

75. Absolute Convergence of Fourier Series

By Masako IZUMI and Shin-ichi IZUMI
Department of Mathematics, The Australian National
University, Canberra, Australia

(Comm. by Zyoiti SUTUNA, M. J. A., May 12, 1969)

1. Introduction and theorems.

1.1. Let f be an even integrable function, with period 2π and its Fourier series be

$$(1) \quad f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx.$$

R. Mohanty [1] has proved the following

Theorem I. *If (I, 1) the function $\log(2\pi/t)f(t)$ is of bounded variation on the interval $(0, \pi)$ and (I, 2) the sequence $(n^{\delta}a_n)$ is of bounded variation for a $\delta > 0$, then $\sum |a_n| < \infty$.*

Later one of us [2] proved

Theorem II. *If (II, 1) f is of bounded variation and $\int_0^{\pi} \log(2\pi/t) \cdot |df(t)| < \infty$ and (II, 2) the sequence $(n^{\delta}\Delta(na_n))$ is of bounded variation for a $\delta > 0$, then $\sum |a_n| < \infty$.*

Recently R.M. Mazhar [3] has proved

Theorem III. *If the condition (II, 1) is satisfied and (III, 2) the sequence*

$$e^{-n^{\alpha}} \sum_{m=1}^n e^{m^{\alpha}} a_m \quad (n=1, 2, \dots)$$

is of bounded variation for an α , $0 < \alpha < 1$, then $\sum |a_n| < \infty$.

The conditions (I, 1) and (II, 1) are mutually exclusive and (I, 2) and (II, 2) are also. The condition (III, 2) is weaker than (II, 2) ([3], Lemma 2) and then Theorem III is a generalization of Theorem II.

1.2. Our object of this paper is partly to prove Theorem I without using Tauberian theorem and partly to generalize the condition (I, 1) as Theorem III, namely:

Theorem 1. *Suppose that the sequence (m_k) is positive and increasing and satisfies the following conditions:*

$$(2) \quad m_{k+1}/m_k \leq A, \quad M_k/m_k \leq Ak^{\delta-\varepsilon} \quad \text{for an } \varepsilon, 0 < \varepsilon < \delta < 1,$$

where $M_k = m_1 + m_2 + \dots + m_k$ and there is an integer p such that

$$(3) \quad |\Delta^{p-1}(M_j \Delta(1/m_j))| \leq A/j \quad \text{for all } j > 1.$$

If (1, 1) f is of bounded variation and $\int_0^{\pi} \log(2\pi/t) |df(t)| < \infty$ and (1, 2) the sequence

$$M_n^{-1} \sum_{k=1}^n m_k(k^\delta a_k) \quad (n=1, 2, \dots)$$

is of bounded variation, then $\sum |a_k| < \infty$.

Theorem 2. *The condition (1, 1) in Theorem 1 can be replaced by the condition (I, 1).*

For the proof of Theorems 1 and 2, we use the method in [2].

The sequence $m_k = e^{k^\alpha}$, $1 - \delta < \alpha < 1$, satisfies the conditions (2) and (3), with $p \geq 1/(1 - \alpha)$.

2. Proof of Theorem 1.

2.1. By (1), $a_n = \frac{2}{\pi} \int_0^\pi f(t) \cos nt \, dt$ and then

$$-\frac{\pi}{2} a_n = \frac{1}{n} \int_0^\pi \sin nt \, df(t) = \frac{1}{n} \int_0^{\pi/n^{\delta'}} + \frac{1}{n} \int_{\pi/n^{\delta'}}^\pi,$$

where $\delta' = \delta'_n$ is taken such that $n^{1-\delta'}$ is an even integer and $\delta'_n \rightarrow \sigma$, σ being a small positive number $< \varepsilon/(p - 1)$, as $n \rightarrow \infty$. We write

$$(4) \quad \begin{aligned} \frac{\pi}{2} \sum_{n=1}^\infty |a_n| &= \sum_{n=1}^\infty \frac{1}{n} \left| \int_0^\pi \sin nt \, df(t) \right| \\ &\leq \sum_{n=1}^\infty \frac{1}{n} \left| \int_0^{\pi/n^{\delta'}} \right| + \sum_{n=1}^\infty \frac{1}{n} \left| \int_{\pi/n^{\delta'}}^\pi \right| = P + Q, \end{aligned}$$

then we have

$$(5) \quad \begin{aligned} P &\leq \sum_{n=1}^\infty \frac{1}{n} \int_0^{\pi/n^{\delta'}} |df(t)| \leq \sum_{n=1}^\infty \frac{1}{n} \sum_{k=\lceil n^{\delta'} \rceil}^\infty \int_{\pi/(k+1)}^{\pi/k} |df(t)| \\ &\leq A \sum_{k=1}^\infty \int_{\pi/(k+1)}^{\pi/k} |df(t)| \sum_{n=1}^{\lceil k^{2/\sigma} \rceil} \frac{1}{n} \leq A \sum_{k=1}^\infty \int_{\pi/(k+1)}^{\pi/k} |df(t)| \log k \\ &\leq A \sum_{k=1}^\infty \int_{\pi/(k+1)}^{\pi/k} \log \frac{2\pi}{t} |df(t)| = A \int_0^\pi \log \frac{2\pi}{t} |df(t)| < \infty \end{aligned}$$

by the condition (1, 1). It remains to prove that Q is finite.

2.2. By (1) and the assumption (1, 1),

$$df(t) \sim \sum_{k=1}^\infty k a_k \sin kt.$$

Since $\sin nt$ vanishes at the point $t = \pi/n^{\delta'}$, the following Parseval formula [4] holds:

$$(6) \quad Q_n = \int_{\pi/n^{\delta'}}^\pi \sin nt \, df(t) = \sum_{k=1}^\infty k a_k \int_{\pi/n^{\delta'}}^\pi \sin nt \sin kt \, dt.$$

For the sake of simplicity, we put $\delta_n = \pi/n^{\delta'}$, then, by Abel's lemma,

$$(7) \quad \begin{aligned} Q_n &= \sum_{k=1}^\infty (k^\delta a_k) m_k \cdot \frac{k^{1-\delta}}{m_k} \int_{\delta_n}^\pi \sin nt \sin kt \, dt \\ &= \sum_{k=1}^\infty A_k \cdot \Delta \left(\frac{k^{1-\delta}}{m_k} \int_{\delta_n}^\pi \sin nt \sin kt \, dt \right), \end{aligned}$$

where $A_k = \sum_{j=1}^k j^\delta a_j m_j$, since, for any fixed n ,

$$\begin{aligned} \left| A_k \cdot \frac{k^{1-\delta}}{m_k} \int_{\delta_n}^{\pi} \sin nt \sin kt \, dt \right| &\leq \frac{A}{k^\delta m_k} \left| \sum_{j=1}^k j^\delta a_j m_j \right| \\ &= A \left| s_k - \frac{1}{k^\delta m_k} \sum_{j=1}^{k-1} s_j ((j+1)^\delta m_{j+1} - j^\delta m_j) \right| \rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

s_k being the k th partial sum of the series $\sum a_j$. Using Abel's lemma again, (7) becomes

$$\begin{aligned} (8) \quad Q_n &= \sum_{k=1}^{\infty} \frac{A_k}{M_k} M_k \cdot \Delta \left(\frac{k^{1-\delta}}{m_k} \int_{\delta_n}^{\pi} \sin nt \sin kt \, dt \right) \\ &= a_1 \sum_{j=1}^{\infty} M_j \cdot \Delta \left(\frac{j^{1-\delta}}{m_j} \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \right) \\ &\quad - \sum_{k=1}^{\infty} \Delta \left(\frac{A_k}{M_k} \right) \sum_{j=k+1}^{\infty} M_j \cdot \Delta \left(\frac{j^{1-\delta}}{m_j} \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \right). \end{aligned}$$

In order to see this, we have to prove that

$$(9) \quad \frac{A_k}{M_k} \sum_{j=k+1}^{\infty} M_j \cdot \Delta \left(\frac{j^{1-\delta}}{m_j} \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \right) \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

for fixed n , which is proved when the series

$$(10) \quad \sum_{j=1}^{\infty} M_j \cdot \Delta \left(\frac{j^{1-\delta}}{m_j} \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \right)$$

is convergent for each fixed n . For this purpose we shall prove that

$$\begin{aligned} (11) \quad &\sum_{j=N}^{N'} \Delta(j^{1-\delta}) \frac{M_j}{m_j} \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \\ &+ \sum_{j=N}^{N'} (j+1)^{1-\delta} M_j \cdot \Delta(1/m_j) \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \\ &+ \sum_{j=N}^{N'} \frac{(j+1)^{1-\delta} M_j}{m_{j+1}} \int_{\delta_n}^{\pi} \sin nt (\sin jt - \sin(j+1)t) \, dt \\ &= R_{N,N'} + S_{N,N'} + T_{N,N'} \rightarrow 0 \quad \text{as } N' > N \rightarrow \infty. \end{aligned}$$

We can suppose $N > n$. Since $n\delta_n$ is even multiple of π ,

$$(12) \quad \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt = - \frac{n \sin j\delta_n}{(j-n)(j+n)}$$

and then, by (2),

$$(13) \quad |R_{N,N'}| \leq A \sum_{j=N}^{N'} \frac{M_j}{j^{2+\delta} m_j} \leq A \sum_{j=N}^{\infty} \frac{1}{j^{2+\epsilon}} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Similarly,

$$(14) \quad |S_{N,N'}| \leq A \sum_{j=N}^{N'} \frac{M_j}{j^{1+\delta}} \Delta \left(\frac{1}{m_j} \right) \leq A \sum_{j=N}^{\infty} \frac{1}{j^{1+\epsilon}} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

By Abel's lemma,

$$\begin{aligned} (15) \quad T_{N,N'} &= \frac{(N+1)^{1-\delta} M_N}{m_{N+1}} \int_{\delta_n}^{\pi} \sin nt \sin Nt \, dt \\ &\quad - \sum_{j=N}^{N'-1} \Delta \left(\frac{(j+1)^{1-\delta} M_j}{m_{j+1}} \right) \int_{\delta_n}^{\pi} \sin nt \sin (j+1)t \, dt \\ &\quad - \frac{(N'+1)^{1-\delta} M_{N'}}{m_{N'+1}} \int_{\delta_n}^{\pi} \sin nt \sin (N'+1)t \, dt \end{aligned}$$

and then, by (2),

$$(16) \quad |T_{N,N'}| \leq \frac{AM_N}{N^{1+\delta}m_{N+1}} + A \sum_{j=N}^{N'} \left(\frac{M_j}{j^{2+\delta}m_{j+1}} + \frac{1}{j^{1+\delta}} + \frac{M_j}{j^{1+\delta}} \Delta \left(\frac{1}{m_{j+1}} \right) \right) + \frac{AM_{N'}}{(N')^{1+\delta}m_{N'+1}} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

By (11), (13), (14) and (16), the series (10) is convergent, and then the formula (8) holds.

2.3. By (4), (6) and (8)

$$Q = \sum_{n=1}^{\infty} \frac{1}{n} |Q_n| \leq A \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{j=1}^{\infty} M_j \cdot \Delta \left(\frac{j^{1-\delta}}{m_j} \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \right) \right| + A \sum_{k=1}^{\infty} \left| \Delta \left(\frac{A_k}{M_k} \right) \right| \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{j=k+1}^{\infty} M_j \cdot \Delta \left(\frac{j^{1-\delta}}{m_j} \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \right) \right|.$$

By the assumption (1, 2),

$$(17) \quad Q \leq A \max_{1 \leq k < \infty} \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{j=k}^{\infty} M_j \cdot \Delta \left(\frac{j^{1-\delta}}{m_j} \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \right) \right| \leq A \max_{1 \leq k < \infty} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \left| \sum_{j=k}^{\infty} \Delta(j^{1-\delta}) \frac{M_j}{m_j} \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \right| + \left| \sum_{j=k}^{\infty} (j+1)^{1-\delta} M_j \cdot \Delta \left(\frac{1}{m_j} \right) \int_{\delta_n}^{\pi} \sin nt \sin jt \, dt \right| + \left| \sum_{j=k}^{\infty} \frac{(j+1)^{1-\delta} M_j}{m_{j+1}} \int_{\delta_n}^{\pi} \sin nt (\sin jt - \sin (j+1)t) \, dt \right| \right\} \leq A \max_{1 \leq k < \infty} (R_k + S_k + T_k).$$

Since

$$(18) \quad \left| \int_{\delta_n}^{\pi} \frac{\sin nt \cos (j \pm 1/2)t}{2 \sin t/2} \, dt \right| \leq \frac{A}{\delta_n |j - n \pm 1/2|},$$

we have, by (2) and Abel's transformation,

$$(19) \quad R_k = \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{j=k}^{\infty} \Delta(j^{1-\delta}) \frac{M_j}{m_j} \int_{\delta_n}^{\pi} \frac{\sin nt}{2 \sin t/2} (\cos (j-1/2)t - \cos (j+1/2)t) \, dt \right| = \sum_{n=1}^{\infty} \frac{1}{n} \left| \Delta(k^{1-\delta}) \frac{M_k}{m_k} \int_{\delta_n}^{\pi} \frac{\sin nt}{2 \sin t/2} \cos (k-1/2)t \, dt - \sum_{j=k}^{\infty} \Delta \left(\Delta(j^{1-\delta}) \frac{M_j}{m_j} \right) \int_{\delta_n}^{\pi} \frac{\sin nt}{2 \sin t/2} \cos (j+1/2)t \, dt \right| \leq \sum_{n=1}^{\infty} \frac{A}{n} \left\{ \left| \frac{M_k}{k^{\delta} m_k \delta_n |k - n - 1/2|} \right| + \left| \sum_{j=k}^{\infty} \left(\frac{M_j}{j^{1+\delta} m_j} + \frac{1}{j^{\delta}} + \frac{M_j}{j^{\delta}} \Delta \left(\frac{1}{m_j} \right) \right) \frac{1}{\delta_n |j - n + 1/2|} \right| \right\} = A \frac{\log k}{k^{1+\epsilon-\sigma}} + A \sum_{j=k}^{\infty} \frac{\log j}{j^{1-\sigma}} \left(\frac{1}{j^{1+\delta} j^{\epsilon-\delta}} + \frac{1}{j^{\delta}} + \frac{1}{j^{\delta} j^{\epsilon-\delta}} \right) = A \frac{\log k}{k^{1+\epsilon-\sigma}} + A \sum_{j=k}^{\infty} \frac{\log j}{j^{1+\epsilon-\sigma}} = A \frac{\log k}{k^{1+\epsilon-\sigma}} + A \frac{\log k}{k^{\epsilon-\sigma}} \leq A.$$

By (2), (3), (17) and $(p-1)$ time use of Abel's lemma,

$$\begin{aligned}
 (20) \quad S_k &= \sum_{n=1}^{\infty} \frac{1}{n} \left| \sum_{j=k}^{\infty} (j+1)^{1-\delta} M_j \Delta \left(\frac{1}{m_j} \right) \right. \\
 &\quad \cdot \left. \int_{\delta n}^{\pi} \frac{\sin nt}{2 \sin t/2} (\cos (j-1/2)t - \cos (j+1/2)t) dt \right| \\
 &\leq A \frac{\log k}{k^{\epsilon-\sigma}} + A \frac{\log k}{k^{1-(p-1)\sigma}} \left(\frac{M_k}{k^{\delta}} + k^{1-\delta} M_k \Delta^2 \left(\frac{1}{m_k} \right) + k^{1-\delta} m_{k+1} \Delta \left(\frac{1}{m_{k+1}} \right) \right) \\
 &\quad + A \sum_{j=k}^{\infty} \frac{\log j}{j^{1-(p-1)\sigma}} \left| \Delta^{p-1} \left((j+1)^{1-\delta} M_j \Delta \left(\frac{1}{m_j} \right) \right) \right| \\
 &\leq A + A \sum_{j=k}^{\infty} \frac{\log j}{j^{1-(p-1)\sigma+\epsilon}} + A \sum_{j=k}^{\infty} \frac{\log j}{j^{\delta-(p-1)\sigma}} \left| \Delta^{p-1} \left(M_j \Delta \left(\frac{1}{m_j} \right) \right) \right| \leq A.
 \end{aligned}$$

Similarly, by (2), (3), (17) and p time use of Abel's lemma,

$$\begin{aligned}
 (21) \quad T_k &\leq A \frac{M_k}{m_k} \frac{\log k}{k^{\delta}} + A \left(k^{1-\delta} \Delta \left(\frac{M_k}{m_k} \right) + \Delta(k^{1-\delta}) \frac{M_{k+1}}{m_{k+1}} \right) \frac{\log k}{k^{1-(p-1)\sigma}} \\
 &\quad + A \sum_{j=k}^{\infty} \frac{\log j}{j^{\delta-(p-1)\sigma}} \Delta^p \left(\frac{M_j}{m_{j+1}} \right) \\
 &\leq A + A \frac{\log k}{k^{\epsilon-(p-1)\sigma}} + A \sum_{j=k}^{\infty} \frac{\log j}{j^{\delta-(p-1)\sigma}} \Delta^{p-1} \left(M_{j+1} \Delta \left(\frac{1}{m_{j+1}} \right) \right) \leq A.
 \end{aligned}$$

Collecting (17), (19), (20) and (21), we get $Q \leq A$, which proves the theorem with (4) and (5).

3. Proof of Theorem 2.

It is sufficient to prove that P in the section 2.1 is finite under the condition $(I, 1)$. Putting $g(t) = \log \frac{2\pi}{t} f(t)$, we get, by integration by parts,

$$\begin{aligned}
 P_n &= \frac{1}{n} \int_0^{\pi/n^{\delta'}} \sin nt \, d f(t) = - \int_0^{\pi/n^{\delta'}} \cos nt \, f(t) \, dt = - \int_0^{\pi/n^{\delta'}} \frac{\cos nt}{\log \frac{2\pi}{t}} g(t) \, dt \\
 &= -g(\pi/n^{\delta'}) \int_0^{\pi/n^{\delta'}} \frac{\cos nu}{\log \frac{2\pi}{u}} \, du + \int_0^{\pi/n^{\delta'}} \left(\int_0^t \frac{\cos nu}{\log \frac{2\pi}{u}} \, du \right) dg(t) \\
 &= -g(\pi/n^{\delta'}) \frac{1}{n} \int_0^{\pi/n^{\delta'}} \frac{\sin nu}{u \left(\log \frac{2\pi}{u} \right)^2} \, du + \int_0^{\pi/n^{\delta'}} \left(\int_0^t \frac{\cos nu}{\log \frac{2\pi}{u}} \, du \right) dg(t).
 \end{aligned}$$

Thus we have

$$|P_n| \leq \frac{A}{n(\log n)^2} + \frac{A}{n} \int_0^{\pi/n^{\delta'}} \frac{|dg(t)|}{\log \frac{2\pi}{t}},$$

since

$$\int_0^{\pi/n^{\delta'}} \frac{\sin nu}{u \left(\log \frac{2\pi}{u} \right)^2} \, du = \int_0^{\pi/n} + \int_{\pi/n}^{\pi/n^{\delta'}} = O \left(\frac{1}{(\log n)^2} \right)$$

by the mean value theorem, and then

$$P = \sum_{n=1}^{\infty} |P_n| \leq A \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2} + A \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=[n^{\delta'}]}^{\infty} \int_{\pi/(k+1)}^{\pi/k} \frac{|dg(t)|}{\log \frac{2\pi}{t}}$$

$$\leq A + \int_0^{\pi} |dg(t)| < A.$$

Thus we get Theorem 2.

References

- [1] R. Mohanty: A criterion for the absolute convergence of a Fourier series. Proc. London Math. Soc., **51** (2), 186-196 (1949).
- [2] S. Izumi: Absolute convergence of some trigonometric series. II. Jour. of Math. Analysis and Applications, **1**, 184-193 (1960).
- [3] R. M. Mazhar: Absolute convergence of Fourier series. Proc. Japan Acad., **44**, 756-761 (1968).
- [4] A. Zygmund: Trigonometric Series. I. Cambridge Press (1959).