Absolute Continuous Bivariate Generalized Exponential Distribution

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Abstract

Generalized exponential distribution has been used quite effectively to model positively skewed lifetime data as an alternative to the well known Weibull or gamma distributions. In this paper we introduce an absolute continuous bivariate generalized exponential distribution by using a simple transformation from a well known bivariate exchangeable distribution. The marginal distributions of the proposed bivariate generalized exponential distributions are generalized exponential distributions. The joint probability density function and the joint cumulative distribution function can be expressed in closed forms. It is observed that the proposed bivariate distribution can be obtained using Clayton copula with generalized exponential distribution as marginals. We derive different properties of this new distribution. It is a five-parameter distribution, and the maximum likelihood estimators of the unknown parameters cannot be obtained in closed forms. We propose some alternative estimators, which can be obtained quite easily, and they can be used as initial guesses to compute the maximum likelihood estimates. One data set has been analyzed for illustrative purposes. Finally we propose some generalization of the proposed model.

Keywords and Phrases: Bivariate exchangeable distribution; Dependence Properties; Clayton Copula; Hazard rate; Maximum likelihood estimators; Pseudo generator.

1 Introduction

Gupta and Kundu [9] proposed the generalized exponential (GE) distribution as an alternative to the well known Weibull or gamma distributions. It is observed that the proposed

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two-parameter GE distribution has several desirable properties and in many situations it may fit better than the Weibull or gamma distribution. Extensive work has been done since then to establish several properties of the generalized exponential distribution. The readers are referred to the recent review article by Gupta and Kundu [10] on a current account of it.

The GE distribution has the following cumulative distribution function (CDF) for $\alpha > 0, \lambda > 0$;

$$F(x;\alpha,\lambda) = \left(1 - e^{-\lambda x}\right)^{\alpha}; \quad \text{if} \quad x > 0, \tag{1}$$

and 0 otherwise. It has the probability density function (PDF)

$$f(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\alpha-1} \quad \text{if} \quad x > 0, \tag{2}$$

and 0 otherwise. Here α and λ are the shape and scale parameters respectively. Form now on a GE distribution with the shape and scale parameters as α and λ respectively will be denoted by GE(α, λ).

The main aim of this paper is to establish a new absolute continuous bivariate generalized exponential distribution whose marginals are generalized exponential distributions. First we introduce the one-parameter exchangeable bivariate distribution on $(0,\infty)\times(0,\infty)$ of Mardia [19]. From now on we call this distribution as the Bivariate Exchangeable (BE) distribution. The BE model can be obtained as a gamma mixture of two independent exponential models also, see for example Lindley and Singpurwalla [18]. From the BE distribution, by simple transformation we obtain an absolute continuous bivariate generalized exponential distribution whose marginals are univariate generalized exponential distributions.

We discuss different properties of the proposed distribution. Estimation of the unknown parameters is an important problem in any statistical inference. The MLEs of the unknown parameters, as expected, cannot be obtained in explicit forms. They have to be obtained by solving non-linear equations. Since the MLEs are difficult to obtain, we propose some alternative estimators, which can be obtained quite easily, and they can be used as initial guesses to compute the MLEs. One data set has been analyzed for illustrative purposes, and finally we propose some generalizations.

Rest of the paper is organized as follows. In section 2 we introduce the bivariate generalized exponential distribution. Its several properties have been discussed in section 3. Maximum likelihood estimation procedure of the unknown parameters is discussed in section 4. In section 5, we provide the analysis of one data set. Generalizations are provided in section 6 and finally we conclude the paper in section 7.

2 Bivariate Generalized Exponential Distribution

In this section we introduce the bivariate generalized exponential distribution and discuss its different properties. First let us consider the following BE random variables (U_1, U_2), which has the following joint PDF for $\alpha > 0$;

$$f_{U_1,U_2}(u_1, u_2) = \frac{\alpha(\alpha+1)}{(1+u_1+u_2)^{\alpha+2}}; \quad u_1 > 0, u_2 > 0, \tag{3}$$

and 0 otherwise, see Mardia [19] also. Note that if (U_1, U_2) has the joint PDF (3), then the marginals, the joint CDF, the joint survival function (SF), the conditional distributions can be obtained in explicit forms.

It has been observed that as α increases, the correlation coefficient of U_1 and U_2 increases first, and then it decreases. The maximum correlation reaches 0.7 at $\alpha = 5.0$ and as α increases, the correlation decreases to 0. Now we define the bivariate absolute continuous generalized exponential random variables, using (U_1, U_2).

Definition: The bivariate random variables (X_1, X_2) is said to be bivariate generalized exponential random variables if, for $\alpha_1 > 0, \lambda_1 > 0, \alpha_2 > 0, \lambda_2 > 0$, (X_1, X_2) has the following relations;

$$U_i = \left(1 - e^{-\lambda_i X_i}\right)^{-\alpha_i} - 1, \quad i = 1, 2.$$

Here the bivariate random variables (U_1, U_2) has the PDF (3).

The joint PDF of (X_1, X_2) for $x_1 > 0, x_2 > 0$, can be easily seen by using transformation technique as

$$f_{X_1,X_2}(x_1,x_2) = \frac{ce^{-\lambda_1 x_1} e^{-\lambda_2 x_2} \left(1 - e^{-\lambda_1 x_1}\right)^{-\alpha_1 - 1} \left(1 - e^{-\lambda_2 x_2}\right)^{-\alpha_2 - 1}}{\left[(1 - e^{-\lambda_1 x_1})^{-\alpha_1} + (1 - e^{-\lambda_2 x_2})^{-\alpha_2} - 1\right]^{\alpha + 2}}, \tag{4}$$

here $c = \alpha(\alpha+1)\alpha_1\alpha_2\lambda_1\lambda_2$. From now on it will be called BVGE distribution, and it will be denoted by BVGE$(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \alpha)$. From the surface plot of the joint PDF of (X_1, X_2), not reported here, it has been observed that it can take different shapes and it is unimodal for different parameter values.

3 Properties

3.1 Joint, Marginal and Conditional PDFs

It may be noted that λ_1 and λ_2 are the scale parameters, and in establishing different properties of BVGE distribution we assume $\lambda_1 = \lambda_2 = 1$ without loss of generality and it will be denoted as BVGE$(\alpha_1, \alpha_2, \alpha)$. We have the following result.

Theorem 3.1 If $(X_1, X_2) \sim$ BVGE$(\alpha_1, \alpha_2, \alpha)$, then

(i) $X_1 \sim$ GE$(\alpha_1\alpha,\, 1)$ and $X_2 \sim$ GE$(\alpha_2\alpha,\, 1)$.

(ii) The joint CDF of (X_1, X_2) is

$$F_{X_1,X_2}(x_1,x_2) = P(X_1 \le x_1, X_2 \le x_2) = \left[(1 - e^{-x_1})^{-\alpha_1} + (1 - e^{-x_2})^{-\alpha_2} - 1\right]^{-\alpha}.$$

(iii) The joint survival function of (X_1, X_2) is

$$\begin{aligned} S_{X_1,X_2}(x_1,x_2) &= P(X_1 \ge x_1, X_2 \ge x_2) \\ &= 1 - (1 - e^{-x_1})^{\alpha\alpha_1} - (1 - e^{-x_2})^{\alpha\alpha_2} + \left[(1 - e^{-x_1})^{-\alpha_1} + (1 - e^{-x_2})^{-\alpha_2} - 1\right]^{-\alpha}. \end{aligned}$$

(iv) The conditional PDF of X_1 given $X_2 = x_2$, is given by

$$f_{X_1|X_2=x_2}(x_1) = g_1(x_1) g_2(x_1, x_2),$$

where $g_1(x)$ is the PDF of GE$(\alpha_1(1+\alpha), 1)$ and

$$g_2(x_1, x_2) = \frac{\left(1-e^{-x_2}\right)^{\alpha_2}}{\left[\left(1-e^{-x_1}\right)^{\alpha_1} + \left(1-e^{-x_2}\right)^{\alpha_2} - \left(1-e^{-x_1}\right)^{\alpha_1}\left(1-e^{-x_2}\right)^{\alpha_2}\right]^{\alpha+2}}.$$

(v) The conditional CDF of X_1 given $X_2 = x_2$ is

$$P(X_1 \le x_1 | X_2 = x_2) = \frac{\left(1-e^{-x_2}\right)^{-\alpha_2(\alpha+1)}}{\left[\left(1-e^{-x_1}\right)^{-\alpha_1} + \left(1-e^{-x_2}\right)^{-\alpha_2} - 1\right]^{\alpha+1}}.$$

PROOF: The proof of Theorem 3.1 can be obtained by using the transformation $u_i = \left(1-e^{-x_i}\right)^{-\alpha_i} - 1$ for $i = 1, 2$ and by routine calculations. ∎

It is interesting to observe from (iv) of Theorem 3.1 that the conditional distribution of X_1 given $X_2 = x_2$, is a weighted GE distribution. Here the weight function for any fixed value of x_2 is a decreasing function of x_1. Therefore, the shape of the conditional PDF of X_1 given $X_2 = x_2$, does not depend on the value of α_2 or x_2, it depends only on the values of α and α_1. It is immediate that if $\alpha_1(1+\alpha) \le 1$, the conditional PDF will be a decreasing function. Otherwise it can be either unimodal or a decreasing function. Different properties of this conditional distribution may be obtained from the general properties of a weighted distribution. Although, it is difficult to obtain different moments in explicit forms, but it may be observed that for fixed α_2, as α_1 increases, the conditional distribution function as provided in (v) of Theorem 3.1 is stochastically increasing.

Note that it is very simple to generate samples from a BVGE distribution. First we can generate (U_1, U_2) by using the conditional distribution function of U_2 given U_1, and then by using the transformation we can generate (X_1, X_2) from BVGE distribution. Finally before finishing this subsection we provide a specific interesting example of BVGE.

Consider, $\alpha\alpha_1 = \alpha\alpha_2 = 1$, *i.e.* $\alpha_1 = 1/\alpha$ and $\alpha_2 = 1/\alpha$, then the PDF of BVGE($1/\alpha$, $1/\alpha$, α) becomes

$$f_{X_1,X_2}(x_1,x_2) = \frac{\alpha+1}{\alpha} \times \frac{e^{-x_1}e^{-x_2}\left(1-e^{-x_1}\right)^{-\frac{1}{\alpha}-1}\left(1-e^{-x_2}\right)^{-\frac{1}{\alpha}-1}}{\left[\left(1-e^{-x_1}\right)^{-\frac{1}{\alpha}}+\left(1-e^{-x_2}\right)^{-\frac{1}{\alpha}}-1\right]^{\alpha+2}}. \tag{5}$$

From Theorem 3.1, it follows that both X_1 and X_2 are exponential random variables with mean 1. Therefore, the joint PDF of X_1 and X_2 as defined by (5) can be considered as bivariate exponential distribution whose marginals are exponential distributions.

3.2 Defining Through Copula

Note that BVGE distribution can be obtained using the copula function also. To every bivariate distribution function, F_{X_1,X_2} with continuous marginals F_{X_1} and F_{X_2}, corresponds a unique function $C : [0,1]^2 \to [0,1]$, called a copula such that

$$F_{X_1,X_2}(x_1,x_2) = C\left\{F_{X_1}(x_1), F_{X_2}(x_2)\right\}, \quad \text{for } (x_1,x_2) \in (-\infty,\infty)\times(-\infty,\infty). \tag{6}$$

Conversely, it is possible to construct a bivariate distribution function having the desired marginal distributions and a chosen description structure, *i.e.* copula. To see this let us consider the following copula

$$C_\theta(u_1,u_2) = \left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-\frac{1}{\theta}} \tag{7}$$

known as Clayton copula, see for example Nelsen [24], with $\theta = \dfrac{1}{\alpha}$, and the two marginals as $X_1 \sim \text{GE}(\alpha\alpha_1,\lambda_1)$ and $X_2 \sim \text{GE}(\alpha\alpha_2,\lambda_2)$ respectively. Therefore, the joint CDF of X_1 and X_2 becomes;

$$\begin{aligned} F_{X_1,X_2}(x_1,x_2) &= C\{(1-e^{-\lambda_1 x_1})^{\alpha\alpha_1}, (1-e^{-\lambda_2 x_2})^{\alpha\alpha_2}\} \\ &= \left[(1-e^{-x_1})^{-\alpha_1}+(1-e^{-x_2})^{-\alpha_2}-1\right]^{-\alpha}. \end{aligned} \tag{8}$$

As a consequence of this relationship (8), many properties of the BVGE distribution are inherited from the well known properties of the Clayton copula, and it will be explored later.

3.3 Hazard Function

In this subsection we provide the bivariate hazard rate of the bivariate generalized exponential distribution. Note that there are several ways of defining the bivariate hazard rates. Basu [2] first defined the bivariate hazard function of an absolute continuous bivariate distribution by simply extending the one-dimensional definition to two-dimension, *i.e.*

$$h_B(x_1, x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{S_{X_1,X_2}(x_1, x_2)}. \tag{9}$$

Unfortunately, the above definition of the hazard function does not uniquely define the joint probability density function. The joint bivariate hazard rate in the sense of Johnson and Kotz [12] is defined as follows;

$$h(x_1, x_2) = \left(-\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}\right) \ln S_{X_1,X_2}(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2)) \quad \text{(say)}. \tag{10}$$

It is well known that the bivariate hazard function $h(x_1, x_2)$ uniquely determines the joint PDF, see Marshall [20]. It may be noted that the hazard function of the generalized exponential distribution can be increasing, decreasing or constant according as the shape parameter greater than, less than or equal to one.

Theorem 3.2: If $(X_1, X_2) \sim \text{BVGE}(\alpha_1, \alpha_2, \alpha)$, then for fixed x_2 (x_1), $h_1(x_1, x_2)$ $(h_2(x_1, x_2))$ is a decreasing function of x_1 (x_2).

Proof: We will show the result for $h_1(x_1, x_2)$, for $h_2(x_1, x_2)$ it will follow along the same line. Let us use the following variables and notation;

$$v = \left(1 - e^{-x_1}\right)^{-\alpha_1}, \quad c_1 = 1 - \left(1 - e^{-x_2}\right)^{\alpha\alpha_2}, \quad c_2 = \left(1 - e^{-x_2}\right)^{-\alpha_2} - 1.$$

Therefore,

$$h_1(x_1, x_2) = -\frac{\partial}{\partial x_1} \ln S_{X_1,X_2}(x_1, x_2) = -\left\{\frac{\partial}{\partial v} \ln\left[c_1 - v^{-\alpha} + \frac{1}{(v + c_2)^{\alpha}}\right]\right\} \times \frac{\partial v}{\partial x_1}. \tag{11}$$

To prove that for fixed x_2, $h_1(x_1, x_2)$ is a decreasing function of x_1, it is enough to prove that for fixed x_2, $g(v)$ and $\frac{\partial v}{\partial x_1}$ both are increasing functions of x_1, where

$$g(v) = \frac{\partial}{\partial v} \ln \left[c_1 - v^{-\alpha} + \frac{1}{(v + c_2)^\alpha} \right].$$

Note that after simplification $g(v)$ can be written as

$$g(v) = \frac{\left(1 + \frac{c_2}{v}\right)^{\alpha+1} - 1}{(v + c_2)\left[c_1(v + c_2)^\alpha - \left(1 + \frac{c_2}{v}\right)^\alpha + 1\right]}.$$

Since the numerator is a decreasing function and the denominator is an increasing function of v, $g(v)$ is a decreasing function of v, *i.e.*, $\frac{\partial g(v)}{\partial v} < 0$. Now

$$\begin{aligned}
\frac{\partial v}{\partial x_1} &= -\alpha_1 \left(1 - e^{-x_1}\right)^{-\alpha_1 - 1} e^{-x_1} < 0, \\
\frac{\partial^2 v}{\partial x_1^2} &= \alpha_1 \left(1 - e^{-x_1}\right)^{-\alpha_1 - 1} e^{-x_1} + \alpha_1(\alpha_1 + 1) \left(1 - e^{-x_1}\right)^{-\alpha_1 - 2} e^{-2x_1} > 0.
\end{aligned}$$

As

$$\frac{\partial g(v)}{\partial x_1} = \frac{\partial g(v)}{\partial v} \times \frac{\partial v}{\partial x_1} > 0, \quad \text{and} \quad \frac{\partial^2 v}{\partial x_1^2} > 0,$$

the result follows. ∎

3.4 Dependency Properties

In this subsection we provide different dependency properties of the bivariate generalized exponential distribution. Let us recall the following definition, see Karlin [15]. A real valued function $K(x, y)$ of two variables ranging over linearly ordered sets A and B, respectively, is said to be *total positivity of order* r (abbreviated by TP_r) if for $1 \leq m \leq r$, and for all $a_1, \cdots, a_m \in A$, $b_1, \cdots, b_m \in B$, such that

$$a_1 < \cdots < a_m, \quad b_1 < \cdots < b_m,$$

then

$$\begin{vmatrix} K(a_1, b_1) & \cdots & K(a_1, b_m) \\ \vdots & \ddots & \vdots \\ K(a_m, b_1) & \cdots & K(a_m, b_m) \end{vmatrix} \geq 0.$$

Here $|\cdot|$ denotes the determinant. Now we have the following dependency result.

THEOREM 3.3: If $(X_1, X_2) \sim \text{BVGE}(\alpha_1, \alpha_2, \alpha)$, and $\alpha_1 = \alpha_2$, then the joint PDF of (X_1, X_2) has TP_2 property.

PROOF: Note that (X_1, X_2) has TP_2 property, if and only if for any $x_{11}, x_{12}, x_{21}, x_{22}$, whenever $0 < x_{11} < x_{12}$, and $0 < x_{21} < x_{22}$, we have

$$f_{X_1,X_2}(x_{11}, x_{21}) f_{X_1,X_2}(x_{12}, x_{22}) \geq f_{X_1,X_2}(x_{12}, x_{21}) f_{X_1,X_2}(x_{11}, x_{22}). \tag{12}$$

If $\alpha_1 = \alpha_2 = \beta$, then after some simplification, it can be easily seen that (12) is equivalent to

$$(u_{22} - u_{21})(u_{12} - u_{11}) \geq 0, \tag{13}$$

where $\left(1 - e^{-x_{ij}}\right)^{-\beta} = u_{ij}$, for $i, j = 1, 2$. Therefore, the result follows. ∎

Since TP_2 is the most stringent dependence property, many other dependency properties follow immediately. For example X_1 and X_2 are positive quadrant dependent, X_1 (X_2) is a positive regression dependent of X_2 (X_1), and X_1 (X_2) is a left tail decreasing in X_2 (X_1).

3.5 Dependency Measures

In this subsection we explicitly compute different measures of dependency, namely Kendall's τ, and the medial correlation. Interestingly, both the measures can be obtained from the Clayton copula. We further provide some dependency measures of extreme events also.

The Kendall's τ defined as the probability of concordance minus the probability of discordance of two pairs (X_1, X_2) and (Y_1, Y_2) of random vectors, having the same joint distribution function, is

$$\tau = P[(X_1 - Y_1)(X_2 - Y_2) > 0] - P[(X_1 - Y_1)(X_2 - Y_2) < 0]. \tag{14}$$

In case of BVGE distribution, we have the following result.

THEOREM 3.4: Let $(X_1, X_2) \sim$ BVGE$(\alpha_1, \alpha_2, \alpha)$, then the Kendall's τ index is given by $\dfrac{\theta}{2+\theta}$, when $\theta = \dfrac{1}{\alpha}$.

PROOF: It is well known that the Kendall's τ index is a copula property, see Domma [6], and

$$\tau = 4\int_0^1 \int_0^1 C(u,v)\frac{\partial^2 C(u,v)}{\partial u \partial v} dudv - 1.$$

Since,

$$\frac{\partial^2 C(u,v)}{\partial u \partial v} = (1+\theta)\frac{u^{-(\theta+1)}v^{-(\theta+1)}}{(u^{-\theta}+v^{-\theta}-1)^{\frac{1}{\theta}+2}},$$

$$\begin{aligned}
\int_0^1 \int_0^1 C(u,v)\frac{\partial^2 C(u,v)}{\partial u \partial v} dudv &= (1+\theta)\int_0^1 \int_0^1 \frac{u^{-(\theta+1)}v^{-(\theta+1)}}{(u^{-\theta}+v^{-\theta}-1)^{\frac{2}{\theta}+2}} dudv \\
&= \frac{(1+\theta)}{\theta^2}\int_1^\infty \int_1^\infty \frac{1}{(x+y-1)^{\frac{2}{\theta}+2}} dxdy \\
&= \frac{(1+\theta)}{\theta^2}\int_0^\infty \int_0^\infty \frac{1}{(x+y+1)^{\frac{2}{\theta}+2}} dxdy \\
&= \frac{(1+\theta)}{\theta^2} \times \frac{\theta^2}{2(2+\theta)} = \frac{(1+\theta)}{2(2+\theta)}.
\end{aligned}$$

Therefore, $\tau = \dfrac{2(1+\theta)}{(2+\theta)} - 1 = \dfrac{\theta}{2+\theta}$. ∎

The population version of the medial correlation coefficient for a pair (X_1, X_2) of continuous random variables was defined by Blomqvist [4]. If M_{X_1} and M_{X_2} denote the medians of X_1 and X_2, respectively, then $M_{X_1X_2}$, the medial correlation of X_1 and X_2 is

$$M_{X_1X_2} = P[(X_1 - M_{X_1})(X_2 - M_{X_2}) > 0] - P[(X_1 - M_{X_1})(X_2 - M_{X_2}) < 0].$$

It has been shown by Nelsen [24] that the median correlation coefficient is also a copula property, and $M_{X_1X_2} = 4C\left(\frac{1}{2}, \frac{1}{2}\right)$. Therefore, for BVGE distribution the medial correlation coefficient between X_1 and X_2 is $4\left(2^{\theta+1}-1\right)^{-\frac{1}{\theta}}$.

The concept of bivariate tail dependence relates to the amount of dependence in the upper quadrant (or lower quadrant) tail of a bivariate distribution, see Joe [11] (page 33).

In terms of the original random variables X_1 and X_2, the upper tail dependence is defined as

$$\chi = \lim_{z \to 1} P(X_2 \geq F_{X_2}^{-1}(z) | X_1 \geq F_{X_1}^{-1}(z)).$$

Intuitively, the upper tail dependence exists, when there is a positive probability that some positive outliers may occur jointly. If $\chi \in (0, 1]$, then X_1 and X_2 are said to be asymptotically dependent, and if $\chi = 0$, they are asymptotically independent. Coles *et al.* [5] showed using the copula function that

$$\chi = \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{1 - u} = \lim_{u \to 1} \left\{ 2 - \frac{\log C(u, u)}{\log u} \right\}.$$

In case of BVGE distribution, it can be shown that $\chi = 0$, *i.e.* X_1 and X_2 are asymptotically independent.

4 Maximum Likelihood Estimation

In this section we describe how to obtain the maximum likelihood estimators of the unknown parameters based on a random sample of size n from BVGE$(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \alpha)$. Based on the sample $\{(x_{11}, x_{12}), \cdots, (x_{n1}, x_{n2})\}$, the log-likelihood function becomes;

$$\begin{aligned} l(\theta) \quad = \quad & -\lambda_1 \sum_{i=1}^{n} x_{i1} - \lambda_2 \sum_{i=1}^{n} x_{i2} - (\alpha_1 + 1) \sum_{i=1}^{n} \ln(1 - e^{-\lambda_1 x_{i1}}) - (\alpha_2 + 1) \sum_{i=1}^{n} \ln(1 - e^{-\lambda_2 x_{i2}}) \\ & -(\alpha + 2) \sum_{i=1}^{n} \ln \left[(1 - e^{-\lambda_1 x_{i1}})^{-\alpha_1} + (1 - e^{-\lambda_2 x_{i2}})^{-\alpha_2} - 1 \right] + n \ln c, \qquad (15) \end{aligned}$$

here $\theta = (\alpha_1, \alpha_2, \lambda_1, \lambda_2, \alpha)$ and $c = \alpha(\alpha + 1)\alpha_1\alpha_2\lambda_1\lambda_2$. The maximum likelihood estimates can be obtained by maximizing (15) with respect to the unknown parameters. As expected, they cannot be obtained in explicit forms. One needs to solve five non-linear equations to compute the MLEs, see the Appendix for details. Note that the Newton-Raphson method or other optimization routine may be used to maximize (15). But to use any optimization routine we need to provide the initial guesses of the parameters and those initial guesses play important roles in any higher dimensional optimization process. Since it is a regular family, the usual asymptotic normality result holds in this case, *i.e.*

$$\sqrt{n}(\widehat{\theta} - \theta) \longrightarrow N_5(0, I^{-1}), \tag{16}$$

here I is the expected Fisher information matrix. Note that the Fisher information matrix cannot be obtained in explicit forms, most of the elements can be obtained only in terms of double integration. We have provided, see in the Appendix, the observed Fisher information matrix, which can be used to compute the asymptotic confidence intervals of the unknown parameters.

Now we discuss how to obtain the initial guesses based on the observed sample. We make the following re-parameterization: $\beta_1 = \alpha\alpha_1$ and $\beta_2 = \alpha\alpha_2$. Since $X_1 \sim \text{GE}(\beta_1, \lambda_1)$ and $X_2 \sim \text{GE}(\beta_2, \lambda_2)$, we fit the generalized exponential distribution to the marginals. We calculate the MLEs of (β_1, λ_1) and (β_2, λ_2) based on the respective marginals. If the MLEs of β_1, λ_1, β_2, λ_2 based on the marginals are denoted by $\widetilde{\beta}_1$, $\widetilde{\lambda}_1$, $\widetilde{\beta}_2$ $\widetilde{\lambda}_2$ respectively, then

$$\widetilde{\beta}_j(\lambda_j) = -\frac{n}{\sum_{i=1}^n \ln(1 - e^{-\lambda_j x_{ij}})}; \quad j = 1, 2,$$

and $\widetilde{\lambda}_j$ can be obtained as a fixed point solution of the following equation

$$h_j(\lambda) = \lambda, \tag{17}$$

where

$$h_j(\lambda) = \left[\frac{\sum_{i=1}^n \frac{x_{ij}e^{-\lambda x_{ij}}}{(1-e^{-\lambda x_{ij}})}}{\sum_{i=1}^n \ln(1 - e^{-\lambda x_{ij}})} + \frac{1}{n}\sum_{i=1}^n \frac{x_{ij}e^{-\lambda x_{ij}}}{(1 - e^{-\lambda x_{ij}})} + \frac{1}{n}\sum_{i=1}^n x_{ij}\right]^{-1},$$

for $j = 1, 2$. Note that (17) is obtained by taking the derivative with respect to λ, of the profile log-likelihood function of the GE distribution. Suppose the initial guess of $\widetilde{\lambda}_j$ is $\lambda_j^{(0)}$, then consider $\lambda_j^{(1)} = h(\lambda_j^{(0)})$, and continue the process until convergence takes place. Once we obtain $\widetilde{\lambda}_j$, then $\widetilde{\beta}_j$ can be obtained as $\widetilde{\beta}_j(\widetilde{\lambda}_j)$. Using $\widetilde{\lambda}_j$ and $\widetilde{\beta}_j$ for $j = 1, 2$, we can plot the profile log-likelihood function of α from (15) and that will provide an initial guess value of α. The details will be illustrated in the data analysis section.

n	α_1	λ_1	α_2	λ_2	α
20	1.1148 (0.0481)	1.0771 (0.0298)	1.1012 (0.0442)	1.0607 (0.0260)	2.0651 (0.0738)
40	1.0592 (0.0273)	1.0319 (0.0147)	1.0603 (0.0253)	1.0397 (0.0135)	2.0461 (0.0476)
60	1.0523 (0.0215)	1.0275 (0.0104)	1.0513 (0.0207)	1.0266 (0.0105)	2.0272 (0.0439)
80	1.0513 (0.0197)	1.0227 (0.0087)	1.0554 (0.0198)	1.0243 (0.0078)	2.0090 (0.0428)

Table 1: The average MLEs and the associated square root of the mean squared errors (within brackets below) are reported for Set 1.

5 Simulation Results and Data Analysis

5.1 Simulation Results

In this subsection we present some simulation results to see how the maximum likelihood estimators behave for different sample sizes and for different parameter values. We have used different sample sizes namely $n = 20$, 40, 60 and 80 and two different sets of parameter values: Set 1: $\alpha_1 = \lambda_1 = \alpha_2 = \lambda_2 = 1$, $\alpha = 2$, and Set 2: $\alpha_1 = \lambda_1 = \alpha_2 = \lambda_2 = 1$, $\alpha = 0.25$. In each case we have computed the maximum likelihood estimators of the unknown parameters by maximizing the log-likelihood function (15). We compute the average estimates and mean squared errors over 1000 replications and the results are reported in Tables 1 and 2.

Some of the points are quite clear from Tables 1 and 2. In all the cases the performances of the maximum likelihood estimates are quite satisfactory. It is observed that as sample size increases the average estimates and the mean squared error decrease for all the parameters, as expected. It also verifies the consistency properties of the MLEs. Moreover, the average biases and MSEs of MLEs of α_i and λ_i for $i = 1, 2$, do not depend on α.

n	α_1	λ_1	α_2	λ_2	α
20	1.0263 (0.0137)	1.0260 (0.0176)	1.0477 (0.0140)	1.0501 (0.0174)	0.2746 (0.0027)
40	1.0312 (0.0119)	1.0267 (0.0157)	1.0234 (0.0110)	1.0410 (0.0151)	0.2710 (0.0021)
60	1.0264 (0.0108)	1.0489 (0.0131)	1.0169 (0.0098)	1.0300 (0.0146)	0.2616 (0.0014)
80	1.0097 (0.0089)	1.0360 (0.0132)	1.0259 (0.0089)	1.0417 (0.0121)	0.2561 (0.0010)

Table 2: The average MLEs and the associated square root of the mean squared errors (within brackets below) are reported for Set 4.

Marginals	Minimum	Maximum	Median	1-st Quartile	3-rd Quartile
X_1	10.79	12.89	11.60	11.22	11.95
X_2	142.72	306.00	164.65	152.37	182.20

Table 3: The basic statistics of X_1 and X_2.

5.2 Data Analysis

In this section for illustrative purposes we have presented the analysis of one bivariate data set. The data set represents the national track records of the 55 different countries for the 1984 Los Angeles Olympics for women. Here X_1 and X_2 represent the track records for 100m flat race and marathon respectively. The data set is originally available in IAAF/ Track and Field Statistics Handbook for the 1984 Los Angeles Olympics, see also Johnson and Wichern [13]. We provide some basic statistics of X_1 and X_2 in Table 3. From the histogram plots and from the scatter plot, not reported here, of X_1 and X_2, it has been observed that both X_1 and X_2 are right skewed and X_1 and X_2 are positively correlated. The sample correlation coefficient between X_1 and X_2 is 0.646. Since both the marginals are positively skewed, and sample correlation coefficient is also within the possible range of the proposed BVGE, we

Model	X_1			X_1		
	MLL	KSD	p	MLL	KSD	p
GE	-33.7369	0.1217	0.3900	-147.1649	0.1155	0.4559
Weibull	-34.1514	0.1124	0.4123	-153.2186	0.1455	0.1948
Log-Normal	-34.1498	0.1121	0.4157	-155.9186	0.1761	0.0659

Table 4: The maximized log-likelihood values, the Kolmogorov-Smirnov distances, and the associated p values for different distribution functions and for two marginals are provided.

use the BVGE distribution to model this data set. Before analyzing the data set we have subtracted 10 and 125 from X_1 and X_2 respectively.

Before progressing further we have fitted the generalized exponential distribution to the marginals and obtained $\widetilde{\beta}_1 = 33.1416$, $\widetilde{\lambda}_1 = 2.5116$, $\widetilde{\beta}_2 = 6.1719$, $\widetilde{\lambda}_2 = 0.0523$. We have fitted different other univariate distributions namely Weibull and log-normal distributions to the marginals. The maximized log-likelihood (MLL) values, the Kolmogorov-Smirnov distances (KSD) and the associated p values for the different distributions and for two marginals are provided in Table 4. In case of GE model, we have also provided the plot of $g(\widehat{F}(x))$ against x in Figure 1. This plot can be used as a goodness of fit for the GE distribution, see Kannan *et al.* [14] It is clear from all these, that GE distribution can be used to fit the marginals reasonably well.

It may be mentioned that although several goodness of fit tests are available for an arbitrary univariate distribution function, but for a general bivariate distribution functions we do not have a satisfactory goodness of fit test. Because of this reason we have tested the marginals only. At least it gives us an indication which bivariate distribution function can be used. Using the above initial guess values of β_1, β_2, λ_1 and λ_2, we plot the approximate profile log-likelihood of α from (15) in the Figure 2. It is clear that the profile log-likelihood function is an unimodal function. From the profile log-likelihood function, we obtain an initial guess value of α as 0.80. Using the above initial estimates, we obtain the MLEs as $\widehat{\alpha}_1 = 41.4916, \widehat{\lambda}_1 = 2.5358, \widehat{\alpha}_2 = 7.7552, \widehat{\lambda}_2 = 0.0526, \widehat{\alpha} = 0.8081$, and the maximized

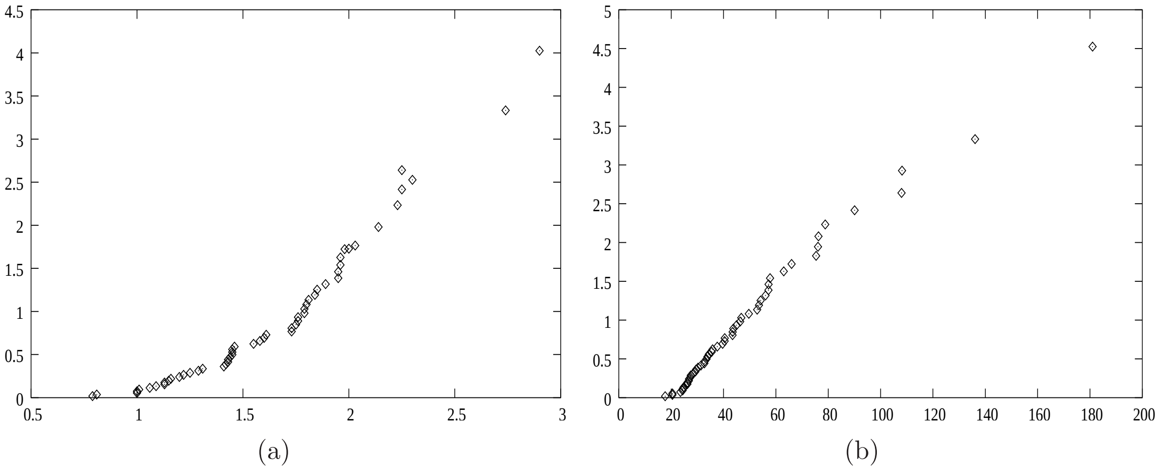

Figure 1: The plot of $g(\widehat{F}(x))$ against x of (a) X_1 (b) X_2

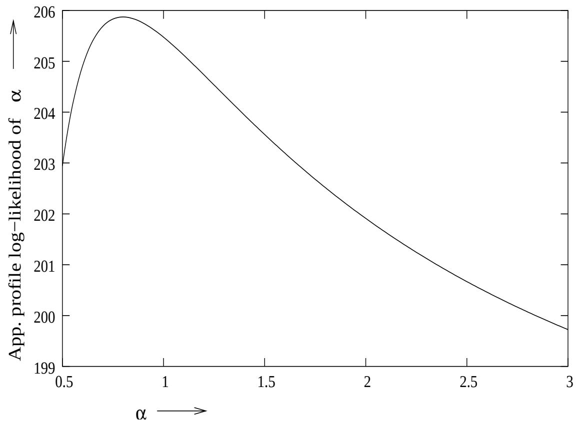

Figure 2: The approximate profile log-likelihood function of α,

log-likelihood value as -268.1177. The 95% confidence intervals of α_1, λ_1, α_2, λ_1 and α are (38.3275, 44.6557), (1.5171, 3.5545), (5.9861, 9.5153), (0.0315, 0.0737), (0.4409, 1.1753) respectively.

Now for comparison purposes, we have fitted bivariate Weibull and bivariate log-normal distributions arising from the same copula, *i.e.* using the marginals as Weibull and log-normal distributions respectively in (6). The corresponding maximized log-likelihood values for bivariate Weibull and bivariate log-normal distributions are -277.8718 and -282.1134 respectively. Although, it may not be always true, but at least in this case it is observed that the proposed BVGE provides a better fit to the given data set, than bivariate Weibull and bivariate log-normal distributions, in terms of the log-likelihood function.

6 Generalization: Bivariate Proportional Reversed Hazard Model

Although, so far we have introduced the bivariate absolute continuous generalized exponential distribution and discussed its several properties, but it is observed that our method can be easily used for a much larger class of distribution functions, namely the Lehmann alternatives class or proportional reversed hazard class. A family of distribution functions, say $\mathcal{F}$, is called a Lehmann alternative class or proportional reversed hazard class if the elements of $\mathcal{F}$ can be expressed as follows:

$$\mathcal{F} = \{F : F(x;\theta) = (F_0(x))^{\theta}, \quad \theta > 0\}, \tag{18}$$

here $F_0(\cdot)$ is a distribution function and it is known as the baseline distribution function. Note that for any $\theta > 0$, $F(x;\theta)$ is a proper distribution function and the support of $F(x;\theta)$ is same as the support of $F_0(x)$.

Lehmann [17] first introduced this class of distribution functions in a testing of hypothesis problem, see for example Gupta, Gupta and Gupta [8]. Recently, proportional reversed hazard model has received considerable attention in the statistical literature. Several proportional reversed hazard models, with different $F_0(\cdot)$ have been introduced and their statistical properties have been studied quite extensively. Those include; the exponentiated Weibull model by Mudholkar *et al.* ([22]), exponentiated Rayleigh by Surles and Padgett ([27]), generalized or exponentiated exponential by Gupta and Kundu [9], exponentiated Pareto by Shawky and Abu-Zinadah [26] and exponentiated gamma by Gupta, Gupta and Gupta [8] were introduced and studied quite extensively by different authors.

Now for any given absolute continuous baseline distribution function $F_0(\cdot)$, with the PDF

$f_0(\cdot)$, we introduce the bivariate proportional reversed hazard model as follows. A bivariate random variable (X_1, X_2) is said to have a bivariate proportional reversed hazard model if the joint PDF of (X_1, X_2) is

$$f_{X_1,X_2}(x_1, x_2) = \frac{c_1(F_0(x_1))^{-\alpha_1-1}(F_0(x_2))^{-\alpha_2-1}f_0(x_1)f_0(x_2)}{[(F_0(x_1))^{-\alpha_1} + (F_0(x_2))^{-\alpha_2} - 1]^{\alpha+2}}, \tag{19}$$

here $\alpha_1 > 0, \alpha_2 > 0, \alpha > 0$ as before and $c_1 = \alpha_1\alpha_2\alpha(\alpha+1)$. Note that (19) is obtained from the joint PDF of (U_1, U_2), (3), by using the transformation

$$U_i = (F_0(X_i))^{-\alpha_i} - 1; \quad i = 1, 2. \tag{20}$$

Now we immediately have the following results which can be easily established.

Theorem 6.1: If (X_1, X_2) has the joint PDF (19), then

(i) X_1 and X_2 have the distribution functions $(F_0(\cdot))^{\alpha\alpha_1}$ and $(F_0(\cdot))^{\alpha\alpha_2}$ respectively.

(ii) The joint CDF of (X_1, X_2) is

$$F_{X_1,X_2}(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2) = \left[(F_0(x_1))^{-\alpha_1} + (F_0(x_2))^{-\alpha_2} - 1\right]^{-\alpha}.$$

(iii) The joint survival function of (X_1, X_2) is

$$\begin{aligned} S_{X_1,X_2}(x_1, x_2) &= P(X_1 \ge x_1, X_2 \ge x_2) \\ &= 1 - (F_0(x_1))^{\alpha\alpha_1} - (F_0(x_2))^{\alpha\alpha_2} + \left[(F_0(x_1))^{-\alpha_1} + (F_0(x_2))^{-\alpha_2} - 1\right]^{-\alpha}. \end{aligned}$$

The conditional PDF and CDF can be obtained very easily. Moreover the generation from a bivariate proportional reversed hazard model is also straight forward. It can be obtained by first generating (U_1, U_2) and then using the inverse transformation of (20), (X_1, X_2) can be obtained. In the general case also it can be easily shown along the same line as Theorem 4.2, that if (X_1, X_2) has the joint PDF (19) and $\alpha_1 = \alpha_2$, then (X_1, X_2) has TP_2 property. If $f_0(\cdot)$ is a decreasing function, then both the components of the joint bivariate hazard function in the sense of Johnson and Kotz [12] are decreasing functions. Moreover, X_1 and

X_2 are positive quadrant dependent, X_1 (X_2) is a regression dependent of X_2 (X_1) and X_1 (X_2) is a left tail decreasing in X_2 (X_1).

7 Conclusions

In this paper we have introduced an absolute continuous bivariate generalized exponential distribution which has generalized exponential marginals. It has been obtained from a BE distribution through a proper transformation. It should be mentioned that this may not be the only way to generate bivariate distribution with a given marginals, see for example Bandyopadhyay and Basu [1], Sankaran and Nair [25], Nayak [23] and the references cited therein.

The proposed bivariate generalized exponential distribution has explicit joint PDF and the joint CDF. Several properties of this distribution have been established. It is further observed that using proper transformation a class of absolute continuous bivariate distributions can be obtained and they have natural extension to the multivariate case also.

Note that we have defined the bivariate proportional reversed hazard model using the transformation

$$U_i = (F_0(X_i))^{-\alpha_i} - 1, \quad i = 1, 2.$$

Similar development is also possible using the transformation

$$U_i = (S_0(X_i))^{-\alpha_i} - 1, \quad i = 1, 2, \tag{21}$$

where $S_0(X_i)$ is any survival function. In this case also the joint PDF and and joint CDF of (X_1, X_2) will be in closed form. Along the same line several other properties also can be established. The work is in progress, it will be reported later.

Acknowledgements

Part of the work of the first author has been supported a grant from the Department of Science an Technology, Government of India. Part of the work of the second author has been supported by a discovery grant from NSERC, CANADA. The authors are thankful to the associate editor and to two unknown reviewers, whose comments had helped to improve the paper significantly.

Appendix: Normal Equations and Observed Fisher Information Matrix

Normal Equations:

We will present the normal equations. We use the following notations;

$$A_{i1} = (1 - e^{-\lambda_1 x_{i1}}), \quad A'_{i1} = x_{i1} e^{-\lambda_1 x_{i1}}, \quad A''_{i1} = -x_{i1}^2 e^{-\lambda_1 x_{i1}},$$

$$A_{i2} = (1 - e^{-\lambda_2 x_{i2}}), \quad A'_{i2} = x_{i2} e^{-\lambda_2 x_{i2}}, \quad A''_{i2} = -x_{i2}^2 e^{-\lambda_2 x_{i2}}.$$

$$\frac{\partial l}{\partial \lambda_1} = -\sum_{i=1}^{n} x_{i1} - (\alpha_1 + 1) \sum_{i=1}^{n} \frac{A'_{i1}}{A_{i1}} + (\alpha + 2) \sum_{i=1}^{n} \frac{\alpha_1 A_{i1}^{-\alpha_1 - 1} A'_{i1}}{A_{i1}^{-\alpha_1} + A_{i2}^{-\alpha_2} - 1} + \frac{n}{\lambda_1} = 0$$

$$\frac{\partial l}{\partial \lambda_2} = -\sum_{i=1}^{n} x_{i2} - (\alpha_2 + 1) \sum_{i=1}^{n} \frac{A'_{i2}}{A_{i2}} + (\alpha + 2) \sum_{i=1}^{n} \frac{\alpha_2 A_{i2}^{-\alpha_2 - 1} A'_{i2}}{A_{i1}^{-\alpha_1} + A_{i2}^{-\alpha_2} - 1} + \frac{n}{\lambda_2} = 0$$

$$\frac{\partial l}{\partial \alpha_1} = -\sum_{i=1}^{n} \ln A_{i1} + (\alpha + 2) \sum_{i=1}^{n} \frac{A_{i1}^{-\alpha_1} \ln A_{i1}}{A_{i1}^{-\alpha_1} + A_{i2}^{-\alpha_2} - 1} + \frac{n}{\alpha_1} = 0$$

$$\frac{\partial l}{\partial \alpha_2} = -\sum_{i=1}^{n} \ln A_{i2} + (\alpha + 2) \sum_{i=1}^{n} \frac{A_{i2}^{-\alpha_2} \ln A_{i2}}{A_{i1}^{-\alpha_1} + A_{i2}^{-\alpha_2} - 1} + \frac{n}{\alpha_2} = 0$$

$$\frac{\partial l}{\partial \alpha} = -\sum_{i=1}^{n} \ln\left(A_{i1}^{-\alpha_1} + A_{i2}^{-\alpha_2} - 1\right) + \frac{n}{\alpha} + \frac{n}{\alpha + 1} = 0$$

Observed Fisher Information Matrix

In this subsection we present the elements of the observed Fisher information matrix.

$$\frac{\partial^2 l}{\partial \lambda_1^2} = -\frac{n}{\lambda_1^2} + (\alpha_1 + 1) \sum_{i=1}^{n} \frac{x_{i1} A'_{i1}}{A_{i1}^2} - (\alpha + 2) \sum_{i=1}^{n} \frac{\alpha_1 x_{i1} A'_{i1} A_{i1}^{-\alpha_1 - 1} C_{i1}}{(A_{i1}^{-\alpha_1} + A_{i2}^{-\alpha_2} - 1)^2}$$

$$
\begin{aligned}
\frac{\partial^2 l}{\partial \lambda_2^2} &= -\frac{n}{\lambda_2^2} + (\alpha_2+1)\sum_{i=1}^{n}\frac{x_{i2}A'_{i2}}{A_{i2}^2} - (\alpha+2)\sum_{i=1}^{n}\frac{\alpha_2 x_{i2}A'_{i2}A_{i2}^{-\alpha_2-1}C_{i2}}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)^2}\\
\frac{\partial^2 l}{\partial \alpha_1^2} &= -\frac{n}{\alpha_1^2} - (\alpha+2)\sum_{i=1}^{n}\frac{A_{i1}^{-\alpha_1}(\ln A_{i1})^2(1-A_{i2}^{-\alpha_2})}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)^2}\\
\frac{\partial^2 l}{\partial \alpha_2^2} &= -\frac{n}{\alpha_2^2} - (\alpha+2)\sum_{i=1}^{n}\frac{A_{i2}^{-\alpha_2}(\ln A_{i2})^2(1-A_{i1}^{-\alpha_1})}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)^2}\\
\frac{\partial^2 l}{\partial \alpha^2} &= -\frac{n}{\alpha^2} - \frac{n}{(\alpha+1)^2}\\
\frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_2} &= -(\alpha+2)\sum_{i=1}^{n}\frac{\alpha_1\alpha_2 A_{i1}^{-\alpha_1-1}A_{i2}^{-\alpha_2-1}A'_{i1}A'_{i2}}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)^2}\\
\frac{\partial^2 l}{\partial \lambda_1 \partial \alpha_1} &= -\sum_{i=1}^{n}\frac{A'_{i1}}{A_{i1}} + (\alpha+2)\sum_{i=1}^{n}\frac{D_{i1}}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)^2}\\
\frac{\partial^2 l}{\partial \lambda_1 \partial \alpha_2} &= -(\alpha+2)\sum_{i=1}^{n}\frac{\alpha_1 A_{i1}^{-\alpha_1-1}A'_{i1}A_{i2}^{-\alpha_2}\ln A_{i2}}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)^2}\\
\frac{\partial^2 l}{\partial \lambda_1 \partial \alpha} &= \sum_{i=1}^{n}\frac{\alpha_1 A_{i1}^{-\alpha_1-1}A'_{i1}}{A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1}\\
\frac{\partial^2 l}{\partial \lambda_2 \partial \alpha_1} &= -(\alpha+2)\sum_{i=1}^{n}\frac{\alpha_2 A_{i2}^{-\alpha_2-1}A'_{i2}A_{i1}^{-\alpha_1}\ln A_{i1}}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)^2}\\
\frac{\partial^2 l}{\partial \lambda_2 \partial \alpha_2} &= -\sum_{i=1}^{n}\frac{A'_{i2}}{A_{i2}} + (\alpha+2)\sum_{i=1}^{n}\frac{D_{i2}}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)^2}\\
\frac{\partial^2 l}{\partial \lambda_2 \partial \alpha} &= \sum_{i=1}^{n}\frac{\alpha_2 A_{i2}^{-\alpha_2-1}A'_{i2}}{A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1}\\
\frac{\partial^2 l}{\partial \alpha_1 \partial \alpha_2} &= -(\alpha+2)\sum_{i=1}^{n}\frac{(A_{i1}^{-\alpha_1}\ln A_{i1})(A_{i2}^{-\alpha_2}\ln A_{i2})}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)^2}\\
\frac{\partial^2 l}{\partial \alpha_1 \partial \alpha} &= \sum_{i=1}^{n}\frac{A_{i1}^{-\alpha_1}\ln A_{i1}}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)}\\
\frac{\partial^2 l}{\partial \alpha_2 \partial \alpha} &= \sum_{i=1}^{n}\frac{A_{i2}^{-\alpha_2}\ln A_{i2}}{(A_{i1}^{-\alpha_1}+A_{i2}^{-\alpha_2}-1)}
\end{aligned}
$$

where

$$
\begin{aligned}
C_{i1} &= (1-A_{i2}^{-\alpha_2})(1+(\alpha_1+1)A_{i1}^{-1}e^{-\lambda_1 x_{i1}}) - A_{i1}^{-\alpha_1}(1+e^{-\lambda_1 x_{i1}}A_{i1}^{-1})\\
C_{i2} &= (1-A_{i1}^{-\alpha_1})(1+(\alpha_2+1)A_{i2}^{-1}e^{-\lambda_2 x_{i2}}) - A_{i2}^{-\alpha_2}(1+e^{-\lambda_2 x_{i2}}A_{i2}^{-1})\\
D_{i1} &= A_{i1}^{-\alpha_1-1}\{(A_{i2}^{-\alpha_2}-1)(1-\alpha_1\ln A_{i1}) + A_{i1}^{-\alpha_1}\}
\end{aligned}
$$

$$D_{i2} = A_{i2}^{-\alpha_2-1}\{(A_{i1}^{-\alpha_1}-1)(1-\alpha_2 \ln A_{i2}) + A_{i2}^{-\alpha_2}\}$$

References

[1] Bandyopadhyay, D. and Basu, A.P. (199), "On a generalization of a model by Lindley and Singpurwalla", *Advances in Applied Probability*, vol. 22, 498-500.

[2] Basu, A.P. (1971), "Bivariate failure rate", *Journal of the American Statistical Association*, vol. 66, 103 - 104.

[3] Block, H., Basu, A. P. (1974), "A continuous bivariate exponential extension", *Journal of the American Statistical Association*, vol. 69, 1031 - 1037.

[4] Blomqvist, N. (1950), "On a measure of dependence between two random variables", *Annals of Mathematical Statistics*, vol. 21, 593 - 600.

[5] Coles, S., Hefferman, J. and Tawn, J. (1999), "Dependence measures for extreme value analysis", *Extremes*, vol 2, 339 - 365.

[6] Domma, F. (2009), "Some properties of the bivariate Burr type III distribution", *Statistics*, to appear, DOI:10.1080/02331880902986547.

[7] Gupta, R.C. and Gupta, R.D. (2007), "Proportional reversed hazard model and its applications", *Journal of Statistical Planning and Inference*, vol. 137, 3525 - 3536.

[8] Gupta, R. C., Gupta, P. L. and Gupta, R. D. (1998), "Modeling failure time data by Lehmann alternatives", *Communications in Statistics - Theory and Methods*, vol. 27, 887 - 904.

[9] Gupta, R. D. and Kundu, D. (1999), "Generalized exponential distribution", *Australian New Zealand Journal of Statistics*, vol. 41, 173-188.

[10] Gupta, R. D. and Kundu, D. (2007), "Generalized exponential distribution; existing methods and some recent developments", *Journal of the Statistical Planning and Inference*, vol. 137, 3537 - 3547.

[11] Joe, H. (1997) *Multivariate model and dependence concept*, Chapman and Hall, London.

[12] Johnson, N.L. and Kotz, S. (1975), "A vector multivariate hazard rate", *Journal of Multivariate Analysis*, vol. 5, 53 - 66.

[13] Johnson, R.A. and Wichern, D.W. (1992), *Applied Multivariate Statistical Analysis*, Third edition, Prentice Hall, New Jersey, USA.

[14] Kannan, N., Kundu, D., Nair, P. Tripathi, R.C. (2010), "The generalized exponential cure rate model with covariates", *Journal of Applied Statistics* , vol. 37, 1625 - 1636.

[15] Karlin, Samuel (1968), *Total Positivity*, Stanford University Press, Stanford, California.

[16] Kundu, D. and Gupta, R.D. (2008), "Bivariate generalized exponential distribution", *Journal of Multivariate Analysis*, vol. 100, 581 - 593.

[17] Lehmann, E.L. (1953), "The power of rank test", *Annals of Mathematical Statistics*, vol. 24, 23 - 42.

[18] Lindley, D.V. and Singpurwalla, N.D. (1986), "Multivariate distributions for the life lengths of a system sharing a common environment", *Journal of Applied Probability*, vol. 23, 418 - 431.

[19] Mardia, K.V. (1962), "Multivariate Pareto distributions", *Annals of Mathematical Statistics*, vol. 33, 1008 - 1015.

[20] Marshall, A.W. (1975), "Some comments on hazard gradient", *Stochastic Process and Applications*, vol. 3, 293 - 300.

[21] Marshall, A.W. and Olkin, I. (1967), "A multivariate exponential distribution", *Journal of the American Statistical Association*, vol. 62, 30 - 44.

[22] Mudholkar, G.S., Srivastava, D.K. and Freimer, M. (1995), "The exponentiated Weibull family: a reanalysis of the bus-motor-failure data", *Technometrics*, vol. 37, 436 - 445..

[23] Nayak, T. (1987), "Multivariate Lomax distribution: Properties and usefulness in reliability theory", *Journal of Applied Probability*, vol. 24, 170 177.

[24] Nelsen, R.B. (2006), *An introduction to copulas*, 2nd-edition, Springer, New York.

[25] Sankaran, P.G. and Nair, N.U. (1993), "A bivariate Pareto model and its applications to reliability", *Naval Research Logistics*, vol. 40, 1013 - 1020.

[26] Shawky, A.I. and Abu-Zinadah, H.H. (2008), "Characterizations of the exponentiated Pareto distribution based on record values", *Applied Mathematical Sciences*, vol. 2, 1283 - 1290.

[27] Surles, J.G. and Padgett, W.J. (2001), "Inference for reliability and stress-strength for a scaled Burr Type X distribution", *Lifetime Data Analysis*, vol. 7, 187-200.