ABSOLUTE CONTINUITY AND HYPONORMAL OPERATORS

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ABSTRACT. Let T be a completely hyponormal operator, with the rectangular

representation $T = A + iB$, on a separable Hilbert space. If 0 is not an

eigenvalue of T^* then T also has a polar factorization $T = UP$, with U

unitary. It is known that A, B and U are all absolutely continuous

operators. Conversely, given an arbitrary absolutely continuous selfadjoint A

or unitary U, it is shown that there exists a corresponding completely

hyponormal operator as above. It is then shown that these ideas can be used to

establish certain known absolute continuity properties of various unitary

operators by an appeal to a lemma in which, in one interpretation, a given

unitary operator is regarded as a polar factor of some completely hyponormal

operator. The unitary operators in question are chosen from a number of sources:

the F. and M. Riesz theorem, dissipative and certain mixing transformations in

ergodic theory, unitary dilation theory, and minimal normal extensions of sub-

normal contractions.

1. INTRODUCTION.

    Let T be a bounded operator, with the rectangular decomposition

$T = A + iB$, on a complex, separable Hilbert space H, and denote its spectrum

and point spectrum by $\sigma(T)$ and $\sigma_p(T)$ respectively. Then T is said to be

hyponormal if

$$T^*T - TT^* = D \geqq 0, \text{ equivalently, } AB - BA = -iC \ (C = \tfrac{1}{2}D \geqq 0), \tag{1.1}$$

and completely hyponormal if, in addition, T has no normal part, that is, if

there is no nontrivial reducing space of T on which it is normal. (For a

recent survey of hyponormal operators, see Clancey [1].) If T is hyponormal

then the sets $\sigma(A)$ and $\sigma(B)$ are just the real projections of $\sigma(T)$ onto the

real and imaginary axes; see Putnam [2], p. 46. It is known that if T is

completely hyponormal then $A = \text{Re}(T)$, as well as $B = \text{Im}(T) = \text{Re}(-iT)$, is

absolutely continuous, that is, if A has the spectral resolution

$$A = \int t dE_t , \tag{1.2}$$

then $\|E_t x\|^2$ is an absolutely continuous function of t for every x in H;

see [2], p. 42.

Since (1.1) holds if T is replaced by its translate $T - zI$, it is clear

that if T is hyponormal then $\sigma_p(T) \subset \sigma_p(T^*)$, and it is not hard to show

that if T is completely hyponormal then $\sigma_p(T)$ is empty. Also, if T is

completely hyponormal then it has a (unique) polar factorization

$$T = UP, \ U = \text{unitary} \quad (\text{hence} \ P = (T^*T)^{\frac{1}{2}}), \tag{1.3}$$

if and only if $0 \notin \sigma_p(T^*)$; see the discussion in Putnam [3], pp. 419-420. If

T is completely hyponormal with the factorization (1.3) then U is also

absolutely continuous; see [2], p. 21 and Lemma 4 of [4]. Thus, if U has

the spectral resolution

$$U = \int_0^{2\pi} e^{it} dG_t \left(\int_0^{2\pi} = \int_0^{2\pi-0} \right) , \tag{1.4}$$

then $\|G_t x\|^2$ is an absolutely continuous function of t for every x in H.

A converse of the above assertions concerning absolute continuity also holds, namely:

(*) Every bounded absolutely continuous selfadjoint operator A is the real part of some completely hyponormal operator.

(**) Every absolutely continuous unitary operator U is the unitary factor in the polar factorization (1.3) of some completely hyponormal operator.

That there exists some nonnormal hyponormal operator T for which $A = \mathrm{Re}(T)$ was shown by Kato [5], and that T can even be constructed so that it is completely hyponormal can be deduced from Kato's results.   A direct proof of (*) will be given below however.

Let E be any bounded subset of $(-\infty, \infty)$ having positive measure and let $T_0 = A_0 + iB_0$, where, for x in $L^2(E)$, $(A_0 x)(t) = tx(t)$ and $(B_0 x)(t) = -(i\pi)^{-1} \int_E (s-t)^{-1} x(s) ds$, the integral being a Cauchy principal value. It is easily verified that $A_0 B_0 - B_0 A_0 = -iC_0$, where $C_0 x = \pi^{-1}(x,1)1$ and 1 denotes the identity function on $L^2(E)$. Since the vectors $\{A_0^n 1\}$, $n = 0,1,2,\ldots$, span $L^2(E)$, it is clear that T_0 is completely hyponormal on $L^2(E)$; see, e.g., Clancey and Putnam [6], p. 452.

It is now easy to prove (*). According to spectral multiplicity theory applied to absolutely continuous operators, A is unitarily equivalent to, and here will simply be assumed to be equal to, the following canonical representation. There exist disjoint Borel sets $\{E_n\}$ on the real line, when $n \in \alpha$ and α is a nonempty subset of $\{1,2,\ldots, \aleph_0\}$, and where each E_n has positive measure and $\bigcup_{n \in \alpha} E_n$ is a bounded set. The operator A is a direct sum $A = \Sigma \oplus A_n$ on $H = \Sigma \oplus H_n$ $(n \in \alpha)$, where each H_n is the direct sum of n copies of $L^2(E_n)$ and $(A_n x_n)(t) = tx_n(t)$ for x_n in H_n. Since each A_n is the real part of the direct sum, T_n , of n copies of $T_0 = A_0 + iB_0$ on the corresponding space H_n , then $A = \mathrm{Re}(T)$, where $T = \Sigma \oplus T_n$ on $H = \Sigma \oplus H_n$ $(n \in \alpha)$. That T is completely hyponormal is a consequence of the properties of T_0 on $L^2(E_n)$

. as described earlier, and so (*) is proved.

Although the selfcommutator $T*T - TT* = D$ of the above operator T need

not be of trace class, it is clear that such a T could be constructed. This

is apparent from the fact that T_n could as well be replaced by $\sum\limits_{k=1}^{n} \oplus (A_0 + it_{nk} B_0)$,

where the t_{nk} $(n \in \alpha, k = 1, \ldots, n)$ are any positive constants whose total sum

is finite.

In order to prove (**), first choose S_0 to be any completely hyponormal

operator on the Hilbert space H with a polar factorization $S_0 = U_0 P_0$, where

U_0 is unitary with simple multiplicity on the entire unit circle $|z| = 1$.

For example, let $\{\lambda_n\}$, $n = 0, \pm 1, \pm 2, \ldots$, be any bounded, strictly increasing

sequence of numbers satisfying, say, $\lim\limits_{n \to -\infty} \lambda_n = 1$ and $\lim\limits_{n \to \infty} \lambda_n = 2$. If H

is the ℓ^2 bilateral sequence Hilbert space, let $A = (a_{ij})$ and $B = (b_{ij})$ be

the doubly infinite matrices defined by $a_{ij} = \delta_{ij} \lambda_j$ and $b_{ij} = \delta_{ij} \lambda_{j-1}$. Then

$A = WBW*$ where W is the unitary bilateral shift $x = (\ldots, x_{-1}, (x_0), x_1, \ldots) \to$

$Wx = (\ldots, x_0, (x_1), x_2, \ldots)$, the parentheses referring to the 0 position. If

$S_0 = U_0 A^{\frac{1}{2}}$, where $U_0 = W*$, then $S_0*S_0 - S_0 S_0* = A - B = D \geqq 0$, and, since

$0 \notin \sigma_p(D)$, S_0 is completely hyponormal. Also, S_0 is nonsingular, that is,

$0 \notin \sigma(S_0)$. In fact, $\sigma(S_0) = \{z : 1 \leqq |z| \leqq \sqrt{2}\}$; see, e.g., Th. 1 and Th. 9

of [3]. In addition, U_0 has the desired properties. (See also section 3 below.)

Next, let U_0 have the spectral resolution $U_0 = \int_0^{2\pi} e^{it} dG_t^0$, and, for

any Borel set $\beta \subset [0, 2\pi]$ of positive measure, consider the unitary operator

$G^0(\beta) U_0$ on the Hilbert space $G^0(\beta) H$. (Note that $G^0(\beta) \neq 0$ if and only if β

has positive measure; see also (3.2) below.) If $S_0(\beta) = G^0(\beta) U_0 [G^0(\beta) S_0* S_0 G^0(\beta)]^{\frac{1}{2}}$,

then a simple calculation shows that $S_0(\beta)$ is hyponormal on $G^0(\beta) H$. In

addition, $S_0(\beta)$ is completely hyponormal on $G^0(\beta) H$; see Lemma 8 of Putnam

[7], pp. 326-327. From here on, the proof of (**) is essentially identical to

that of (*) but where unitary operators now play the role formerly played by

selfadjoint ones. Also, corresponding to a similar argument above, the
completely hyponormal operator T of (**) can be chosen so that T*T - TT* is
of trace class.

Absolutely continuous selfadjoint and unitary operators occur in various
connections. The results of (*) and (**) indicate that the study of such
operators can be reduced, at least in principle, to the study of the real part,
or of the unitary factor in the polar factorization, of an appropriate
completely hyponormal operator. In the sections below this (somewhat tendentious)
point of view will be exploited to investigate certain unitary operators arising
in the following situations: bilateral shifts and their role in the F. and
M. Riesz theorem and in dissipative and certain mixing transformations in
ergodic theory; unitary dilations of contractions; minimal normal extensions of
subnormal contractions.

2. A LEMMA.

The following lemma (I) asserts some absolute continuity properties of
operators effecting the unitary equivalence of two selfadjoint operators having
a positive difference. As will be noted below, it concerns the unitary operators
of polar factorizations of certain corresponding completely hyponormal operators.
It will play a key role in later applications.

LEMMA (I) Let J and D be bounded, selfadjoint operators on a Hilbert
space H and suppose that $D \geqq 0$ and $\neq 0$. Let U be unitary with the
spectral resolution (1.4) and suppose that

$$J - UJU^* = D \qquad\qquad (2.1)$$

(i) If

H is the least space reducing both J and U and
containing the range of D,   (2.2)

then U is absolutely continuous.

(ii) _If_ β _is any Borel subset of_ $[0,2\pi]$ _for which_ $G(\beta) = I$ _(and whether or not (2.2) is satisfied) then_

$$2\pi\|D\| \leqq d(J)|\beta| \text{ , } \underline{\text{where}} \quad d(J) = \max \sigma(J) - \min \sigma(J) \text{ .} \tag{2.3}$$

Here, $\max \sigma(J)$ and $\min \sigma(J)$ denote the maximum and minimum points of $\sigma(J)$ and $|\beta|$ denotes linear Lebesgue measure of β. (The proof of a slightly different formulation of (I) is given in Putnam [8] under the added hypothesis that $0 \notin \sigma_p(D)$, in which case, condition (2.2) is automatically assured.) Note that $\|D\| \leqq d(J)$ and that, in (ii), if there exists a J for which $d(J) = \|D\| > 0$, or even if there exists a sequence J_1 , J_2 ,..., and a corresponding sequence D_1 , D_2 ,..., for which $d(J_n) = \lim\|D_n\| > 0$ (finite), then $|\beta| = 0$ whenever β is any Borel subset of $[0,2\pi]$ for which $G(\beta) = 0$.

In order to prove (i), note that J can be replaced by $J - kI$ for any real constant k and hence there is no loss of generality in supposing that $J \geqq$ const > 0. Then, in particular, $T = UJ^{\frac{1}{2}}$ satisfies (1.1) by virtue of (2.1), so that T is hyponormal. That T is completely hyponormal is a consequence of (2.2) and, as noted in section 1 above, U is absolutely continuous.

The proof of (ii) essentially has been given in [8]. An alternate proof can be obtained from an inequality concerning hyponormal operators derived in [4]. In fact, if T is any completely hyponormal operator with factorization (1.3) and if, for $0 \leqq t < 2\pi$,

$$\alpha_t = \sigma(T) \cap \{z: \arg z = t\} \quad \text{and} \quad M(t) = \int_{\alpha_t} rdr \text{ ,} \tag{2.4}$$

then, for any Borel set β of $[0,2\pi)$,

$$\pi\|G(\beta)DG(\beta)\| \leqq \int_{\beta} M(t)dt \text{ .} \tag{2.5}$$

See Theorem 1 of [4], where the additional hypothesis that T be completely hyponormal is imposed. This restriction is not necessary, however. In fact, if T is normal, $D = 0$ and (2.5) becomes trivial. If T is not normal, then one

need only split off its normal part and then apply the result in [4] to the remaining completely hyponormal part of T. (Note that the spectrum of the restriction of T to a reducing subspace is a subset of the spectrum of T on the entire space.) If $T = UJ^{\frac{1}{2}}$ as defined above, then clearly $\sigma(T)$ is a subset of the annulus centered at the origin with inner radius $(\min \sigma(J))^{\frac{1}{2}}$ and outer radius $(\max \sigma(J))^{\frac{1}{2}}$. Consequently, $M(t) \leqq \frac{1}{2}(\max \sigma(J) - \min \sigma(J))$ and (2.3) follows from (2.5).

3. BILATERAL SHIFTS

Let U be unitary on H and let $K \neq 0$ be a subspace of H for which

$$K \perp U^n K \quad \text{for} \quad n = \pm 1, \ \pm 2, \ \ldots, \quad \text{and} \quad H = \sum_{-\infty}^{\infty} \oplus U^n K . \qquad (3.1)$$

Then U is a bilateral shift on H of (constant) multiplicity = dim K. (A space K satisfying the first part of (3.1) is called a wandering subspace of U.) If a bilateral shift U has the spectral resolution (1.4) then it is known that for any Borel subset β of $[0, 2\pi]$,

$$G(\beta) = 0 \quad \text{if and only if} \quad |\beta| = 0. \qquad (3.2)$$

This fact can be derived from (I) as follows. Let $\{\lambda_n\}$, $n = 0, \pm 1, \pm 2, \ldots$, be any bounded, nondecreasing sequence of real numbers. For any x in H one has the (unique) representation $x = \Sigma x_n$ where $x_n \in U^n K$. Let the selfadjoint operator J on H be defined by $Jx = \Sigma \lambda_n x_n$. It is seen that $UJU^*x = \Sigma \lambda_{n-1} x_n$, and so (2.1) holds with $Dx = \Sigma(\lambda_n - \lambda_{n-1})x_n$. Clearly, $D \geqq 0$. If now $\{\lambda_n\}$ is strictly increasing, then $0 \notin \sigma_p(D)$, and hence (2.2) holds. The first half of (3.2) now follows from (i) of (I). Next, if the $\{\lambda_n\}$ sequence is chosen so that, say, $\lambda_n = 0$ for $n \leqq 0$ and $\lambda_n = 1$ for $n \geqq 1$, then the corresponding operators J and D satisfy $d(J) = \|D\| = 1$ and so the second part of (3.2) follows from (ii) of (I).

The F. and M. Riesz theorem [9] furnishes an interesting instance of how

bilateral shifts can arise. Let λ be normalized linear Lebesgue measure and

μ any nonzero, finite, complex Borel measure on the unit circle $C: |z| = 1$.

The F. and M. Riesz theorem is the assertion that

$$\int_C z^n d\mu = 0 \quad \text{for} \quad n = 1, 2, \ldots,$$ (3.3)

implies

$$|\mu| \ll \lambda \ll |\mu| ,$$ (3.4)

that is, $|\mu|$ and λ are equivalent measures on C. If U is the unitary

operator on $L^2(|\mu|)$ of multiplication by z and if M is the subspace of

$L^2(|\mu|)$ spanned by $\{z, z^2, \ldots\}$ then

$$M = \sum_{n=0}^{\infty} \oplus U^n K ,$$ (3.5)

where K is the one-dimensional wandering subspace $M \ominus UM$ and $L^2(|\mu|) = \sum_{-\infty}^{\infty} \oplus U^n K$. See Sarason [10], pp. 14-18, Sarason [11], and Putnam [8]. It turns

out ([8]) that (3.4) is simply an interpretation of (3.2).

It may be noted that the representation (3.5) is a form of what is usually

called the Wold decomposition for isometries. (See Sz. Nagy and Foiaş [12], pp.

3, 51 for a statement of the theorem and some of its history. Perhaps it

should be referred to as the von Neumann-Wold-Halmos decomposition.)

Another instance where the bilateral shift occurs comes from ergodic theory.

Here, one has a system (X, S, m, ϕ) consisting of a measure space, X, of points

x and a measure, m, defined on some σ-algebra, S, of subsets of X, where

$0 < m(X) \leq \infty$. In addition, ϕ is here assumed to be an invertible measure

preserving transformation of X onto itself. (See e.g., Brown [13], Halmos [14].)

Let U be the unitary operator defined by

$$U: f(x) \to f(\phi(x)) \quad \text{on} \quad L^2(X, dm).$$ (3.6)

In case there exists a set $A \subset S$ for which $m(A) > 0$ and the images

$A_n = \phi^n(A)$ $(n = 0, \pm 1, \pm 2, \ldots)$ are disjoint and $X = \overset{\infty}{\underset{\infty}{\bigcup}} A_n$, then A is a generating set of X and ϕ is said to be dissipative; see Halmos [14], p. 11, Hopf [15], p. 46. In case ϕ is dissipative, then $m(X) = \infty$ and U is a bilateral shift. In fact, if $K = L^2(A, dm)$ then K is a wandering subspace of U, so that (3.1) holds with $H = L^2(X, dm)$.

In case $m(X) < \infty$, the unitary operatory U of (3.6) has 1 in its point spectrum, the characteristic function of X being an eigenvector. However, even in this case it may happen that U is a bilateral shift on the orthogonal complement $L^2(X, dm) \ominus R$, where R denotes the eigenspace corresponding to the eigenvalue 1. This happens, for instance, if (X, S, m, ϕ) is a Kolmogorov (or K-) system, in which case ϕ is a special type of mixing transformation called a K-automorphism. For the definitions of these concepts, see Arnold and Arez [16], pp. 32-35, 153-157, Brown [13], p. 39, problem 25. Then R is the space of constants in $L^2(X, dm)$ and the space $L^2(X, dm) \ominus R$ admits a decomposition of the form (3.1) for some wandering subspace K. Incidentally, in this case, U may be a bilateral shift of infinite multiplicity; cf. [16], pp. 156-157.

4. UNITARY DILATIONS

Let T be a contraction $(\|T\| \leq 1)$ on the Hilbert space H. It is a well-known result of B. Sz.-Nagy that there exists a Hilbert space $K \supset H$ and a unitary operator U on K with the property that

$$T^n = PU^n|H , \qquad n = 0, 1, 2, \ldots, \qquad (4.1)$$

where P is the orthogonal projection $P:K \to H$. Such a unitary U is called a unitary dilation of T. (It is easily verified that also $T^{*n} = PU^{*n}|H$ for $n = 1, 2, \ldots$.) The minimal unitary dilation, which is known to be unique, is then obtained by restricting U to the least subspace of K which reduces U and contains H. (A brief discussion of unitary dilation theory, its history, applications, and references, can be found in Halmos [17]. For an extensive

treatment of the subject see Sz.-Nagy and Foias [12] and the more recent survey

article of Douglas [18].)

Suppose now that the contraction T is completely nonunitary (c.n.u.), so

that there is no nontrivial reducing space of T on which its restriction is

unitary. Thus, either $T^*T \neq I$ or $TT^* \neq I$. If the minimal unitary dilation,

U, of such a contraction has the spectral resolution (1.4) then it is an

important property of U that it satisfies (3.2), a result first established by

B. Sz.-Nagy and C. Foias [19]. In general, however, U is itself not a bilateral

shift, although (3.2) does imply that it always contains such a shift. More

precisely, there is a subspace of K which reduces U and on which it is a

bilateral shift. This is clear if one notes that (3.2) implies that U has a

spectral multiplicity of at least 1 at every point of the unit circle $|z| = 1$.

It will be shown below how the assertion (3.2) for the minimal unitary

dilation of a c.n.u. contraction can be deduced from the lemma (I) of section 2.

First, a few preliminaries will be discussed. For the given contraction, T,

the so-called defect operators are defined by $D_T = (I - T^*T)^{\frac{1}{2}}$ and

$D_{T^*} = (I - TT^*)^{\frac{1}{2}}$, and it is fairly easy to show that

$$TD_T = D_{T^*}T \quad \text{and} \quad T^*D_{T^*} = D_T T^* \; ; \tag{4.2}$$

see, e.g., [18], p. 165. Let

$$K_1 = \ldots \oplus H \oplus (H) \oplus H \oplus \ldots, \tag{4.3}$$

where the parentheses refer to the 0 position. If one defines the operator

matrix

$$U_1 = \begin{pmatrix} \ddots & & & & \\ & \ddots & I & & 0 \\ & & D_T & -T^* & \\ & & (T) & D_{T^*} & \\ 0 & & & I & \ddots \\ & & & & \ddots \end{pmatrix}, \tag{4.4}$$

where (T) refers to the $(0,0)$ element, then U_1 is readily verified to be a

unitary dilation of T on the space K_1 of (4.3). This explicit construction

of a unitary dilation is due to J. J. Schaffer [20]. In order to obtain the

minimal unitary dilation one need only restrict U_1 to the least subspace of

K_1 which reduces U_1 and contains H.

That even U_1 is absolutely continuous (that is, the "if" part of (3.2) if

$\{G_t\}$ refers now to the spectral family of U_1) will be obtained as an

application of (i) of (I) of section 2. Let $\{\lambda_n\}$, $n = 0, \pm1, \pm2,\ldots,$ be a

real sequence satisfying

$$\ldots \geqq \lambda_{-2} \geqq \lambda_{-1} \geqq \lambda_0 \geqq \lambda_1 \geqq \lambda_2 \geqq \ldots \text{ and } |\lambda_n| \leqq \text{const} < \infty , \qquad (4.5)$$

and define the selfadjoint diagonal operator matrix J on K_1 of (4.3) by

$$J = \ldots \oplus \lambda_{-1}I \oplus (\lambda_0 I) \oplus \lambda_1 I \oplus \ldots . \qquad (4.6)$$

A straightforward calculation using (4.2) and the (crucial) equality $\lambda_0 = \lambda_1$

of (4.5) shows that

$$J - UJU^* = D = \ldots \oplus (\lambda_{-1} - \lambda_0)I \oplus (0) \oplus (\lambda_1 - \lambda_2)I \oplus \ldots . \qquad (4.7)$$

Also, in view of (4.5), $D \geqq 0$.

By choosing the λ_n's so that $\lambda_0 = \lambda_1$ and $\lambda_{n-1} > \lambda_n$ for $n \neq 1$, it is

clear from (4.7) that if M is the least subspace of K_1 which reduces both

J and U and contains the range of D then $M^{\perp} \subset H$. Since $M^{\perp}$ reduces U

and since T is c.n.u., then $M^{\perp} = 0$, that is, $M = K_1$. Thus, (2.2) holds

with the H occurring there now replaced by K_1 , and so U_1 of (4.4) is

absolutely continuous. This implies, of course, that its restriction to any

nontrivial reducing subspace, and, in particular, the minimal unitary dilation,

must also be absolutely continuous.

Moreover, corresponding to a similar argument in section 3, one may choose

$\lambda_n = 1$ if $k \leqq 1$ and $\lambda_n = 0$ if $n \geqq 2$. Then (4.5) holds and if the

corresponding J is defined by (4.6), with D given by (4.7), then clearly

$1 = d(J) = \|D\|$. It follows from (ii) of the lemma (I) that $|\beta| = 0$ whenever

$G(\beta) = 0$, β being a Borel subset of $[0,2\pi]$ and $\{G_t\}$ being the spectral

family of U_1 of (4.4). However, the corresponding assertion for the minimal

unitary dilation does not immediately follow.

Thus, so far, (3.2) has been established when $\{G_t\}$ is the spectral family

of U_1 but only the "if" part of (3.2) has been proved when $\{G_t\}$ is the

spectral family of the minimal unitary dilation U (T being a c.n.u. contraction).

It is not hard to establish the "only if" part too however. In fact, let K be

the subspace of K_1 of those vectors $x = \ldots + x_{-1} + (x_0) + x_1 + \ldots$, where

x_0 is in H and x_n is in the closure of $D_T H$ for $n \leq -1$ and in the closure

of $D_{T*} H$ for $n \geq 1$. Then K is the least subspace of K_1 reducing U and

containing H, so that $U = U_1 | K$ is the minimal unitary dilation of T. (See

[12], pp. 17-18.) Since T is c.n.u., either $D_T \neq 0$ or $D_{T*} \neq 0$. Suppose,

for definiteness, that $D_T \neq 0$. (If $D_T = 0$ and $D_{T*} \neq 0$, the treatment is

similar.) Note that J of (4.6) leaves K invariant and J is now restricted

to this space. If one chooses the sequence of (4.5) so that $\lambda_n = 1$ for

$n \leq -1$ and $\lambda_n = 0$ for $n \geq 0$, then clearly $d(J) = \|D\| = 1$. Hence, by (ii)

of the lemma (I), if β is any Borel subset of $[0,2\pi]$ and if the minimal

unitary dilation $U = U_1 | K$ has the spectral resolution (1.4), then $|\beta| = 0$

whenever $G(\beta) = 0$. This establishes (3.2) where the minimal unitary dilation

has the spectral resolution (1.4).

Incidentally, it can be shown that if U on K is the minimal unitary

dilation of a c.n.u. contraction T on H ($\subseteq K$) then $L = $ closure of $(U - T)H$

and $L* = $ closure of $(U* - T*)H$ are wandering subspaces of U and

$$K = \ldots \oplus U^{*2}L* \oplus U*L* \oplus L* \oplus H \oplus L \oplus UL \oplus U^2 L \oplus \ldots . \qquad (4.8)$$

See [12], p. 57. The argument applied above to U_1 on K_1 and then to its

restriction $U = U_1 K$ can now be applied directly to U on K of (4.8) in

order to establish, again, the relation (3.2) for U.

5. SUBNORMAL CONTRACTIONS

Let T be subnormal on H so that there exists a Hilbert space $K' \supset H$
and a normal operator N on K' such that $NH \subset H$ and $T = N|H$. Then N is
a normal extension of T. It is known that there is a unique minimal normal
extension, and N will henceforth be assumed to be of this type. A subnormal
T is said to be completely subnormal if T has no normal part. It may be noted,
incidentally, that the subnormal operators constitute a proper subset of the set
of hyponormal operators. (Concerning the basic properties of subnormal operators,
see [17]. A more elaborate treatment of subnormal operator theory can be found
in the recent tract of Conway and Olin [21].

Suppose now that T is a completely subnormal contraction on H with the
minimal normal extension N on K' and with the minimal unitary dilation U
of K. (The space H is regarded as a subspace of K and of K'.) Let P'
and P denote the orthogonal projections $P':K' \to H$ and $P:K \to H$, and let N
and U have the spectral resolutions $N = \int z dE_z \, (= \int_{|z| \leq 1} z dE_z)$ and
$U = \int_C z dG_z$ $(C = \{z: |z| = 1\})$. Then, for x in H and $n = 1, 2, \ldots$, one
has $T^n x = N^n x = PU^n x$ and $T^{*n} x = P'N^{*n} x = PU^{*n} x$. Hence, if f is analytic on
$|z| < 1$ and continuous on $|z| \leq 1$, then, for x in H, $\int f(z) dE_z x =$
$P \int_C f(z) dG_z x$ and $P' \int \bar{f}(z) dE_z x = P \int_C \bar{f}(z) dG_z x$. Consequently, if h is any
(real) function harmonic in $|z| < 1$ and continuous on $|z| \leq 1$, then

$$P' \int_{|z| \leq 1} h dE_z x = P \int_C h dG_z x , \quad x \text{ in } H \qquad (5.1)$$

Let β be any closed subset of the unit circle C and let ϕ be a real
continuous function on C such that $\phi = 1$ on β and $0 \leq \phi < 1$ on $C - \beta$.
(Clearly, such functions exist.) Then it is well-known that there exists an h
as described above (which can be given explicitly by the Poisson integral) for

which h = φ on C. In view of (5.1), one has

$$P'(\int_{|z|<1} h dE_z + \int_C \phi dE_z)x = P \int_C \phi dG_z x \quad \text{for x in H .} \tag{5.2}$$

From basic max-min properties of harmonic functions it is clear that

$0 \leqq h \leqq 1$ in $|z| < 1$, and so, on forming inner products of the vectors in

(5.2) with x, one obtains $\int_C \phi d\|E_z x\|^2 \leqq \int_C \phi d\|G_z x\|^2$. On choosing a sequence

of functions $\phi = \phi_n$ (n = 1,2,...) converging to the characteristic function of

β one obtains

$$\|E(\beta)x\| \leqq \|G(\beta)x\| , \quad x \text{ in H.} \tag{5.3}$$

Since (5.3) holds for an arbitrary closed set β on C it clearly holds

also for any Borel set β on C. Let $|\beta|$ denote Lebesgue arc length measure

on C. In particular, if $|\beta| = 0$, then, in view of the absolute continuity of

the minimal unitary dilation of a c.n.u. contraction (cf. section 4 above),

$G(\beta) = 0$ (on K), and so $E(\beta)P' = 0$ (on K'). Hence $0 = N^{*n}E(\beta)P' = E(\beta)N^{*n}P'$

for n = 0,1,2,... . Since N is the minimal normal extension of T , K' is

spanned by the vectors {N*Nx: x ∈ H, n = 0,1,2,...}, and so $E(\beta) = 0$.

Thus, if β is any Borel subset of C for which $|\beta| = 0$ then also

E(β) = 0. In other words, the minimal normal extension of a completely subnormal

contraction is absolutely continuous on the unit circle $|z| = 1$. Other proofs

of this last assertion can be found in Conway and Olin [21], p. 35, Olin [22],

and Putnam [23].

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