Mathematical Problems in Engineering,
Volume 7, pp. 413–431
Reprints available directly from the publisher
Photocopying permitted by license only

Absolute Stability of Nonlinear Systems with Time Delays and Applications to Neural Networks

XINZHI LIU[a], XUEMIN SHEN[b] and YI ZHANG[c]

[a]*Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1;* [b]*Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1;* [c]*Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*

(Received 5 January 2000)

In this paper, absolute stability of nonlinear systems with time delays is investigated. Sufficient conditions on absolute stability are derived by using the comparison principle and differential inequalities. These conditions are simple and easy to check. In addition, exponential stability conditions for some special cases of nonlinear delay systems are discussed. Applications of those results to cellular neural networks are presented.

Keywords: Absolute stability; Nonlinear; Delay; Neural network

AMS (MOS) Subject Classifications: 34K35, 34H05, 49J25

1 INTRODUCTION

Since its inception in the 1940s, the concept of absolute stability has attracted the attention of many researchers including mathematicians and engineers, and numerous results have been published in the literature, [1–5]. The significance of this concept is that it does not require very precise information on certain nonlinear portion of a

control system. However, most of the results are with respect to linear control systems or systems described by ordinary differential equations (ODEs), while the real control systems in general are nonlinear systems with time delays. In this paper, absolute stability of nonlinear system with time delays is studied. By constructing suitable Lyapunov functions, sufficient conditions to guarantee absolute stability of the systems are derived. These conditions are simple and easy to check. In addition, existence of a unique equilibrium point and its exponential stability for some special cases of the system are discussed.

The remainder of this paper is organized as follows. Section 2 gives the definitions of absolute stability and equilibrium points. In section 3, sufficient conditions on absolute stability as well as those on exponential stability for some special cases are derived in detail. Applications of those conditions to cellular neural networks are presented in section 4. Conclusions are given in section 5.

2 PRELIMINARIES

Consider a nonlinear system with time delays given as follows

$$\begin{cases} \psi_i' &= -a_i\psi_i + \sum_{j=1}^n f_{ij}(\psi_j) + \sum_{j=1}^n g_{ij}(\psi_j(t-\tau_{ij}(t))) + d_i\phi \\ \phi' &= h(\delta), \quad \delta = \sum_{i=1}^n p_i\psi_i - r\phi, \quad i = 1,\ldots,n, \end{cases} \tag{2.1}$$

where a_i, d_i, p_i and r are constants, $f_{ij}, g_{ij}, \tau_{ij} \in C^1(R,R)$, $f_{ij}(0) = g_{ij}(0) = 0$ and $0 \le \tau_{ij}(t) \le \tau, \tau > 0$.

Let

$$U = \{h : |h \in C(R,R), h(0) = 0, h(\delta)\delta > 0, \delta \neq 0\}.$$

Definition 2.1 The trivial solution of (2.1) is called globally asymptotically stable if it is stable and all solutions of (2.1) satisfy

$$\lim_{t\to\infty} \psi_i(t) = 0, \quad \lim_{t\to\infty} = 0, \quad i = 1,2,\ldots,n.$$

Definition 2.2 System (2.1) is called absolutely stable if for any $h \in U$ and any $\tau \ge 0$, it is globally asymptotically stable.

When $h(\delta) = 0$, ϕ becomes a constant. Let $I_i = d_i\phi$ and $x_i = \psi_i$, $i = 1,2,\ldots,n$ (2.1) becomes

$$x_i' = -a_i x_i + \sum_{j=1}^{n} f_{ij}(x_j) + \sum_{j=1}^{n} g_{ij}(x_j(t - \tau_{ij}(t))) + I_i. \tag{2.2}$$

Definition 2.3 A point $x^* = (x_1^*, \ldots, x_n^*)^T \in R^n$ is called an equilibrium point of (2.2) if

$$a_i x_i^* = \sum_{j=1}^{n} f_{ij}(x_j^*) + \sum_{j=1}^{n} g_{ij}(x_j^*) + I_i, \quad i = 1, 2, \ldots, n.$$

Definition 2.4 The equilibrium point $x^* = (x_1^*, \ldots, x_n^*)^T \in R^n$ of system (2.2) is called exponentially stable if there exists a $\lambda > 0$ such that

$$|x_i - x_i^*| \le M\left(\sum_{j=1}^{n} \sup_{t_0 - \tau \le t \le t_0} |x_j - x_j^*|\right) e^{-\lambda(t - t_0)}, \quad i = 1, 2, \ldots, n.$$

Definition 2.5 [8] A real n × n matrix Λ with nonnegative diagonal and nonpositive off-diagonal elements is called M-matrix if all its eigenvalues have a nonnegative real parts or its principal minors are positive. If Λ has all eigenvalues with positive real parts, then it is called a nonsingular M-matrix. Usually, $\aleph$ is used to denote the class of all nonsingular M-matrices.

LEMMA 2.1 [8] *If $A = (a_{ij}) \in \aleph$, then there exists a positive diagonal matrix P such that $A^T P$ is strictly diagonally dominant, i.e.,*

$$a_{ii} p_i + \sum_{j=1, j \ne i} p_j a_{ji} > 0, \quad i = 1, 2, \ldots, n.$$

LEMMA 2.2 [7] *Let $g \in C(R^+ \times R, R)$, $g(t, 0) = 0$ and $r(t) = r(t, t_0, u_0)$ be the maximal solution of*

$$u' = g(t, u), \quad u(t_0) = u_0 \ge 0,$$

existing on $J = [t_0, t_0 + \alpha)$, $0 < \alpha \le \infty$. if $m \in C(R^+, R^+)$ and

$$Dm(t) \le g(t, m(t)), \quad t \in J,$$

where D is any fixed Dini derivative, then $m(t_0) \le u_0$ implies

$$m(t) \le r(t), \quad t \in J.$$

LEMMA 2.3 *Let E be a complete matric space and $T : E \to E$ be a contraction mapping, i.e. $|T(x) - T(y)| \le \alpha |x - y|$, $\alpha \in [0, 1)$, $x, y \in E$, then T has unique fixed point.*

3 MAIN RESULTS

This section establishes some sufficient conditions on absolute stability for system (2.1), and on exponential stability for some special case of (2.1). Two examples are given to illustrate the derived conditions.

THEOREM 3.1 *The system (2.1) is absolutely stable if the following two conditions are satisfied.*

$$(A_1)\colon\ |f'_{ij}(s)| \le m_{ij},\ |g'_{ij}(s)| \le M_{ij},\ |\tau'_{ij}(s)| \le 1;$$
$$(A_2)\colon\ -\Omega \in \aleph,$$

where

$$\Omega = (\omega_{ij}) = \begin{pmatrix} -a_1 + m_{11} + M_{11} & m_{12} + M_{12} & \dots & m_{1n} + M_{1n} & |d_1| \\ m_{21} + M_{21} & -a_2 + m_{22} + M_{22} & \dots & m_{2n} + M_{2n} & |d_2| \\ \dots & \dots & & & \\ m_{n1} + M_{n1} & m_{n2} + M_{n2} & \dots & -a_n + m_{nn} + M_{nn} & |d_n| \\ |p_1| & |p_2| & \dots & |p_n| & -r \end{pmatrix}$$

Proof Let

$$\begin{cases} x_i = -a_i\psi_i + \sum_{j=1}^{n} f_{ij}(\psi_j) + \sum_{j=1}^{n} g_{ij}(\psi_j(t - \tau_{ij}(t))) + d_i\phi \\ \delta = \sum_{i=1}^{n} p_i\psi_i - r\phi, \quad i = 1, \dots, n, \end{cases} \tag{3.3}$$

then system (2.1) becomes

$$\begin{cases} x'_i = -a_i x_i + \sum_{j=1}^{n} \dfrac{d}{ds} f_{ij}(s)x_j \\ \qquad + \sum_{j=1}^{n} \dfrac{d}{ds} g_{ij}(s)x_j(t - \tau_{ij}(t))(1 - \tau'_{ij}(t)) + d_i h(\delta) \\ \delta' = \sum_{i=1}^{n} p_i x_i - rh(\delta), \quad i = 1, 2, \dots, n. \end{cases} \tag{3.4}$$

Since $-\Omega \in \aleph$, by Lemma 2.1, there exist positive real numbers $\beta_1, \beta_2, \dots, \beta_{n+1}$ such that

$$\sum_{j=1}^{n+1} \beta_j \omega_{ji} < 0, \quad i = 1, 2, \ldots, n+1. \tag{3.5}$$

Define a Lyapunov functional $v(x, x_t, \delta)$ by

$$v(x, x_t, \delta) = \sum_{i=1}^{n} \beta_i \left[|x_i| + \sum_{j=1}^{n} M_{ij} \int_{t-\tau_{ij}(t)}^{t} |x_j(s)| ds \right] + \beta_{n+1} |\delta|. \tag{3.6}$$

It is clear that

$$v(x, x_t, \delta) \geq \sum_{i=1}^{n} \beta_i |x_i| + \beta_{n+1} |\delta|.$$

The generalized derivative $D^+ v$ along with system (3.4) is

$$\begin{aligned}
D^+ v \leq \sum_{i=1}^{n} \beta_i \Bigg\{ & - a_i |x_i| + \sum_{j=1}^{n} \left| \frac{d}{ds} f_{ij}(s) x_j \right| \\
& + \sum_{j=1}^{n} \left| \frac{d}{ds} g_{ij}(s) x_j(t - \tau_{ij}(t)) \right| (1 - \tau'_{ij}(t)) + |d_i h(\delta)| \\
& + \sum_{j=1}^{n} M_{ij} [|x_j| - |x_j(t - \tau_{ij}(t))| (1 - \tau'(t))] \Bigg\} \\
& + \sum_{i=1}^{n} \beta_{n+1} |p_i x_i| - \beta_{n+1} r |h(\delta)| \\
\leq \sum_{i=1}^{n} \beta_i \Bigg\{ & - a_i |x_i| + \sum_{j=1}^{n} m_{ij} |x_j| \\
& + \sum_{j=1}^{n} M_{ij} |x_j(t - \tau_{ij}(t))| (1 - \tau'_{ij}(t)) + |d_i h(\delta)| \\
& + \sum_{j=1}^{n} M_{ij} [|x_j| - |x_j(t - \tau_{ij}(t))| (1 - \tau'(t))] \Bigg\} \\
& + \sum_{i=1}^{n} \beta_{n+1} |p_i x_i| - \beta_{n+1} r |h(\delta)|
\end{aligned}$$

$$\leq \sum_{i=1}^{n} \beta_i \Bigg\{ - a_i|x_i| + \sum_{j=1}^{n}(m_{ij} + M_{ij})|x_j| + |d_i h(\delta)| \Bigg\} + \sum_{i=1}^{n} \beta_{n+1}|p_i x_i| - \beta_{n+1} r|h(\delta)|$$

$$\leq \sum_{i=1}^{n} \Bigg\{ - \beta_i a_i + \beta_{n+1}|p_i| + \sum_{j=1}^{n} \beta_j(m_{ji} + M_{ji}) \Bigg\}|x_i| + \left(\sum_{i=1}^{n} \beta_i|d_i| - \beta_{n+1} r \right)|h(\delta)| \leq 0.$$

With (3.5), we conclude that

$$\lim_{t\to\infty} x_i(t) = 0, \quad \lim_{t\to\infty} \delta(t) = 0. \tag{3.7}$$

On the other hand, from (3.3), we have

$$\phi = \sum_{i=1}^{n} \frac{p_i}{r} \psi_i - \frac{\delta}{r}, \tag{3.8}$$

where δ satisfy the condition $\lim_{t\to\infty} \delta(t) = 0$.

Next, we show that

$$\lim_{t\to\infty} \psi_i(t) = 0, \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = 0. \tag{3.9}$$

Let the auxiliary function

$$w = \sum_{i=1}^{n} \beta_i|\psi_i|,$$

then along with system (2.1), we get

$$D^+ w \leq \sum_{i=1}^{n} \beta_i \Bigg\{ - a_i|\psi_i| + \sum_{j=1}^{n} |f_{ij}(\psi_j)| + \sum_{j=1}^{n} |g_{ij}(\psi_j(t - \tau_{ij}(t)))| + |d_i \phi| \Bigg\}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{n} \beta_i \Bigg\{ - a_i|\psi_i| + \sum_{j=1}^{n} m_{ij}|\psi_j| \\
&\qquad + \sum_{j=1}^{n} M_{ij}|\psi_j(t-\tau_{ij}(t))| + |d_i\phi| \Bigg\} \\
&= \sum_{i=1}^{n} \beta_i \Bigg\{ - a_i|\psi_i| + \sum_{j=1}^{n} m_{ij}|\psi_j| \\
&\qquad + \sum_{j=1}^{n} M_{ij} \Bigg| \psi_j(t) - \int_{t-\tau_{ij}(t)}^{t} \psi_j'(s)ds \Bigg| + |d_i\phi| \\
&\le \sum_{i=1}^{n} \beta_i \Bigg\{ - a_i|\psi_i| + \sum_{j=1}^{n} \{m_{ij} + M_{ij}\}|\psi_j| \\
&\qquad + \sum_{j=1}^{n} M_{ij} \int_{t-\tau}^{t} |x_j|ds + \left| d_i \left(\sum_{j=1}^{n} \frac{p_j}{\tau}\psi_j - \frac{\delta}{\tau} \right) \right| \Bigg\} \\
&\le \sum_{j=1}^{n} \Bigg\{ - \beta_i a_j + \sum_{i=1}^{n} \beta_i [m_{ij} + M_{ij}] + \left| \frac{p_j}{r} \right| \sum_{i=1}^{n} \beta_i |d_i| \Bigg\} |\psi_j| \\
&\quad + \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \beta_i M_{ij} \right) \int_{t-\tau}^{t} |x_j|ds + \sum_{j=1}^{n} \beta_i |d_i| \left| \frac{\delta}{\tau} \right| \\
&\le -c \sum_{j=1}^{n} \beta_j |\psi_j| + F(t) \\
&= -cw + F(t)
\end{aligned}
$$

where

$$
\begin{aligned}
-c &= \max_{1 \le j \le n} \left\{ \beta_j a_j + \sum_{i=1}^{n} \beta_i [m_{ij} + M_{ij}] + \left| \frac{p_j}{r} \right| \sum_{i=1}^{n} \beta_i |d_i| \right\} / \beta_j \\
&\le \max_{1 \le j \le n} \left\{ \sum_{i=1}^{n+1} \beta_i \omega_{ij} + \left| \frac{p_j}{r} \right| \left\{ \sum_{i=1}^{n+1} \beta_i \omega_{i(n+1)} \right\} \right\} / \beta_j < 0,
\end{aligned}
$$

and

$$
F(t) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \beta_i M_{ij} \right) \int_{t-\tau}^{t} |x_j|ds + \sum_{j=1}^{n} \beta_i |d_i| \left| \frac{\delta}{r} \right|.
$$

By Lemma 2.2, $w(t) \le r(t)$, where $r(t)$ is the maximal solution of

$$u' = -cu + F(t), \quad u(t_0) = w(t_0).$$

It can be seen that, for any $\tau, \lim_{t\to\infty} F(t) = 0$ since $\lim_{t\to\infty}|x_i| = 0$ and $\lim_{t\to\infty}|\delta(t)| = 0$. This implies that $\lim_{t\to\infty}|r(t)| = 0$ and hence $\lim_{t\to\infty}|w(t)| = 0$. Thus (3.9) is true in view of (3.8). The proof is complete.

In the following, we study two special cases of system (2.1).

Case 1 If $f_{ij}(\psi_j) = a_{ij}\psi_j$, $g_{ij}(\psi_j) = b_{ij}\psi_j$, the system becomes a linear system of the form

$$\begin{cases} \psi_i' = -a_i\psi_i + \sum_{j=1}^n a_{ij}\psi_j + \sum_{j=1}^n b_{ij}\psi_j(t-\tau_{ij}(t)) + d_i\phi, \\ \phi' = h(\delta), \quad \delta = \sum_{i=1}^n p_i\psi_i - r\phi, \end{cases} \tag{3.10}$$

where a_i, d_i, p_i, r, I_i are constants and $0 \le \tau_{ij}(t) \le \tau = const.$

COROLLARY 3.1 *If $-\Omega \in \aleph$, then system (3.10) is absolutely stable, where*

$$\Omega = (\omega_{ij}) = \begin{pmatrix} -a_1 + |a_{11}| + |b_{11}| & |a_{12}| + |b_{12}| & \cdots & |a_{1n}| + |b_{1n}| & |d_1| \\ |a_{21}| + |b_{21}| & -a_2 + |a_{22}| + |b_{22}| & \cdots & |a_{2n}| + |b_{2n}| & |d_2| \\ \cdots\cdots & \cdots\cdots & & & \\ |a_{n1}| + |b_{n1}| & |a_{n2}| + |b_{n2}| & \cdots & -a_n + |a_{nn}| + |b_{nn}| & |d_n| \\ |p_1| & |p_2| & \cdots & |p_n| & -r \end{pmatrix}.$$

Example 3.1 Consider the following system

$$\begin{cases} \psi_1' = -3\psi_1 + \psi_2 + \psi_1(t - \cos t) - \psi_2(t - \sin t) \\ \psi_2' = \psi_1 - 6\psi_2 + 2\psi_1(t - \cos(t/2)) + \psi_2(t - \sin(t/2)) + 2\phi \\ \phi' = h(\delta), \quad \delta = \psi_1 - \psi_2 - 5\phi \end{cases} \tag{3.11}$$

In this system,

$$\Omega = (\omega_{ij}) = \begin{pmatrix} -3 & 2 & 0 \\ 3 & -5 & 2 \\ 1 & 1 & -5 \end{pmatrix}.$$

It can be seen that $-\Omega \in \aleph$, and hence system (3.11) is absolutely stable.

Case 2 If $n = 1$, system (2.1) reduces to

$$\begin{cases} \psi' = -a\psi + f_1(\psi) + g_1(\psi(t - \tau(t))) + d\phi \\ \phi' = f(\delta),\ \delta = p\psi - r\phi \end{cases} . \tag{3.12}$$

COROLLARY 3.2 *If* $|f_1'| \le m_1$, $|g_1'| \le M_1, |\tau'(t)| \le 1$, *and there exist* $\beta_1, \beta_2 > 0$, *such that*

$$\beta_1(-a + m_1 + M_1) + \beta_2|p| < 0; \quad \beta_1|d| < \beta_2 r,$$

then system (3.12) is absolutely stable.

Example 3.2 Consider the following system

$$\begin{cases} \psi'(t) = -5\psi(t) + \sin\psi(t) + \psi(t - \cos t) + 2\phi, \\ \phi'(t) = f(\delta), \quad \delta(t) = 2\psi(t) - 3\phi(t). \end{cases} \tag{3.13}$$

Here, $a = 5, m_1 = 1, M_1 = 1, |d| = 2, |p| = 2, r = 3, |\tau'(t)| \le 1$. Since $-a + m_1 + M_1 + |p| = -1 < 0$ and $|d| - r = -1 < 0$, by Theorem 2.2, system (3.13) is absolutely stable.

On the other hand, system (2.1) can be generalized to a highly nonlinear system of the following form

$$\begin{cases} \xi_i' = -a_i\xi_i + g_i(\xi_1, \ldots, \xi_n, \xi_1(t - \tau_{i1}(t)), \ldots, \xi_n(t - \tau_{in}(t))) + b_i\eta \\ \eta' = f(\sigma), \quad \sigma = \sum_{i=1}^n p_i\xi_i - r\sigma, \quad i = 1, 2, \ldots, n, \end{cases} \tag{3.14}$$

where $g_i, \tau_{ij} \in C^1, g_i(0, \ldots, 0) = 0$, and $f \in U$ $(i, j = 1, 2, \ldots, n)$.

The next result establishes absolute stability for system (3.14) under suitable conditions on the function g_i.

THEOREM 3.2 *If*

$$(A_3): \quad \left|\frac{\partial g_i}{\partial \xi_j}(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_n)\right| \le m_{ij},$$

$$\left|\frac{\partial g_i}{\partial \zeta_j}(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_n)\right| \le M_{ij}, \quad |\tau_{ij}'(t)| \le 1,$$

then $-\Omega \in \aleph$ *implies absolute stability of system (3.14), where* Ω *is the same of Theorem 3.1.*

Proof Let

$$\begin{cases} x_i = -a_i\xi_i + g_i(\xi_1, \ldots, \xi_n, \xi_1(t-\tau_{i1}(t)), \ldots, \xi_n(t-\tau_{in}(t))) + b_i\eta \\ \delta = \sum_{i=1}^n p_i\xi_i - r\sigma, \quad i = 1, 2, \ldots, n. \end{cases}$$

then the system (3.14) become

$$\begin{cases} x_i' = -a_i x_i + \sum_{j=1}^n \frac{\partial g_i}{\partial \xi_j}(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_n)x_j \\ \qquad + \sum_{j=1}^n \frac{\partial g_i}{\partial \zeta_j}(\xi_1, \ldots, \xi_n, \zeta_1, \ldots, \zeta_n)(x_j(t-\tau_{ij}(t)))(1-\tau_{ij}'(t)) + d_i f(\sigma) \\ \sigma' = \sum_{i=1}^n p_i x_i - rf(\sigma), \quad i = 1, 2, \ldots, n. \end{cases} \tag{3.15}$$

Since $-\Omega \in \aleph$, then there exist $\beta_1, \beta_2, \ldots, \beta_{n+1}$ such that

$$\sum_{j=1}^{n+1} \beta_j \omega_{ji} < 0, \quad i = 1, 2, \ldots, n+1. \tag{3.16}$$

Define a Lyapunov functional v by

$$v(x, x_t, \delta) = \sum_{i=1}^n \beta_i \left[|x_i| + \sum_{j=1}^n M_{ij} \int_{t-\tau_{ij}(t)}^t |x_j(s)| ds \right] + \beta_{n+1}|\delta|. \tag{3.17}$$

Using the similar proof of Theorem 3.1, we can obtain

$$\lim_{t\to\infty} |\xi_i| = 0 \quad \text{and} \quad \lim_{t\to\infty} \eta = 0.$$

The proof is complete.

Example 3.3 Consider the following system

$$\begin{cases} \xi_1' = -4\xi_1 + g_1(\xi_1, \xi_2, \xi_1(t-\tau_{11}(t)), \quad \xi_2(t-\tau_{12}(t))) + \eta \\ \xi_2' = -5\xi_2 + g_2(\xi_1, \xi_2, \xi_1(t-\tau_{21}(t)), \quad \xi_2(t-\tau_{22}(t))) + 2\eta \\ \eta' = f(\sigma), \quad \sigma = -\xi_1 + 2\xi_2 - 6\sigma, \end{cases} \tag{3.18}$$

where $g_1 = 1 - \cos(\xi_1 + \xi_2(t - \tau_{12}(t)) + \sin(\xi_2 + \xi_1(t - \tau_{11}(t))), g_2 = \ln(1 + \xi_1^2 + \xi_2^2(t - \tau_{22}(t)))$; $\tau_{11}, \tau_{12}, \tau_{22} \in C', f \in U$ and $|\tau'_{11}(t)| \leq 1, |\tau'_{12}(t)| \leq 1, |\tau'_{22}(t)| \leq 1$. It can be shown that

$$\left|\frac{\partial g_1}{\partial \xi_i}(\xi_1, \xi_2, \zeta_1, \zeta_2)\right| \leq 1 = m_{1i}, \quad \left|\frac{\partial g_1}{\partial \zeta_i}(\xi_1, \xi_2, \zeta_1, \zeta_2)\right| \leq 1 = M_{1i}, \quad i = 1, 2;$$

$$\left|\frac{\partial g_2}{\partial \xi_1}(\xi_1, \xi_2, \zeta_1, \zeta_2)\right| \leq 1 = m_{21}, \quad \left|\frac{\partial g_2}{\partial \xi_2}(\xi_1, \xi_2, \zeta_1, \zeta_2)\right| = 0 = m_{22}$$

$$\left|\frac{\partial g_2}{\partial \zeta_1}(\xi_1, \xi_2, \zeta_1, \zeta_2)\right| = 0 = M_{21}, \quad \left|\frac{\partial g_2}{\partial \zeta_2}(\xi_1, \xi_2, \zeta_1, \zeta_2)\right| \leq 1 = M_{22}$$

and

$$\Omega = \begin{pmatrix} -2 & 2 & 1 \\ 1 & -4 & 2 \\ 1 & 2 & -6 \end{pmatrix}$$

Since $-\Omega \in \aleph$, by Theorem 3.2, system (3.18) is absolutely stable.

In system (2.2), assume there exist $m_{ij} > 0, M_{ij} > 0$ and $\tau > 0$ such that

$$|f_{ij}(x) - f_{ij}(y)| \leq m_{ij}|x - y|, \quad |g_{ij}(x) - g_{ij}(y)| \leq M_{ij}|x - y|,$$
$$0 \leq \tau_{ij}(t) \leq \tau < \infty, \quad i, j = 1, 2, \ldots, n.$$

For any $x = (x_1, \ldots, x_n)^T \in R^n$, we define the norm of x as

$$\|x\| = \sum_{i=1}^{n} |x_i|.$$

THEOREM 3.3 *If*

$$(A4): \sum_{i=1}^{n} [m_{ij} + M_{ij}]/a_i < 1, \quad i, j = 1, 2, \ldots, n,$$

then system (2.2) has a unique equilibrium.

Proof For $x \in R^n$, we define a mapping $T: R^n \to R^n$ by

$$Tx_i = \left[\sum_{j=1}^{n} f_{ij}(x_j) + \sum_{j=1}^{n} g_{ij}(x_j) + I_i\right] \Big/ a_i, \quad i = 1, 2, \ldots, n.$$

It is obvious that T is continuous for any $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T \in R^n$,

$$\begin{aligned}
\|Tx - Ty\| &= \sum_{i=1}^{n} |Tx_i - Ty_i| \\
&\leq \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} [(|f_{ij}(x_j) - f_{ij}(y_j)|) + |g_{ij}(x_j) - g_{ij}(y_j)|] \Big/ a_i \right\} \\
&\leq \sum_{i=1}^{n} \left\{ \sum_{j=1}^{n} [(m_{ij} + M_{ij})(|x_j - y_j|)] \Big/ a_i \right\} \\
&\leq \sum_{j=1}^{n} \left\{ \left[\sum_{i=1}^{n} (m_{ij} + M_{ij})/a_i \right] (|x_j - y_j|) \right\} \\
&\leq \delta \sum_{j=1}^{n} |x_j - y_j| = \delta \|x - y\|,
\end{aligned}$$

where

$$\delta = \max_{1 \leq j \leq n} \left[\sum_{i=1}^{n} (m_{ij} + M_{ij})/a_i \right] < 1.$$

This indicates that T is a contraction mapping and hence, by Lemma 2.3, T has a unique fixed point, that is, there exists $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in R^n$ such that

$$x_i^* = \left[\sum_{j=1}^{n} f_{ij}(x_j^*) + \sum_{j=1}^{n} g_{ij}(x_j^*) + I_i \right] / a_i, \quad i = 1, 2, \ldots, n.$$

The proof is complete.

THEOREM 3.4 *If condition* (A_4) *is satisfied, then the equilibrium point of system (2.2) is exponentially stable.*

Proof Assume that $x^* = (x_1^*, \ldots, x_n^*) \in R^n$ is the unique equilibrium of system (2.2). Take $c > 0$ such that

$$\delta(c) = \max_{1 \leq j \leq n} \left\{ \sum_{i=1}^{n} [m_{ij} + M_{ij} e^{c\tau}]/(a_i - c) < 1 \right\}.$$

Let

$$P_i(t, x_i) = |x_i - x_i^*| e^{\epsilon(t-t_0)}.$$

then

$$\begin{aligned} D^+ P_i(t, x_i) \leq\ & \epsilon e^{\epsilon(t-t_0)} |x_i - x_i^*| + e^{\epsilon(t-t_0)} \Bigg[- a_i |x_i - x_i^*| \\ & + \sum_{j=1}^{n} |f_{ij}(x_j) - f_{ij}(x_j^*)| \\ & + \sum_{j=1}^{n} |g_{ij}(x_j(t - \tau_{ij}(t))) - g_{ij}(x_j^*)| \Bigg] \\ \leq\ & (-a_i + \epsilon) P_i + \sum_{j=1}^{n} m_{ij} P_j \\ & + \sum_{j=1}^{n} M_{ij} e^{\epsilon\tau} P_j(t - \tau_{ij}(t), x_i(t - \tau_{ij}(t))). \end{aligned}$$

For any $M > 1$, we claim that

$$P_i(t, x_i) \leq MK = M \max_{1 \leq i \leq n} \left\{ \sup_{t_0 - \tau \leq t \leq t_0} P_i(t, x_i) \right\}$$

for $t \geq t_0 - \tau$ and $i = 1, 2, \ldots, n$.

In fact, if it were not true, then there would exist i and $t_1 > t_0$ such that

$$P_i(t, x_i) \begin{cases} < & MK \ t < t_1 \\ = & MK \ t = t_1 \end{cases}$$

$$P_j(t, x_j) \leq MK, \quad t \leq t_1, \ j \neq i,$$

i.e., $D^+ P_i(t_1, x_i) \geq 0$. However,

$$\begin{aligned} D^+ P_i(t_1, x_i(t_1)) \leq\ & (-a_i + \epsilon) P_i(t_1, x_i(t_1)) + \sum_{j=1}^{n} m_{ij} P_j(t_1, x_j(t_1)) \\ & + \sum_{j=1}^{n} M_{ij} e^{\epsilon\tau} P_j(t_1 - \tau_{ij}(t_1), x_j(t_1 - \tau_{ij}(t_1))) \\ \leq\ & (-a_i + \epsilon) MK + \sum_{j=1}^{n} m_{ij} MK + \sum_{j=1}^{n} M_{ij} e^{\epsilon\tau} MK \\ \leq\ & \left[(-a_i + \epsilon) + \sum_{j=1}^{n} (m_{ij} + M_{ij} e^{\epsilon\tau}) \right] MK < 0. \end{aligned}$$

This contracdict with $D^+P_i(t_1, x_i(t_1)) \geq 0$ and hence

$$P_i(t, x_i) \leq MK, \quad (i = 1, 2, \ldots, n)$$

for all $t \geq t_0$. Since $P_i(t, x_i) = |x_i(t) - x_i^*|e^{\epsilon(t-t_0)}$, we have

$$|x_i(t) - x_i^*| = P_i(t, x_i)e^{-\epsilon(t-t_0)} \leq MKe^{-\epsilon(t-t_0)}.$$

The proof is completed.

4 APPLICATIONS TO CELLULAR NEURAL NETWORKS

In this section, applications of the obtained stability conditions in Section 3 to cellular neural networks (CNNs) are presented. CNNs represent a new paradigm for nonlinear analog signal processing and its applications for various practical problems have been demonstrated [10, 11]. The basic circuit unit of cellular neural networks is called a *cell*. It contains linear and nonlinear circuit elements, which typically are linear capacitors, linear resistors, linear and nonlinear controlled sources, and independent sources .Any cell in a cellular neural network is connected only to its neighbor cells. The adjacent cells can interact directly with each other. Cells not directly connected together may affect each other indirectly because of the propagation effects of the *comtinuous*-time dynamics of cellular neural networks. Nonlinear and delay-type CNNs (DCNNs) were introduced recently in [6] and have found applications in the areas of classification of patterns and reconstruction of moving images. In general, the dynamic behavior of a DCNN can be described by the following system [6, 9]

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{n} a_{ij}f(x_j(t)) + \sum_{j=1}^{n} b_{ij}f(x_j(t - \tau_{ij})) + u_i, \quad i = 1, 2, \ldots, n, \tag{4.19}$$

where $x(.) = \{x_1(.), \ldots, x_n(.)\}^T$ is the input state vector, $f(x(.)) = \{f(x_1(.)), \ldots, f(x_n(.))\}$ is the output vector, $f(x) \approx [|x + 1| - |x - 1|]/2; 0 \leq \tau_{ij} \leq \tau < \infty$ is a delay of the interaction from cell j onto

the cell i. $A = \{a_{ij}\}$ is the feedback matrix, $B = \{b_{ij}\}$ is the delayed feedback matrix, $u = (u_1, \ldots, u_n)^T$ is an external input. When used as a pattern classifier, the DCNN is required to possess a unique and globally asymptotically stable equilibrium point independently of the initial conditions [13]. Note that (4.19) is a special case of (2.2) where $a_i = 1, f_{ij}(s) = g_{ij}(s) = f(s) \approx [|s+1| + |s-1|]/2, i,j = 1,2,\ldots,n$.

Since

$$U^* = \{f \mid f \in C(R,R), |f(x) - f(y)| \leq |x-y|, \\ (f(x) - f(y))(x-y) \geq 0\},$$

it can be seen that if $f(x) = [|x+1| - |x-1|]/2$, $f \in U^*$. By Theorems 3.3 and 3.4, the following results can be obtained.

THEOREM 4.1 *If $f \in U$ and*

$$\sum_{j=1}^{n} (|a_{ji}| + |b_{ji}|) < 1, \quad i = 1,2,\ldots,n,$$

then network (4.19) has an equilibrium point which is globally exponentially stable.

Remark

(i) In Theorem 4.1, $f(x)$ is not required to be exactly equal to $(|x+1| - |x-1|)/2$ and hence, Theorem 4.1 is more general and has some robustness;

(ii) Since one function $f \in U^*$ of networks (4.19) is used only for the equilibrium point in Theorem 4.1, more useful results can be obtained as follows.

THEOREM 4.2 *If $f \in U^*$ and*

$$-1 + \sum_{i=1}^{n} (a_{ij}^* + |b_{ij}|) < 0,$$

where

$$a_{ij}^* = \begin{cases} a_{ii}, & i = j \\ |a_{ij}|, & i \neq j, \end{cases}$$

then network (4.19) has a unique equilibrium point which is globally exponentially stable.

Proof Let $x^* = (x_1^*, \ldots, x_n^*)^T$ be an equilibrium of (4.19). Rewrite (4.19) as

$$(x_i(t) - x_i^*)' = -(x_i(t) - x_i^*) + \sum_{j=1}^{n} a_{ij}(f(x_j(t)) - f(x_j^*)) + \sum_{j=1}^{n} b_{ij}(f(x(t-\tau_{ij})) - f(x_j^*)), \quad i = 1, 2, \ldots, n. \tag{4.20}$$

Define a Lyapunov functional V_i by

$$V_i = |x_i - x_i^*| + \sum_{j=1}^{n} |b_{ij}| \int_{t-\tau_{ij}}^{t} |f(x_j(s)) - f(x_j^*)| ds,$$

then along with system (4.20), we have

$$\begin{aligned} D^+ V_i &\le -|x_i - x_i^*| + \sum_{j=1}^{n} a_{ij}^* |f(x_j) - f(x_j^*)| \\ &\quad + \sum_{j=1}^{n} |b_{ij}| |f(x_j(t-\tau_{ij})) - f(x_j^*)| \\ &\quad + \sum_{j=1}^{n} |b_{ij}| [|f(x_j) - f(x_j^*)| - |f(x_j(t-\tau_{ij})) - f(x_j^*)|] \\ &= -|x_i - x_i^*| + \sum_{j=1}^{n} [a_{ij}^* + |b_{ij}|] |f(x_j) - f(x_j^*)|, \end{aligned}$$

where $f \in U^*$. Let $V = \sum_{i=1}^{n} V_i$, then $V \ge \sum_{i=1}^{n} |x_i - x_i^*|$ and

$$\begin{aligned} D^+ V &\le -\sum_{i=1}^{n} |x_i - x_i^*| + \sum_{i=1}^{n} \sum_{j=1}^{n} [a_{ij}^* + |b_{ij}|] |f(x_j) - f(x_j^*)| \\ &\le -\sum_{i=1}^{n} |x_i - x_i^*| + \sum_{j=1}^{n} \left[\sum_{i=1}^{n} (a_{ij}^* + |b_{ij}|) \right] |f(x_j) - f(x_j^*)| \end{aligned}$$

$$
\begin{aligned}
&\leq -\sum_{j=1}^{n}\{|x_j - x_j^*| - |f(x_j) - f(x_j^*)|\} \\
&\quad + \sum_{j=1}^{n}\left[-1 + \sum_{i=1}^{n}(a_{ij}^* + |b_{ij}|)\right]|f(x_j) - f(x_j^*)| \\
&\leq -\sum_{j=1}^{n}\{|x_j - x_j^*| - |f(x_j) - f(x_j^*)|\} - \lambda\sum_{i=1}^{n}|f(x_j) - f(x_j^*)| \\
&\leq -\lambda\sum_{j=1}^{n}|x_j - x_j^*| \leq -\lambda V,
\end{aligned}
$$

where

$$
-\lambda = \max_{1\leq j\leq n}\left[-1 + \sum_{i=1}^{n}(a_{ij}^* + |b_{ij}|)\right] < 0.
$$

By Lemma 2.2, we have $V \leq r(t, t_0, r_0)$, here $r(t, t_0, r_0) = r_0 e^{-\lambda(t-t_0)}$ is the maximal solution of

$$
u' = -\lambda u, \quad u(t_0) = r_0.
$$

Let $r_0 = \sup_{t_0-\tau\leq s\leq t_0}\sum_{i=1}^{n}|x_i - x_i^*|$, we have

$$
\sum_{i=1}^{n}|x_i(t) - x_i^*| \leq V \leq \sup_{t_0-\tau\leq s\leq t_0}\sum_{i=1}^{n}|x_i - x_i^*|e^{-\lambda(t-t_0)}.
$$

The proof is completed.

Example 4.1 Consider the following system

$$
\begin{cases}
x_1' = -x_1 - 3f(x_1(t)) - 2f(x_2(t)) - f(x_1(t-\tau_{11})) + f(x_3(t-\tau_{13})) + u_1 \\
x_2' = -x_2 + f(x_1(t)) - 4f(x_2(t)) + 2f(x_2(t-\tau_{22})) + 3f(x_3(t-\tau_{33})) + u_2 \\
x_3' = -x_3 + f(x_1(t)) - 6f(x_3(t)) + 2f(x_3(t-\tau_{33})) + u_3,
\end{cases}
\tag{4.21}
$$

where $f \in U^*$. Compare with (4.19), we have

$$
\begin{array}{lllllll}
a_{11}^* = -3, & a_{12}^* = -2, & a_{13}^* = 0, & b_{11} = -1, & b_{12} = 0, & b_{13} = 1 \\
a_{21}^* = 1, & a_{22}^* = -4, & a_{23}^* = 0, & b_{21} = 0, & b_{22} = 2, & b_{23} = 3, \\
a_{31}^* = 1, & a_{32}^* = 0, & a_{33}^* = -6, & b_{31} = 0, & b_{32} = 0, & b_{33} = 2.
\end{array}
$$

It can be seen that all conditions of Theorem 4.2 are satisfied. Hence, the unique equilibrium point of (4.21) is globally exponentially stable.

It is worth to mention that the property of exponential stability of this example can not be obtained by [9–12].

Next, we consider a model of bidirectional associative memory neural network with delays

$$\begin{cases} x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{q} a_{ij} g_j(y_j(t - \tau_{ij}^1(t))) + I_i, & i = 1, 2, \ldots, p \\ y_j'(t) = -b_j y_j(t) + \sum_{i=1}^{p} b_{ij} g_i(x_i(t - \tau_{ji}^2(t))) + J_j, & j = 1, 2, \ldots, q \end{cases} \tag{4.22}$$

where $x = (x_1, \ldots, x_p) \in R^p$, $y = (y_1, \ldots, y_q) \in R^p$, $a_i, b_j > 0$, $g_i, g_j \in C'$ and $|g_i'(s)| \le m_i$, $|s_j'(s)| \le m_j$, $0 \le \tau_{ij}^1(t) \le \tau$, $0 \le \tau_{ji}^2(t) \le \tau$. Note that

$$\Omega_1 = \begin{pmatrix} -a_1 & 0 & \cdots & 0 & |a_{11}|m_1 & \cdots & & |a_{1q}|m_q & |I_1| \\ \cdots\cdots & \cdots\cdots & & & & & & & \\ 0 & \cdots & 0 & -a_p & |a_{p1}|m_1 & \cdots & & |a_{pq}|m_p & |I_p| \\ |b_{11}|m_1 & \cdots & & |b_{1p}|m_p & -b_1 & 0 & \cdots & 0 & |J_1| \\ \cdots\cdots & & \cdots\cdots & & & & & & \\ |b_{q1}|m_1 & & \cdots & |b_{qp}|m_p & 0 & \cdots & 0 & -b_q & |J_q| \end{pmatrix}$$

Using Theorems 3.3 and 3.4, we can conclude that if $-\Omega_1 \in \aleph$, system (4.22) has an equilibrium point which is globally exponentially stable.

5 CONCLUSIONS

In this paper, we have studied the problem of absolute stability for nonlinear systems described by differential equations with time delays. In addition, we have investigated the existence of unique equilibrium point and its global exponential stability for some special cases. Our approaches have utilized the method of Lyapunov functions, fixed point theorem, the comparison principle and the techniques of differential inequalities. The stability results may be generalized to other systems and may be more applicable in real world applications.

Acknowledgment

This work has been supported by the Natural Sciences and Engineering Research Council of Canada under grants No OG-PIN108310 and No RGPIN203560.

References

[1] Lur'e, A. I. and Postnikov, V. N., (1944). "On the Theory of Stability of Control Systems", *Prikl. Matem. i Mehk.*, **VIII**(3), 246–248, (in Russian).
[2] Lur'e, A. I. (1951). "On the Problem of Stability of Regulating Systems", *Prikl. Matem. i Mehk.*, **XV**(1), 69–74, (in Russian).
[3] LaSalle, J. P. and Lefschetz, S. (1961). "*Stability by Liapunov's Direct Method with Applications*", Academic Press, New York, London.
[4] Wang, L. and Wang, M. (1991). "Lecture Notes in Contemporary Mathematics", **2** (1991), 235–247.
[5] Wang, L. Zhang, Y. and Zhang, Y. (1993). "Absolute Stability of a Type Control System with Delay", *Chinese Science Bulletin*, **38**(16), 1445–1448.
[6] Roska, T. and Chua, L. O. (1990). "Cellular Neural Networks with Nonlinear and Delay-Type Template Elements", *Proc. IEEE Int. Workshop on Cellular Networks and Their Applications*, Dec 16–19, 1990, Budapest (CNNA-90),12–25.
[7] Lakshmikantham, V. and Leela, S. (1969). "*Differential and Integral Inequalities*", Vol.1 and 2, Academic Press, New York.
[8] Berman, A. and Plemmons, R. J. (1979). "*Nonnegative Matrices in the Mathematical Sciences*", Academic Press, New York.
[9] Roska, T. Wu, C. W. Balsi, M. and Chua, L. O. (1992). "Stability and Dynamics of Delay Type General and Cellular Neural Networks", *IEEE Trans. Circuits Sys.*, **39**, 487–490.
[10] Chua, L. O. and Yang, L. (1988). "Cellular Neural Networks: Theory", *IEEE Trans. Circuits Sys.*, **35**, 1257–1272.
[11] Chua, L. O. and Yang, L. (1988). "Cellular Neural Networks: Applications", *IEEE Trans. Circuits Sys.*, **35**, 1273–1290.
[12] Arik, S. and Tavsanoglu, V. (1998). "Equilibrium Analysis of Delayed CNN's", *IEEE Trans. Circuits Sys.*, **45**(2), 168–171.
[13] Roska, T. Wu, C. W. Balsi, M. and Chua, L. O. (1993). "Stability of Cellular Neural Networks with Dominant Nonlinear and Delay-type Templates", *IEEE Trans. Circuits Sys.*, **40**, 270–272.
[14] Zhang, Y. "*Qualitative Analysis of Bidirectional Associative Memory Neural Networks with Delays*", to appear in Computer Study and Development.