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Absence of Singular Continuous Spectrum for Certain Self-Adjoint Operators

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Abstract. We give a sufficient condition for a self-adjoint operator to have the following properties in a neighborhood of a point E of its spectrum:
a) its point spectrum is finite;
b) its singular continuous spectrum is empty;
c) its resolvent satisfies a class of a priori estimates.

Notations, Definitions, and Main Theorem

Let H be a self-adjoint operator on a Hilbert space $\mathscr{H}$. We will denote by $\mathscr{H}_n (n \in \mathbf{Z})$ the Hilbert space constructed from the spectral representation for H with the scalar product:

$$(\Phi|\Psi)_n = \int (\lambda^2+1)^{n/2} (\Phi|P_H(d\lambda)\Psi).$$

For functions $P \in L^\infty(\mathbf{R})$, P_H will denote the associated operator given by the usual functional calculus.

$P_H(E,\delta)$ will denote the spectral projection for H onto the interval $(E-\delta, E+\delta)$. P_H^p and P_H^c will denote the spectral projectors respectively onto the point spectrum and the continuous spectrum of H; $\sigma_c(H) = \mathbf{R}/\{E \in \mathbf{R} | E$ is an eigenvalue of $H\}$.

If A is a self-adjoint operator and $D(A) \cap D(H)$ is dense in $\mathscr{H}$, $i[H,A]$ will denote the symmetric form on $D(A) \cap D(H)$ given by

$$(\Phi|i[H,A]\Psi) = i\{(H\Phi|A\Psi) - (A\Phi|H\Psi)\}$$

for $\Psi, \Phi \in D(A) \cap D(H)$. If this form is bounded below and closeable, $i[H,A]^0$ will denote the self-adjoint operator associated to the closure [1].

1. Definition. Let H be a self-adjoint operator on a Hilbert space with domain $D(H)$; a self-adjoint operator A is a conjugate operator for H at a point $E \in \mathbf{R}$ if and only if the following conditions hold:

(a) $D(A) \cap D(H)$ is a core for H.

(b) $e^{+iA\alpha}$ leaves the domain of H invariant and for each $\Psi \in D(H)$

$$\sup_{|\alpha|<1} \|He^{+iA\alpha}\Psi\| < \infty.$$

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(c) The form $i[H,A]=i(HA-AH)$ defined on $D(A)\cap D(H)$ is bounded below and closeable; moreover, the self-adjoint operator $i[H,A]^0$ associated to its closure admits a domain containing $D(H)$.

(d) The form defined on $D(A)\cap D(H)$ by $[[H,A]^0, A]$ is bounded as a map from $\mathscr{H}_{+2}$ into $\mathscr{H}_{-2}$.

(e) There exist strictly positive numbers α and δ and a compact operator K on $\mathscr{H}$, so that:

$$P_H(E,\delta)\, i[H,A]^0 P_H(E,\delta) \geqq \alpha P_H(E,\delta) + P_H(E,\delta) K P_H(E,\delta).$$

Theorem. *Let H be a self-adjoint operator, having a conjugate operator A at the point $E\in\mathbf{R}$, (i.e. suppose H and A satisfy conditions* (a)–(e) *above). Then there is a neighborhood $(E-\delta, E+\delta)$ of E so that:*

1. *In $(E-\delta,E+\delta)$ the point spectrum of H is finite.*
2. *For each closed interval $[a,b]\subset(E-\delta,E+\delta)\cap\sigma_c(H)$, there exists a finite constant c_0 so that:*

$$\sup_{\substack{\operatorname{Re} z\in[a,b]\\ \operatorname{Im} z\neq 0}} \| |A+i|^{-1}(H-z)^{-1}|A+i|^{-1}\| \leqq c_0 .$$

Remark. The above theorem gives a method for obtaining a priori estimates of Agmon type [2] for certain self-adjoint operators, following from the existence of the conjugate operator A of H in the neighborhood of some point.

The essential condition in the definition of conjugate operator is condition (e); the other conditions justify the algebraic manipulations. To obtain the a priori estimates on $(H-z)^{-1}$ when z approaches a point $E\in\sigma_c(H)$, we prove a priori estimates, uniform in ε and z, on the operator $(H-z-i\varepsilon B^*B)^{-1}$. Here ε and $\operatorname{Im} z$ have the same sign, $\operatorname{Re} z\in(E-\delta_0, E+\delta_0)$, and $B^*B=P_H(E,2\delta_0)\, i[H,A]\, P_H(E,2\delta_0)$. This estimate is obtained by proving a differential inequality of the form:

$$\left\| \frac{d}{d\varepsilon} F_z(\varepsilon) \right\| \leqq K(\varepsilon, \|F_z(\varepsilon)\|)$$

for $F_z(\varepsilon)=|A+i|^{-1}(H-z-i\varepsilon B^*B)^{-1}|A+i|^{-1}$.

In Sect. I, we give examples and applications. As new results we obtain the absence of singular continuous spectrum and a priori estimates in the following two cases:

(a) Relatively compact perturbations of certain pseudo-differential operators.

(b) Three-body Schrödinger operators with long-range two-body forces.

In Sect. II we give the proof of the main theorem.

I. Examples and Applications

1. The Laplacian

Let $\mathscr{H}=L^2(\mathbf{R}^n, d^n x)$, $H=H_0=-\Delta$ and

$$A=\tfrac{1}{4}(x\cdot p+p\cdot x) \qquad p=-i\nabla.$$

A is the generator of the dilations introduced by Combes and used in [3].

$-\Delta$ and A are defined on $\mathscr{S}$, the $\mathscr{C}^\infty$ functions of rapid decrease. $\mathscr{S}$ is a core for

H. The explicit formula:

$$e^{+iA\alpha}(H_0+i)^{-1}=(e^{-\alpha}H_0+i)^{-1}e^{+iA\alpha}$$

shows that $e^{+iA\alpha}$ leave $D(H)$ invariant. $\mathscr{S}$ is invariant under the dilation group and $i[-\Delta, A]=-\Delta$ in the sense of quadratic forms on $\mathscr{S}$. By Proposition II.1, condition (c) holds on $D(A)\cap D(H)$ and $i[H,A]^0=-\Delta$. Condition (d) then reduces to condition (c). Condition (e) is trivially satisfied at any point $E\neq 0$ by choosing $\delta<\frac{|E|}{2}$.

2. Two-Body Schrödinger Operators

Let

$$\mathscr{H}=L^2(\mathbf{R}^n, d^n x), \qquad H=-\Delta+V.$$

We will often write H_0 for $-\Delta$. Much work has been done on these operators and we refer the reader to [4] for detailed references. Moreover, recently a very intuitive method has been introduced by Enss to prove asymptotic completeness for such systems [5].

We shall suppose that:

(i) V is H_0 compact;

(ii) the operator $i\left\{V\frac{xp+px}{4}-\frac{xp+px}{4}V\right\}$ is defined on $\mathscr{S}$ and coincides on $\mathscr{S}$ with an H_0 compact operator B.

(iii) B admits a decomposition: $B=B_s+B_l$ where $B_s^*|x|$ and $|x|B_s$ are H_0 bounded operators, and $[B_l, xp+px]$ coincides on $\mathscr{S}$ with a form coming from an H_0 compact operator.

Remark. When V is the operator of multiplication by a function $v(x)$, $[V, xp+px]=2ix\cdot\nabla v$, so that condition (ii) is satisfied if $x\cdot\nabla v$ is H_0 compact. Condition (iii) is satisfied if there is a smooth function $j(x)$ of compact support such that the operators $x_i\frac{\partial}{\partial x_i}\left\{(1-j(x))x_j\frac{\partial v}{\partial x_j}\right\}$ are H_0 compact for all i,j.

Theorem I.1. *If V is a symmetric operator satisfying hypotheses* (i)...(iii), *then the operator* (sgn E) A *is conjugate to $H=H_0+V$ at all $E\neq 0$. ($A=\frac{1}{4}(xp+px)$.)*

If $E<0$, then 0 *and* $\mathbb{1}$ *are also conjugate operators to H at E.*

Proof. Since V is H_0 compact, $D(H)=D(H_0)$. By Example 1, $D(H_0)$ and therefore $D(H)$ is left invariant by $e^{+iA\alpha}$. By hypothesis (ii) the form $i[H,A]$ coincides on $\mathscr{S}$ with the form associated to the symmetric operator H_0+B on $\mathscr{S}$, hence by Proposition II.1, condition (c) holds with $i[H,A]^0=H_0+B$.

To show that condition (d) holds, we write:

$$[A, i[H,A]^0]=[A, B_s]+[A, H_0+B_l]$$

the first term is bounded as a map from $\mathscr{H}_{+2}$ into $\mathscr{H}_{-2}$ by hypotheses (iii), the second coincides on $\mathscr{S}$ with the quadratic form of an H_0 bounded, self-adjoint operator.

Let us verify condition (e).

$$P_H(E,\delta)i[H,A]^0 P_H(E,\delta)=P_H(E,\delta)\{H-V+B\}P_H(E,\delta).$$

Since V and $B=i[V,A]$ are H compact operators, by taking $\delta<\frac{|E|}{2}$ we have, letting $P_H(E,\delta)=P_H$,

$$P_H i[H,A]^0 P_H \geqq \frac{E}{2}P_H+P_H K P_H \quad \text{if} \quad E>0.$$

If E is negative, we can see that the following two relations hold

$$P_H i[H,-A]^0 P_H \geqq \frac{|E|}{2}P_H+P_H-KP_H$$

$$P_H i[H,A]^0 P_H=P_H(H_0+B)P_H.$$

Adding them, we see that 0 and therefore $\mathbb{1}$ are both conjugate operators for H at energy $E<0$.

Remarks. As a consequence of Theorem I.1, we proved that the eigenvalues of H can only accumulate at $E=0$, and are of finite multiplicity; outsided of them, the resolvent $(H-z)^{-1}$ satisfies a priori estimate of Agmon's type [2].

3. Perturbations of Pseudo-Differential Operators

In [6], among the extensions of the method introduced in [5], the author proves similar results for short-range perturbations of pseudo-differential operators.

Let $\mathscr{H}=L^2(\mathbf{R}^n,d^nx)$ and denote by $L^2(\mathbf{R}^n,d^np)$ the Hilbert space obtained by Fourier transformation.

Let $h_0(p)$ be a measurable function from R^n to R and h_0 the associated multiplication operator on $L^2(\mathbf{R}^n,d^np)$. Suppose that:

$$\lim_{|p|\to\infty}|h_0(p)|=\infty.$$

Definition. $E\in\mathbf{R}$ is a regular point of h_0 if and only if there is a neighborhood $(E-\delta_0,E+\delta_0)$ of E so that on

$$O(E,\delta_0)=\{p\in\mathbf{R}^n\,|\,|h_0(p)-E|<\delta_0\}.$$

h_0 is $\mathscr{C}^m$ for an $m\geqq 3$ and

$$\sum_{i=1}^{n}\left(\frac{\partial h_0}{\partial p_i}\right)^2(p)\geqq\alpha>0, \qquad p\in O(E,\delta_0).$$

Definition. h_0+V is a regular perturbation of h_0 if V satisfies the following conditions.

1. V is a symmetric h_0-compact operator.
2. For all real valued $g \in \mathscr{C}_0^m(\mathbf{R}^n)$, the $\mathscr{C}^m$ functions of compact support, the operators

$$B_i = (x_i g(p) + g(p) x_i) V - V(x_i g(p) + g(p) x_i)$$

are defined on $\mathscr{S}$ and extended to bounded, h_0-compact operators.
3. $[x_j g(p) + g(p) x_j, B_i]$ is bounded as a map from $\mathscr{H}_{+2}$ to $\mathscr{H}_{-2}$.

Theorem I.2. *Let $H = h_0 + V$ be a regular perturbation of h_0. For each regular point E of h_0, there is an operator A conjugate to H at E.*

Corollary I.3. *Let $h_0 + V$ be a regular perturbation of h_0. For each regular point E of h_0, there is a neighborhood $(E-\delta, E+\delta)$ so that*
1. *the point spectrum of $h_0 + V$ is finite in $(E-\delta, E+\delta)$.*
2. *For all $[a,b] \subset (E-\delta, E+\delta) \cap \sigma_c(H)$ there is a finite constant c_0 so that:*

$$\sup_{\substack{\operatorname{Re} z \in [a,b] \\ \operatorname{Im} z \neq 0}} \|(1+|x|)^{-1}(H-z)^{-1}(1+|x|)^{-1}\| \leqq c_0 .$$

Proof. Since $|h_0(p)| \to \infty$ as $|p| \to \infty$, $O(E, \delta_0)$ is a bounded subset of $\mathbf{R}^n$, so that we can find a $\mathscr{C}^{m-1}$ vector field $g_i(p) i \in \{1, \ldots, n\}$ of compact support in $\mathbf{R}^n$, with

$$g_i(p) = \frac{\partial h_0}{\partial p_i}(p) \quad \text{if} \quad p \in O(E, \delta_0)$$

$$g_i(p) = 0 \quad \text{if} \quad |h_0(p)| > M_0 .$$

Let $\hat{A}$ the formally symmetric operator defined on $L^2(\mathbf{R}^n, d^n p)$ by

$$\hat{A} = \sum_{i=1}^{n} g_i(p) i \frac{\partial}{\partial p_i} + \frac{i}{2} \frac{\partial g_i}{\partial p_i}(p) = \frac{1}{2} \sum_i (g_i x_i + x_i g_i) .$$

By the commutator theorem [4] it is easily seen that $\hat{A}$ is essentially self-adjoint on the domain of $x^2 = \sum_{i=1}^{n} x_i^2$.

Let A be the self-adjoint extension so obtained. Since $D(x^2) \cap D(h_0)$ is a core for h_0, $D(A) \cap D(h_0)$ is a core for h_0. One can easily see (cf. Appendix A.1) that the unitary group $e^{+iA\alpha}$ is actually the group of unitary transformations on $L^2(\mathbf{R}^n, d^n p)$ associated with the group of diffeomorphisms $\Gamma_\alpha : \mathbf{R}^n \mapsto \mathbf{R}^n$ determined by the differential equation:

$$\frac{d}{d\alpha} \Gamma_\alpha^i(p) = g_i(\Gamma_\alpha(p))$$

$$\Gamma_0(p) = p .$$

It follows that $e^{+iA\alpha}$ leaves invariant the functions $\Psi(p)$ with support contained in $\{p \in \mathbf{R}^n \mid |h_0(p)| > M_0\}$, and hence $e^{iA\alpha}$ leaves $D(h_0)$ invariant. Conditions (c) and (d) are satisfied because of the regularity assumptions (2) and (3) on V. (These hypotheses can be easily verified for a class of long range potentials with sufficient regularity at infinity.)

Let us verify property (e). By hypothesis there exist $\alpha>0$, $\delta_0>0$ such that

$$P_{h_0}(E,\delta_0)i[h_0,A]^0 P_{h_0}(E,\delta_0) \geqq \alpha P_{h_0}(E,\delta_0).$$

For any smooth function $\tilde{P}$ such that $\tilde{P}=1$ on $(E-\delta, E+\delta)$ $\delta<\delta_0$ and $\tilde{P}=0$ on $\mathbf{R}/(E-\delta_0, E+\delta_0)$, we have:

$$\tilde{P}_{h_0} i[h_0,A]^0 \tilde{P}_{h_0} \geqq \alpha \tilde{P}_{h_0}^2 \quad \text{and} \quad P(E,\delta)=P(E,\delta)\tilde{P}.$$

Note that $\tilde{P}_H-\tilde{P}_{h_0}$ is a compact operator since V is h_0 compact and $\tilde{P}(\lambda)$ is a smooth function of compact support.

Then:

$$\begin{aligned} &P_H(E,\delta)i[h_0,A]^0 P_H(E,\delta) \\ &\quad = P_H(E,\delta)\tilde{P}_H \sum_i g_i^2(p)\tilde{P}_H P_H(E,\delta) \\ &\quad = P_H(E,\delta)\tilde{P}_{h_0} \sum_i g_i^2(p)\tilde{P}_{h_0} P_H(E,\delta) + P_H(E,\delta)K' P_H(E,\delta) \\ &\quad \geqq \alpha P_H(E,\delta)\tilde{P}_{h_0}^2 P_H(E,\delta) + P_H(E,\delta)K' P_H(E,\delta) \\ &\quad \geqq \alpha P_H^2(E,\delta) + P_H(E,\delta)K'' P_H(E,\delta). \end{aligned}$$

By hypothesis (2) $[V,A]$ is h_0 compact, hence there exist numbers α, $\delta>0$ and a compact operator K so that condition (e) holds. This proves Theorem I.2. The Corollary I.3 follows from Theorem I.2 and the abstract theorem since $D(A)$ contains $D(|x|)$, and hence $A(1+|x|)^{-1}$ is a bounded operator.

4. Three-Body Schrödinger Operators

Let x_i, m_i be the coordinates and mass of the i-th particle where $x_i \in \mathbf{R}^n$, $i \in \{1,2,3\}$. For each pair of particles $(i,j)=\alpha$ (such pairs are always denoted by Greek letters), we will denote

$$x_\alpha = x_i - x_j\,; \qquad y_\alpha = x_k - \frac{m_i x_i + m_j x_j}{m_i + m_j} \qquad k \notin \alpha$$

$$m_\alpha^{-1} = m_i^{-1} + m_j^{-1}$$

$$n_\alpha^{-1} = m_k^{-1} + (m_i + m_j)^{-1}$$

when one removes the center of mass of the system, the Hilbert space is then

$$\mathscr{H} = L^2(\mathbf{R}^{2n}, d^n x_\alpha d^n y_\alpha) \qquad \forall \alpha.$$

k_α and p_α will denote $-i\nabla_{x_\alpha}$ and $-i\nabla_{y_\alpha}$.

In $\mathscr{H}$, the Hamiltonian of the system is written

$$H = H_0 + V$$

$$H_0 = \frac{1}{2m_\alpha} k_\alpha^2 + \frac{1}{2n_\alpha} p_\alpha^2 \qquad \forall \alpha.$$

The dilation group acts in the same way independently of the representation $L^2(d^n x_\alpha, d^n y_\alpha)$ of $\mathscr{H}$. Let A be its generator normalized so that $i[H_0, A] = H_0$. We have $A = A_\alpha^1 + A_\alpha^2$ where A_α^1 and A_α^2 are the generators of the dilation group on $L^2(d^n x_\alpha)$ and $L^2(d^n y_\alpha)$, respectively.

Hypotheses on the potential V

Suppose that $V = \sum_\alpha v_\alpha$ where, for each α, v_α is an operator acting on $L^2(d^n x_\alpha)$ and satisfying hypotheses (i)–(iii) of Example 2.

We will further denote:

$$H_\alpha = H_0 + v_\alpha = h_\alpha + \frac{p_\alpha^2}{2n_\alpha}; \qquad h_\alpha = \frac{k_\alpha^2}{2m_\alpha} + v_\alpha .$$

By Theorem I.1, the eigenvalues of h_α have finite multiplicity and can only accumulate at 0.

Theorem I.3. *Let $H = H_0 + V$ on $L^2(d^n x_\alpha, d^n y_\alpha)$ where V is a symmetric operator satisfying the above hypotheses. Then $A = A_\alpha^1 + A_\alpha^2$ is a conjugate operator for H at all $E \in \mathbf{R}$ with*

$$E \notin \bigcup_\alpha \sigma_p(h_\alpha) \cup \{0\} .$$

Corollary I.4. 1. *The point spectrum of $H = H_0 + \sum_\alpha v_\alpha$ can accumulate only at 0 or at eigenvalues of subsystems.*

2. *For all intervals $[a,b] \subset \mathbf{R} \setminus \left\{ \sigma_p(H) \bigcup_\alpha \sigma_p(h_\alpha) \cup \{0\} \right\}$, there is a c_0 so that*

$$\sup_{\substack{\mathrm{Re}\, z \in [a,b] \\ \mathrm{Im}\, z \neq 0}} \|(1+|x|)^{-1}(H-z)^{-1}(1+|x|)^{-1}\| \leqq c_0 .$$

Under the hypotheses made on the two-body potential v_α, conditions (a)–(d) are satisfied in the same way that they were in the two-body problem. Let us now prove that condition (e) holds.

Proposition 4.1. *Let $E \in \mathbf{R}$, and let c_α be an h_α-compact operator in $L^2(\mathbf{R}^n, d^n x_\alpha)$. Then for every $\varepsilon > 0$ there is $\delta_0 > 0$, a finite rank spectral projection $e_\alpha^{N_0}$ for h_α and an operator K compact in $\mathscr{H} = L^2(\mathbf{R}^{2n}, d^n x_\alpha\, d^n y_\alpha)$ so that*

$$P_H c_\alpha P_H = P_H E_\alpha^N c_\alpha E_\alpha^N P_H + P_H K P_H + o(\varepsilon),$$

where:

(i) $E_\alpha^N = e_\alpha^N \otimes \mathbb{1} y_\alpha$ *where e_α^N is a finite rank spectral projection for h_α that contains $e_\alpha^{N_0}$,*

(ii) P_H *is any spectral projection for H onto any Borel set contained in $(E-\delta_0, E+\delta_0)$;*

(iii) $\|o(\varepsilon)\| \leqq \frac{\varepsilon}{6}$.

Proof. Since c_α is an h_α-compact operator, we can find $e_\alpha^{N_0}$ so that

$$\|e_\alpha^{N_0} c_\alpha e_\alpha^{N_0} - P_{h_\alpha}^p c_\alpha P_{h_\alpha}^p\| \leqq \frac{\varepsilon}{12} .$$

Furthermore, from general properties of the continuous spectrum, one can find a $\delta_0>0$ and a smooth function $\tilde{P}$ with $\tilde{P}=1$ on $(E-\delta_0, E+\delta_0)$ and 0 on $\mathbf{R}\backslash(E-2\delta_0, E+2\delta_0)$ so that

$$\|\tilde{P}_{H_\alpha}\{c_\alpha - P^p_{h_\alpha} c_\alpha P^p_{h_\alpha}\}\tilde{P}_{H_\alpha}\| \leqq \frac{\varepsilon}{12}.$$

Hence for all $\delta \leqq \delta_0$ and all spectral projections P_H on $(E-\delta, E+\delta)$ we have

$$P_H c_\alpha P_H = P_H E^N_\alpha c_\alpha E^N_\alpha P_H + P_H\{c_\alpha - P^p_{h_\alpha} c_\alpha P^p_{h_\alpha}\} P_H + o_1(\varepsilon)$$

with $\|o_1(\varepsilon)\| \leqq \frac{\varepsilon}{12}$.

On the other hand $P_H = P_H \tilde{P}_H$ and thus

$$\begin{aligned} P_H\{c_\alpha - P^p_{h_\alpha} c_\alpha P^p_{h_\alpha}\} P_H &= P_H(\tilde{P}_H - \tilde{P}_{H_\alpha})\{c_\alpha - P^p_{h_\alpha} c_\alpha P^p_{h_\alpha}\} P_H \\ &\quad + P_H \tilde{P}_{H_\alpha}\{c_\alpha - P^p_{h_\alpha} c_\alpha P^p_{h_\alpha}\}(\tilde{P}_H - \tilde{P}_{H_\alpha}) P_H \\ &\quad + P_H \tilde{P}_{H_\alpha}\{c_\alpha - P^p_{h_\alpha} c_\alpha P^p_{h_\alpha}\} \tilde{P}_{H_\alpha} P_H, \end{aligned}$$

where the first two terms on the right hand side are compact operators in $\mathscr{H}$ and the last has norm less than $\frac{\varepsilon}{12}$.

Proposition 4.2. *For all $\varepsilon>0$, we can find $\delta_0>0$, $E^{N_0}_\alpha = e^{N_0}_\alpha \otimes \mathbb{1}_{y_\alpha}$, and a compact operator K so that:*

$$\begin{aligned} P_H i\Big[H_0 + \sum_\alpha v_\alpha, A\Big] P_H &= P_H\Big(1 - \sum_\alpha E^{N_0}_\alpha\Big) H_0 \Big(1 - \sum_\alpha E^{N_0}_\alpha\Big) P_H \\ &\quad + \sum_\alpha P_H E^{N_0}_\alpha \{H_0 + i[v_\alpha, A^1_\alpha]\} E^{N_0}_\alpha P_H \\ &\quad + o(\varepsilon) + P_H K P_H \end{aligned}$$

with $\|o(\varepsilon)\| < \varepsilon$, for any spectral projection P_H onto an interval contained in $(E-\delta_0, E+\delta_0)$.

Proof. We have

$$\begin{aligned} H_0 &= \Big(1 - \sum_\alpha E^N_\alpha\Big) H_0 \Big(1 - \sum_\alpha E^N_\alpha\Big) + \sum_\alpha E^N_\alpha H_0 E^N_\alpha \\ &\quad + \sum_\alpha \{E^N_\alpha H_0 (1 - E^N_\alpha) + (1 - E^N_\alpha) H_0 E^N_\alpha\} \\ &\quad - \sum_{\alpha \neq \beta} \sum E^N_\alpha H_0 E^N_\beta. \end{aligned}$$

The terms in the last sum are all compact operators in $\mathscr{H}$ and $E^N_\alpha H_0 (1 - E^N_\alpha) = -E^N_\alpha v_\alpha (1 - E^N_\alpha)$ since E^N_α commutes with $H_\alpha = H_0 + v_\alpha$. We consider spectral projections e^N_α for h_α so that

$$\sum_\beta E^N_\beta H_0 (1 - E^N_\beta) = \sum_\beta P^p_{h_\beta}(-v_\beta) P^c_{h_\beta} + o(\varepsilon)$$

with $\|o(\varepsilon)\| < \frac{\varepsilon}{2}$.

Next, we apply Proposition 4.1 to each of the operators

$$c_\alpha = i[v_\alpha, A_\alpha^1] - P_{h_\alpha}^p v_\alpha P_{h_\alpha}^c - P_{h_\alpha}^c v_\alpha P_{h_\alpha}^p.$$

By Proposition 4.1, we can find $E_\alpha^{N_0}$ and $\delta_0 > 0$ satisfying Proposition 4.2.

Proposition 4.3. *Let* $\alpha_0 = \operatorname{dist}\left(E, \{0\} \bigcup_\alpha \sigma_p(h_\alpha)\right)$. *We can find* δ_0 *so that*

$$\sum_\alpha P_H E_\alpha^N \{H_0 + i[v_\alpha, A_\alpha^1]\} E_\alpha^N P_H \geqq \sum_\alpha \frac{\alpha_0}{2} P_H E_\alpha^N P_H + P_H K P_H ; P_H = P_H(E, \delta_0)$$

Proof. If we choose δ_0 so that

$$\delta_0 \leqq \frac{1}{4} \inf_\alpha \inf_{i \neq j} |\lambda_\alpha^i - \lambda_\alpha^j|$$

$$\delta_0 \leqq \frac{\alpha_0}{4}.$$

λ_α^i, being the eigenvalues of $h_\alpha e_\alpha^N$.

If we pick a function $\tilde{P}$ equal to 1 on $(E - \delta_0, E + \delta_0)$ and 0 on $\mathbf{R} \setminus (E - 2\delta_0, E + 2\delta_0)$,

$$\tilde{P}_{H_\alpha} E_\alpha^i \{H_0 + i[v_\alpha, A_\alpha^1]\} E_\alpha^j \tilde{P}_{H_\alpha} = 0 \quad \text{if} \quad i \neq j$$

since $E_\alpha^j \tilde{P}_{H_\alpha}$ and $E_\alpha^i \tilde{P}_{H_\alpha}$ viewed as functions of p_α^2 have support in disjoint intervals $\left(E_\alpha^i \tilde{P}(H_\alpha) = \tilde{P}\left(\lambda_\alpha^i + \frac{p_\alpha^2}{2n_\alpha}\right) E_\alpha^i\right)$. Furthermore, by the Virial Theorem,

$$\begin{aligned}
&\tilde{P}_{H_\alpha} E_\alpha^N \{H_0 + i[v_\alpha, A_\alpha^1]\} E_\alpha^N \tilde{P}_{H_\alpha} \\
&= \sum_i \tilde{P}_{H_\alpha} E_\alpha^i i[h_\alpha, A_\alpha^1] E_\alpha^i \tilde{P}_{H_\alpha} \\
&\quad + \sum_i \tilde{P}_{H_\alpha} E_\alpha^i \frac{p_\alpha^2}{2n_\alpha} E_\alpha^i \tilde{P}_{H_\alpha} \\
&= \sum_i \tilde{P}_{H_\alpha} E_\alpha^i \frac{p_\alpha^2}{2n_\alpha} E_\alpha^i \tilde{P}_{H_\alpha} \\
&\geqq \frac{\alpha_0}{2} \tilde{P}_{H_\alpha} E_\alpha^N \tilde{P}_{H_\alpha}.
\end{aligned}$$

Propositions 4.2 and 4.3 enable us to find, for all $\varepsilon > 0$, (e_α^N) and $\delta_0 > 0$ so that

$$\begin{aligned}
&P_H(E, \delta) i[H, A]^0 P_H(E, \delta) \\
&\geqq P_H\left(1 - \sum_\alpha E_\alpha^N\right) H_0 \left(1 - \sum_\alpha E_\alpha^N\right) P_H \\
&\quad + \frac{\alpha_0}{2} \sum_\alpha P_H E_\alpha^N P_H \\
&\quad + P_H K P_H + P_H o(\varepsilon) P_H,
\end{aligned}$$

where $\|o(\varepsilon)\| < \varepsilon$, for all $\delta < \delta_0$.

To verify condition (e), since $\varepsilon > 0$ is arbitrary, it now suffices to show that there is a finite constant c_0 so that

$$P_H \leqq c_0 \left\{ P_H \left(1 - \sum_\alpha E_\alpha^N\right) H_0 \left(1 - \sum_\alpha E_\alpha^N\right) P_H + \sum_\alpha P_H E_\alpha^N P_H \right\}$$

which is immediate if $E \neq 0$; the constant c_0 evidently does not depend on N and δ.

II. Proof of Theorem I

We start the proof of the abstract theorem by the following proposition which is useful in applications to verify the hypothesis (c) when $D(A) \cap D(H)$ is not explicitly known.

Proposition II.1. *Let H and A be self-adjoint operators that satisfy conditions* (a), (b) *and the following conditions* (c′).

(c′) *There is a set $\mathscr{S} \subset D(A) \cap D(H)$ such that*

i) $e^{+iA\alpha} \mathscr{S} \subset \mathscr{S}$,

ii) *$\mathscr{S}$ is a core for H,*

iii) *the form $i[H, A]$ on $\mathscr{S}$ is bounded below and closeable, and the associated self-adjoint operator $i[H, A]_{\mathscr{S}}^0$ satisfies*

$$D(i[H, A]_{\mathscr{S}}^0) \supset D(H)$$

then for all Φ, $\Psi \in D(A) \cap D(H)$

$$(\Phi | i[H, A] \Psi) = (\Phi | i[H, A]_{\mathscr{S}}^0 \Psi)$$

and hence the form $i[H, A]$ on $D(A) \cap D(H)$ is closeable and the associated self-adjoint operator satisfies:

$$i[H, A]^0 = i[H, A]_{\mathscr{S}}^0 .$$

Proof. It suffices to check that for each $\Phi, \Psi \in D(A) \cap D(H)$

$$(\Phi | i[H, A] \Psi) = (\Phi | i[H, A]_{\mathscr{S}}^0 \Psi) .$$

By hypothesis (b), the operators $He^{+iA\alpha}(H+i)^{-1}$ are closed and everywhere defined, hence bounded by the closed graph theorem. For each $\Psi \in \mathscr{H}$, by (b) $\sup_{\alpha \in [-1, +1]} \|He^{+iA\alpha}(H+i)^{-1}\Psi\| < \infty$ and by the principle of uniform boundedness in Banach spaces, this family of operators is uniformly bounded: there is a $c_0 < \infty$ such that:

$$\sup_{\alpha \in [-1, +1]} \|He^{+iA\alpha}(H+i)^{-1}\| \leqq c_0 . \tag{II.1}$$

Consequently, for each $\Phi, \Psi \in D(A) \cap D(H)$, $(H(\alpha) = e^{-iA\alpha} H e^{+iA\alpha})$,

$$\lim_{\alpha \to 0} \frac{1}{\alpha} (\Phi | (H(\alpha) - H) \Psi)$$

$$= \lim_{\alpha \to 0} \frac{1}{\alpha} (\Phi | (e^{-iA\alpha} - 1) He^{+iA\alpha} \Psi) + \frac{1}{\alpha} (\Phi | H(e^{+iA\alpha} - 1) \Psi)$$

$$= (\Phi | i[H, A] \Psi) .$$

Since $He^{+iA\alpha}\Psi$ is uniformly bounded in α, this family of vectors converges weakly to $H\Psi$ when $\alpha \to 0$.

For each $\Phi, \Psi \in D(H)$ there are sequences u_n and v_n such that

$$\|(H+i)(u_n-\Phi)\| \to 0, \qquad \|(H+i)(v_n-\Psi)\| \to 0$$

with $u_n, v_n \in \mathcal{S}$. Thus:

$$\frac{1}{\alpha}(\Phi|(H(\alpha)-H)\Psi) = \lim_{n\to\infty} \frac{1}{\alpha}(u_n|(H(\alpha)-H)v_n).$$

By hypothesis (c′), the derivative

$$\frac{d}{d\alpha}(u_n|H(\alpha)v_n) = (u_n|e^{-iA\alpha}i[H,A]^0_{\mathcal{S}}e^{+iA\alpha}v_n)$$

is a continuous function: one can then use the mean value theorem to obtain:

$$\frac{1}{\alpha}(\Phi|(H(\alpha)-H)\Psi) = \lim_{n\to\infty}(u_n|e^{-iA\alpha_n}i[H,A]^0_{\mathcal{S}}e^{+iA\alpha_n}v_n),$$

where $\alpha_n \in [0,\alpha]$. Since $D(i[H,A]^0_{\mathcal{S}}) \supset D(H)$, (II.1) assures that as $n\to\infty$, $\alpha \to 0$

$$\begin{aligned}(\Phi|i[H,A]\Psi) &= \lim_{\alpha\to 0}\frac{1}{\alpha}(\Phi|(H(\alpha)-H)\Psi) \\ &= (\Phi|i[H,A]^0_{\mathcal{S}}\Psi).\end{aligned}$$

Proposition II.2. *Suppose that the two self-adjoint operators H and A satisfy conditions* (a)–(c). *Then $(H-z)^{-1}$ leaves $D(A)$ invariant for all $z \notin \sigma(H)$.*

Proof. Since A is self-adjoint, it suffices to show that the family of operators

$$e^{-iA\alpha}(H-z)^{-1}(A+i)^{-1} = (H(\alpha)-z)^{-1}e^{-iA\alpha}(A+i)^{-1}$$

is strongly differentiable; it suffices to show that the family $H(\alpha)(H-z)^{-1}$ is strongly differentiable, or equivalently to show that for each $\Psi \in D(H)$

$$\lim_{\alpha\to 0}\left\|\frac{H(\alpha)-H}{\alpha}\Psi - i[H,A]^0\Psi\right\| = 0.$$

Let $\Psi_n \in D(A)\cap D(H)$ so that $\|(H+i)(\Psi_n-\Psi)\| \to 0$. Then

$$\frac{H(\alpha)-H}{\alpha}\Psi - i[H,A]^0\Psi = \lim_{n\to\infty}\frac{H(\alpha)-H}{\alpha}\Psi_n - i[H,A]^0\Psi_n$$

exactly as in Proposition II.1. Since $e^{+iA\alpha}$ leaves $D(A)\cap D(H)$ invariant for each $\Phi \in D(A)\cap D(H)$, $\|\Phi\|=1$, there exist $\alpha_{n,\Phi} \in [0,\alpha]$ so that

$$\left(\Phi\left|\frac{H(\alpha)-H}{\alpha}\Psi_n\right.\right) = (\Phi|e^{-iA\alpha_{n,\varphi}}i[H,A]^0 e^{+iA\alpha_{n,\Phi}}\Psi_n).$$

Bound (II.1) and the hypothesis that $D(H) \subset D(i[H,A]^0)$, together imply

$$\|(H(\alpha)-H)\Psi\| \leqq \alpha c_0 \|(H+i)\Psi\| \qquad \text{(II.2)}$$

for all $\Psi \in D(H)$. Furthermore,

$$\left|\left(\Phi\left|\frac{H(\alpha)-H}{\alpha}\Psi_n\right.\right)-(\Phi|i[H,A]^0\Psi_n)\right|$$
$$\leqq c\|(H+i)(\Psi_n-\Psi)\|+\|(\Phi|\{e^{-iA\alpha_{n,\Phi}}i[H,A]^0e^{+iA\alpha_{n,\Phi}}-i[H,A]^0\}\Psi)\|$$
$$\leqq o\left(\frac{1}{n}\right)+\sup_{\alpha'\in[0,\alpha]}\|\{e^{-iA\alpha'}i[H,A]^0e^{+iA\alpha'}-i[H,A]^0\}\Psi\|$$
$$\leqq o\left(\frac{1}{n}\right)+\sup_{\alpha'\in[0,\alpha]}\|i[H,A]^0(e^{+iA\alpha'}-\mathbb{1})\Psi\|+\|(e^{-iA\alpha'}-\mathbb{1})i[H,A]^0\Psi\|$$
$$\leqq o\left(\frac{1}{n}\right)+o(\alpha)+\sup_{\alpha'\in[0,\alpha]}c_0\|H(e^{+iA\alpha'}-1)\Psi\|.$$

But finally

$$\|H(e^{+iA\alpha'}-1)\Psi\|=\|(H(\alpha')-e^{-iA\alpha'}H)\Psi\|$$
$$\leqq\|(H(\alpha')-H)\Psi\|+\|(1-e^{-iA\alpha'})H\Psi\|$$

which goes to zero as $\alpha\to 0$ by (II.2).

Proposition II.3. *If the operators H, A satisfy conditions* (a)–(c), *then $(A\pm i\lambda)^{-1}$ leaves $D(H)$ invariant for sufficiently large λ. Further $(H+i)i\lambda(A+i\lambda)^{-1}(H+i)^{-1}$ converges strongly to 1 as $|\lambda|\to\infty$.*

Proof. By Proposition II.2, we have in the operator sense

$$(A+i\lambda)^{-1}(H+i)^{-1}-(H+i)^{-1}(A+i\lambda)^{-1}$$
$$=(A+i\lambda)^{-1}\{(H+i)^{-1}A-A(H+i)^{-1}\}(A+i\lambda)^{-1}$$
$$=(A+i\lambda)^{-1}(H+i)^{-1}[A,H](H+i)^{-1}(A+i\lambda)^{-1},$$

where the last equality holds in the sense of quadratic form on $\mathscr{H}$. By condition (c), there is a bounded operator $B(\lambda)=[A,H]^0(H+i)^{-1}(A+i\lambda)^{-1}$ with $\|B(\lambda)\|\to 0$ as $|\lambda|\to\infty$ such that

$$(A+i\lambda)^{-1}(H+i)^{-1}(1-B(\lambda))=(H+i)^{-1}(A+i\lambda)^{-1}.$$

This proves Proposition II.3 since when $|\lambda|$ is sufficiently large, $1-B(\lambda)$ is invertible and $i\lambda(A+i\lambda)^{-1}(1-B(\lambda))^{-1}$ converges strongly to $\mathbb{1}$ as $|\lambda|\to\infty$.

Proposition II.4 (The Virial Theorem). *Let H and A be two self-adjoint operators satisfying conditions* (a)–(c). *Then*

1. *For all $\Psi\in D(H)$*

$$[H,A]^0\Psi=\lim_{|\lambda|\to\infty}[H,Ai\lambda(A+i\lambda)^{-1}]\Psi.$$

2. *If Ψ is an eigenvector of H, we have*

$$(\Psi|[H,A]^0\Psi)=0.$$

Proof. Let $\Psi \in D(H)$, $\Phi \in D(A) \cap D(H)$. By Propositions II.2 and II.3, for sufficiently large $|\lambda|$,

$$\begin{aligned}
&(\Phi|[H, Ai\lambda(A+i\lambda)^{-1}]\Psi) \\
&= (\Phi|\{HAi\lambda(A+i\lambda)^{-1} - Ai\lambda(A+i\lambda)^{-1}H\}\Psi) \\
&= (\Phi|(HA-AH)i\lambda(A+i\lambda)^{-1}\Psi) \\
&\quad + (A\Phi|\{Hi\lambda(A+i\lambda)^{-1} - i\lambda(A+i\lambda)^{-1}H\}\Psi) \\
&= (\Phi|[H,A]^0 i\lambda(A+i\lambda)^{-1}\Psi) \\
&\quad + (\Phi|A(A+i\lambda)^{-1}[H,A]^0 i\lambda(A+i\lambda)^{-1}\Psi).
\end{aligned} \tag{II.3}$$

Since $[A,H]^0 i\lambda(A+i\lambda)^{-1}\Psi \to [A,H]^0\Psi$ by Proposition II.3 and condition (c), and since $A(A+i\lambda)^{-1} \xrightarrow{s} 0$, Proposition II.3 implies that

$$\lim_{|\lambda|\to\infty} [H, Ai\lambda(A+i\lambda)^{-1}]\Psi = [H,A]^0\Psi.$$

Proving (1). Finally, if Ψ is an eigenvector for H, $\Psi \in D(H)$ and $H\Psi = E\Psi$, so that

$$(\Psi|[H,A]^0\Psi) = \lim_{|\lambda|\to\infty} (\Psi|[H, Ai\lambda(A+i\lambda)^{-1}]\Psi) = 0.$$

Proof of Part (1) *of Theorem 1*

If one supposes that the self-adjoint operators H, A satisfy conditions (a)–(c), and if furthermore they satisfy condition (e) at $E \in \mathbf{R}$ then the point spectrum in $(E-\delta, E+\delta)$ is finite. Suppose not. Then there is a sequence Ψ_n of orthonormal eigenvectors $H\Psi_n = E_n\Psi_n$. By Proposition II.4

$$\begin{aligned}
0 &= (\Psi_n|i[H,A]^0\Psi_n) = (\Psi_n|P_H(E,\delta)i[H,A]^0 P_H(E,\delta)\Psi_n) \\
&\geqq \alpha\|\Psi_n\|^2 + (\Psi_n|K\Psi_n).
\end{aligned}$$

Since the Ψ_n are orthonormal, $\Psi_n \xrightarrow{w} 0$ in $\mathscr{H}$ and since K is compact $\lim\limits_{n\to\infty} (\Psi_n|i[H,A]^0\Psi_n) \geqq \alpha$ which is impossible.

Proposition II.5 (Quadratic Estimate). *Let H be a self-adjoint operator with domain $D(H)$ and B^*B a bounded positive operator on $\mathscr{H}$. Then*

1. *$H - z - i\varepsilon B^*B$ is invertible if* $\operatorname{Im} z$ *and ε have the same sign.*
2. *If* $\operatorname{Im} z$ *and ε have the same sign, let*

$$G_z(\varepsilon) = (H - z - i\varepsilon B^*B)^{-1}.$$

Let B' an operator with $B'^*B' \leqq B^*B$ and C any bounded self-adjoint operator on $\mathscr{H}$, then:

$$\|B'G_z(\varepsilon)C\| \leqq \frac{1}{\sqrt{\varepsilon}}\|CG_z(\varepsilon)C\|^{1/2}.$$

Proof. Since B^*B is bounded $H-z-i\varepsilon B^*B$ is a closed operator on $D(H)$. When $\Psi \in D(H)$ and ε and $\operatorname{Im} z$ have the same sign, we have

$$\begin{aligned}\|(H-z-i\varepsilon B^*B)\Psi\|^2 &= \|(H-\operatorname{Re} z)\Psi\|^2+\|(\operatorname{Im} z+\varepsilon B^*B)\Psi\|^2\\ &\quad -2\operatorname{Im}((H-\operatorname{Re} z)\Psi|\varepsilon B^*B\Psi)\\ &\geqq (\operatorname{Im} z)^2\|\Psi\|^2 .\end{aligned} \tag{II.4}$$

From this inequality and the fact that $H-z-i\varepsilon B^*B$ is a closed operator, it follows that $H-z-i\varepsilon B^*B$ is injective with closed range in $\mathscr{H}$. By the open mapping theorem, its inverse exists as a bounded operator from $\operatorname{Rang}(H-z-i\varepsilon B^*B)$ into $\mathscr{H}_{+2}$. But $\operatorname{Rang}(H-z-i\varepsilon B^*B)=\mathscr{H}$ since if $\Phi_0 \in \mathscr{H}$ is orthogonal to this range, then $\Phi_0 \in D(H)$ and $(H-\bar{z}+i\varepsilon B^*B)\Phi_0=0$ which by (II.4) implies $\Phi_0=0$. Finally:

$$\begin{aligned}\|B'G_z(\varepsilon)C\|^2 &= \|CG_z^*(\varepsilon)B'^*B'G_z(\varepsilon)C\|\\ &\leqq \frac{1}{\varepsilon}\|C(H-\bar{z}+i\varepsilon B^*B)^{-1}(\operatorname{Im} z+\varepsilon B^*B)(H-z-i\varepsilon B^*B)^{-1}C\|\\ &\leqq \frac{1}{2\varepsilon}\|C(G_z^*(\varepsilon)-G_z(\varepsilon))C\|\\ &\leqq \frac{1}{\varepsilon}\|CG_z(\varepsilon)C\| = \frac{1}{\varepsilon}\|CG_z^*(\varepsilon)C\| .\end{aligned}$$

Proof of Part (2) *of Theorem 1*

We will prove the following lemma which clearly implies statement (2) of Theorem 1.

Lemma. *Let H be a self-adjoint operator with conjugate operator A in a neighborhood of E, i.e. suppose H, A, and E satisfy conditions* (a)–(e). *Then for any $E' \in (E-\delta, E+\delta)\cap\sigma_c(H)$, there is a neighborhood (a,b) of E' and a constant c_0 so that*

$$\sup_{\substack{\operatorname{Re} z\in[a,b]\\ \operatorname{Im} z\neq 0}} \||A+i|^{-1}(H-z)^{-1}|A+i|^{-1}\| \leqq c_0 .$$

Proof. By hypothesis (e), there are numbers α, $\delta>0$ and a compact operator K on $\mathscr{H}$ such that

$$P_H(E,\delta)i[H,A]^0P_H(E,\delta) \geqq \alpha P_H^2(E,\delta)+P_H(E,\delta)KP_H(E,\delta),$$

where $P_H(E,\delta)$ is the spectral projector of H onto the interval $(E-\delta, E+\delta)$. By hypothesis $E' \in \sigma_c(H)$, hence the spectral projector for H onto $(E'-\varepsilon, E'+\varepsilon)$ converges weakly to zero as $\varepsilon \to 0$. Hence one can find $\delta'>0$ and a smooth function $P \leqq 1$, $P=1$ on $(E'-\delta', E'+\delta')$, $P=0$ on $\mathbf{R}/(E-\delta, E+\delta)$ so that (denoting by P_H the operator associated to this P)

$$\pm P_HKP_H \leqq \frac{\alpha}{2}P_H^2$$

and hence

$$P_H i[H,A]^0 P_H \geqq \frac{\alpha}{2} P_H^2 .$$

Let $B^*B = P_H i[H,A]^0 P_H$.

By Proposition II.5, $G_z(\varepsilon) = (H - z - i\varepsilon B^*B)^{-1}$ exists if $\operatorname{Im} z$ and ε have the same sign. Let

$$F_z(\varepsilon) = |A+i|^{-1} G_z(\varepsilon) |A+i|^{-1} .$$

We have by Proposition II.5

$$\|P_H G_z(\varepsilon)|A+i|^{-1}\| \leqq \frac{c}{\sqrt{\varepsilon}} \|F_z(\varepsilon)\|^{1/2} . \tag{II.5}$$

Furthermore,

$$\begin{aligned} &\|(1-P_H)G_z(\varepsilon)|A+i|^{-1}\| \\ &\leqq \|(1-P_H)G_z(0)\| \, \|(1-i\varepsilon B^*BG_z(\varepsilon))|A+i|^{-1}\| \\ &\leqq c\|(1-P_H)G_z(0)\| . \end{aligned} \tag{II.6}$$

Remark. (II.5) and (II.6) remain true if one replaces P_H and $(1-P_H)$ by $(H+i)P_H$ and $(H+i)(1-P_H)$. If we restrict $\operatorname{Re} z$ to a closed interval $[a,b]$ strictly contained in $(E'-\delta', E'+\delta')$, $(1-P_H)G_z(0)$ is uniformly bounded, and there is a constant c so that:

$$\|F_z(\varepsilon)\| \leqq \frac{c}{\varepsilon} \qquad \operatorname{Re} z \in [a,b] . \tag{II.7}$$

Furthermore

$$\frac{d}{d\varepsilon} F_z(\varepsilon) = |A+i|^{-1} G_z(\varepsilon) P_H i[H,A]^0 P_H G_z(\varepsilon)|A+i|^{-1} .$$

We can write

$$\begin{aligned} P_H[H,A]^0 P_H = {} & [H,A]^0 - (1-P_H)[H,A]^0 P_H \\ & - P_H[H,A]^0(1-P_H) - (1-P_H)[H,A]^0(1-P_H) \end{aligned}$$

so that by Eqs. (II.5) and (II.6) and the remarks following them, there are constants c_1, c_2 so that

$$\begin{aligned} \left\| \frac{d}{d\varepsilon} F_z(\varepsilon) \right\| \leqq {} & \| |A+i|^{-1} G_z(\varepsilon) i[H,A]^0 G_z(\varepsilon) |A+i|^{-1} \| \\ & + c_1 + c_2 \frac{1}{\sqrt{\varepsilon}} \|F_z(\varepsilon)\|^{1/2} . \end{aligned} \tag{II.8}$$

By condition (d) and Proposition II.6 (see the appendix), $G_z(\varepsilon): D(A) \to D(A) \cap D(H)$ and $[B^*B, A]$ is bounded as a map from $\mathscr{H}_{+2}$ into $\mathscr{H}_{-2}$. Hence in (II.8), we can write $[H,A]^0$ as $[H - z - i\varepsilon B^*B, A] + i\varepsilon[B^*B, A]$. Substituting this relation into

(II.8), we find that

$$\left\| \frac{d}{d\varepsilon} F_z(\varepsilon) \right\| \leqq \tilde{c}_1 + \tilde{c}_2 \frac{1}{\sqrt{\varepsilon}} \|F_z(\varepsilon)\|^{1/2} + \tilde{c}_3 \|F_z(\varepsilon)\|$$

for constants $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ independent of ε and z such that $\mathrm{Re}\, z \in [a,b]$ and $\mathrm{Im}\, z$ and ε with the same sign.

This differential inequality together with the relation (II.7) shows that there exists a constant c_0 so that

$$\|F_z(\varepsilon)\| \leqq c_0$$

for all z with $\mathrm{Re}\, z \in [a,b]$, $\mathrm{Im}\, z \neq 0$ and $\mathrm{Im}\, z$, ε having the same sign.

Appendix I

Let $\{g_i(p)\}\, i \in \{1, \ldots, n\}$ be a $\mathscr{C}^2$ vector field, and let $\hat{A}$ be the symmetric operator defined on $L^2(\mathbf{R}^n, d^n p)$ by

$$\begin{aligned} \hat{A} &= \sum_{i=1}^{n} g_i(p) i \frac{\partial}{\partial p_i} + \frac{i}{2} \frac{\partial g_i}{\partial p_i}(p) \\ &= \frac{1}{2} \sum_i (g_i x_i + x_i g_i)\,. \end{aligned}$$

If each g_i is $\mathscr{C}^2$ the quadratic form defined by $\hat{A}$ admits a form domain containing the form domain of $x^2 = \sum_{i=1}^{n} x_i^2$, the same holds for the quadratic form $\hat{A}x^2 - x^2\hat{A}$. By the commutator theorem ([4, Vol. II]), $\hat{A}$ defines a self-adjoint operator A which is essentially self-adjoint on any core for x^2. On the other hand, the system of differential equations

$$\frac{d}{d\alpha} \Gamma_\alpha^i(p) = g_i(\Gamma_\alpha(p))$$

$$\Gamma_0(p) = p$$

defines a group of homeomorphism $\Gamma_\alpha : \mathbf{R}^n \mapsto \mathbf{R}^n$ and the following group of unitary transformations on $L^2(\mathbf{R}^n, d^n p)$

$$(U_\alpha \Psi)(p) = \left| \det\left(\frac{\partial \Gamma_\alpha^i}{\partial p_j}(p) \right) \right|^{1/2} \Psi(\Gamma_\alpha(p))$$

we then have

$$\begin{aligned} \frac{d}{d\alpha}(U_\alpha \Psi)_{\alpha=0}(p) &= \sum_i g_i(p) \frac{\partial \Psi}{\partial p_i}(p) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial g_i}{\partial p_i}(p) \cdot \Psi(P) \\ &= -i(A\Psi)(p)\,, \end{aligned}$$

where A is the self-adjoint extension of $\hat{A}$.

Let us finally note that $D(A)$ contains $D(|x|)$.

Appendix II

Proposition II.6. *Let H, A be operators that satisfy conditions* (a)...(d). *Then:*

1. *Let g be any function with $t\hat{g}(t) \in L^1(\mathbf{R}, dt)$, then*

$$g(H): D(A) \cap D(H) \to D(A).$$

2. *Let $B^*B = P_H i[H, A]^0 P_H$ as defined in the lemma of Sect. II. Then $[B^*B, A]$ is a bounded map from $\mathscr{H}_{+2}$ into $\mathscr{H}_{-2}$.*

3. $G_z(\varepsilon): D(A) \to D(A) \cap D(H)$.

Proof. Let $\Psi \in D(A) \cap D(H)$, $A(\lambda) = Ai\lambda(A + i\lambda)^{-1}$ for some sufficiently large $|\lambda|$. Then

$$\|\{A(\lambda)e^{-iHt} - e^{-iHt}A(\lambda)\}\Psi\| \leqq \sup_{\substack{\Phi \in D(H) \\ \|\Phi\| = 1}} \left| \int_0^t (\Phi | e^{+i(s-t)H}[H, A(\lambda)] e^{-isH}\Psi)\, ds \right|.$$

Since e^{-iHs} leaves $D(H)$, and also $A(\lambda)$ by Proposition II.3, we then have

$$\|\{A(\lambda)e^{-iHt} - e^{-iHt}A(\lambda)\}\Psi\| \leqq |t| \sup_{|s| \leqq |t|} \sup_{\substack{\Phi' \in D(A) \cap D(H) \\ \|\Phi'\| = 1}} |(\Phi' | [H, A(\lambda)] e^{-isH}\Psi)|.$$

By Eq. (II.3) in Propositions II.4 and II.3, one then sees that

$$\begin{aligned} \|Ae^{-iHt}\Psi\| &\leqq \lim_{|\lambda| \to \infty} \|A(\lambda)e^{-iHt}\Psi\| \\ &\leqq c|t| \, \|(H+i)\Psi\| + \|A\Psi\|. \end{aligned}$$

It is now enough to use the identity $g(H) = \int_{-\infty}^{+\infty} \hat{g}(t) e^{-iHt}\, dt$ to see that

$$g(H): D(A) \cap D(H) \to D(A) \quad \text{if} \quad |t|\hat{g}(t) \in L^1(\mathbf{R}, dt),$$

and that

$$\|\{Ag(H) - g(H)A\}\Psi\| \leqq c\|(H+i)\Psi\| \int_{-\infty}^{+\infty} |t|\, |\hat{g}(t)|\, dt. \tag{II.9}$$

Let $B^*B = P_H i[H, A]^0 P_H$. Since $P(\lambda)$ is smooth, its Fourier transform decays rapidly. Hence P_H takes $D(A) \cap D(H)$ into $D(A) \cap D(H)$ and so $[B^*B, A]$ in the sense of quadratic forms on $D(A) \cap D(H)$ can be written:

$$[B^*B, A] = [P_H, A][H, A]^0 P_H + P_H[[H, A]^0, A] P_H + P_H[H, A]^0 [P_H, A].$$

By hypothesis (d) and the relation (II.9), the form $[B^*B, A]$ on $D(A) \cap D(H)$ is bounded as a map from $\mathscr{H}_{+2}$ into $\mathscr{H}_{-2}$ and in particular if

$$\begin{aligned} &\Psi \in D(H) \, \|[(H - z - i\varepsilon B^*B), A(\lambda)]\Psi\|_{-2} \\ &\leqq \sup_{\substack{\Phi \in D(A) \cap D(H) \\ \|\Phi\|_{+2} = 1}} \{|(\Phi | [H - z - i\varepsilon B^*B, A] i\lambda(A + i\lambda)^{-1}\Psi)| \\ &\quad + |(\Phi | A(A + i\lambda)^{-1}[H - z - i\varepsilon B^*B, A] i\lambda(A + i\lambda)^{-1}\Psi)|\}. \end{aligned}$$

By Proposition II.3, the operators $\lambda(A+i\lambda)^{-1}$ and $A(A+i\lambda)^{-1}=1-i\lambda(A+i\lambda)^{-1}$ are uniformly bounded from $\mathscr{H}_{+2}$ into $\mathscr{H}_{+2}$ for λ large enough. It follows that $[H-z-i\varepsilon B^*B, A(\lambda)]$ are uniformly bounded (in λ) from $\mathscr{H}_{+2}$ into $\mathscr{H}_{-2}$. It follows that $G_z(\varepsilon)=(H-z-i\varepsilon B^*B)^{-1}$ preserves $D(A)$ and hence:

$$G_z(\varepsilon): D(A)\to D(A)\cap D(H).$$

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References

1. Kato, T.: Perturbation theory for linear operators. Berlin, Heidelberg, New York: Springer 1966
2. Agmon, S.: Ann. Scuola Norm. Sup. Pisa, Ser. 4, **2**, 151–218 (1975)
3. Aguilar, J., Combes, J.M.: Commun. Math. Phys. **22**, 269–279 (1971)
   Balslev, E., Combes, J.M.: Commun. Math. Phys. **22**, 280–294 (1971)
4. Reed, M., Simon, B.: Methods of modern mathematical physics. Tomes II and III. New York: Academic Press 1979
5. Enss, V.: Commun. Math. Phys. **61**, 285 (1978)
6. Simon, B.: Duke Math. J. **46**, 119–168 (1979)

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