welded interface,

$y_4 = 0$ and y_1, y_2, y_5, y_6 continuous at each solid–fluid interface,

y_1, y_2, y_5, y_6 continuous at each fluid–fluid interface.

At the pole $r = 0$, the regularity of the solution has to be imposed (see Crossley 1975; Denis 1970 and also Dunford & Schwartz 1963).

Recall that the coefficients A, C, F, L, N are those of the tensor b^{ijkl} (see 50) which is equal to d^{ijkl} in the hydrostatic case. Thus, in this case, equations (64) correspond to those obtained by Takeuchi & Saito (1972). In the case of an orthotropic pre-stress, the coefficients A, C, L, N are still related to wavefront velocity (49, 50) and taking into account orthotropy of the pre-stress practically amounts to the introduction of the term $(l-1)(l+2)(\sigma_T - \sigma_N)/r$ in the coefficient of y_1 in the second equation of (65b).

Let u represent a displacement field and s the associated 'stress vector' (52). Expanding u and s on the spherical harmonic vector basis as in (66) and taking into account the orthogonality properties of this basis, (64b) easily yields:

$$(A(u)\,|\,u) = \int_0^b [\![\, |y_1|^2\{(4(A - N - F^2/C) - 4\rho\,|g_0|\,r - (l-1)(l+2)(\sigma_N - \sigma_T)\}$$
$$+ |ry_2|^2/C + l(l+1)\,|y_3|^2\,\{l(l+1)(A - F^2/C) - 2N\} + l(l+1)\,|ry_4|^2/L$$
$$+ l(l^2-1)(l+2)N|y_7|^2 + l(l+1)\,|ry_8|^2/L$$
$$+ R\,\{2l(l+1)(\rho\,|g_0|\,r - 2(A - N - F^2/C))y_1\bar{y}_3 - \rho y_1\bar{y}_6 r^2 - l(l+1)\,\rho y_3\bar{y}_5\, r\}]\!]\, dr, \tag{68}$$

where $R\ \{\ \}$ denotes the real part.

This provides the means to estimate to the first order the effect of the deviatoric part of pre-stress upon the spheroidal eigenfrequencies. Indeed, with the help of the so-called 'Rayleigh Principle' (see, e.g. Woodhouse & Dahlen 1978), it is easily derived from (68) that the relative perturbation of a spheroidal eigenfrequency with angular order l due to the deviatoric part of pre-stress may be expressed as:

$$\frac{\delta\omega}{\omega} = \frac{(l-1)(l+2)}{2\omega^2} \int_0^b (\sigma_T - \sigma_N)y_1^2\, dr \Big/ \!\! \int_0^b \{y_1^2 + l(l+1)y_3^2\}\,\rho r^2\, dr, \tag{69}$$

where y_1 and y_3 are the components (66) of the corresponding eigenfunction for the reference hydrostatic model.

Let $K(r)$ denote the kernel of expression (69), that is the kernel related to the logarithmic derivative of ω with respect to $(\sigma_T - \sigma_N)$:

$$K(r) = \frac{(l-1)(l+2)}{2\omega^2}\, y_1^2(r) \Big/ \!\! \int_0^b \{y_1^2 + l(l+1)y_3^2\}\,\rho r^2\, dr. \tag{70}$$

Figs 1 and 2 show the kernel $K(r)$ as a function of depth, corresponding respectively to the fundamental spheroidal mode and the first overtone for $l = 20$, 50 and 100. They have been obtained with the use of Wiggins's (1976) computational algorithm and earth model 1066B (Gilbert & Dziewonski 1975).

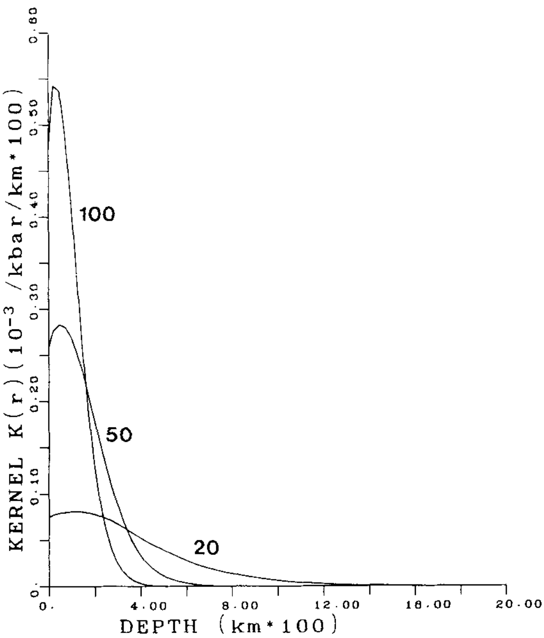

Figure 1. Kernel $K(r)$ as a function of depth, corresponding to the fundamental spheroidal mode for l = 20, 50 and 100, obtained with the use of Wiggins's (1976) computational algorithm and earth model 1066 B (Gilbert & Dziewonski 1975).

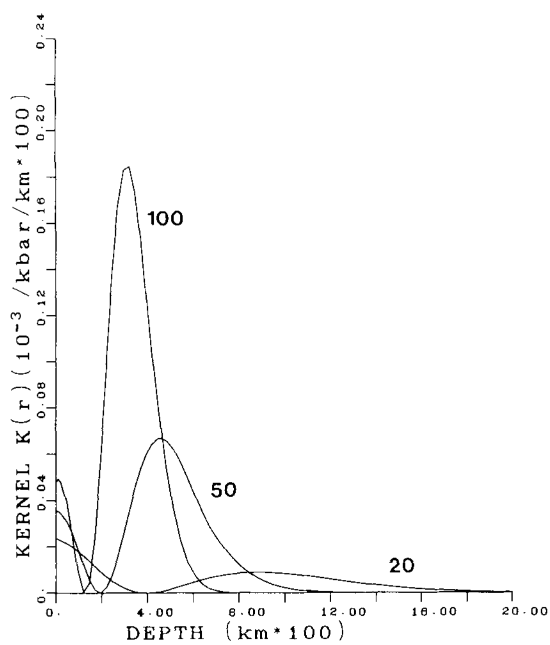

Figure 2. Same as Fig. 1 except for the first overtone.

Considering these graphs and the poorness of our knowledge of the state of the stress inside the Earth, it is believed that there is no reason to ignore, *a priori*, the deviatoric part of pre-stress in global or regional earth models.

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Appendix

(1) Let i, h and ϕ be respectively the specific internal energy, entropy and free energy and Υ the temperature field of the considered material. Using normal state variables, the following expressions stand:

$$h = -\frac{\partial\phi}{\partial\Upsilon}, \quad i = \phi + \Upsilon h.$$

The first two principles of thermodynamics, the virtual power principle and expression (2) yield for any reversible virtual evolutions around the reference state (for more details about the subject of this appendix see, e.g. Bamberger 1981; Malvern 1969; Truesdell 1972 or Marsden & Hugues 1978):

$$\delta_\Upsilon\phi = \delta i - \Upsilon\delta h = \delta w_S = \frac{1}{\rho}\sigma^{pq}\delta\epsilon_{pq} = \frac{1}{\rho_t}T^{ij}\delta e_{ij}.$$

Then one can see that the knowledge of the stress-strain relation results in the knowledge of specific free energy.

In the case of solid elasticity, the normal variables are the temperature Υ and the Lagrangian finite deformation tensor ϵ; thus:

$$\sigma^{pq} = \rho\frac{\partial\phi}{\partial\epsilon_{pq}}. \tag{A1.1}$$

Developing ϕ to the second order around the equilibrium state leads to:

$$\rho\phi = \rho\phi_0 - \rho h_0(\Upsilon - \Upsilon_0) + \sigma_0^{ij}\epsilon_{ij} + \frac{1}{2}\{A(\Upsilon - \Upsilon_0)^2 + 2(\Upsilon - \Upsilon_0)\alpha^{ij}\epsilon_{ij} + \lambda^{ijkl}\epsilon_{ij}\epsilon_{kl}\}$$

where $h_0 = -(\partial\phi/\partial\Upsilon)_0$ is the specific entropy in the reference state, σ_0^{ij} the pre-stress tensor, and where from the symmetry of ϵ_{ij} the tensors λ^{ijkl} and α^{ij} may be chosen with the following symmetries:

$$\alpha^{ij} = \alpha^{ji}, \quad \lambda^{ijkl} = \lambda^{jikl} = \lambda^{klij}. \tag{A1.2}$$

Then (A1.1) may be rewritten:

$$\sigma^{ij} = \sigma_0^{ij} + (\Upsilon - \Upsilon_0)\alpha^{ij} + \lambda^{ijkl}\epsilon_{kl}.$$

Furthermore, assuming that the evolution is adiabatic, and thus isentropic ($h = h_0$):

$$\Upsilon - \Upsilon_0 = -\frac{1}{A}\alpha^{ij}\epsilon_{ij}, \quad (\delta_t)_\Upsilon\phi = \delta_t i,$$

where $\delta_l i$ is the Lagrangian variation of specific internal energy. Thus:

$$\sigma^{ij} = \sigma_0^{ij} + c^{ijkl}\,\epsilon_{kl},$$

where the tensor $c^{ijkl} = \lambda^{ijkl} - (\alpha^{ij}\alpha^{kl})/A$, which contains the same symmetries as λ given by (A1.2), corresponds to the quadratic coefficients in the development of $\rho\delta_l i$, with respect to ϵ, related to isentropic evolution:

$$\rho\delta_l i = \rho\{(\phi - \phi_0) + h_0(\Upsilon - \Upsilon_0)\} = \sigma_0^{ij}\,\epsilon_{ij} + \frac{1}{2}c^{ijkl}\,\epsilon_{ij}\,\epsilon_{kl}.$$

Restricting now the development of σ to the first order in Du and that of $\delta_l i$ to the second order leads to:

$$\sigma^{ij} = \sigma_0^{ij} + c^{ijkl}D_k u_l,$$

$$\rho\delta_l i = \sigma_0^{ij}D_i u_j + \frac{1}{2}(c^{ijkl} + \sigma_0^{ik}g^{jl})D_i u_j D_k u_l, \tag{A1.3}$$

where the presence of σ_0 in the quadratic terms comes from the quadratic ones in ϵ.

In the case of a perfect fluid, the normal state variables reduce to the temperature Υ and the density ρ. Then taking into account mass conservation:

$$\rho_t\delta_\Upsilon\phi = -pg^{ij}\delta e_{ij} \quad \text{with } p = \rho_t^2\,\frac{\partial\phi}{\partial\rho_t},$$

and thus:

$$T^{ij} = -pg^{ij}.$$

If we consider an isentropic evolution, and then suppose that p is locally a function of the density, we obtain to the first order:

$$p - p_0 = \left(\frac{\partial p}{\partial\rho}\right)_h(\rho_t - \rho) = -\rho\left(\frac{\partial p}{\partial\rho}\right)_h\theta/(1+\theta),$$

where θ is the volumetric dilatation: $\theta = \rho/\rho_t - 1$.
Taking into account expression (4) of θ yields to the first order:

$$p - p_0 = -p_0\gamma\,\mathrm{div}(u), \tag{A1.4}$$

where

$$\gamma = \frac{\rho}{p_0}\left(\frac{\partial p}{\partial\rho}\right)_h$$

is the adiabatic index of the fluid.

The Lagrangian variation of the specific internal energy corresponding to an isentropic evolution may be expressed as:

$$\delta_l i = (\delta_l)_\Upsilon\phi = \int_0^t \frac{\partial\phi}{\partial\rho_t}\,\frac{\partial\rho_t}{\partial t}\,dt = -\frac{1}{\rho}\int_0^t p\partial_t\theta\,dt.$$

Thus taking into account expressions (4) and (A1.4), we obtain to the second order:

$$\rho\delta_l i = p_0\left\{-\,\mathrm{div}(u) + \frac{1}{2}\,\{(\gamma-1)\,\mathrm{div}(u)^2 + D_i u^j D_j u^i\}\right\}. \tag{A1.5}$$

Let us now recall that the internal energy is not additive. Indeed, if we consider the evolution of two systems S_1, S_2, the variation of internal energy of the whole system $S_1 \cup S_2$ is:

$$\delta I = \delta I_1 + \delta I_2 - W_M,$$

where W_M is the mutual work caused by the action of S_1 upon S_2 and vice versa. In the case of the Earth, we deduce:

$$\delta I = \int_V \rho \delta_l i \, dV - W_M,$$

and here W_M is the work due to non-local gravitational effects and possible slipping at inner interface.

(2) Let W denote the Weingarten operator, that is the self-adjoint operator defined at each point of the surface Σ, assumed to be sufficiently regular, on the space tangent $T_a(\Sigma)$, by:

$$\forall u \in T_a(\Sigma), \quad W(u) = D_u(n) \in T_a(\Sigma),$$

div_Σ and grad_Σ the usual divergence and gradient surface operators:

$$\mathrm{div}\, u = \mathrm{div}_\Sigma\{u - (u \cdot n)n\} + Du(n) \cdot n + (c_1 + c_2)u \cdot n$$

$$\mathrm{grad}\, p_0 = \mathrm{grad}_\Sigma p_0 + (\mathrm{grad}\, p_0 \cdot n)n$$

where c_1 and c_2 are the principal curvatures of Σ at the considered point, i.e. the eigenvalues of W.

Taking into account (26), it is deduced from (27):

$$[\tau^*(n)] = -(p_0 \,\mathrm{div}_\Sigma [u] + \mathrm{grad}_\Sigma p_0 \cdot [u])n - p_0 W[u]$$

$$= -\mathrm{div}_\Sigma (p_0 [u])n - p_0 W[u].$$

From (26) and the definition of W, one may also deduce:

$$[Du(n) \cdot n]\, n = [Du^*(n)] + W[u],$$

so that:

$$[\tau^*(n)] = -n \,\mathrm{div}_\Sigma (p_0 [u]) + p_0 [Du^*(n) - \{Du(n) \cdot n\}n].$$

From (30) it is then straightforward that:

$$[\tau'^*(n)] = [\tau^*(n)] - p_0 [Du^*(n) - \mathrm{div}(u)n],$$

$$= \{-\mathrm{div}_\Sigma (p_0 [u]) + p_0 \,\mathrm{div}_\Sigma [u]\} n$$

$$= -([u] \cdot \mathrm{grad}_\Sigma p_0)n.$$

(3) From the definition (23) of the operator A, it is first reduced:

$$(A(u) \mid v) = \int_V - D_i\{(c^{ijkl} + \sigma_0^{ik} g^{jl}) D_k u_l\} \bar{v}_j + \rho\{g - g_0 - \Omega \times (\Omega \times u)\} \cdot \bar{v} \; dV.$$

With the help of Stoke's formula and taking into account expressions (19a) and (23b) it follows:

$$(A(u)\mid v)=\int_V (c^{ijkl}+\sigma_0^{ik}g^{jl})D_iu_jD_k\bar{v}_l+\rho\{(\Omega\cdot u)(\Omega\cdot\bar{v})-|\Omega|^2u\cdot\bar{v}\}\ dV$$

$$+\frac{G}{2}\int_{V\times V}\left\{\frac{(u'-u)\cdot(\bar{v}'-\bar{v})}{|a'-a|^3}-3\,\frac{\{(a'-a)\cdot(u'-u)\}\{(a'-a)\cdot(\bar{v}'-\bar{v})\}}{|a'-a|^5}\right\}dm\,dm'$$

$$+\int_\Sigma [\tau^*(n)\cdot\bar{v}]\ d\Sigma \tag{A3.1}$$

where Σ comprises (33, 34) all solid–fluid (or fluid–fluid) interfaces, with unit normal n oriented following the jump through Σ.

The surface integral in (A3.1) is now to be evaluated. With the convention that the minus sign refers always to a fluid side, we deduce with the use of (35a) and (23b):

$$[\tau^*(n)\cdot\bar{v}]=[\tau^*(n)]\cdot\bar{v}_+ +\tau_-(n)\cdot[\bar{v}],$$

$$\begin{aligned}[\tau^*(n)\cdot\bar{v}] &= -(\bar{v}\cdot n)\,\mathrm{div}_\Sigma(p_0[u])+p_0\ (n\cdot\{Du_+(\bar{v}_+)-Du_-(\bar{v}_+)\}\\ &\quad-(\bar{v}\cdot n)\{Du_+(n)\cdot n-Du_-(n)\cdot n\}+n\cdot\{Du_-(\bar{v}_+)-Du_-(\bar{v}_-)\})\\ &=-\mathrm{div}_\Sigma\{p_0[u](\bar{v}\cdot n)\}+p_0\ [\![[u]\cdot\mathrm{grad}_\Sigma(\bar{v}\cdot n)+n\cdot[Du\{\bar{v}-(\bar{v}\cdot n)n\}]]\!].\end{aligned}$$

Taking into account the following relation:

$$\begin{aligned}[v]\cdot\mathrm{grad}_\Sigma(u\cdot n) &= [v-(v\cdot n)n]\cdot\mathrm{grad}_\Sigma(u\cdot n)\\ &= n\cdot[Du\{v-(v\cdot n)n\}]+[W\{v-(v\cdot n)n\}\cdot u]\end{aligned}$$

and the fact that the inner interfaces are closed surfaces, by Stoke's formula it is finally deduced:

$$\int_\Sigma[\tau^*(n)\cdot v]\,d\Sigma=\int_\Sigma p_0\,[\![[u]\cdot\mathrm{grad}_\Sigma(\bar{v}\cdot n)+[\bar{v}]\cdot\mathrm{grad}_\Sigma(u\cdot n)-$$

$$[W\{u-(u\cdot n)n\}\cdot\{\bar{v}-(\bar{v}\cdot n)n\}]]\!]\ d\Sigma,$$

which substituted in (A3.1) leads to (36).

(4) Let σ_N be any regular scalar function the value of which is $-p_0$ in V_F and 0 on ∂V. Let Σ denote all the interfaces where a fluid is involved and $S\ \{\}$ the symmetric part: $S\{a(u,v)\}=\{a(u,v)+\overline{a(v,u)}\}/2$.

Then:

$$\int_V[S\{\mathrm{div}\,(\sigma_N(D\bar{v}(u)-u\,\mathrm{div}\,(\bar{v}))\}\}]\ dV$$

$$=\int_V\ [\sigma_N\{D_iu^jD_j\bar{v}^i-(D_iu^i)(D_j\bar{v}^j)\}+S\{(D\bar{v}(u)-u\,\mathrm{div}(\bar{v}))\cdot\mathrm{grad}\,\sigma_N\}]\ dV$$

$$=\int_\Sigma p_0S\{[D\bar{v}^*(n)\cdot u-(u\cdot n)\,\mathrm{div}\,(\bar{v})]\}\ d\Sigma$$

$$=\int_\Sigma\ p_0\,[\![[\bar{v}]\,.\,\mathrm{grad}_\Sigma(u\,.\,n)+[u]\,.\,\mathrm{grad}_\Sigma(\bar{v}\,.\,n)-[W\{u-(u\,.\,n)n\}\,.\,\{\bar{v}-(\bar{v}\,.\,n)n\}]]\!]$$

$$+S\{(u\,.\,n)[\bar{v}]\,.\,\mathrm{grad}_\Sigma p_0\}\quad d\Sigma. \tag{A4.1}$$

Furthermore, let us recall that:

$$-\int_V \{Dg_0'(u) + \text{grad}\,\psi(u)\}\cdot\bar{v}\,dm$$

$$= \frac{G}{2}\int_{V\times V}\left\{\frac{(u'-u)\cdot(\bar{v}'-\bar{v})}{|a'-a|^3} - 3\,\frac{\{(a'-a)\cdot(u'-u)\}\{(a'-a)\cdot(\bar{v}'-\bar{v})\}}{|a'-a|^5}\right\} dmdm'$$

$$+\int_V \{\Omega\times(\Omega\times u)\}\,.\,\bar{v}\,dm$$

$$= -\int_V Dg_0'(u)\,.\,\bar{v}\,dm - \int_V \frac{\text{grad}\,\psi(u)\cdot\text{grad}\,\psi(\bar{v})}{4\pi G}dV - \int_{\Sigma_\rho}\left[\rho\bar{v} - \frac{\text{grad}\,\psi(\bar{v})}{4\pi G}\right]\cdot n\,\psi(u)\,d\Sigma$$

$$= -\int_V Dg_0'(u)\,.\,\bar{v}\,dm - \int_E \frac{\text{grad}\,\psi(u)\cdot\text{grad}\,\psi(\bar{v})}{4\pi G}dV, \tag{A4.2}$$

where Σ_ρ consists of all the interfaces where ρ has a discontinuity and where the boundary conditions (20) have been taken into account.

With the help of these relations (A4.1, A4.2), expression (36) for $(A(u)\mid v)$ yields (38a). Then, in order to obtain (38b) we just have to observe that:

$$\text{div}\{\rho(u\,.\,g_0')\bar{v}\} = \rho(u\,.\,g_0')\,\text{div}(\bar{v}) + (u\,.\,g_0')\bar{v}\,.\,\text{grad}\,\rho + \rho Dg_0'(u)\,.\,\bar{v} + \rho g_0'\,.\,Du(\bar{v}).$$

(5) First of all recall that since the first coordinate is orthogonal to the other two:

$$g^{\beta\alpha}\Gamma^\gamma_{\alpha 1} = g^{\alpha\gamma}\Gamma^\beta_{\alpha 1} \tag{A5.1}$$

$$g^{\beta\alpha}\Gamma^1_{\alpha i} = -g^{11}\Gamma^\beta_{1i}. \tag{A5.2}$$

(A5.1 may be easily obtained from the expression of the Christoffel symbols:

$\Gamma^k_{ij} = \dfrac{1}{2}g^{kl}\,(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij})$, and (A5.2) comes from: $D_i g^{1\alpha} = \partial_i g^{1\alpha} = 0$.)

It is convenient to rewrite equations (22, 23):

$D_i\tau^{ij} = \rho\delta_l\Psi^j$, where (18):

$\delta_l\Psi = \partial^2_{tt}u + 2\Omega\times\partial_t u - Dg_0'(u) - \text{grad}\,\psi.$

This yields:

$$g^{11}D_1\tau_1{}^\beta = \rho\delta_l\Psi^\beta - \nabla_\alpha\tau^{\alpha\beta} - \Gamma^\alpha_{1\alpha}\tau^{1\beta} - \Gamma^\beta_{1\alpha}\tau^{\alpha 1} \tag{A5.3a}$$

$$D_1\tau_1{}^1 = \rho\delta_l\Psi_1 - g_{11}D_\alpha\tau^{\alpha 1}. \tag{A5.3b}$$

Taking into account (53), (47) may be rewritten:

$$\tau^{\alpha\beta} = g^{\alpha\beta}\left\{\left(A' - 2N' - \frac{F'^2}{C'+\sigma_N}\right)(\nabla_\gamma u^\gamma + \Gamma^\gamma_{1\gamma}u^1) + \frac{F'}{C'+\sigma_N}\tau_1{}^1\right\}$$

$$+ N'g^{\beta\gamma}(\nabla_\gamma u^\alpha + \Gamma^\alpha_{1\gamma}u^1) + (N'+\sigma_T)g^{\alpha\gamma}(\nabla_\gamma u^\beta + \Gamma^\beta_{1\gamma}u^1).$$

Thus:

$$\nabla_\alpha \tau^{\alpha\beta} = g^{\alpha\beta} \left\{ \left(A' - 2N' - \frac{F'^2}{C' + \sigma_N} \right) \nabla_\alpha (\nabla_\gamma u^\gamma + \Gamma^\gamma_{1\gamma} u^1) + \frac{F'}{C' + \sigma_N} \nabla_\alpha \tau_1{}^1 \right\}$$

$$+ N' g^{\beta\gamma} \nabla_\alpha (\nabla_\gamma u^\alpha + \Gamma^\alpha_{1\gamma} u^1) + (N' + \sigma_T) g^{\alpha\gamma} \nabla_\alpha (\nabla_\gamma u^\beta + \Gamma^\beta_{1\gamma} u^1), \quad \text{(A5.4)}$$

(Note that since the first coordinate is orthogonal to the other two $\Gamma^\alpha_{1\beta}$ is a tensor of second order over the surfaces $x^1 = c^t$.)

Furthermore (see Choquet-Bruhat *et al.* 1982; Doubrovine *et al.* 1982):

$$\nabla_\alpha \nabla_\gamma u^\alpha = \nabla_\gamma \nabla_\alpha u^\alpha - R_{\delta\gamma} u^\delta, \quad \text{(A5.5)}$$

where $R_{\delta\gamma} = R^\alpha{}_{\delta\alpha\gamma}$ is the Ricci tensor obtained by contraction of the Riemann tensor. From the expression of the Riemann tensor:

$$R^{ijkl} = \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^i_{pl} \Gamma^p_{jk} - \Gamma^i_{pk} \Gamma^p_{jl},$$

and the flatness of Euclidian space one deduces:

$$R_{\delta\gamma} = \Gamma^\alpha_{1\alpha} \Gamma^1_{\delta\gamma} - \Gamma^\alpha_{1\gamma} \Gamma^1_{\delta\alpha}. \quad \text{(A5.6)}$$

Substituting (A5.4) into (A5.3a) and taking into account (A5.1, 5.2, 5.5 and 5.6) leads to (55a).

Expressions (47) lead to:

$$\tau^{\alpha 1} = \frac{L'}{L' + \sigma_N} \tau^{1\alpha} + \left(\frac{L' \sigma_N}{L' + \sigma_N} + \sigma_T \right) g^{\alpha\beta} D_\beta u^1.$$

This equality is not tensorial. But taking into account:

$$D_\alpha \tau^{\alpha 1} = \partial_\alpha \tau^{\alpha 1} + \Gamma^\alpha_{\alpha\beta} \tau^{\beta 1} + \Gamma^1_{1\alpha} \tau^{\alpha 1} + \Gamma^\alpha_{\alpha 1} \tau^{11} + \Gamma^1_{\alpha\beta} \tau^{\alpha\beta},$$

and:

$$\partial_\alpha (D_\beta u^1) + \Gamma^1_{\alpha 1} D_\beta u^1 - \Gamma^\gamma_{\alpha\beta} D_\gamma u^1 = \nabla_\alpha \nabla_\beta u^1 + \nabla_\alpha (\Gamma^1_{\beta\gamma} u^\gamma + \Gamma^1_{\beta 1} u^1)$$

$$+ \Gamma^1_{1\alpha} (\nabla_\beta u^1 + \Gamma^1_{\beta i} u^i),$$

permits us to obtain (55b), via (A5.3b).

(6) Taking into account equilibrium of the reference configuration (57), it is not difficult to deduce:

$$S\{-\rho D g_0'(u) \,.\, \bar{v} + \{D\bar{v}(u) - u \operatorname{div}(\bar{v})\} \,.\, \operatorname{grad} \sigma_N\}$$

$$= S\{-\rho D g_0'(u) \,.\, \bar{v} - \rho g_0' \cdot D\bar{v}(u) + \rho g_0' \cdot u \operatorname{div}(\bar{v})\}$$

$$- (\sigma_N - \sigma_T) \operatorname{tr}(W) S\{D_i \bar{v}^1 u^i - u^1 D_i \bar{v}^i\}$$

$$= \rho S\{g_0' \,.\, u \operatorname{div}(\bar{v}) - \bar{v} \,.\, \operatorname{grad}(u \,.\, g_0')\} - (\sigma_N - \sigma_T) \operatorname{tr}(W) S\{D_i \bar{v}^1 u^i - u^1 D_i \bar{v}^i\}. \quad \text{(A6.1)}$$

Using this relation and (50b), expression (38a) yields:

$$(A(u)|v) = \int_V b^{ijkl} D_i u_j D_k \bar{v}_l + \rho S\{u \,.\, g_0' \operatorname{div}(\bar{v}) - \bar{v} \,.\, \operatorname{grad}(u \,.\, g_0')\}$$

$$+ (\sigma_N - \sigma_T)(D_\alpha u^\beta D_\beta \bar{v}^\alpha - g^{\alpha\beta} D_\alpha u^1 D_\beta \bar{v}_1 - D_\alpha u^\alpha D_\beta \bar{v}^\beta$$

$$+ \operatorname{tr}(W) S\{u^1 D_\beta \bar{v}^\beta - D_\beta u^1 \bar{v}^\beta\} \quad dV - \int_E \frac{\operatorname{grad} \psi(u) \cdot \operatorname{grad} \psi(\bar{v})}{4\pi G} dV. \quad \text{(A6.2)}$$

Furthermore from (A5.1, 5.2, 5.5, 5.6) and (61) it may be deduced that:

$$D_\alpha u^\beta D_\beta \bar{v}^\alpha - g^{\alpha\beta} D_\alpha u^1 D_\beta \bar{v}_1 - D_\alpha u^\alpha D_\beta \bar{v}^\beta + \mathrm{tr}(W) S\{u^1 D_\beta \bar{v}^\beta - D_\beta u^1 \bar{v}^\beta\}$$

$$= \nabla_\alpha (u^\beta \nabla_\beta \bar{v}^\alpha - u^\alpha \nabla_\beta \bar{v}^\beta) + \Gamma^\beta_{1\alpha}\Gamma^\alpha_{1\beta} u^1 \bar{v}^1 - g^{\alpha\beta}\nabla_\alpha u^1 \nabla_\beta \bar{v}_1$$

$$+ S\{\nabla_\alpha(2\Gamma^\alpha_{1\beta} u^\beta \bar{v}^1 - \Gamma^\beta_{1\beta}\bar{v}^\alpha u^1) - u^1 \bar{v}^\beta \nabla_\beta(\Gamma^\alpha_{1\alpha})\}.$$

Thus, making use of Stoke's formula and taking into account the lateral invariance of $(\sigma_N - \sigma_T)$ it follows that:

$$\int_V (\sigma_N - \sigma_T) [\![D_\alpha u^\beta D_\beta \bar{v}^\alpha - g^{\alpha\beta} D_\alpha u^1 D_\beta \bar{v}_1 - D_\alpha u^\alpha D_\beta \bar{v}^\beta$$

$$+ \mathrm{tr}(W) S\{u^1 D_\beta \bar{v}^\beta - D_\beta u^1 \bar{v}^\beta\}]\!] \, dV$$

$$= \int_V (\sigma_\pi - \sigma_\tau) [\![\mathrm{tr}(W^2) u_N \bar{v}_N - \mathrm{grad}_T u_N \cdot \mathrm{grad}_T \bar{v}_N - S\{u_N \bar{v}_T \cdot \mathrm{grad}_T\, \mathrm{tr}(W)\}]\!] \, dV.$$

Taking into account this latter relation, (A6.2) yields (64a).
Let us now consider the tensor: $\tau''^{ij} = b^{ijkl} D_k u_l$. From (50) and by analogy with (46) it is clear that:

$$\tau''^{\alpha\beta} = \{(A - 2N)\eta^\gamma{}_\gamma + F\eta^1{}_1\}\, g^{\alpha\beta} + 2N\eta^{\alpha\beta}, \quad \tau''^{\alpha 1} = 2L\eta^{\alpha 1},$$

$$\tau'^{11} = F\eta^\gamma{}_\gamma g^{11} + C\eta^{11}, \tag{A6.3}$$

where the tensor η_{ij} is defined by $\eta_{ij} = (D_i u_j + D_j u_i)/2$ and (see 52) $s^j = \tau''^j_1$. It follows that:

$$b^{ijkl} D_i u_j D_k \bar{u}_l = \tau''^{\alpha\beta}\bar{\eta}_{\alpha\beta} + 2\tau''^{1\alpha}\bar{\eta}_{1\alpha} + \tau''^{11}\bar{\eta}_{11}$$

$$= (A - 2N - F^2/C)\,|\,\mathrm{div}_T u_T + \mathrm{tr}(W) u_N|^2 + |s_N|^2/C$$

$$+ |s_T|^2/L + 2N\eta^{\alpha\beta}\bar{\eta}_{\alpha\beta}. \tag{A6.4}$$

Furthermore:

$$2\eta^{\alpha\beta}\bar{\eta}_{\alpha\beta} = \nabla_\gamma\{(g^{\alpha\gamma}u^\beta + g^{\beta\gamma}u^\alpha)(\nabla_\alpha \bar{u}_\beta - \Gamma^1_{\alpha\beta}\bar{u}_1)\} + \nabla_\alpha(2g^{\alpha\gamma}\Gamma^\beta_{1\gamma}u^1\bar{u}_\beta)$$

$$- u^\beta g^{\alpha\gamma}\nabla_\gamma\nabla_\alpha \bar{u}_\beta - u^\alpha \nabla_\gamma \nabla_\alpha \bar{u}^\gamma + (g^{\alpha\gamma}u^\beta + g^{\beta\gamma}u^\alpha)\nabla_\gamma(\Gamma^1_{\alpha\beta}\bar{u}_1)$$

$$- 2\bar{u}_\beta \nabla_\alpha(g^{\alpha\gamma}\Gamma^\beta_{1\gamma}u^1) + 2\Gamma^\beta_{1\gamma}\Gamma^\gamma_{1\beta}u^1\bar{u}_1.$$

With the help of (A5.1, 5.2, 5.5, 5.6), (61) and Stoke's formula we deduce that:

$$\int_V 2N\eta^{\alpha\beta}\bar{\eta}_{\alpha\beta}\, dV = -\int_V N\, [\![u_T \cdot \Delta_T \bar{u}_T + u_T \cdot \mathrm{grad}_T \mathrm{div}_T \bar{u}_T + \det(W)\, u_T \cdot \bar{u}_T$$

$$- 2\,\mathrm{tr}(W^2)\, u_\pi \bar{u}_\pi + 4R\{u_N \bar{u}_T \cdot \mathrm{grad}_T\{\mathrm{tr}(W)\} + W(\bar{u}_T) \cdot \mathrm{grad}_T u_\pi\}]\!] \, dV, \tag{A6.5}$$

where $R\ \{\ \}$ denotes the real part.
Taking into account that:

$$\int_E \frac{|\,\mathrm{grad}\,\psi\,|^2}{4\pi G}\, dV = \int_V \rho\, \mathrm{grad}\,\psi \cdot \bar{u}\, dV$$

(see Appendix 4), and with the help of (A6.4, 6.5) and (59), (64b) may now be derived from (64a).