Absolutely Continuous Invariant Measures for One-Parameter Families of One-Dimensional Maps

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Abstract. Given a one-parameter family $f_\lambda(x)$ of maps of the interval $[0, 1]$, we consider the set of parameter values λ for which f_λ has an invariant measure absolutely continuous with respect to Lebesgue measure. We show that this set has positive measure, for two classes of maps: i) $f_\lambda(x) = \lambda f(x)$ where $0 < \lambda \leq 4$ and $f(x)$ is a function C^3-near the quadratic map $x(1-x)$, and ii) $f_\lambda(x) = \lambda f(x)$ (mod 1) where f is C^3, $f(0) = f(1) = 0$ and f has a unique nondegenerate critical point in $[0, 1]$.

0. Introduction

Dynamical systems generated by noninvertible maps of an interval into itself have been intensely studied recently. The most widely considered was the family $f_\lambda : x \to \lambda x(1-x)$, $x \in [0, 1]$, $0 \leq \lambda \leq 4$.

It is well-known that if f_λ has an attracting periodic orbit $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ then all probabilisitic f_λ-invariant measures are singular with respect to a Lebesgue measure dx, and the iterations $f_{\lambda *}^n dx$ converge in the weak *-topology to the discrete invariant measure supported by $\bar{\alpha}$.

It is probable (but not proved) that this situation is typical from the topological point of view, i.e. for a general one-parameter family of smooth mappings $f_\lambda : I \to I$, $\lambda \in \Lambda$, there is an open and dense subset Λ_0 of Λ such that for $\lambda \in \Lambda_0$, the set of limit points for $f_{\lambda *}^n dx$ consists of a finite number of measures supported by periodic attracting orbits.

We show in the present paper that this is not so from the metric point of view. Namely we prove for a certain class of one-parameter families f_λ that the set

$\Lambda_1 = \{\lambda : f_\lambda$ has an invariant finite measure μ_λ absolutely continuous with respect to dx $(\mu_\lambda < dx)\}$

has a positive measure in Λ.

In the classical case $x \to 4x(1-x)$ considered by Ulam and von Neumann in [1], the invariant measure $\mu(dx)$ has density $\varrho(x) = \dfrac{1}{\pi \sqrt{x(1-x)}}$. In [2] Bunimovič

constructed absolutely continuous measures for the piecewise smooth mappings $x \to n\sin\pi x(\bmod 1)$, $n \in \mathbb{Z}$. Ruelle in [3] considered $f_\lambda : x \to \lambda x(1-x)$ and proved that an invariant measure $\mu_\lambda < dx$ exists for $\lambda = 3,678\ldots$ – chosen in such a way that the third iterate of the critical point, $f_\lambda^3(\frac{1}{2})$, falls into the unstable fixed point $x = 1 - \dfrac{1}{\lambda}$.

Bowen in [4] found sufficient conditions for the existence of an invariant measure $\mu_\lambda < dx$ for $f_\lambda(x) = \lambda x(1-x)$, when $\frac{1}{2}$ is a preimage of a periodic unstable point. In [5] it was shown that the cardinality of $\{\lambda : f_\lambda$ has an invariant measure $\mu_\lambda < dx\}$ is that of the continuum for the family $x \to \lambda x(1-x)$ and any C^2-family f_λ sufficiently close to $\lambda x(1-x)$. Similar results were obtained by Misiurewicz [6] and Szlenk [7] for a class of mappings with negative Schwarzian derivative. Ognev in [8] proved for $x \to \lambda x(1-x)$ that if $\frac{1}{2}$ is a preimage of a periodic unstable point, then the density of the invariant measure is analytic. Ito, Tanaka, Nakada in [9] studied the space of parameters of unimodal linear transformations and found explicitly the densities of the invariant measures.

   Collet and Eckmann in [10] proved for a particular family $f_\delta(x)$ that f_δ has sensitive dependence with respect to initial conditions in the sense of Guckenheimer [11] for a set of δ of positive measure. The mappings f_γ obtained with our construction are also sensitive dependent. It is unknown whether sensitive dependence implies existence of absolutely continuous invariant measure.

   We shall consider two kinds of one-parameter families $f_\lambda(x)$.

   1. Piecewise smooth families $x \mapsto \lambda f(x)$ (mod 1), where $f(x) : [0,1] \to [0,1]$ is a C^3-map with a single nondegenerate critical point, $f(0) = f(1) = 0$, and λ is a big parameter.

   2. Smooth families $x \mapsto \lambda x(1-x) 0 \leq \lambda \leq 4$, and $\lambda \cdot f(x)$ with $f(x)$ sufficiently close to $x(1-x)$ in $C^3([0,1], [0,1])$.

   We formulate now our main results.

Theorem A. Let $f_\lambda : x \to \lambda f(x)$ (mod 1) *be a piecewise smooth family. There exists* $T_0 > 0$, *such that for any* $\varepsilon > 0$ *there is an* $L(\varepsilon)$, *so that if* $L \geq L(\varepsilon)$ *then the interval* $[L, L + T_0]$ *on the* λ-*axis contains a set* $\mathcal{M}$ *satisfying*
   i) $\operatorname{mes} \mathcal{M} > T_0 - \varepsilon$;
   ii) $\forall \lambda \in \mathcal{M}$ f_λ *admits an invariant measure* $\mu_\lambda < dx$.

Theorem B. Let $f_\lambda(x)$ *be one of the smooth families mentioned above. Then there is a set of positive measure* Λ_1 *so that for* $\lambda \in \Lambda_1$ f_λ *admits an invariant measure* $\mu_\lambda < dx$.

Remark. The parameter values λ_1 such that the critical point of f_{λ_1} is contained in the preimage of an unstable periodic orbit (e.g. $\lambda_1 = 4$ for $\lambda \cdot x(1-x)$, or in the preimage of a certain invariant unstable Cantor set (see [5])) turn out to be one-sided Lebesgue points of Λ_1, i.e. $\forall \varepsilon > 0 \exists \delta > 0$, such that

$$\operatorname{mes}\{\lambda \in \Lambda_1 : \lambda_1 \geq \lambda \geq \lambda_1 - \delta\} > \delta(1-\varepsilon).$$

   In Sects. 1–12 we prove Theorem A for the family $x \to \lambda x(1-x)$ (mod 1). In Sect. 13 we point out modifications concerning the case of an arbitrary family $x \to \lambda \cdot f(x)$ (mod 1) and show how to reduce the proof of Theorem B to the proof of Theorem A.

1. Idea of Proof

The number T_0 for the family $f_\lambda : x \to \lambda x(1-x) \pmod 1$ equals 4: as λ varies from L to $L+4$, the image of the critical point $f_\lambda(\frac{1}{2}) = \dfrac{\lambda}{4} \pmod 1$ passes over the entire interval $[0,1]$). In order to prove Theorem A we must find for a given $\varepsilon > 0$ an $L(\varepsilon)$ such that, if $L \geq L(\varepsilon)$ then the interval $[L, L+4]$ contains a set $\mathscr{M}$ so that $\mathrm{mes}\,\mathscr{M} > 4 - \varepsilon$ and for any $\lambda \in \mathscr{M}\, f_\lambda$ has an invariant measure $\mu_\lambda < dx$. Without loss of generality we can assume that λ varies from $N_0 = 4k_0$ to $N_0 + 4$, $k_0 \in \mathbb{Z}_+$. For a smooth map $g(\lambda, x)$ we shall use the notation Dg, D^2g for $\dfrac{\partial g(\lambda, x)}{\partial x}, \dfrac{\partial^2 g(\lambda, x)}{\partial x^2}$.

The central part of the proof of Theorem A is the construction for $\lambda \in \mathscr{M}$ of a special partition ξ_λ of $[0, 1]$. The elements of ξ_λ are intervals $\varDelta_i(\lambda)$, $i \in \mathbb{Z}_+$, which satisfy the following conditions:

   i) $\mathrm{int}\,\varDelta_i(\lambda) \cap \mathrm{int}\,\varDelta_j(\lambda)) = \emptyset$.
   ii) $\forall i \exists n_i \in \mathbb{Z}_+$ such that $f_\lambda^{n_i}$ maps $\varDelta_i(\lambda)$ diffeomorphically onto $[0, 1]$.
   iii) $\inf\limits_{\varDelta_i \in \xi_\lambda} \min\limits_{x \in \varDelta_i} |Df_\lambda^{n_i}(x)| > \lambda^{c_0}$ for some $c_0 > 0$ (λ is a big parameter here, so $\lambda \gg 0$).
   iv) $\sup\limits_{\varDelta_i \in \xi_\lambda} \max\limits_{x \in \varDelta_i} \left| \dfrac{D^2 f_\lambda^{n_i}(x)}{Df_\lambda^{n_i}(x)} \right| \cdot |\varDelta_i(\lambda)| < 1 + \lambda^{-t_1}$, for some $t_1 > 0$.

Let $\mathscr{X}(\lambda)$ be the union of all elements $\varDelta_i(\lambda)$ of ξ_λ. Then $\mathscr{X}(\lambda) = [0, 1] \pmod 0$.

The set $\mathscr{M}$ and the sets $\mathscr{X}(\lambda)$ for $\lambda \in \mathscr{M}$ are constructed by induction. $\mathscr{M}$ is obtained as an intersection $\mathscr{M} = \bigcap\limits_{n=0}^{\infty} \mathscr{M}_n$, where

$$\mathscr{M}_0 = [N_0, N_0 + 4], \qquad \mathscr{M}_{n+1} \subset \mathscr{M}_n,$$

$$\mathrm{mes}\,\mathscr{M}_{n+1} > (1 - \varepsilon_{n+1})\,\mathrm{mes}\,\mathscr{M}_n, \qquad \sum_{n=1}^{\infty} \varepsilon_n = O(\lambda^{-t_2}), \qquad t_2 > 0.$$

At the nth induction step, we define for any $\lambda \in \mathscr{M}_{n-1}$ a set $\mathscr{X}_n(\lambda) \subset [0, 1]$ which is the union of a countable number of intervals $\varDelta_i^{(k)}(\lambda)$, $k = 1, \dots, n$. The intervals constructed at step k do not change at the next steps. The sets $\mathscr{X}_n(\lambda)$ satisfy the following properties:

$$\mathscr{X}_n(\lambda) \subset \mathscr{X}_{n+1}(\lambda); \qquad \mathrm{mes}\,\mathscr{X}_n(\lambda) > 1 - \lambda^{-t_3 n}, \qquad t_3 > 0.$$

Finally we set $\mathscr{X}(\lambda) = \bigcup\limits_{n=1}^{\infty} \mathscr{X}_n(\lambda)$. Any element $\varDelta_i(\lambda)$ of ξ_λ coincides with one of $\varDelta_i^{(n)}(\lambda)$.

Let us define the map $T_\lambda : \mathscr{X}(\lambda) \to [0, 1]$ by $T_\lambda | \varDelta_i(\lambda) = f_\lambda^{n_i}$. The results of Adler [12] and Walters [13] imply the existence and the uniqueness of a T_λ-invariant measure $\nu_\lambda < dx$. The endomorphism $([0,1], T_\lambda, \nu_\lambda)$ is exact, and its natural extension is a Bernoulli shift. The f_λ-invariant measure μ_λ is constructed from ν_λ.

2. First Steps of the Inductive Construction

The graph of the map f_λ consists of a lot of monotone branches which we denote by $f(\lambda, x)$ and the middle parabola denoted by $h(\lambda, x)$. The domains of $f(\lambda, x)$ and

$h(\lambda, x)$ depend continuously on λ. When $\lambda = 4k_0$, a new middle branch is born, which exists for $\lambda \leq 4(k_0 + 1)$ and then breaks up into two monotone branches.

We shall denote by $\Delta f(\lambda, x)$ the domain of $f(\lambda, x)$, by $x_{min}(\lambda)$ the endpoint nearest to $\frac{1}{2}$ of the interval $\Delta f(\lambda, x)$, and by $x_{max}(\lambda)$ the other endpoint of $\Delta f(\lambda, x)$. We shall distinguish $[a, b]$ from $[b, a]$ according to its position relative to $\frac{1}{2}$ and not according to its orientation.

We fix a positive number $s < \frac{1}{13}$.

Step 1

Pick the branch $f'(\lambda, x)$ of f_λ whose domain $\Delta f'(\lambda, x) = \Delta'(\lambda) = [x'_{min}(\lambda), x'_{max}(\lambda)]$ is contained in $[0, \frac{1}{2}]$ and is closest to $\frac{1}{2}$, subject to the condition

$$|x'_{min}(\lambda) - \tfrac{1}{2}| > \lambda^{-s} \quad \text{for all } \lambda \in \mathcal{M}_0.$$

Denote by $\Delta''(\lambda) = \Delta f''(\lambda, x)$ the analogous interval in $[\frac{1}{2}, 1]$. Define $\delta_1(\lambda) = [x'_{min}(\lambda), x''_{min}(\lambda)]$, noting that $\delta_1(\lambda)$ has the form

$$\delta_1(\lambda) = [\tfrac{1}{2} - r_1(\lambda), \tfrac{1}{2} + r_1(\lambda)], \qquad r_1(\lambda) > \lambda^{-s} \tag{2.1}$$

and let $\mathcal{X}_1(\lambda) = [0, 1] \backslash \delta_1(\lambda)$. Thus,

$$[0, 1] = \mathcal{X}_1(\lambda) \bigcup \delta_1(\lambda).$$

Both $\mathcal{X}_1(\lambda)$ and $\delta_1(\lambda)$ are the union of several domains of branches, $\Delta f(\lambda, x)$, varying continuously with λ.

Since

$$|Df(\lambda, x)| = 2\lambda |x - \tfrac{1}{2}|,$$

we have

$$|\Delta'(\lambda)| < \tfrac{1}{2} \lambda^{-1+s}$$

and

$$\left| \frac{dx'_{min}(\lambda)}{d\lambda} \right| = \left| \frac{\partial f(\lambda, x)/\partial \lambda}{\partial f(\lambda, x)/\partial x} \right|_{x = x'_{min}(\lambda)} < \frac{1}{8\lambda^{1-s}}.$$

This implies

$$\frac{1}{\lambda^s} < r_1(\lambda) < \frac{1}{\lambda^s} + \frac{1}{\lambda^{1-s}} = \frac{1}{\lambda^s} \left[1 + \frac{1}{\lambda^{1-2s}} \right]. \tag{2.1a}$$

In order to construct the set $\mathcal{M}_1$ we consider the domains $\Delta f(\lambda, x) = [x_{min}(\lambda), x_{max}(\lambda)]$ satisfying

$$|x_{min}(\lambda) - \tfrac{1}{2}| > 1/\lambda^{s/2}.$$

We obtain as above that for any such domain

$$|\Delta f(\lambda, x)| < \tfrac{1}{2} \lambda^{-1+s/2}$$

$$\left| \frac{dx_{min}(\lambda)}{d\lambda} \right| < \tfrac{1}{8} \lambda^{-1+s/2}.$$

The top of the graph, $h(\lambda, \frac{1}{2})$, moves with velocity

$$\frac{dh(\lambda, \frac{1}{2})}{d\lambda} = \frac{1}{4}. \tag{2.1b}$$

A comparison of velocities shows that to each branch $f_i(\lambda, x)$ with domain $\Delta_i(\lambda)$ there corresponds a uniquely defined interval $\mathscr{J}_i = \mathscr{J}(\Delta_i)$ of λ-values such that, as λ ranges over $\mathscr{J}_i$, the top $h(\lambda, \frac{1}{2})$ ranges over $\Delta_i(\lambda)$ and its image $f_i(\lambda, h(\lambda, \frac{1}{2}))$ ranges over $[0, 1]$.

So we define $\mathscr{M}_1$ as the union of these $\mathscr{J}_i$:

$$\mathscr{M}_1 = \bigcup \{ \mathscr{J}_i = \mathscr{J}(\Delta_i) | (\forall \lambda \in \mathscr{M}_0) |x_{\min}(\lambda) - \tfrac{1}{2}| > 1/\lambda^{s/2} \}.$$

It follows from the estimates (2.1), (2.1a) and (2.1b) that

$$\operatorname{mes} \mathscr{M}_1 > 4 \left[1 - \max_{N_0 \leq \lambda \leq N_0 + 4} \operatorname{mes} \mathscr{X}_1(\lambda) \right] > 4 \left[1 - \frac{2(1 + \gamma_1)}{N_0^{s/2}} \right], \tag{2.2}$$

where

$$\gamma_1 < 1/N_0^{1-s}.$$

Step 2. Construction of $\mathscr{X}_2(\lambda)$

Let us denote by f_1 the branches $f(\lambda, x)$ such that $\Delta f \subset \mathscr{X}_1(\lambda)$ and by g the branches with $\Delta g \subset \delta_1(\lambda)$. Let us consider compositions $f_1 \circ g$. Any domain Δg can be represented in the form

$$\Delta g = \bigcup \Delta(f_1 \circ g) \cup \bigcup g^{-1}(\delta_1). \tag{2.3}$$

Choose an interval

$$\delta_2(\lambda) = \left[\frac{1}{2} - \frac{c_{21}}{\lambda^{2s}}, \frac{1}{2} + \frac{c_{22}}{\lambda^{2s}} \right], \quad 1 < c_{21}, c_{22} < 1 + O(1/\lambda^{1-3s})$$

which is a union of domains $\Delta(f_1 \circ g)$ and $g^{-1}\delta_1$. We shall use g_1 to denote $g|\delta_1 \backslash \delta_2$ and f_{21} to denote $f_1 \circ g_1$. Then (2.3) implies

$$\delta_1 = \bigcup \Delta f_{21} \cup \bigcup g_1^{-1} \delta_1 \cup \delta_2. \tag{2.4}$$

For any particular branch $\tilde{g}_1$ we have

$$\tilde{g}_1^{-1}(\delta_1) = \bigcup \tilde{g}_1^{-1}(\Delta f_{21}) \cup \bigcup \tilde{g}_1^{-1} \circ g_1^{-1}(\delta_1) \cup \tilde{g}_1^{-1} \delta_2,$$

where the large unions are over all f_{21} and g_1 respectively. Denote the branches $f_{21} \circ g_1$ by f_{22}. Since $\Delta(f_{21} \circ g_1) = g_1^{-1}(\Delta f_{21})$, we can rewrite (2.4) as

$$\delta_1 = \bigcup \Delta f_{21} \cup \bigcup \Delta f_{22} \cup \bigcup g_1^{-2}(\delta_1) \cup \bigcup g_1^{-1}(\delta_2) \cup \delta_2, \tag{2.5}$$

where g_1^{-2} denotes any composition of the form $\tilde{g}_1^{-1} \circ \tilde{g}_1^{-1}$. Proceeding in the same way we obtain the representation

$$\delta_1 = \bigcup \Delta f_{21} \cup \bigcup \Delta f_{22} \cup \ldots \cup \bigcup \Delta f_{2k} \cup \bigcup g_1^{-(k-1)}(\delta_2)$$
$$\cdot \cup \ldots \cup \bigcup g_1^{-1}(\delta_2) \cup \delta_2 \cup \bigcup g_1^{-k} \delta_1, \tag{2.6}$$

where

$$f_{2\ell} = f_{21} \circ g_{1i_1} \circ \ldots \circ g_{1i_{\ell-1}}$$
$$g_1^{-r} = g_{1i_r}^{-1} \circ \ldots \circ g_{1i_1}^{-1}.$$

Any branch g_1 satisfies

$$|Dg_1| > 2\lambda^{1-2s}$$
$$|D^2 g_1| = 2\lambda$$
(2.7)

from which it follows (see for example [11]) that

$$\lim_{k \to \infty} \mathrm{mes}\left[\bigcup g_1^{-k}(\delta_1)\right] = 0.$$

Therefore, we can write

$$\delta_1 = \bigcup_{k=1}^{\infty} (\Delta f_{2k}) \cup \bigcup_{k=1}^{\infty} g_1^{-k}(\delta_2) \cup \delta_2 (\mathrm{mod}\, 0),$$
(2.8)

where mod 0 means we neglect sets with zero Lebesgue measure. (Hereafter, in analogous equalities, "mod 0" will be understood.) Using the notation f_2 for all the f_{2k}, $k = 1, 2, \ldots$, we obtain

$$[0,1] = \bigcup \Delta f_1 \cup \bigcup \Delta f_2 \cup \bigcup_{k=1}^{\infty} g_1^{-k}(\delta_2) \cup \delta_2$$
(2.9)

or

$$[0,1] = \mathscr{X}_2(\lambda) \cup \bigcup_{k=1}^{\infty} g_1^{-k}(\delta_2) \cup \delta_2,$$
(2.10)

where by construction $\mathscr{X}_2(\lambda)$ is partitioned by the various domains Δf_1 and Δf_2 constructed in steps 1 and 2. These domains will be elements of the partition ξ_λ.

Now (2.3) and (2.8) induce an analogous structure inside δ_2:

$$\delta_2 = \bigcup \Delta(f_1 \circ g) \cup \bigcup_{k=1}^{\infty} \Delta(f_{2k} \circ g) \cup \bigcup_{n=0}^{\infty} g^{-1} \circ g_1^{-n}(\delta_2).$$
(2.11)

Notice that one of the g's in (2.11) stands for h. Suppose $h(\frac{1}{2}) \in \Delta \tilde{f}_1$. Then for any other branch $f_1 \neq \tilde{f}_1$ either $f_1 \circ h$ has two monotone branches or none; similarly h^{-1} on δ_2 has two or no monotone branches. The only branch of parabolic type in (2.11) is $\tilde{f}_1 \circ h$.

We see from (2.10) that $\mathscr{X}_2(\lambda)$ is the complement (mod 0) of the preimages of δ_2 under the various branches $g_1^k (k \geq 0)$. At the end of the next section, we will see that $\mathscr{M}_2$ is the set of those $\lambda \in \mathscr{M}_1$ for which the appropriate branch f_1 takes the critical value $h(\lambda, \frac{1}{2})$ into the complement of the g_1^k-preimages of an interval $\hat{\delta}_2$ which is also small but much larger than δ_2.

3. Step $n+1$. Geometrical Part

We assume after step n that the set $\mathscr{M}_n$ has been defined and for every $\lambda \in \mathscr{M}_n$ the set $\mathscr{X}_n(\lambda)$ has been constructed. Every $\mathscr{X}_n(\lambda)$ is a countable union of domains $\Delta f_k(\lambda, x)$,

$k = 1, 2, \ldots, n$, where we use f_k to denote a branch constructed at step k. The interval $[0, 1]$ can be represented (mod 0) in the following form:

$$[0, 1] = \left[\bigcup_{k=1}^{n} (\bigcup \Delta f_k) \right] \cup \left[\bigcup_{m=1}^{\infty} (\bigcup \delta_n^{-m}) \right] \cup \delta_n. \tag{3.1}$$

Here the interval

$$\delta_n = \delta_n(\lambda) = \left[\frac{1}{2} - \frac{c_{n1}}{\lambda^{sn}}, \frac{1}{2} + \frac{c_{n2}}{\lambda^{sn}} \right], \quad 1 \leqq c_{n1}, c_{n2} \leqq 1 + O\left(\frac{1}{\lambda^{tn}} \right), \quad t = \frac{\alpha}{10},$$

and δ_n^{-m} are various diffeomorphic preimages of δ_n. We shall denote by $G_n : \delta_n^{-m} \to \delta_n$ the corresponding diffeomorphisms without pointing out their dependence on m; if $m = 0$, $G_n = \mathrm{Id}$.

In order to describe the representation of δ_n analogous to (3.1) we need some additional notation. Let F_{n-1} be a composition of maps f_k constructed at the previous steps:

$$F_{n-1} = f_{i_{n-1}} \circ f_{i_{n-2}} \circ \ldots \circ f_{i_2} \circ f_{i_1}, \quad i_1 = 1, \quad i_2 \in [1, 2], \ldots, i_{n-1} \in [1, n-1].$$

We shall distinguish two kinds of branches for various powers of f with domains inside δ_n: the first have the form $F_{n-1} \circ g(\lambda, x)$ ($F_{n-1} \circ h(\lambda, x)$ for the central branch) where g denotes the initial map $x \to \lambda x (1 - x)$: and the second kind are all the remaining branches, mapping their domains diffeomorphically onto $[0, 1]$, and denoted by $\hat{f}_n(\lambda, x)$. So we assume δ_n has the following representation after Step n:

$$\delta_n = (\bigcup \Delta F_{n-1} \circ g) \cup (\bigcup \Delta \hat{f}_n) \cup \left[\bigcup_{m=m_n}^{\infty} (\bigcup \delta_n^{-m}) \right]. \tag{3.2}$$

Now for any $\lambda \in \mathcal{M}_n$ we describe the construction of $\mathscr{X}_{n+1}(\lambda)$. The estimates which allow us to realize this construction are adduced in subsequent sections.

a) We consider the compositions $f_k \circ F_{n-1} \circ g$ and $f_k \circ \hat{f}_n$ for all $f_k (k \in [1, n])$, $F_{n-1} \circ g$, and $\hat{f}_n$. Then the domains $\Delta F_{n-1} \circ g$ and $\Delta \hat{f}_n$ have the following representations

$$\left. \begin{aligned} \Delta F_{n-1} \circ g &= \left[\bigcup_{k=1}^{n} (\bigcup \Delta f_k \circ F_{n-1} \circ g) \right] \cup \left[\bigcup_{m=0}^{\infty} (\bigcup (F_{n-1} \circ g)^{-1} (\delta_n^{-m})) \right] \\ \Delta \hat{f}_n &= \left[\bigcup_{k=1}^{n} (\bigcup \Delta f_k \circ \hat{f}_n) \right] \cup \left[\bigcup_{m=0}^{\infty} (\bigcup \hat{f}_n^{-1} (\delta_n^{-m})) \right] \end{aligned} \right\}. \tag{3.3}$$

Notice that the representation (3.3) for $\Delta F_{n-1} \circ h$ contains only the members corresponding to Δf_k and δ_n^{-m} which lie in the image of $F_{n-1} \circ h$.

b) In (3.3) some new preimages of δ_n arose, namely $(F_{n-1} \circ g)^{-1} \delta_n^{-m}$ and $\hat{f}_n^{-1} \delta_n^{-m}$. We still denote them δ_n^{-m}, but the corresponding diffeomorphisms $G_n \circ F_{n-1} \circ g$ and $G_n \circ \hat{f}_n$ will be denoted by G'_n. Let us rewrite (3.3) in the form

$$\left. \begin{aligned} \Delta F_{n-1} \circ g &= (\bigcup \Delta f_k \circ F_{n-1} \circ g) \cup (\bigcup \delta_n^{-m}) \\ \Delta \hat{f}_n &= (\bigcup \Delta f_k \circ \hat{f}_n) \cup (\bigcup \delta_n^{-m}) \end{aligned} \right\}. \tag{3.4}$$

Now we choose an interval $\delta_{n+1}(\lambda)$ composed of whole elements of the partition generated in (3.2) and (3.4):

$$\delta_{n+1}(\lambda) = \left[\frac{1}{2} - \frac{c_{n+1,1}(\lambda)}{\lambda^{s(n+1)}}, \frac{1}{2} + \frac{c_{n+1,2}(\lambda)}{\lambda^{s(n+1)}}\right], \qquad 1 \le c_{n+1,i} \le 1 + 0\left(\frac{1}{\lambda^{t(n+1)}}\right) \qquad (3.5)$$

c) We shall distinguish the maps with domains in $\delta_n \backslash \delta_{n+1}$, thus we use some additional notation.

Let $g_n = \lambda x(1-x) \pmod{1}|\delta_n \backslash \delta_{n+1}$. We shall use $f_{n+1\,1}$ to denote the branches $f_k \circ F_{n-1} \circ g_n$ and $f_k \circ \hat{f}_n|\delta_n \backslash \delta_{n+1}$. Finally, we shall use $\tilde{G}_n$ to denote the G_n or G'_n with domain inside $\delta_n \backslash \delta_{n+1}$. Using (3.2) and (3.4) we obtain the following representation of δ_n:

$$\delta_n = (\bigcup \Delta f_{n+1\,1}) \cup \left(\bigcup_{m=m_n}^{\infty} \delta_n^{-m}\right) \cup \delta_{n+1}. \qquad (3.6)$$

Let us define recurrently the branches $f_{n+1\,k}$, $k = 2, 3, \ldots$. If $\Delta f_{n+1\,k-1} \subset \delta_n \backslash \delta_{n+1}$ and $\tilde{G}_n : \delta_n^{-m} \to \delta_n$, then $f_{n+1\,k} = f_{n+1\,k-1} \circ \tilde{G}_n$. Any branch f_{n+1k} maps $\tilde{G}_n^{-1}(\Delta f_{n+1k-1})$ onto $[0, 1]$. For any given $N \in \mathbb{Z}_+$ we can rewrite (3.6) proceeding as in Sect. 2:

$$\delta_n = \left[\bigcup_{k=1}^{N}(\bigcup \Delta f_{n+1\,k})\right] \cup \left(\bigcup_{m=m_n}^{\infty}\delta_{n+1}^{-m}\right) \cup \left(\bigcup_{m=N\cdot m_n}^{\infty}\delta_n^{-m}\right) \cup \delta_{n+1} \qquad (3.7)$$

The preimages δ_{n+1}^{-m} and δ_n^{-m} in (3.7) have the form $(\tilde{G}_{n_1} \circ \tilde{G}_{n_2} \circ \ldots \circ \tilde{G}_{n_p})^{-1}\delta_{n+1}$ (respectively δ_n) and the branches $f_{n+1\,k}$ have the form

$$f_{n+1\,k} = f_{n+1\,1} \circ \tilde{G}_{n_1} \circ \tilde{G}_{n_2} \circ \ldots \circ \tilde{G}_{n_p}.$$

If $n > I$, there is an infinite number of $\tilde{G}_n$, and there is no uniform estimate $|D^2 \tilde{G}_n| < \text{const}$. However using a generalization of one result of [14] (see Lemma 1 below) we obtain

$$\lim_{N \to \infty} \text{mes}\left(\bigcup_{m=N\cdot m_n}^{\infty}\delta_n^{-m}\right) = 0. \qquad (3.8)$$

This implies

$$\delta_n = \left[\bigcup_{k=1}^{\infty}(\bigcup \Delta f_{n+1\,k})\right] \cup \left[\bigcup_{m=m_n}^{\infty}(\bigcup \delta_{n+1}^{-m})\right] \cup \delta_{n+1}. \qquad (3.9)$$

Apart from $\delta_n^{-m} \subset \delta_n \backslash \delta_{n+1}$ we have $\delta_n^{-m} \subset [0,1] \backslash \delta_n$ and $\delta_n^{-m} \subset \delta_{n+1}$ (domains of G_n and G'_n from (3.1), (3.2), (3.4)). Then (3.9) induces in any such domain $\delta_n^{-m} = G_n^{-1}\delta_n$ the corresponding decomposition

$$\delta_n^{-m} = (\bigcup \Delta f_{n+1\,k} \circ G_n) \cup (\bigcup \delta_{n+1}^{-m}), \qquad (3.10)$$

where $\delta_{n+1}^{-m} = G_n^{-1} \circ \tilde{G}_{n_p}^{-1} \circ \ldots \circ \tilde{G}_{n_1}^{-1}\delta_{n+1}$.

We shall use f_{n+1k} to denote $f_{n+1k-1} \circ G_n$ for any G_n with domain $\delta_n^{-m} \subset [0,1] \backslash \delta_n$; f_{n+1} to denote f_{n+1k} for any k; F_n to denote $f_k \circ F_{n-1}$; $\hat{f}_{n+1}$ to denote $f_k \circ \hat{f}_n$ for $\hat{f}_n$ such that $\Delta \hat{f}_n \subset \delta_{n+1}$, and also $\hat{f}_{n+1}$ to denote $f_{n+1} \circ G_n$ and

$f_{n+1} \circ G_n'$ with ΔG_n (respectively $\Delta G_n') \subset \delta_{n+1}$; G_{n+1} to denote any composition of the form $\tilde{G}_{n_1} \circ \tilde{G}_{n_2} \circ \ldots \circ \tilde{G}_{n_p} | \delta_{n+1}^{-m}$, or $\tilde{G}_{n_1} \circ \ldots \circ \tilde{G}_{n_p} \circ G_n | \delta_{n+1}^{-m}$ or $\tilde{G}_{n_1} \circ \ldots \circ \tilde{G}_{n_p} \circ G_n' | \delta_{n+1}^{-m}$.

With these notations we have:

$$[0,1] = \left[\bigcup_{k=1}^{n+1} (\bigcup \Delta f_k) \right] \cup \left[\bigcup_{m=1}^{\infty} (\bigcup \delta_{n+1}^{-m}) \right] \cup \delta_{n+1} \qquad (3.11)$$

and

$$\delta_{n+1} = (\bigcup \Delta F_n \circ g) \cup (\bigcup \Delta \hat{f}_{n+1}) \cup \left[\bigcup_{m=m_{n+1}}^{\infty} (\bigcup \delta_{n+1}^{-m}) \right]. \qquad (3.12)$$

(3.11) and (3.12) correspond to (3.1) and (3.2) with n replaced by $n+1$. So we have described Step $n+1$ on the x-axis for any $\lambda \in \mathcal{M}_n$.

d) According to the induction hypothesis $\mathcal{M}_n$ is the union of a countable set of closed intervals with disjoint interiors and some set $\mathcal{F}_n$ consisting of limit points of such intervals.

$$\mathcal{M}_n = (\bigcup \mathcal{J}_n) \bigcup \mathcal{F}_n.$$

We assume inductively that $\mathcal{F}_n \subset \mathcal{M}$, and define $\mathcal{M}_{n+1} \cap \mathcal{J}_n$ for all $\mathcal{J}_n$. We fix some positive $\alpha \leq s/4$. As λ varies over $\mathcal{J}_n$, the top of the central branch $F_{n-1} \circ h(\lambda, \frac{1}{2})$ varies over some Δf_{k_0} and $f_{k_0} \circ F_{n-1} \circ h(\lambda, \frac{1}{2})$ varies over $[0,1]$. Moreover when λ varies in $\mathcal{J}_n$ all the maps $F, G, f, \hat{f}$ constructed at previous steps vary continuously. Let $\mathcal{J}_n'$ be one of these components of $\mathcal{M}_n$. In order to construct the set $\mathcal{M}_{n+1} \cap \mathcal{J}_n'$ we shall point out the admissible positions for the top $f_{k_0} \circ F_{n-1} \circ h(\lambda, \frac{1}{2})$. Let $\mathcal{J}_n' = [a_n, b_n]$. When constructing $\delta_{n+1}(\lambda)$, we shall choose it varying continuously when $\lambda \in \mathcal{J}_n'$ and still satisfying (3.5). Then we shall expand $\delta_{n+1}(\lambda)$ almost homothetically and obtain an interval $\hat{\delta}_{n+1}(\lambda)$ varying continuously with $\lambda \in \mathcal{J}_n'$, composed of whole domains Δf_k and δ_{n+1}^{-m} and satisfying for $\lambda \in \mathcal{J}_n'$ the following

$$\lambda^{\alpha(n+1)} |\delta_{n+1}(\lambda)| \leq |\hat{\delta}_{n+2}(\lambda)| \leq \lambda^{\alpha(n+1)} \left(1 + 0 \left(\frac{1}{\lambda^{t(n+1)}} \right) \right) |\delta_{n+1}(\lambda)| \qquad (3.13)$$

For any preimage $\delta_{n+1}^{-m} = G_{n+1}^{-1} \delta_{n+1} \subset [0,1] \backslash \delta_{n+1}$ the corresponding domain $\hat{\delta}_{n+1}^{-m} = G_{n+1}^{-1} \hat{\delta}_{n+1}$ turns out to be defined and the lengths of δ_{n+1}^{-m} and $\hat{\delta}_{n+1}^{-m}$ are still related by (3.13). Then we define

$$\mathcal{M}_{n+1} \cap \mathcal{J}_n = \left\{ \lambda : f_{k_0} \circ F_{n-1} \circ h(\lambda, \frac{1}{2}) \in [0,1] \backslash \bigcup_m \bigcup \hat{\delta}_{n+1}^{-m}(\lambda) \right\}.$$

The condition $f_{k_0} \circ F_{n-1} \circ h(\lambda, \frac{1}{2}) \in \hat{\delta}_{n+1}^{-m}$ defines an interval in $\mathcal{J}_n$. Thus $\mathcal{M}_{n+1} \cap \mathcal{J}_n$ is the complement of the union of these intervals. $\mathcal{M}_{n+1} \cap \mathcal{J}_n$ consists of intervals $\mathcal{J}_{nk}' = \{ \lambda : f_{k_0} \circ F_{n-1} \circ h(\lambda, \frac{1}{2}) \in \Delta f_k(\lambda) \}$ and of a limit set $\mathcal{F}_{n+1}(\mathcal{J}_n)$. As λ varies over $\mathcal{J}_{nk}'$, $f_k \circ f_{k_0} \circ F_{n-1} \circ h(\lambda, \frac{1}{2})$ varies over $[0,1]$.

So we have

$$\mathcal{M}_{n+1} \cap \mathcal{J}_n = \left(\bigcup_k \mathcal{J}_{nk}' \right) \cup \mathcal{F}_{n+1}(\mathcal{J}_n) \qquad (3.14)$$

and finally

$$\mathcal{M}_{n+1} = \left(\bigcup_{\mathcal{J}_n} (\mathcal{M}_{n+1} \cap \mathcal{J}_n) \right) \cup \mathcal{F}_n. \qquad (3.15)$$

4. Estimates for Fluctuation of Derivative

Let $f:\Delta \to I$ be a C^2-diffeomorphism of some closed interval. Then by differentiating $\log|Df(z)|$, we see that

$$\max_{x,y\in\Delta}\left|\frac{Df(x)}{Df(y)}\right| \leq \exp\left(\max_{z\in\Delta}\left|\frac{D^2f(z)}{Df(z)}\right|\cdot|\Delta|\right). \tag{4.1}$$

We shall use the notation $\mu(f,\Delta) = \max_{x\in\Delta}\left|\frac{D^2f(x)}{Df(x)}\right|\cdot|\Delta|$ and when there is no doubt about the domain of f, we shall often write $\mu(f)$. Let $f_1:\Delta_1 \xrightarrow{\text{onto}} I$,
$f_2:\Delta_2 \xrightarrow{\text{onto}} J \supset \Delta_1$ be as above, $\Delta_{12}=f_2^{-1}\Delta_1 \subset \Delta_2$. Then $f_1\circ f_2(\Delta_{12})=I$. Using the mean value theorem and (4.1) we obtain

$$\begin{aligned}
\mu(f_1\circ f_2, \Delta_{12}) &= \max_{x\in\Delta_{12}}\left|\frac{D^2(f_1\circ f_2)(x)}{D(f_1\circ f_2)(x)}\right||\Delta_{12}| \\
&= \max_{x\in\Delta_{12}}\left|\frac{D^2f_1(f_2(x))\cdot[Df_2(x)]^2 + Df_1(f_2(x))\cdot D^2f_2(x)}{Df_1(f_2(x))\cdot Df_2(x)}\right||\Delta_{12}| \\
&\leq \left[\max_{y\in\Delta_1}\left|\frac{D^2f_1(y)}{Df_1(y)}\right||\Delta_1|\right]\cdot\left[\max_{x\in\Delta_{12}}|Df_2(x)|\cdot\frac{|\Delta_{12}|}{|\Delta_1|}\right] \\
&\quad + \max_{x\in\Delta_{12}}\left|\frac{D^2f_2(x)}{Df_2(x)}\right|\cdot|\Delta_2|\cdot\frac{|\Delta_{12}|}{|\Delta_2|} \\
&\leq \mu(f_1)\cdot\max_{x,\theta\in\Delta_{12}}\left|\frac{Df_2(x)}{Df_2(\theta)}\right| + \mu(f_2)\cdot\frac{|\Delta_{12}|}{|\Delta_2|}. \tag{4.2}
\end{aligned}$$

Since by (4.1)

$$\max_{x,\theta\in\Delta_{12}}\left|\frac{Df_2(x)}{Df_2(\theta)}\right| \leq \exp\left[\mu(f_2)\cdot\frac{|\Delta_{12}|}{|\Delta_2|}\right]$$

and

$$\frac{|\Delta_{12}|}{|\Delta_2|} = \left|\frac{Df_2(\eta_2)}{Df_2(\eta_{12})}\right|\frac{|\Delta_1|}{|J|} \leq [\exp\mu(f_2)]\frac{|\Delta_1|}{|J|}$$

we obtain

$$\max_{x,\theta\in\Delta_{12}}\left|\frac{Df_2(x)}{Df_2(\theta)}\right| \leq \exp\left[\mu(f_2)\{\exp\mu(f_2)\}\frac{|\Delta_1|}{|J|}\right]. \tag{4.3}$$

Consequently

$$\begin{aligned}
\mu(f_1\circ f_2\cdot\Delta_{12}) &\leq \mu(f_1)\exp\left[\{\mu(f_2)\exp\mu(f_2)\}\cdot\frac{|\Delta_1|}{|J|}\right] \\
&\quad + \{\mu(f_2)\exp\mu(f_2)\}\cdot\frac{|\Delta_1|}{|J|}. \tag{4.4}
\end{aligned}$$

Using the notation $v(f, \Delta) = \mu(f, \Delta)\exp\mu(f, \Delta)$, (4.4) is equivalent to

$$\mu(f_1 \circ f_2, \Delta_{12}) \leqq \mu(f_1)\exp\left[v(f_2)\cdot\frac{|\Delta_1|}{|J|}\right] + v(f_2)\cdot\frac{|\Delta_1|}{|J|}. \tag{4.5}$$

Let $h(x) = ax^2$, and let Δ denote an interval in $\mathbb{R}_+$; let H denote the distance from Δ to 0, so that $\Delta = (H, H + |\Delta|)$, and suppose $f : \Delta \to I$ is a C^2 diffeomorphism. Let $\delta = [x_{\min}, x_{\max}] \subset \mathbb{R}_+$, be one of the two diffeomorphic preimages of $\Delta : d = h^{-1}(\Delta) \cap \mathbb{R}_+$. We obtain as above

$$\mu(f \circ h, \delta) \leqq \mu(f)\cdot\max_{x, y \in \delta}\left|\frac{Dh(x)}{Dh(y)}\right| + |\Delta|\max_{x, y \in \delta}\left|\frac{D^2 h(x)}{[Dh(y)]^2}\right|$$

and thus

$$\mu(f \circ h, \delta) \leqq \mu(f)\frac{x_{\max}}{x_{\min}} + \frac{|\Delta|}{2ax_{\min}^2}.$$

Since $ax_{\max}^2 = H + |\Delta|$, and $ax_{\min}^2 = H$, we have

$$\frac{x_{\max}}{x_{\min}} = \sqrt{1 + \frac{|\Delta|}{H}} < 1 + \frac{|\Delta|}{2H}.$$

This implies

$$\mu(f \circ h, \delta) < \mu(f)\left(1 + \frac{|\Delta|}{2H}\right) + \frac{|\Delta|}{2H} \tag{4.6}$$

or

$$\mu(f \circ h, \delta) < \mu(f)\left(1 + \frac{|\Delta|}{2ax_{\min}^2}\right) + \frac{|\Delta|}{2ax_{\min}^2} \tag{4.7}$$

5. Preliminary Lemma

We shall use the following several times

Lemma 1. *Let $I \cup J = N$ be an interval, $I = \bigcup\limits_{i=1}^{\infty} \Delta\varphi_i$, where*

1) φ_i *are C^2-diffeomorphisms from their domains onto N;*
2) $\mathrm{int}\,\Delta\varphi_i \cap \mathrm{int}\,\Delta\varphi_j = \emptyset$, $i \neq j$;
3) $|D\varphi_i| > \bar{c}_1 < 1$;
4) $\mu(\varphi_i) < \bar{c}_2$;
5) $\mathrm{mes}\,J > 0$; $\mathrm{mes}\,I \cap J = 0$.

Then $I = \bigcup\limits_{k=1}^{\infty} \varphi^{-k}J \pmod{0}$, where $\varphi^{-k}J = \bigcup\limits_{i_1\ldots i_k} \varphi_{i_1}^{-1} \circ \ldots \circ \varphi_{i_k}^{-1}J$.

Proof. Since φ_i is onto, $\Delta\varphi_i = \varphi_i^{-1}J \cup \varphi_i^{-1}I$. Thus

$$I = \bigcup_i \Delta\varphi_i = \bigcup_i (\varphi_i^{-1}J \cup \varphi_i^{-1}I) = \varphi^{-1}J \cup \left[\bigcup_{i_1}\varphi_{i_1}^{-1}\left(\bigcup_{i_2}\varphi_{i_2}^{-1}J \cup \varphi_{i_2}^{-1}I\right)\right]$$

$$= \varphi^{-1}J \cup \varphi^{-2}J \cup \varphi^{-2}I.$$

In a similar way we obtain for any N

$$I = \bigcup_{k \leqq N} \varphi^{-k} J \cup \varphi^{-N} I. \tag{5.1}$$

For any $i_1, \ldots, i_k$,

$$\varphi_{i_1}^{-1} \circ \ldots \circ \varphi_{i_k}^{-1} I = \left[\bigcup_i \varphi_{i_1}^{-1} \circ \ldots \circ \varphi_{i_k}^{-1} \circ \varphi_i^{-1} I \right] \cup \left[\bigcup_i \varphi_{i_1}^{-1} \circ \ldots \circ \varphi_{i_k}^{-1} \circ \varphi_i^{-1} J \right]. \tag{5.2}$$

Suppose there were a constant $\theta > 0$ independent of k such that for any $i_1, \ldots, i_k$

$$\frac{\operatorname{mes} \varphi_{i_1}^{-1} \circ \ldots \circ \varphi_{i_k}^{-1} J}{\operatorname{mes} \varphi_{i_1}^{-1} \circ \ldots \circ \varphi_{i_k}^{-1} I} > \theta. \tag{5.3}$$

Then it would follow from (5.2) that

$$\operatorname{mes} \varphi^{-(k+1)} I < (1 + \theta)^{-1} \operatorname{mes} \varphi^{-k} I,$$

thus $\lim_{k \to \infty} \operatorname{mes} \varphi^{-k} I = 0$, and in view of (5.1) this would prove Lemma 1. Note that for $k = 1$, (5.3) follows from hypothesis (5).

Consider a C^2 diffeomorphism $\varphi^k = \varphi_{i_k} \circ \ldots \circ \varphi_{i_1} : \varphi_{i_1}^{-1} \circ \ldots \circ \varphi_{i_k}^{-1} N \to N$. By the mean value theorem and by (4.1), a proof of (5.3) would follow from a uniform upper bound on the quantities $\mu(\varphi^n)$ independent of n. We will show

$$\mu(\varphi^n) < \left(\sum_{i=0}^{\infty} \frac{\bar{c}_2 \exp \bar{c}_2}{\bar{c}_1^i} \right) \exp \left(\sum_{i=1}^{\infty} \frac{\bar{x}_2 \exp \bar{c}_2}{\bar{c}_1^i} \right). \tag{5.4}$$

We prove (5.4) by induction. From (4.5),

$$\mu(\varphi^n) = \mu(\varphi^{n-1} \circ \varphi) \leqq \mu(\varphi^{n-1}) \exp \left[v(\varphi) \frac{|\Delta \varphi^{n-1}|}{|N|} \right] + v(\varphi) \frac{|\Delta \varphi^{n-1}|}{|N|}. \tag{5.5}$$

According to hypotheses 3 and 4

$$v(\varphi) < \bar{c}_2 \exp \bar{c}_2$$

and

$$|\Delta \varphi^{n-1}| < |N| / \bar{c}_1^{n-1}.$$

Thus

$$\mu(\varphi^n) \leqq \mu(\varphi^{n-1}) \exp[v(\varphi) / \bar{c}_1^{n-1}] + v(\varphi) / \bar{c}_1^{n-1}. \tag{5.6}$$

Suppose for $k \leqq n-1$ that

$$\mu(\varphi^k) \leqq \left(\sum_{i=0}^{k-1} v(\varphi) / \bar{c}_1^i \right) \exp \left(\sum_{i=1}^{k-1} v(\varphi) / \bar{c}_1^i \right).$$

(Note that for $k = 1$, the second factor above equals 1 and this becomes the obvious inequality $\mu(\varphi) < v(\varphi)$.) Then, using (5.6),

$$\mu(\varphi^n) \leqq \left(\sum_{i=0}^{n-2} \frac{v(\varphi)}{\bar{c}_1^i} \right) \exp \left(\sum_{i=1}^{n-2} \frac{v(\varphi)}{\bar{c}_1^i} \right) \exp \left(\frac{v(\varphi)}{\bar{c}_1^{n-1}} \right) + \frac{v(\varphi)}{\bar{c}_1^{n-1}}$$

$$\leqq \left(\sum_{i=0}^{n-2} \frac{v(\varphi)}{\bar{c}_1^i} \right) \exp \left(\sum_{i=1}^{n-1} \frac{v(\varphi)}{\bar{c}_1^i} \right) + \frac{v(\varphi)}{\bar{c}_1^{n-1}} \exp \left(\sum_{i=1}^{n-1} \frac{v(\varphi)}{\bar{c}_1^i} \right)$$

$$\leqq \left(\sum_{i=0}^{n-1} \frac{v(\varphi)}{\bar{c}_1^i} \right) \exp \left(\sum_{i=1}^{n-1} \frac{v(\varphi)}{\bar{c}_1^i} \right)$$

and (5.4) is proved.

6. Transition from n to $n+1$, I. Hypotheses of Induction. Estimates of Derivatives

(3.1) and (3.2) give us the following representation of $[0,1]$ after Step n:

$$[0,1] = \left[\bigcup_{k=1}^{n} \left(\bigcup \Delta f_k\right)\right] \cup \left[\bigcup_{m=1}^{\infty} \left(\bigcup \delta_n^{-m}\right)\right] \cup \left(\bigcup \Delta F_{n-1} \circ g\right) \cup \left(\bigcup \Delta \hat{f}_n\right) \qquad (6.1)$$

All domains in (6.1) depend on λ which varies in $\mathscr{I}_n$, but throughout Sects. 6 and 7. λ will be fixed. Any δ_n^{-m} in (6.1) is a preimage of δ_n under some diffeomorphism denoted by G_n. For given δ_n^{-m} let $p = \max\left\{k : \delta_n^{-m} \subset \left[\frac{1}{2} - \frac{1}{\lambda^{sk}}, \frac{1}{2} + \frac{1}{\lambda^{sk}}\right]\right\}$. Then we shall use the notation $G_{n,p}$ for G_n.

Let $0 < s \leq \frac{1}{13}$, $1 < \alpha \leq s/4$ be constants defined in Sects. 2 and 3, $c_0 = 1 - s$, $c_1 = 1 - 2s$, $c_2 = 1 - s + \alpha$, $\gamma = 1 - 3s$, $t = \alpha/10$, $v = \dfrac{2(s-\alpha)}{c_0}$. Now we formulate the hypotheses of induction.

    a) *Hypotheses on derivatives:*

$\left. \begin{array}{l} a_{1n}^1) \;\; |Df_k| > 2^k \cdot \lambda^{c_1 k} \\[4pt] a_{1n}^2) \;\; |Df_k| > 2\lambda^{c_0} \end{array} \right\} \; k = 1, \ldots, n$

$a_{2n})\;\; |DF_{n-1}| > 2^{n-1} \lambda^{c_0(n-1)}$

$a_{3n})\;\; |D\hat{f}_n| > 2^n \cdot \lambda^{c_1 n}$

$a_{4n}^1)\;\; |DG_{n,p}| > \lambda^{s(1-v)p}$

$a_{4n}^2)\;\; |DG_n| > 2\lambda^{c_2/2}$.

    b) *Hypotheses on μ:*

$b_{1n})\;\; \mu(f_k) < \left(\sum_{i=1}^{k} \frac{1}{2^i \cdot \lambda^{\gamma i}}\right) \cdot \prod_{i=1}^{k} \left(1 + \frac{1}{2^i \cdot \lambda^{\gamma i}}\right) \cdot \exp\left(\sum_{i=1}^{k} \frac{1}{2^i \cdot \lambda^{\gamma i}}\right), \qquad k = 1, 2, \ldots, n.$

$b_{2n})\;\; \mu(F_{n-1}) < \left(\sum_{i=1}^{n-1} \frac{1}{2^i \cdot \lambda^{\gamma i}}\right) \cdot \prod_{i=1}^{n-1} \left(1 + \frac{1}{2^i \cdot \lambda^{\gamma i}}\right) \cdot \exp\left(\sum_{i=1}^{n-1} \frac{1}{2^i \cdot \lambda^{\gamma i}}\right)$

$b_{3n})\;\; \mu(\hat{f}_n) < \left(\sum_{i=1}^{n} \frac{1}{2^i \cdot \lambda^{\gamma i}}\right) \cdot \prod_{i=1}^{n} \left(1 + \frac{1}{2^i \cdot \lambda^{\gamma i}}\right) \cdot \exp\left(\sum_{i=1}^{n} \frac{1}{2^i \cdot \lambda^{\gamma i}}\right)$

$b_{4n})\;\; \mu(G_n) < \dfrac{1}{\lambda^{\alpha n}}.$

We suppose a_{in}, b_{in} to be true and we have to prove a_{in+1}, b_{in+1}.

Remark VI/1. At the beginning of Step $n+1$ we constructed some new preimages δ_n^{-m} with corresponding maps denoted by G_n' ($G_n' : \delta_n^{-m} \to \delta_n$, see Sect. 3). We have to prove that G_n' also satisfy the conditions a_{4n}, b_{4n} which we denote in this case a_{4n}', b_{4n}'.

Remark VI/2. Some additional induction hypotheses related to the variation of λ will be formulated below. In particular the possibility of choice of intervals δ_n, $\hat{\delta}_n$ will be proved, and estimates of sizes of these intervals and their preimages will be given in Sect. 10. Now we shall use (3.5) and (3.13) with n instead of $n+1$ (this is assumed inductively) and with $n+1$ (this will be proved in Sect. 10). One easily checks there is no vicious circle here.

$a_{1\,n+1}$) According to the construction of Sect. 3, $\{f_{n+1}\} = \bigcup_{i=1}^{\infty} \{f_{n+1\,i}\}$ where

$f_{n+1\,1} = f_k \circ F_{n-1} \circ g_n \left(\text{with } g_n = \lambda \cdot x^2 \left| \left\{ |x| > \dfrac{1}{\lambda^{s(n+1)}} \right\} \right. \text{ in local coordinates near } \tfrac{1}{2} \right)$, or

$f_{n+1\,1} = f_k \circ \hat{f}_n$. In the first case a_{1n}, a_{2n} and the form of g_n above imply

$$|Df_{n+1\,1}| \geqq 2\lambda^{c_0} \cdot 2^{n-1} \cdot \lambda^{c_0(n-1)} \cdot 2\lambda \cdot \frac{1}{\lambda^{s(n+1)}} > 2^{n+1} \cdot \lambda^{c_1(n+1)}.$$

In the second case a_{1n}, a_{3n} imply

$$|Df_{n+1\,1}| \geqq 2\lambda^{c_0} \cdot 2^n \cdot \lambda^{c_1 n} > 2^{n+1} \cdot \lambda^{c_1(n+1)}.$$

Thus $a_{1\,n+1}^1$ is true for $f_{n+1\,1}$. The choice of s implies $2c_1 > c_0$, hence a_{1n}^1 implies a_{1n}^2 for $n \geqq 2$. All $f_{n+1\,k}$, $k \geqq 2$ are compositions of the form $f_{n+1\,k} = f_{n+1\,k-1} \circ G_n$ or $f_{n+1\,k} = f_{n+1\,k-1} \circ G_n'$ with $\Delta G_n' \subset \delta_n \backslash \delta_{n+1}$. According to a_{4n}^2, $|DG_n| > 1$. $(a_{4n}^2)'$ proved below is much stronger than $|DG_n| > 1$, and G_n' under consideration satisfies $|DG_n'| > 2^n \cdot \lambda^{c_1 n}$. Indeed, $G_n' = G_n \circ F_{n-1} \circ g_n$ or $G_n' = G_n \circ \hat{f}_n$. In both cases a_{2n} and a_{3n} imply as above $|DG_n'| > 2^n \cdot \lambda^{c_1 n}$. $\square$

$a_{2\,n+1}$) $F_n = f_k \circ F_{n-1}$. Hence a_{1n} and a_{2n} imply $a_{2\,n+1}$. $\square$

$a_{4n}^{2\,'}$) We consider $G_n : \delta_n^{-m} \to \delta_n$, $G_n' = G_n \circ F_{n-1} \circ g$ or $G_n' = G_n \circ \hat{f}_n$ and their domains $\delta_n^{-N} = (F_{n-1} \circ g)^{-1}(\delta_n^{-m})$ or $\delta_n^{-M} = \hat{f}_n^{-1}(\delta_n^{-m})$. The most complicated is the case of central branch $F_{n-1} \circ h$. We omit indices and use δ to denote δ_n^{-m} (if $m = 0$, $\delta = \delta_n$), G to denote G_n (if $m = 0$, $G = \text{id}$), ℓ to denote $(F_{n-1} \circ h)^{-1}\delta$. We estimate $|D(G \circ F_{n-1} \circ h)|$. Let $H = \text{dist}(\delta, F_{n-1} \circ h(\tfrac{1}{2}))$. The induction construction of Step n implies that the top $F_{n-1} \circ h(\tfrac{1}{2})$ lies outside an interval $\hat{\delta}$ corresponding to δ. Thus (see (3.13) with n instead of $n+1$)

$$H \geqq (\lambda^{an} - 1) \cdot |\delta|/2.$$

Let $H_1 = \text{dist}(F_{n-1}^{-1}\delta, h(\tfrac{1}{2}))$. It follows from (4.1) and b_{2n} that

$$H_1 > (\lambda^{an} - 1) \cdot |F_{n-1}^{-1}\delta| \cdot 2^{-1} \cdot \exp\left(-\frac{1 + \varepsilon_{6.2}}{2\lambda^\gamma} \right), \quad \text{where } \varepsilon_{6.2} = O(\lambda^{-\gamma}) \qquad (6.2)$$

Remark VI/3. Several constants $0 \leqq \varepsilon_{i,k} < \lambda^{-t}$ are indexed according to the numbers of inequalities in which they occur.

Let $\ell = [x_{\min}, x_{\max}]$. We have, using the local coordinate,

$$h(x_{\min}) = \lambda \cdot x_{\min}^2 = H_1, \qquad x_{\min}\sqrt{H_1 \lambda^{-1}}, \qquad |Dh|_\ell \geqq 2\lambda |x_{\min}| = 2\sqrt{\lambda \cdot H_1}.$$

In consequence of $|\delta| = |F_{n-1}^{-1}\delta| \cdot |DF_{n-1}(\theta)|$, for some $\theta \in F_{n-1}^{-1}\delta$, we obtain

$$|Dh|_\ell \geqq \sqrt{\frac{2 \cdot \lambda^{an+1}|\delta|}{|DF_{n-1}(\theta)|}} (1 - \varepsilon_{6.3}) \qquad (6.3)$$

Since $|D(F_{n-1} \circ h)| = |DF_{n-1}| \cdot |Dh|$, we have, using (4.1) and b_{2n}, for any $x \in \Delta F_{n-1}$

$$|D(F_{n-1} \circ h)|_\ell \geqq \sqrt{2 \cdot \lambda^{an+1} \cdot |DF_{n-1}(x)| \cdot |\delta|} (1 - \varepsilon_{6.4}) \qquad (6.4)$$

If $\delta = \delta_n$, then (6.4), $|\delta_n| > 2 \cdot \lambda^{-sn}$ and a_{2n} imply

$$|D(F_{n-1} \circ h)|_\ell| \geqq (\sqrt{2} \cdot \lambda^{\frac{1}{2}(c_1 + \alpha)})^n \cdot \sqrt{2\lambda^s} \cdot (1 - \varepsilon_{6.5}). \tag{6.5}$$

If $\delta = \delta_n^{-m} = G^{-1}\delta_n$ we obtain, using a_{4n}

$$|D(G \circ F_{n-1} \circ h)|_\ell| \geqq (\sqrt{2} \cdot \lambda^{\frac{1}{2}(c_1 + \alpha)})^n \sqrt{\lambda^{s + c_2/2}} \cdot 2(1 - \varepsilon_{6.6}). \tag{6.6}$$

(6.5) and (6.6) imply $(a_{4n}^2)'$ for $G_n' = G_n \circ F_{n-1} \circ n$. In the case $G_n' = G_n \circ F_{n-1} \circ g$ we have in (6.2) $H_1 > \frac{1}{2} \cdot |\Delta F_{n-1}| \exp\left(-\frac{1 + \varepsilon_{6.2}}{2\lambda^\gamma}\right)$, which leads to better estimates. In the case $G_n' = G_n \circ \hat{f}_n$ $(a_{4n}^2)'$ is obvious because of a_{3n} and a_{4n}^2. $\square$

a_{4n+1}^2) Follows from a_{4n}^2, $(a_{4n}^2)'$ and the definition:

$G_{n+1} = G_{n_1} \circ \ldots \circ G_{n_p}$. $\square$

a_{3n+1}) If $\hat{f}_{n+1} = f_{n+1} \circ G_{n_1} \circ \ldots \circ G_{n_p}$, a_{3n+1} follows from a_{1n+1}, a_{4n}^2 and $(a_{4n}^2)'$. If $\hat{f}_{n+1} = f_k \circ \hat{f}_n$, a_{3n+1} follows from a_{1n} and a_{3n}. $\square$.

Remark VI/4. The inequalities (6.5), (6.6) show that the derivatives of G_n' grow exponentially with n, but this is not sufficient to prove $(a_{4n}^1)'$. Indeed, let n_1 be so that $F_{n_1-1} \circ h(\lambda, \frac{1}{2})$ may lie in the domain δ_1. As δ_1 contains Δf_2 of arbitrary small diameter, the interval $\Delta(f_2 \circ F_{n_1-1} \circ h) = \Delta(F_{n_1} \circ h)$ may also be arbitrarily small and the corresponding $\delta_{n_1}^{-M} = (F_{n_1} \circ h)^{-1}\delta_{n_1}$ is contained in δ_N with arbitrarily large N. However $|DF_{n_1}|$ turns out to be very large in this case, which implies $(a_{4n}^1)'$.

a_{4n}^1) We use the notation introduced in the proof of $(a_{4n}^2)'$. According to the definition, the domain $\ell = (F_{n-1} \circ h)^{-1}\delta$ of $G_{n,p}'$ is so that $\ell \subset (\frac{1}{2} - \lambda^{-sp}, \frac{1}{2} + \lambda^{-sp})$, but $\ell \not\subset (\frac{1}{2} - \lambda^{-s(p+1)}, \frac{1}{2} + \lambda^{-s(p+1)})$. Let $\ell \subset (\frac{1}{2}, \frac{1}{2} + \lambda^{-sp})$. Then $H_1 = \lambda x_{\min}^2 < \lambda^{1-2sp}$. It follows from (6.2)

$$\frac{\lambda}{\lambda^{2sp}} > \frac{\lambda^{\alpha n}}{2} \cdot \frac{|\delta|(1 - \varepsilon_{6.7})}{|DF_{n-1}(\theta_1)|}. \tag{6.7}$$

(6.7) together with b_{2n} imply for any $x \in \Delta F_{n-1}$

$$|DF_{n-1}(x)| > \frac{\lambda^{\alpha n - 1} \cdot \lambda^{2sp}}{2} \cdot |\delta|(1 - \varepsilon_{6.8}). \tag{6.8}$$

Thus we can rewrite (6.4) as

$$|D(F_{n-1} \circ h)|_\ell| \geqq \lambda^{\alpha n} \cdot \lambda^{sp} \cdot |\delta| \cdot (1 - \varepsilon_{6.9}). \tag{6.9}$$

From $|\delta| = \frac{|\delta_n|}{|DG_n(\theta_2)|}$, and $|\delta_n| > 2 \cdot \lambda^{-sn}$ we obtain using b_{4n}

$$|DG_{n,p}| = |D(G_n \circ F_{n-1} \circ h)|_\ell| \geqq 2 \cdot \lambda^{\alpha n} \cdot \lambda^{s(p-n)} \cdot (1 - \varepsilon_{6.10}). \tag{6.10}$$

Let us compare p and n. Let $\mathscr{D}_{n-1} = (\frac{1}{2} - u_{n-1}^1, \frac{1}{2} + u_{n-1}^2)$ be the domain of $F_{n-1} \circ h$, and $p_1 = \max\{q : u_{n-2}^2 < \lambda^{-sq}\}$. Then $p \geqq p_1$. We have in the local coordinate system, using a_{2n},

$$2^{n-1} \cdot \lambda^{c_0(n-1)+1} \cdot (u_{n-1}^2)^2 < F_{n-1} \circ h(\frac{1}{2}) \leqq 1.$$

Thus

$$\lambda^{-s(p_1+1)} < u_{n-1}^2 < [(\sqrt{2})^{n-1}\lambda^{(c_0 n + s)1/2}]^{-1}. \tag{6.11}$$

(6.11) implies $n < 2s(p_1 + \frac{1}{2})/c_0$, which gives for v a somewhat worse estimate than $2(s-\alpha)/c_0$. We prefer to improve it instead of taking a different v. It suffices to make $F_{n-1} \circ h(\lambda, \frac{1}{2})$ lie outside $(\frac{1}{2} - \lambda^{-s/2}, \frac{1}{2} + \lambda^{-s/2})$ for the first two steps. This gives an extra factor $\lambda^{-s/2}$ on the right side of (6.11). Hence

$$n < \frac{2s}{c_0} p_1. \tag{6.12}$$

Remark VI/5. For a given n_0 we may introduce the additional condition

$$F_{n-1} \circ h(\lambda, \tfrac{1}{2}) \in [0,1] \setminus [\tfrac{1}{2} - \lambda^{-s/2}, \tfrac{1}{2} + \lambda^{-s/2}] \qquad n \leqq n_0$$

(above $n_0 = 2$). This simplifies the estimates, but as follows from Sect. 11, it gives an extra factor of $(1 - 2N_0^{-s/2})^{n_0}$ in the estimate of $\mathcal{M}$. However, this factor can be made arbitrarily close to 1 by taking N_0 sufficiently large.

As $p \geqq p_1$, (6.10) and (6.12) imply

$$|DG'_{n,p}| > \lambda^{ps\left(1 - \frac{2(s-\alpha)}{c_0}\right)} = \lambda^{ps(1-v)} \tag{6.13}$$

which finishes the proof of $(a_{4n}^1)'$ for $\ell = (F_{n-1} \circ h)^{-1} \delta$.

If $G'_{n,p} = G_n \circ F_{n-1} \circ g$, the estimate of H_1 (see the proof of $(a_{4n}^2)'$) implies that (6.10) turns out into $|DG'_{n,p}| > \lambda^{sp}(1 - \varepsilon_{6.10})$. Finally when $G'_{n,p} = G_n \circ \hat{f}_n$, notice that any $\hat{f}_n$ is a composition of the form $\varphi \circ G_k$, $k \leqq n-1$, where G_k satisfies a_{4k}^1, and $|D\varphi| > 1$.   □

a_{4n+1}^1)   Follows from a_{4n}^1, a_{4n}^2, $(a_{4n}^1)'$, $(a_{4n}^2)'$ and the definition of G_{n+1}.   □

7. Transition from n to $n+1$, II. Estimates of μ

b_{4n+1})   Let $G'_n = G_n \circ F_{n-1} \circ h : I \to \delta_n$, where

$$G_n : \Delta G_n = \delta_n^{-m} \to \delta_n, \quad \text{and} \quad \ell = (F_{n-1} \circ h)^{-1} \delta_n^{-m}.$$

We estimate $\mu(G'_n, \ell)$ first.

According to (4.5)

$$\mu(G_n \circ F_{n-1}, F_{n-1}^{-1} \Delta G_n) \leqq \mu(G_n) \exp(v(F_{n-1}) \cdot |\Delta G_n|) + v(F_{n-1}) \cdot |\Delta G_n|.$$

We have

$$|\delta_n| = |\Delta G'_n| \cdot |DG_n(\theta_0)|, \quad |\delta_n| < 2 \cdot \lambda^{-sn}(1 + O(\lambda^{-tn})), \quad |DG_n| > 2\lambda^{c_2/2}.$$

In consequence of b_{2n}, $v(F_{n-1}) = 1 + O(\lambda^{-\gamma})$. Thus we obtain, using a_{2n}, b_{4n}:

$$\mu(G_n \circ F_{n-1}, F_{n-1}^{-1} \Delta G_n) < \lambda^{-\alpha n} \cdot \exp\left[\frac{2(1 + O(\lambda^{-\gamma}))}{|DG_n(\theta_0)| \lambda^{c_0(n-1)+sn} \cdot 2^{n-1}}\right]$$
$$+ \frac{2(1 + O(\lambda^{-\gamma}))}{|DG_n(\theta_0)| \lambda^{c_0(n-1)+sn} \cdot 2^{n-1}} = \frac{1 + \varepsilon_{7.1}}{\lambda^{\alpha n}}. \tag{7.1}$$

Proceeding along the line of the proof of (6.2), and using (7.1) and (4.6) with $\Delta = F_{n-1}^{-1} \Delta G_n$, $H > \frac{1}{2}(1 - O(\lambda^{-\gamma})) \cdot \lambda^{\alpha n} \cdot |\Delta|$ we obtain

$$\mu(G_n \circ F_{n-1} \circ h, \ell) < \frac{1 + \varepsilon_{7.1}}{\lambda^{\alpha n}} \left(1 + \frac{1 + O(\lambda^{-\gamma})}{\lambda^{\alpha n}}\right) + \frac{1 + O(\lambda^{-\gamma})}{\lambda^{\alpha n}} < \frac{2(1 + \varepsilon_{7.2})}{\lambda^{\alpha n}}. \quad (7.2)$$

The proof for $G_n' = G_n \circ F_{n-1} \circ g$ and $G_n' = G_n \circ \hat{f}_n$ is similar and gives a better estimate

$$\mu(G_n') < (1 + \varepsilon_{7.3}) \lambda^{-\alpha n}. \quad (7.3)$$

Then we consider $G_{n+1} = \tilde{G}_{n_1} \circ \dots \circ \tilde{G}_{n_s} : \delta_n^{-M} \to \delta_n$, $\delta_n^{-M} \subset \delta_n \setminus \delta_{n+1}$. When estimating $\mu(G_{n+1}, \delta_n^{-M})$ we use the proof of Lemma 1 with $\varphi_i = \tilde{G}_{n_i}$, $\bar{c}_2 = (1 + \varepsilon_{7.3}) \lambda^{-\alpha n}$, according to (7.3) and b_{4n}, and $\bar{c}_1 = \max(\lambda^{c_2/2}, \lambda^{s[n(1-\nu)-1]})$, according to a_{4n}. Then (5.4) gives

$$\mu(G_{n+1}, \delta_n^{-M}) < (1 + \varepsilon_{7.4}) \lambda^{-\alpha n}. \quad (7.4)$$

The estimates (3.5) of $|\delta_n|$ and $|\delta_{n+1}|$ imply

$$\frac{|\delta_{n+1}|}{|\delta_n|} < \frac{1 + \varepsilon_{7.5}}{\lambda^s}. \quad (7.5)$$

Considering $\delta_{n+1}^{-M} = G_{n+1}^{-1} \delta_{n+1}$ and applying (7.4), (7.5) we obtain

$$\mu(G_{n+1}, \delta_{n+1}^{-M}) \leq \mu(G_{n+1}, \delta_n^{-M}) \cdot \frac{|\delta_{n+1}^{-M}|}{|\delta_n^{-M}|} \leq \mu(G_{n+1}, \delta_n^{-M}) \cdot \frac{|\delta_{n+1}|}{|\delta_n|}$$

$$\cdot \exp \mu(G_{n+1}, \delta_n^{-M}) < \frac{(1 + \varepsilon_{7.4})(1 + \varepsilon_{7.5})(1 + \varepsilon_{7.6})}{\lambda^{\alpha n + s}} < \frac{1}{\lambda^{\alpha(n+1)}} \quad (7.6)$$

which proves b_{4n+1} for $\delta_{n+1}^{-M} \subset \delta_n \setminus \delta_{n+1}$.

Any $G_{n+1} : \delta_{n+1}^{-N} \to \delta_{n+1}$ for $\delta_{n+1}^{-N} \subset [0,1] \setminus (\delta_n \setminus \delta_{n+1})$ is either a restriction of $G_n : \delta_n^{-N} \to \delta_n$ on $\delta_{n+1}^{-N} \subset \delta_n^{-N}$, or a composition of the form $\tilde{G}_{n+1} \circ G_n$ or $\tilde{G}_{n+1} \circ G_n'$, where $\mu(\tilde{G}_{n+1})$ satisfies (7.6), $\mu(G_n)$ satisfies b_{4n} and $\mu(G_n')$ satisfies (7.2). The case of restriction is treated along the lines of (7.5), (7.6). In the other cases, (4.5) together with a_{4n}^2 imply

$$\mu(G_{n+1}, \delta_{n+1}^{-N}) \leq \prod_{i=4}^{6} (1 + \varepsilon_{7.1}) \cdot \frac{1}{\lambda^{\alpha n + s}} \exp\left(\frac{\exp(3 \cdot \lambda^{-\alpha n})}{2\lambda^{\alpha n + c_2/2}}\right) + \frac{\exp(3 \cdot \lambda^{-\alpha n})}{2\lambda^{\alpha n + c_2/2}} \quad (7.7)$$

which proves b_{4n+1}. $\square$

b_{1n+1}) $\{f_{n+1}\} = \bigcup_{k=1}^{\infty} \{f_{n+1k}\}$, where $f_{n+11} = f_k \circ F_{n-1} \circ g_n$, $k \in [1, n]$, or $f_{n+11} = f_k \circ \hat{f}_n$, and f_{n+1k} are obtained from f_{n+11} using consecutive compositions with different sorts of $\tilde{G}_n$ and G_n. Let us begin with $f_{n+11} = f_k \circ F_{n-1} \circ g_n$. (4.5) implies:

$$\mu(f_k \circ F_{n-1}) \leq \mu(f_k) \exp(\nu(F_{n-1}) \cdot |\Delta f_k|) + \nu(F_{n-1}) \cdot |\Delta f_k|.$$

We have $\nu(F_{n-1}) = \frac{1 + \varepsilon_{7.8}}{2\lambda^\gamma}$ (in consequence of b_{2n}), $|\Delta f_k| < \frac{1}{2^k \lambda^{c_1 k}}$ (in consequence of a_{1n}), thus

$$\mu(f_k \circ F_{n-1}) \leq \mu(f_k) \exp\left(\frac{1 + \varepsilon_{7.8}}{2^{k+1} \cdot \lambda^{c_1 k + \gamma}}\right) + \frac{1 + \varepsilon_{7.8}}{2^{k+1} \cdot \lambda^{c_1 k + \gamma}}. \quad (7.8)$$

Let Δ be the domain of $f_k \circ F_{n-1}$. Then (3.5) and (4.7) used with $a = \lambda$, imply

$$\mu(f_k \circ F_{n-1} \circ g_n) \leq \mu(f_k \circ F_{n-1})(1 + |\Delta| \cdot \lambda^{2s(n+1)-1}) + |\Delta| \cdot \lambda^{2s(n+1)-1}.$$

We have $|\Delta| < 2^{-n} \cdot \lambda^{-c_0 n}$, because of a_{1n}^2 and a_{2n}, and thus

$$\mu(f_k \circ F_{n-1} \circ g_n) \leq \mu(f_k \circ F_{n-1})\left(1 + \frac{1}{\lambda^s \cdot 2^n \cdot \lambda^{(c_0 - 2s)(n+1)}}\right) + \frac{1}{\lambda^s \cdot 2^n \cdot \lambda^{(c_0 - 2s)(n+1)}}. \quad (7.9)$$

In a similar way one verifies using a_{3n} and b_{3n} that $\mu(f_{n+1\,1} = f_k \circ \hat{f}_n)$ also satisfies (7.8).

Using b_{1n}, (7.8) and (7.9), we obtain

$$\mu(f_{n+1\,1}) \leq \left[\left(\sum_{i=1}^{k} \frac{1}{2^i \lambda^{\gamma i}}\right) \cdot \prod_{i=1}^{k} \left(1 + \frac{1}{2^i \lambda^{\gamma i}}\right) \cdot \left(\exp \sum_{i=1}^{k} \frac{1}{2^i \lambda^{\gamma i}}\right)\right.$$

$$\left. \cdot \exp\left(\frac{1 + \varepsilon_{7.8}}{2^{k+1} \lambda^{c_1 k + \gamma}}\right) + \frac{1 + \varepsilon_{7.8}}{2^{k+1} \lambda^{c_1 k + \gamma}}\right]$$

$$\cdot \left(1 + \frac{1}{\lambda^s \cdot 2^n \cdot \lambda^{(c_0 - 2s)(n+1)}}\right) + \frac{1}{\lambda^s \cdot 2^n \cdot \lambda^{(c_0 - 2s)(n+1)}}. \quad (7.10)$$

Since $c_1 - s = c_0 - 2s = \gamma$, we have

$$(1 + \varepsilon_{7.8}) \cdot 2^{-(k+1)} \cdot \lambda^{-c_1 k - \gamma} = (1 + \varepsilon_{7.8}) \cdot 2^{-(k+1)} \cdot \lambda^{-\gamma(k+1)} \cdot \lambda^{-sk} \ll 2^{-(k+1)} \cdot \lambda^{-\gamma(k+1)}$$

and

$$\lambda^{-s} \cdot 2^{-n} \cdot \lambda^{-(c_0 - 2s)(n+1)} = (2\lambda^{-s}) \cdot 2^{-(n+1)} \cdot \lambda^{-\gamma(n+1)} < 2^{-(n+1)} \cdot \lambda^{-\gamma(n+1)}.$$

Therefore

$$\mu(f_{n+1\,1}) < \left(\sum_{i=1}^{k} \frac{1}{2^i \lambda^{\gamma i}}\right) \cdot \left(\exp \sum_{i=1}^{k+1} \frac{1}{2^i \lambda^{\gamma i}}\right) \cdot \left(\prod_{i=1}^{k+1} \left(1 + \frac{1}{2^i \lambda^{\gamma i}}\right)\right)$$

$$+ \frac{1}{2^{k+1} \cdot \lambda^{\gamma(k+1)}}\left[\frac{(1 + \varepsilon_{7.8})[1 + (2\lambda^{-s}) \cdot 2^{-(n+1)} \cdot \lambda^{-\gamma(n+1)}]}{\lambda^{sk}} + \frac{2}{\lambda^s \cdot 2^{n-k} \cdot \lambda^{\gamma(n-k)}}\right]. \quad (7.11)$$

Since $k \leq n$, the factor in square brackets is less than 1, which implies $b_{1\,n+1}$ for $f_{n+1\,1}$.

If $f_{n+1\,k} = f_{n+1\,1} \circ \tilde{G}_{n+1} = f_{n+1\,1} \circ \tilde{G}_{n_1} \circ \ldots \circ \tilde{G}_{n_p}$ we have using (4.5), (3.5), (7.4) and $a_{1\,n+1}^1$

$$\mu(f_{n+1\,k}) \leq \mu(f_{n+1\,1}) \exp\left(\frac{1 + \varepsilon_{7.12}}{2^{n+1} \cdot \lambda^{s + \alpha n + \gamma(n+1)}}\right) + \frac{1 + \varepsilon_{7.12}}{2^{n+1} \cdot \lambda^{s + \alpha n + \gamma(n+1)}}. \quad (7.12)$$

Substituting (7.10) in (7.12) we obtain $b_{1\,n+1}$ as above. The same reasoning proves $b_{1\,n+1}$ for $f_{n+1\,k} = f_{n+1\,1} \circ \tilde{G}_{n+1} \circ G_n$. $\square$

$b_{2\,n+1}$) The proof is similar to the above proof of $b_{1\,n+1}$. $\square$

$b_{3\,n+1}$) For $\hat{f}_{n+1} = f_k \circ \hat{f}_n$ with $\Delta \hat{f}_n \subset \delta_{n+1}$ and for $\hat{f}_{n+1} = f_{n+1} \circ G_n$ the proof is similar. For $\hat{f}_{n+1} = f_{n+1} \circ G_n'$ (7.2) is applied. $\square$

8. Measure of Holes After Step $n+1$

For any $\lambda \in \mathcal{M}_n$ we estimate the measure of the union $\delta_n(\lambda) \cup \bigcup_{m=1}^{\infty} (\bigcup \delta_n^{-m}(\lambda))$, where $\delta_n^{-m}(\lambda) \subset [0,1] \setminus \delta_n(\lambda)$.

Lemma 2. *There exists an $\varepsilon < \lambda^{-t}$ so that for any $k \in \mathbb{Z}_+ \setminus \{0\}$*

$$\mathrm{mes} \left[\delta_k \cup \bigcup_{m=1}^{\infty} (\bigcup \delta_k^{-m}) \right] < \frac{(1+\varepsilon)^k}{\lambda^{sk}}.$$

Proof. We proceed by induction and assume that after Step n:
   i) The estimate of Lemma 2 holds for $k=n$;
   ii) To any hole δ_n^{-m} there corresponds a unique hole $\delta_{n-1}^{-m} \supset \delta_n^{-m}$ and a set $K_{n,m} = K_{n,m}(\delta_n^{-m}) \subset \delta_{n-1}^{-m}$, such that $K_{n,m} \subset \mathcal{X}_n$ and for some $\varepsilon_{8.1} = O(\lambda^{-t})$

$$\frac{\mathrm{mes}\, \delta_n^{-m}}{\mathrm{mes}\, K_{n,m}} < \frac{1 + \varepsilon_{8.1}}{\lambda^s}. \tag{8.1}$$

Remark VIII/1. The proof of Lemma 4 in Sect. 10 implies ii above. However we prove ii here in order to separate the proof of Lemma 2.

Remark VIII/2. We shall use here that the intervals δ_n, δ_{n+1} constructed in Sect. 10 are chosen so as to have $\delta_{n-1}^{-m} \subset \delta_n \setminus \delta_{n+1}$ for the holes δ_{n-1}^{-m} corresponding to holes $\delta_n^{-m} \subset \delta_n \setminus \delta_{n+1}$.
   We began Step $n+1$ by taking compositions $f_k \circ (F_{n-1} \circ g)$ or $f_k \circ \hat{f}_n$ and creating new holes of the form $(F_{n-1} \circ g)^{-1} \delta_n^{-m}$, $\hat{f}_n^{-1} \delta_n^{-m}$. Let $\delta_{nn+1} = \delta_n \setminus \delta_{n+1}$. There are holes δ_n^{-m} of two kinds inside δ_{nn+1}: the old ones $\delta_n^{-m} \subset \delta_{n-1}^{-m}$, and the new ones $\tilde{\delta}_n^{-M} = (F_{n-1} \circ g_n)^{-1} \delta_n^{-m}$, or $\tilde{\delta}_n^{-M} = \hat{f}_n^{-1} \delta_n^{-m}$ for $\varDelta \hat{f}_n \subset \delta_{nn+1}$, $m = 0, 1, \ldots$. Let

$$p_{nn+1} = \mathrm{mes}\,[(\bigcup \delta_n^{-m}) \cap \delta_{nn+1}], \quad \tilde{p}_{nn+1} = \mathrm{mes}\,[(\bigcup \tilde{\delta}_n^{-m}) \cap \delta_{nn+1}].$$

Then (8.1) implies

$$p_{nn+1} < |\delta_{nn+1}| \cdot (1 + \varepsilon_{8.1}) \cdot \lambda^{-s}. \tag{8.2}$$

One obtains similarly to (7.9) $\mu(F_{n-1} \circ g_n) < 1 + \lambda^{-\gamma}$. Then i) implies

$$\tilde{p}_{nn+1} < |\delta_{nn+1}| \cdot (\exp \mu(F_{n-1} \circ g_n)) \cdot [(1 + \varepsilon_{8.1}) \cdot \lambda^{-s}]^n$$
$$< |\delta_{nn+1}| \cdot (1 + \lambda^{-\gamma}) \cdot [(1 + \varepsilon_{8.1}) \cdot \lambda^{-s}]^n. \tag{8.3}$$

The construction of Sect. 3 implies the one-to-one correspondence between δ_{n+1}^{-m} and corresponding $\delta_n^{-m}(\delta_{n+1}^{-m} \subset \delta_n^{-m} \subset \delta_{nn+1})$. We have, according to the construction, $\mathrm{mes}\,(\bigcup \varDelta f_{n+1\,1}) \geqq (\mathrm{mes}\, \delta_{nn+1}) - p_{nn+1} - \tilde{p}_{nn+1}$. Now, we let $K_{n+1} = K_{n+1,0} = \bigcup \varDelta f_{n+1\,1}$ correspond to δ_{n+1}. In consequence of (8.2) and (8.3) we have

$$\frac{\mathrm{mes}\, \delta_{n+1}}{\mathrm{mes}\, K_{n+1}} < \frac{2(1 + o(\lambda^{-t(n+1)}))}{\lambda^{s(n+1)}}$$
$$: \left[\frac{2}{\lambda^{sn}} \left(1 - \frac{1 + o(\lambda^{-t(n+1)})}{\lambda^s} \right) \cdot \left(1 - \frac{1 + \varepsilon_{8.1}}{\lambda^s} - (1 + \lambda^{-\gamma}) \left(\frac{1 + \varepsilon_{8.1}}{\lambda^s} \right)^n \right) \right]. \tag{8.4}$$

The right part of (8.4) is less than $(1 + \varepsilon_{8.1}) \lambda^{-s}$ for a suitable $\varepsilon_{8.1} = O(\lambda^{-r})$, $s > r > t$.

We let $K_{n+1,m} = G_{n+1}^{-1}(K_{n+1})$ correspond to $\delta_{n+1}^{-m} = G_{n+1}^{+1}(\delta_{n+1})$. We have

$$\frac{\operatorname{mes}\delta_{n+1}^{-m}}{\operatorname{mes}K_{n+1,m}} < \frac{\operatorname{mes}\delta_{n+1}}{\operatorname{mes}K_{n+1}} \cdot \exp\mu(G_{n+1}). \tag{8.5}$$

Because of b_{4n+1}, the right side of (8.4) with the additional factor $\exp\mu(G_{n+1})$ is still less than $(1+\varepsilon_{8.1})\lambda^{-s}$ and (8.1) is proved for $k=n+1$. Lemma 2 with $\varepsilon=\varepsilon_{8.1}$ follows now from

$$\operatorname{mes}(\bigcup\delta_{n+1}^{-m}) < (1+\varepsilon)\lambda^{-s}\cdot\operatorname{mes}(\bigcup K_{n,m}) < (1+\varepsilon)\lambda^{-s}\cdot\operatorname{mes}(\bigcup\delta_n^{-m}) < \left(\frac{1+\varepsilon}{\lambda^s}\right)^{n+1}.$$

The estimates of Sects. 6–8 prove the following

Proposition 1. *Let $\lambda\in[N_0, N_0+4]$ be so that for any $n=1,2,...,F_{n+1}$*
$$\circ h(\lambda,\tfrac{1}{2})\in[0,1]\Big\backslash \bigcup_{m=0}^{\infty}(\bigcup\hat{\delta}_n^{-m}).\ \textit{Then a partition } \xi_\lambda \textit{ as in Sect. 1 exists.}$$

Remark VIII/3. Notice that if λ is such that at step n $F_{n-1}\circ h(\lambda,\tfrac{1}{2})$ falls into a limit set $\mathscr{F}_n$ defined in Sect. 3 the condition of Proposition 1 will be satisfied. It is certainly so at Step n, and at subsequent steps the holes $\hat{\delta}_p^{-m}$ lie either in $\hat{\delta}_n^{-m}$, or in the intervals $F_{n-1}\circ g$, $\hat{f}_n$ constructed at Step n (there is no middle branch $F_p\circ h$ for $p\geq n$). The estimates of Sects. 6–8 are even better in this case.

Remark VIII/4. If we suppose $F_{n-1}\circ h(\lambda,\tfrac{1}{2})$ is outside $\hat{\delta}_1(\lambda)=(\tfrac{1}{2}-\lambda^{-(s-\alpha)}, \tfrac{1}{2}+\lambda^{-(s-\alpha)})$ for all n, the above condition of Proposition 1 will be satisfied. In particular, if $h(\lambda,\tfrac{1}{2})$ falls into some f_λ-invariant set (e.g. periodic orbit or invariant Cantor set of [5]) lying outside $\hat{\delta}_1(\lambda)$, λ satisfies this condition. Thus card$\{\lambda$ satisfying Proposition 1$\}$ equals the continuum. One can check however, using estimates of Sect. 11, that $\operatorname{mes}\{\lambda:F_{n-1}\circ h(\lambda,\tfrac{1}{2})\in[0,1]\backslash\hat{\delta}_1(\lambda)\}=0$.

9. Velocities of Endpoints of Domains $\Delta f_n(\lambda)$

Let f_n be one of the maps constructed at step n, with domain $\Delta f_n=[x_{1n},x_{2n}]$. In this section we prove the following

Lemma 3. *There is an $\varepsilon=O(\lambda^{-s(1-\nu)})$ such that for $i=1,2$*

$$\left|\frac{dx_{in}(\lambda)}{d\lambda}\right| < \frac{\lambda^{sn}(1+\varepsilon)}{8\lambda}.$$

Proof. Any $x_{ik}(\lambda)$ satisfies $f_k(\lambda,x_{ik}(\lambda))=0$ or 1. Thus

$$\left|\frac{dx_{ik}(\lambda)}{d\lambda}\right| = \left|\frac{\partial f_k(\lambda,x_{ik}(\lambda))/\partial\lambda}{\partial f_k(\lambda,x_{ik}(\lambda))/\partial x}\right|.$$

We proceed by induction as in the main construction. Consider the maps $f_k(2\leq k \leq n)$, $G_n:\delta_n^{-m}\to\delta_n$, and $\hat{f}_n$. Assume inductively that the following estimates hold:

$$c_{1k})\ \left|\frac{\partial f_k(\lambda,x)/\partial\lambda}{\partial f_k(\lambda,x)/\partial x}\right| < \frac{\lambda^{sk}}{8\lambda}\left[1 + \sum_{i=1}^{k-1}\frac{1+\varepsilon}{\lambda^{s(1-\nu)i}}\right].$$

c_{2n}) Let H_n denote either G_n or $\hat{f}_n$, and pick p so that, if $p \leq n$, then $\Delta H_n \subset [0, 1] \setminus \delta_p$, while if $p > n$, then $\text{dist}(\Delta H_n, \frac{1}{2}) > \lambda^{-sp}$. Let $N = \max(n, p)$. Then

$$\left| \frac{\partial H_n(\lambda, x)/\partial \lambda}{\partial H_n(\lambda, x)/\partial x} \right| < \frac{\lambda^{sN}}{8\lambda} \left[1 + \sum_{i=1}^{n-1} \frac{1+\varepsilon}{\lambda^{s(1-v)i}} \right].$$

For $k = 1$, these estimates are proven in Sect. 2. We will prove $c_{1\,n+1}$ in the various cases that arise from the construction. $c_{2\,n+1}$ is similar. In particular, c_{1n} implies Lemma 3.

Suppose $\varphi_i(\lambda, x)$, $i = 1, \ldots, n$ are C^1 functions, and define

$$F(\lambda, x) = \varphi_n(\lambda, \varphi_{n-1}(\lambda, \ldots, \varphi_1(\lambda, x)\ldots)).$$

One sees that

$$\frac{\partial F}{\partial \lambda} = \sum_{k=1}^{n} \left[\left(\frac{\partial \varphi_k}{\partial \lambda} \right) \prod_{i=k+1}^{n} \frac{\partial \varphi_i}{\partial x} \right]$$

so that, at any point (λ_0, x_0) in the domain of F,

$$\frac{\partial F/\partial \lambda}{\partial F/\partial x} = \sum_{k=1}^{n} \left[\left(\frac{\partial \varphi_k/\partial \lambda}{\partial \varphi_k/\partial x} \right) \cdot \left(\prod_{i=1}^{k-1} \frac{\partial \varphi_i}{\partial x} \right)^{-1} \right], \tag{9.1}$$

where the partials of φ_i are evaluated at $(\lambda_0, \varphi_{i-1}(\lambda_0, \ldots, \varphi_1(\lambda_0, x_0)\ldots))$.

To prove $c_{1\,n+1}$, we first consider the case

$$f_{n+1\,1} = f_{i_n} \circ f_{i_{n-1}} \circ \ldots \circ f_{i_1} \circ g_n(\lambda, x).$$

Since

$$g_n(\lambda, x) = \lambda x(1-x) \quad |x - \tfrac{1}{2}| > \lambda^{-s(n+1)},$$

we have

$$|\partial g_n/\partial \lambda| < \tfrac{1}{4}, |\partial g_n/\partial x| > 2\lambda/\lambda^{s(n+1)}.$$

Using (9.1), a_{1n}^2 and $c_{1k}(k = i_1, \ldots, i_n)$, we obtain

$$\left| \frac{\partial f_{n+1\,1}/\partial \lambda}{\partial f_{n+1\,1}/\partial x} \right| < \frac{\lambda^{s(n+1)}}{2\lambda} \left[\frac{1}{4} + \frac{\lambda^s(1+\varepsilon)}{8\lambda} \left(1 + \frac{1}{\lambda^{s(1-v)} - 1} \right) \right.$$

$$\left. \cdot \left(1 + \frac{\lambda^s}{2\lambda^{co}} + \ldots + \left(\frac{\lambda^s}{2\lambda^{co}} \right)^{n-1} \right) \right]$$

$$< \frac{\lambda^{s(n+1)}}{8\lambda} \left[1 + \frac{1+\varepsilon_{9.2}}{2\lambda^{co}} \right]. \tag{9.2}$$

This proves $c_{1\,n+1}$ in case $f_{n+1\,1} = f_k \circ F_{n-1} \circ g_n$.

In case $f_{n+1\,1} = f_k \circ \hat{f}_n$, (9.1), c_{1k} and c_{2n}, and a_{3n} imply

$$\left| \frac{\partial f_{n+1\,1}/\partial \lambda}{\partial f_{n+1\,1}/\partial x} \right| < \frac{\lambda^{s(n+1)}}{8\lambda} \left[1 + \sum_{i=1}^{n-1} \frac{1+\varepsilon}{\lambda^{s(1-v)i}} + \frac{1}{\lambda^{\gamma n}} \right]. \tag{9.3}$$

Similarly, if $F_{n-1} = f_{i_{n-1}} \circ \ldots \circ f_{i_1}$ and $|x - \tfrac{1}{2}| > \lambda^{-sp}$, then

$$\left| \frac{\partial(F_{n-1} \circ g)/\partial \lambda}{\partial(F_{n-1} \circ g)/\partial x} \right| < \frac{\lambda^{sp}}{8\lambda} \left(1 + \frac{1+\varepsilon_{9.4}}{2\lambda^{co}} \right). \tag{9.4}$$

Now let $G'_n = G_n \circ F_{n-1} \circ g$, where $\Delta(F_{n-1} \circ g) \subset [0,1] \backslash \left(\frac{1}{2} - \frac{1}{\lambda^{sp}}, \frac{1}{2} + \frac{1}{\lambda^{sp}} \right)$, and $\Delta G_n \subset [0,1] \backslash \delta_n$. Using (9.1), c_{2n}, a_{2n} and a_{4n}^2, we see that

$$\left| \frac{\partial G'_n/\partial \lambda}{\partial G'_n/\partial x} \right| < \frac{\lambda^{sp}}{8\lambda} \left(1 + \frac{1 + \varepsilon_{9.4}}{2\lambda^{co}} \right) + \frac{\lambda^{sn}(1+\varepsilon)}{(8\lambda)(2\lambda)^{co(n-1)}(2\lambda^{1-sp})} \sum_{i=0}^{n-1} \lambda^{-s(1-v)i}$$

$$= \frac{\lambda^{sp}}{8\lambda} \left(1 + \frac{1 + \varepsilon_{9.5}}{2\lambda^{co}} \right). \tag{9.5}$$

On the other hand, for $G'_n = G_n \circ \hat{f}_n$ where $\Delta \hat{f}_n \subset [0,1] \backslash \left(\frac{1}{2} - \frac{1}{\lambda^{sp}}, \frac{1}{2} + \frac{1}{\lambda^{sp}} \right)$, we obtain an estimate similar to (9.3):

$$\left| \frac{\partial G'_n/\partial \lambda}{\partial G'_n/\partial x} \right| < \frac{\lambda^{sp}}{8\lambda} \left(1 + \sum_{i=1}^{n-1} \frac{1 + \varepsilon}{\lambda^{s(1-v)i}} + \frac{1}{\lambda^{yn}} \right). \tag{9.6}$$

Finally, let $\tilde{G}_n = G_n$ or G'_n, $\Delta \tilde{G}_n \subset \delta_n \backslash \delta_{n+1}$. Then in c_{2n}, (9.4) and (9.6), $p = n+1$. Now a_{4n}^1 implies $|D\tilde{G}_n| > \lambda^{ns(1-v)}$. Hence using (9.1), (9.2) or (9.3) and c_{2n}, (9.5) and (9.6), we obtain for $f_{n+1\,k} = f_{n+1\,1} \circ \tilde{G}_{n_1} \circ \dots \circ \tilde{G}_{n_{k-1}}$

$$\left| \frac{\partial f_{n+1\,k}/\partial \lambda}{\partial f_{n+1\,k}/\partial x} \right| < \frac{\lambda^{s(n+1)}}{8\lambda} \left[1 + \sum_{i=1}^{n-1} \frac{1 + \varepsilon}{\lambda^{s(1-v)i}} + \frac{1}{\lambda^{yn}} \right] \left[1 + \frac{1}{\lambda^{s(1-v)n}} + \dots \right.$$

$$\left. + \frac{1}{\lambda^{s(1-v)(k-1)n}} \right]$$

$$< \frac{\lambda^{s(n+1)}}{8\lambda} \left[1 + \sum_{i=1}^{n-1} \frac{1 + \varepsilon}{\lambda^{s(1-v)i}} + \frac{1}{\lambda^{yn}} \right] \left[1 + \frac{1}{\lambda^{s(1-v)n} - 1} \right]$$

$$< \frac{\lambda^{s(n+1)}}{8\lambda} \left[1 + \sum_{i=1}^{n} \frac{1 + \varepsilon}{\lambda^{s(1-v)i}} \right] \tag{9.7}$$

for a suitable $\varepsilon = O(\lambda^{-s(1-v)})$.

This proves $c_{1\,n+1}$; the proof of $c_{2\,n+1}$ is similar.

10. Construction of $\delta_{n+1}(\lambda)$ and $\hat{\delta}_{n+1}(\lambda)$. Structure of $\mathcal{X}_{n+1}$ in a $\lambda^{(-s+2\alpha)(n+1)}$-Neighborhood of δ_{n+1}

a) Recall that at step $n+1$ of the induction construction, we consider λ contained in an interval $\mathcal{J}_n = [\lambda_{0n}, \lambda_{1n}]$. As λ varies in $\mathcal{J}_n$, all the maps under consideration together with their domains vary continuously with λ.

The induction hypotheses a_{in} in Sect. 6 imply the following estimates on the diameters of the domains appearing at step $n+1$:

$$\left. \begin{array}{l} |\Delta f_k \circ \hat{f}_n| < \lambda^{-s} \cdot (2\lambda^{c_1})^{-(n+1)} \\ |\delta_{np_0}^{-m}| < |\delta_n| \cdot \lambda^{-s(1-v)p_0} \\ |\Delta f_k \circ F_{n-1} \circ g| < (2\lambda^{c_1})^{-(n+1)}. \end{array} \right\} \tag{10.1}$$

In the second estimate of (10.1), we write δ_n^{-m} as $\delta_{np_0}^{-m}$, where p_0 denotes the minimum integer p such that

$$\delta_n^{-m} \subset [\tfrac{1}{2} - \lambda^{-sp}, \tfrac{1}{2} + \lambda^{-sp}].$$

In the third estimate, recall that $\Delta F_{n-1} \circ g = [x_{\min}, x_{\max}]$ with

$$|x_{\min} - \tfrac{1}{2}| > \lambda^{-s(n+1)}.$$

For any $G_n : \delta_n^{-k} \to \delta_n$ we have, according to b_{4n} of Sect. 6, that $\mu(G_n) < \lambda^{-\alpha n}$. But actually for $\delta_n^{-k} \subset [0, 1] \setminus \delta_{n+1}$, we can strongly enlarge δ_n^{-k} and still have the maps G_n defined with $\mu(G_n)$ small. Let us consider the homothetic transformation

$$\psi_n(\lambda) : x \mapsto \tfrac{1}{2} + (x - \tfrac{1}{2})\lambda^{2\alpha n}.$$

It follows from the condition $\alpha \leqq s/4$ that for $n \geqq 3$ one can define

$$q(n) = \max\{q : \psi_n(\lambda)\delta_n(\lambda) \subset \delta_q(\lambda)\} \geqq 1.$$

Remark X/1. For $n = 1$, the endpoints of $\psi_1(\lambda)\delta_1(\lambda)$ belong to $\bigcup \Delta f_1$. We define $\delta_{q(1)}(\lambda)$ for all $\lambda \in \mathscr{I}_n$ as the minimal interval containing $\psi_1(\lambda)\delta_1(\lambda)$ of the form $[x_{1\max}(\lambda), x_{2\max}(\lambda)]$, where $x_{1\max}(\lambda) \in [0, \tfrac{1}{2}]$ and $x_{2\max}(\lambda) \in [\tfrac{1}{2}, 1]$ are endpoints of domains Δf_1. We define $\delta_{q(2)}(\lambda)$ in an analogous way whenever $\psi_2(\lambda)\delta_2(\lambda) \not\subset \delta_1(\lambda)$.

It follows from the construction of Sect. 3 that for every interval $G_n^{-1}\delta_n$ (or $(G_n')^{-1}\delta_n$) which lies outside the domain $\Delta F_{q(n)} \circ h$, the corresponding preimage $G_n^{-1}\delta_{q(n)}$ is defined. Indeed, the maps G_n under consideration are those compositions of $G_{q(n)}$ and $F_k \circ g$ or $\hat{f}_k$, $q(n) \leqq k \leqq n$, which map their domains onto $[0, 1]$. Using Lemma 1 and following the proof of b_{4n}, we get for some $\varepsilon_{10.2} < \lambda^{-t}$

$$\mu(G_n, \delta_{q(n)}^{-k}) < (1 + \varepsilon_{10.2})\lambda^{-\alpha q(n)}. \tag{10.2}$$

From the definition of $q(n)$ for $n \geqq 3$ it follows that

$$q(n) \geqq \max\left\{q : \left(1 - 2\frac{\alpha}{s}\right)n > 1\right\}.$$

Since $2\dfrac{\alpha}{s} \leqq \tfrac{1}{2}$, we get

$$q(n) \geqq \max\left\{q : \frac{n}{2} > q\right\} = \begin{cases} \dfrac{n}{2} - 1 & \text{for} \quad n \text{ even} \\[2mm] \dfrac{n}{2} - \dfrac{1}{2} & \text{for} \quad n \text{ odd}. \end{cases}$$

In particular, we always have

$$q(n) \geqq \frac{n}{2} - 1. \tag{10.3}$$

We shall show that for $n \geqq 3$

$$\Delta F_{q(n)} \circ h \subset \left(\frac{1}{2} - \frac{1}{\lambda^{s(n+1)}}, \frac{1}{2} + \frac{1}{\lambda^{s(n+1)}}\right). \tag{10.4}$$

Let $\Delta F_{q(n)} \circ h = [\frac{1}{2} - v_{1n}, \frac{1}{2} + v_{2n}]$, $v_{in} > 0$. In a way analogous to (6.11) we get

$$v_{in} < 1/[(\sqrt{2})^{q(n)} \lambda^{(c_0 q(n) + 1)/2}]. \tag{10.5}$$

From (10.3) and (10.5) we obtain that for (10.4) it is enough to have $\dfrac{c_0}{2}\left(\dfrac{n}{2} - 1\right) + \dfrac{1}{2}$

$> s(n+1)$, or taking into account that $c_0 = 1 - s$, $\left(\dfrac{1-s}{4} - s\right) n > \dfrac{s}{2}$. This holds for

$s \leqq \frac{1}{13}$, $n \geqq 1$.

From the fact that for $s \leqq \frac{1}{13}$ the domain of the central branch $\Delta h \subset \delta_6$, it follows that for $n \leqq 5$ if $G_n^{-1} \delta_n \subset \delta_n \backslash \delta_{n+1}$, then $G_n^{-1}[0, 1]$ is defined.

In such a way, for all $n \geqq 1$ and for all domains

$$G_n^{-1} \delta_n \subset [0, 1] \bigg\backslash \left(\frac{1}{2} - \frac{1}{\lambda^{s(n+1)}}, \frac{1}{2} + \frac{1}{\lambda^{s(n+1)}}\right)$$

the preimage

$$G_n^{-1} \delta_{q(n)} \supset G_n^{-1}(\psi_n(\lambda) \delta_n(\lambda))$$

is defined.

b) Let us estimate the length of $\mathscr{J}_n$. When λ varies in $\mathscr{J}_n$

$$f_{i_{n-1}} \circ f_{i_{n-2}} \circ \ldots \circ f_{i_1} \circ h(\lambda, \tfrac{1}{2})$$

varies in one of Δf_{i_n} and $f_{i_n} \circ f_{i_{n-1}} \ldots f_{i_1} \circ h(\lambda, \tfrac{1}{2})$ varies in $[0, 1]$. We have

$$\frac{\partial(f_{i_n} \circ \ldots \circ f_{i_1} \circ h(\lambda, \tfrac{1}{2}))}{\partial \lambda} = \prod_{k=1}^{n} \frac{\partial f_{i_k}}{\partial x} \left[\frac{1}{4} + \sum_{j=1}^{n} \frac{\partial f_{i_j}/\partial \lambda}{\partial f_{i_j}/\partial x} \left(\prod_{\ell=1}^{j-1} \frac{\partial f_{i_\ell}}{\partial x}\right)^{-1}\right], \tag{10.6}$$

where the arguments of $f_{i_p}(\lambda, x)$ are $x = f_{i_{p-1}} \circ f_{i_{p-2}} \circ \ldots \circ f_{i_1} \circ h(\lambda, \tfrac{1}{2})$. In consequence of c_{1k} and a_{1n}^2, the sum in brackets is larger than $\dfrac{1}{4} - \dfrac{(1 + \varepsilon_{10.6})\lambda^s}{8\lambda} > \dfrac{1}{4}(1 - \lambda^{-c_0})$. We shall use $v_n(\lambda)$ to denote the velocity of the top. We have

$$v_n(\lambda) = \left|\frac{\partial(f^n \circ \ldots \circ f^1 \circ h(\lambda, \tfrac{1}{2}))}{\partial \lambda}\right| > (2\lambda^{c_0})^n \cdot \frac{1}{4}\left(1 - \frac{1}{\lambda^{c_0}}\right). \tag{10.7}$$

Thus

$$|\mathscr{J}_n| < 4 \cdot (1 - \lambda_{0n}^{-c_0})^{-1} \cdot (2\lambda_{0n}^{c_0})^{-n}. \tag{10.8}$$

We formulate the induction conditions on the choice of $\delta_n(\lambda)$.

i) The interval $\delta_n(\lambda)$ is of the form:

$$\delta_n(\lambda) = (\tfrac{1}{2} - c_{n1}(\lambda) \cdot \lambda^{-sn}, \tfrac{1}{2} + c_{n2}(\lambda) \cdot \lambda^{-sn}), \quad 1 \leqq c_{ni}(\lambda) < 1 + o(\lambda^{-tm}). \tag{10.9}$$

ii) If for some δ_n^{-k} $\delta_{q(n)}^{-k} \cap \delta_n \neq \emptyset$, then $\delta_{q(n)}^{-k} \subset \delta_n$.

iii) If a_n is an endpoint of δ_n, then a_n coincides with a common endpoint of two intervals: some Δf_n exterior to δ_n and some $\Delta F_{n-1} \circ g$ or $\Delta \hat{f}_n$ interior to δ_n.

According to the construction of Sect. 3 we consider intervals $\Delta f_k \circ F_{n-1} \circ g$, $\Delta f_k \circ \hat{f}_n$, $(G_n')^{-1} \delta_n$, and have to choose an interval $\delta_{n+1}(\lambda)$ satisfying the above conditions and varying continuously with $\lambda \in [\lambda_{0n}, \lambda_{1n}]$.

Consider the point $\xi_{0n}=\frac{1}{2}-(1+\lambda_{0n}^{-\frac{\alpha}{8}(n+1)})\lambda_{0n}^{-s(n+1)}$. For $k=q(n)$, (ii) implies that if two intervals $\delta_{q(n)}^{-m}$ intersect, then one of them contains the other. Let $\bar{\delta}_{q(n)}^{-m}$ be the maximal interval containing ξ_{0n}. Then we replace ξ_{0n} by ξ_{1n} which is the endpoint of $\bar{\delta}_{q(n)}^{-m}$. If ξ_{0n} is not contained in any $\delta_{q(n)}^{-m}$, but is inside some interval $\Delta f_k \circ F_{n-1} \circ g$ or $\Delta f_k \circ \hat{f}_n$, we let ξ_{1n} be the right endpoint of this interval. If ξ_{0n} is a limit point of δ_n^{-m} we obtain any of the previous cases with an arbitrary small perturbation of ξ_{0n}. The estimates (10.1), c_{2n}, (10.8), (10.9) show that when $\lambda \geq \lambda_{0n}$ varies in $\mathcal{I}_n$

$$\frac{1}{2}-(1+\lambda_{0n}^{-\frac{\alpha}{8}(n+1)})\cdot\lambda_{0n}^{-s(n+1)}-(1+\varepsilon_{10.10})\lambda_{0n}^{-s}(2\lambda_{0n}^{c_1})^{-(n+1)}<\xi_{1n}(\lambda)$$

$$<\frac{1}{2}-(1+\lambda_{0n}^{-\frac{\alpha}{8}(n+1)})\cdot\lambda_{0n}^{-s(n+1)}+2(1+\varepsilon_{10.10})\lambda_{0n}^{-[sq(n)+s(1-v)]n}$$

$$+(1+\varepsilon_{10.10})\lambda_{0n}^{-s}(2\lambda_{0n}^{c_1})^{-(n+1)}. \tag{10.10}$$

We shall show that for $n \geq 7$ $\lambda_{0n}^{-[sq(n)+s(1-v)n]} \ll \lambda_{0n}^{-\left(\frac{\alpha}{8}+s\right)(n+1)}$. For this it is enough to have $sq(n)+s(1-v)n>\left(s+\frac{\alpha}{8}\right)(n+1)$. Since

$$q(n) \geq \frac{n}{2}-1, \quad \alpha \leq \frac{s}{4}, \quad v=\frac{2(s-\alpha)}{1-s}<\frac{2s}{1-s},$$

we get the inequality $n\left(\frac{1}{2}-\frac{2s}{1-s}-\frac{1}{32}\right)>2+\frac{1}{32}$, which holds for $n \geq 7$, $s \leq \frac{1}{13}$.

For $n \leq 6$ the check that for $\delta_{q(n)}^{-1} \subset \delta_n \setminus \delta_{n+1}$, $|\delta_{q(n)}^{-1}| \ll \frac{1}{\lambda^{s(n+1)}}$ is straightforward. The worst estimates correspond to $n=6$. Since $q(6) \geq 2$ and $Dh|_{\delta_6 \setminus \delta_7}>2\lambda^{1-7s}$, we get

$$|h^{-1}\delta_2|<\frac{1+\varepsilon}{\lambda^{1-5s}} \ll \frac{1}{\lambda^{7s}}$$

for $s \leq \frac{1}{13}$.

Taking into account (10.8) and the formula $c_1=1-2s \geq 11s$, we obtain from (10.10)

$$\frac{1}{2}-(1+2\cdot\lambda^{-\frac{\alpha}{8}(n+1)})\lambda^{-s(n+1)}<\xi_{1n}(\lambda)<\frac{1}{2}-\lambda^{-s(n+1)}$$

and we can make ξ_{1n} the left endpoint of $\delta_{n+1}(\lambda)$. The analogous choice of the right endpoint gives us

$$\delta_{n+1}(\lambda)=(\frac{1}{2}-(1+c_{n+1\,1})\lambda^{-s(n+1)},\frac{1}{2}+(1+c_{n+1\,2})\lambda^{-s(n+1)})c_{n+1\,i}=o(\lambda^{-t(n+1)}). \tag{10.11}$$

One easily checks that $\delta_{n+1}(\lambda)$ also satisfies (ii) and (iii).

c) We then construct an enlarged interval $\hat{\delta}_{n+1}(\lambda)$. We begin by expanding $\delta_{n+1}(\lambda_{0n})$ with a homothetic transformation

$$\varphi_{n+1}:x \to \frac{1}{2}+(x-\frac{1}{2})\lambda_{0n}^{\alpha(n+1)}(1+\lambda_{0n}^{-\frac{\alpha}{8}(n+1)}).$$

Then we proceed with the endpoints of $\varphi_{n+1}\delta_{n+1}(\lambda_{0n})$ as above, i.e. using a small perturbation we make the endpoints of $\varphi_{n+1}\delta_{n+1}(\lambda_{0n})$ coincide with endpoints of

some interval Δf_k, $k \leq n$. One checks as above, that this can be done so that the interval $\hat{\delta}_{n+1}(\lambda)$ satisfies for all $\lambda \in \mathscr{J}_n = [\lambda_{0n}, \lambda_{1n}]$ the inequalities:

$$\lambda^{\alpha(n+1)}|\delta_{n+1}(\lambda)| < |\hat{\delta}_{n+1}(\lambda)| < \lambda^{\alpha(n+1)}|\delta_{n+1}(\lambda)|(1 + O(\lambda^{-\frac{\alpha}{8}(n+1)})). \qquad (10.12)$$

As $\hat{\delta}_{n+1}(\lambda) \subset \delta_{q(n+1)}(\lambda)$, for any $\delta_{n+1}^{-k}(\lambda) = G_{n+1}^{-1}\delta_{n+1}(\lambda)$ the corresponding interval $\hat{\delta}_{n+1}^{-k} = G_{n+1}^{-1}\hat{\delta}_{n+1}(\lambda)$ is defined. Taking into account an additional factor $\exp \mu(G_{n+1}, \hat{\delta}_{n+1}^{-k}) < (1+\varepsilon)\lambda^{-(s-\alpha)(n+1)}$ we still have

$$\lambda^{\alpha(n+1)}|\delta_{n+1}^{-k}(\lambda)| < |\hat{\delta}_{n+1}^{-k}(\lambda)| < \lambda^{\alpha(n+1)} \cdot (1 + o(\lambda^{-t(n+1)})) \cdot |\delta_{n+1}^{-k}(\lambda)| \qquad (10.13)$$

d) When estimating $\mathrm{mes}\,\mathscr{M}_{n+1}$ we shall use the following

Lemma 4. *For any n there is a set $L_n \subset \mathscr{X}_n$ corresponding to δ_n, and for any $\delta_n^{-k} \subset [0,1] \backslash \delta_n$ there is a corresponding set $L_n^{-k} \subset \mathscr{X}_n$, such that*

(a) *if $\delta_n^{-k_1} \neq \delta_n^{-k_2}$ then $L_n^{-k_1} \cap L_n^{-k_2} = \emptyset$*

and

(b) *$\mathrm{mes}(L_n^{-k}) > (1 - \varepsilon_0)\lambda^{2\alpha n}\,\mathrm{mes}(\delta_n^{-k})$, with $\varepsilon_0 = O(\lambda^{-2\alpha})$.*

Proof. In addition to the estimate $\alpha \leq s/4$, we will suppose that α has the form

$$\alpha = s/2k_0,$$

where k_0 is an integer ≥ 2. This assumption is not really necessary, but it simplifies the notation.

If an interval δ with center x_0 and a number $c > 0$ are given, we shall denote by $c \cdot \delta$ the image of δ under the homothetic transformation $x \to x_0 + (x - x_0) \cdot c$. Further, we shall use $\delta_n^{(r)}$ to denote the set $\lambda^{2\alpha r} \cdot \delta_n \backslash \lambda^{2\alpha(r-1)} \cdot \delta_n$.

Let $\varepsilon_1 = 3 \cdot \lambda^{-2\alpha}$, $\psi_0 = 0$, $\psi_i = (2 \cdot \lambda^{-(s-2\alpha)})^i$, $i \geq 1$, $c_n = \prod_{i=0}^{n-1}(1 + \psi_i)$, $c = \lim_{n \to \infty} c_n$.

We prove Lemma 4 by induction. We assume that L_n, L_n^{-k} are constructed and consist of Δf_r, $r \leq n$, and that the following property holds: For any δ_n^{-k}, $k = 0, 1, \dots$ there exists an increasing sequence of intervals $(\lambda^{2\alpha r} \cdot \delta_n)^{-k}$, $r = 0, 1, \dots, R = R(\delta_n^{-k}) \geq n$, such that

$$\frac{\mathrm{mes}(\lambda^{2\alpha(r-1)} \cdot \delta_n)^{-k}}{\mathrm{mes}(L_n^{-k} \cap \delta_n^{(r)})} < \frac{c_n(1 + \varepsilon_1)}{\lambda^{2\alpha}}. \qquad (10.14)$$

We define L_{n+1}^{-m} corresponding to δ_{n+1}^{-m} and prove (10.14) for $n+1$. Then Lemma 4 follows with $1 - \varepsilon_0 = c^{-1} \cdot (1 + \varepsilon_1)^{-1}$.

Consider $\lambda^{2\alpha n} \cdot \delta_n \subset \delta_{q(n)}$. Condition ii and the construction of δ_{n+1} imply $\delta_{q(n)}^{-m} \subset \delta_n \backslash \delta_{n+1}$ for $\delta_n^{-m} \subset \delta_n \backslash \delta_{n+1}$. Considering maximal elements $\hat{\delta}_{q(n)}^{-m}$ among $\{\hat{\delta}_{q(n)}^{-m} \subset \delta_n \backslash \delta_{n+1}\}$ and the corresponding diffeomorphisms $\bar{G}_n^{-m}$, we transmit the structure from $\lambda^{2\alpha n} \cdot \delta_n$ into each $\hat{\delta}_{q(n)}^{-m}$ and obtain that corresponding to any $\delta_n^{-m} \subset \delta_n \backslash \delta_{n+1}$ one can pick $L_n^{-m} \subset \delta_{q(n)}^{-m} \subset \delta_n \backslash \delta_{n+1}$ so that $L_n^{-m} \cap L_n^{-r} = \emptyset$, if $\delta_n^{-m} \neq \delta_n^{-r}$ and (10.14) multiplied by an additional factor $\exp(\lambda^{-\alpha q(n)})$ holds for L_n^{-m}.

Let us consider the domain $V_{n+1} = \lambda^{2\alpha n} \cdot \delta_n \backslash \lambda^{2\alpha(n+1)} \cdot \delta_{n+1}$. Taking into account (10.9), (10.11), and $s = 2k_0\alpha$, $k_0 \geq 2$, we obtain

$$\mathrm{mes}(V_{n+1} \triangle (\lambda^{2\alpha n} \cdot \delta_n \backslash \lambda^{2\alpha[n-(k_0-1)]} \cdot \delta_n)) = o(\lambda^{-(s-2\alpha)(n+1)}).$$

Together with (10.14) this implies

$$c_n(1+\varepsilon_1)(1+\chi_{1\,n+1})\cdot \text{mes}(L_n\cap V_{n+1}) > \lambda^{2\alpha}\cdot \text{mes}(\lambda^{2\alpha(n+1)}\cdot \delta_{n+1}) \qquad (10.15)$$

(here and below $\chi_{i\,n+1}=o(\lambda^{-t(n+1)})$).

For $\delta_{n+1}^{-m} \subset \delta_n^{-m} \subset \delta_{q(n)}^{-m} \subset \delta_n\backslash\delta_{n+1}$ the corresponding set V_{n+1}^{-m} is defined and

$$c_n(1+\varepsilon_1)(1+\chi_{2\,n+1})\cdot \text{mes}(L_n^{-m}\cap V_{n+1}^{-m}) > \lambda^{2\alpha}\cdot \text{mes}(\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m}. \qquad (10.16)$$

We define $\bar{L}_{n+1}=\bigcup(L_n^{-m}\cap V_{n+1}^{-m})$ where the sum is taken over all $L_n^{-m}\subset \delta_n\backslash\delta_{n+1}$. For any $r\geq 1$ such that $\delta_n \supset \lambda^{2\alpha r}\cdot \delta_{n+1}$ consider $\delta_{q(n)}^{-m}\subset \delta_{n+1}^{(r)}$ and corresponding $(\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m}$, $V_{n+1}^{-m}\subset \delta_{q(n)}^{-m}$. Since the dimensions of $\delta_{q(n)}^{-m}$ are small compared to $\delta_{n+1}^{(r)}$ (see the proof of 10.10 above) we obtain from (10.16) that

$$\frac{\text{mes}(\lambda^{2\alpha(r-1)}\cdot \delta_{n+1})}{\text{mes}(\bar{L}_{n+1}\cap \delta_{n+1}^{(r)})} < \frac{1}{\lambda^{2\alpha}}\cdot \frac{(1-\lambda^{-2\alpha})^{-1}}{1-\dfrac{c_n\cdot(1+\varepsilon_1)(1+\chi_{2n})}{\lambda^{2\alpha}}}. \qquad (10.17)$$

Besides, for any $\delta_n^{-m}\subset \delta_n\backslash\delta_{n+1}$,

$$(L_n^{-m}\cap[(\lambda^{2\alpha n}\cdot \delta_n)^{-m}\backslash V_{n+1}^{-m}]) = (L_n^{-m}\cap(\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m}).$$

All L_n^{-m} consist of domains $\Delta\hat{f}_n$ and $\Delta(F_{n-1}\circ g)$. At Step $n+1$ when constructing $\Delta f_{n+1\,1}$ we reproduce the structure from $[0,1]\backslash\delta_n$ on each $\Delta\hat{f}_n$ or $\Delta(F_{n-1}\circ g_n)$ using respectively $\hat{f}_n^{-1}$ or $(F_{n-1}\circ g_n)^{-1}$. We denote by $(\delta_n^{-m})'$, $(\delta_{q(n)}^{-m})'$ the new preimages of δ_n, $\delta_{q(n)}$ under $\hat{f}_n^{-1}$, $(F_{n-1}\circ g_n)^{-1}$. The estimate of $\text{mes}\left[\bigcup\limits_{m=0}^{\infty}\bigcup\delta_n^{-m}\right]$ from Sect. 8 together with the estimate of $\mu(G_{q(n)}, [0,1]\backslash\delta_{n+1})$ show that after excluding the set $\bigcup\limits_{m=0}^{\infty}\bigcup(\delta_{q(n)}^{-m})'$ from each $\Delta\hat{f}_n$ or $\Delta(F_{n-1}\circ g_n)$ the measures of $\bar{L}_{n+1}$ and of any $L_n^{-m}\cap(\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m}$ are multiplied by a factor larger than $1-(2\cdot\lambda^{-(s-2\alpha)})^n$. This factor implies the passage from ε_n to ε_{n+1} in estimates (10.14) for $(\lambda^{2\alpha r}\cdot \delta_{n+1})^{-m}$. We let $(L_n^{-m})'\cap[(\lambda^{2\alpha(n+1)}\delta_{n+1})^{-m}]$ correspond to $(\delta_n^{-m})'$. Thus to each δ_{n+1}^{-m}, $(\delta_n^{-m})'\subset \delta_n\backslash\delta_{n+1}$ uniquely corresponds its $\lambda^{2\alpha(n+1)}$ – enlargement which does not intersect $\bar{L}_{n+1}\backslash\bigcup(\delta_{q(n)}^{-m})'$. We now set

$$L_{n+1}=\begin{cases} L_n & \text{outside } \delta_n \\ \bar{L}_{n+1}\backslash\bigcup(\delta_{q(n)}^{-m})' & \text{inside } \delta_n\backslash\delta_{n+1}. \end{cases}$$

Notice that $L_{n+1}\cap\delta_n\backslash\delta_{n+1}$ consists of $\Delta f_{n+1\,1}$. (10.17) together with the estimate of $\bigcup(\delta_{q(n)}^{-m})'$ gives

$$\frac{\text{mes}(\lambda^{2\alpha(\mu-1)}\cdot \delta_{n+1})}{\text{mes}(L_{n+1}\cap\delta_{n+1}^{(\mu)})} < \lambda^{-2\alpha}(1+2.5\cdot\lambda^{-2\alpha}) \qquad (10.18)$$

and (10.14) follows for δ_{n+1}. The maps G_n^{-1}, $G_n'^{-1}$ and their compositions transmit (10.18) on $(\delta_{n+1}^{(r)})^{-m}\subset \delta_n^{-m}\backslash\delta_{n+1}^{-m}$ with an additional factor $\exp\lambda^{-\alpha q(n)}$. Joining it to the above estimate of

$$\text{mes}[(L_n^{-m}\cap(\lambda^{2\alpha(n+1)}\cdot \delta_{n+1})^{-m})\backslash\bigcup(\delta_{q(n)}^{-m})']$$

finishes the proof of (10.14) and of Lemma 4.

Remark X/2. The above construction is similar to one used in Sect. 8 in order to estimate the measure of holes at Step $n+1$.

Remark X/3. R which bounds r in (10.14), may be much larger than n. For example, the construction implies that the consecutive $\lambda^{2\alpha r}$-enlargements of δ_n are taken until we obtain the whole interval $[0, 1]$.

11. The Positivity of Measure

Remember that at step $n+1$ we consider $\lambda \in \mathcal{J}_n = [\lambda_{0n}, \lambda_{1n}]$. As λ varies in $\mathcal{J}_n$, $F_{n-1} \circ h(\lambda, \frac{1}{2})$ traverses some Δf_n and $f_n \circ F_{n-1} \circ h(\lambda, \frac{1}{2}) = F_n \circ h(\lambda, \frac{1}{2})$ traverses $[0, 1]$. The set $\mathscr{X}_{n+1}(\lambda) = \bigcup\limits_{k=1}^{n+1} \Delta f_k$ is defined for all $\lambda \in \mathcal{J}_n$, and all the domains $\Delta f_k = (\Delta f_k)_\lambda$ as well as the holes $\delta_{n+1}^{-m}(\lambda)$ and their enlargements $\hat{\delta}_{n+1}^{-m}(\lambda)$ vary continuously with $\lambda \in \mathcal{J}_n$. We then define $\mathscr{M}_{n+1} \cap \mathcal{J}_n$ as the set consisting of those $\lambda \in \mathcal{J}_n$ for which

$$F_n \circ h(\lambda, \tfrac{1}{2}) \in \mathscr{X}_{n+1}(\lambda) \Big\backslash \bigcup_{m=0}^{\infty} \bigcup \hat{\delta}_{n+1}^{-m}(\lambda).$$

We saw in Sect. 10 that the velocity of the top satisfies

$$v_n(\lambda) = \left| \frac{\partial}{\partial \lambda} F_n \circ h(\lambda, \tfrac{1}{2}) \right| > (2\lambda^{c_0})^n [4(1 + \varepsilon_{11.1})]^{-1}. \tag{11.1}$$

At the same time the endpoints $x_k(\lambda)$ of $\Delta f_k(\lambda)$, $k \leq n+1$, move with velocities

$$\left| \frac{dx_k}{d\lambda} \right| < \frac{(1 + \varepsilon_{11.2})}{8\lambda} \cdot \lambda^{s(n+1)}. \tag{11.2}$$

(11.1) and (11.2) imply that for any Δf_k, $k \leq n+1$, the condition $F_n \circ h(\lambda, \frac{1}{2}) \in \Delta f_k(\lambda)$ defines an interval $\mathcal{J}(\Delta f_k) \subset \mathcal{J}_n$, as does the condition $F_n \circ h(\lambda, \frac{1}{2}) \in \delta_{n+1}^{-m}(\lambda)$.

A priori the condition

$$\text{mes} \bigcup_{m=0}^{\infty} \bigcup \hat{\delta}_{n+1}^{-m}(\lambda) < [(1 + \varepsilon)\lambda^{-(s-\alpha)}]^{n+1}$$

does not imply the predominance of

$$\left\{ \lambda \in \mathcal{J}_n : F_n \circ h(\lambda, \tfrac{1}{2}) \in [0, 1] \Big\backslash \bigcup_{m=0}^{\infty} \bigcup \hat{\delta}_{n+1}^{-m}(\lambda) \right\}$$

in $\mathcal{J}_n$, and we have to do some additional estimates. In consequence of Lemma 4 for $k = n+1$, to any $\delta_{n+1}^{-k} = G_{n+1}^{-1} \delta_{n+1}$ there corresponds uniquely a set $L_{n+1}^{-k} = G_{n+1}^{-1} L_{n+1} \subset \mathscr{X}_{n+1} \cap \delta_{q(n+1)}^{-k}$ such that

$$\text{mes } L_{n+1}^{-k} > (1 - \varepsilon)\lambda^{2\alpha(n+1)} |\delta_{n+1}^{-k}|.$$

We define $\hat{L}_{n+1}^{-k} = (L_{n+1}^{-k} \backslash \hat{\delta}_{n+1}^{-k})$. Thus for any $\lambda \in \mathcal{J}_n$ the following estimate holds:

$$\text{mes } \hat{L}_{n+1}^{-k} > (1 - \varepsilon_{11.3})\lambda^{\alpha(n+1)} |\hat{\delta}_{n+1}^{-k}|, \qquad k = 0, 1, \dots, \varepsilon_{11.3} = O(\lambda^{-2\alpha}) \tag{11.3}$$

Let $\mathscr{J} = \mathscr{J}(\delta_{q(n+1)}^{-k}) = [\lambda_0, \lambda_1]$ be an interval on the λ-axis such that $F_n \circ h(\lambda, \frac{1}{2}) \in \delta_{q(n+1)}^{-k}$ when $\lambda \in \mathscr{J}$. Because of the definition of $q(n)$, $|\delta_{q(n+1)}^{-k}|$ $< (1 + o(\lambda^{-t(n+2)})) \cdot \lambda^{2\alpha(n+1)} \cdot \lambda^{2s} \cdot |\delta_{n+1}^{-k}|$. Then the comparison of velocities (11.1) and (11.2) implies

$$|\mathscr{J}| < \frac{4 \cdot \lambda_0^{2\alpha(n+1)+2s}}{(2\lambda_0^{c_0})^n} (1 + \varepsilon_{11.4}) \cdot |\delta_{n+1}^{-k}(\lambda_0)|. \tag{11.4}$$

When λ passes $\mathscr{J}$, the measures of $\hat{\delta}_{n+1}^{-k}$ and $\hat{L}_{n+1}^{-k}$ vary in particular because of the variation of $\partial/\partial x(G_{n+1}^{-1})$. We shall show this variation is small.

a)

Lemma 5. *Let Γ_p denote δ_p if $p \leq n$, and $(\frac{1}{2} - \lambda^{-sp}, \frac{1}{2} + \lambda^{-sp})$ if $p > n$. Let $F_\lambda(x)$ be one of the diffeomorphisms $G_n(\lambda, x)$, $\hat{f}_n(\lambda, x)$, or $f_n(\lambda, x)$, and suppose $\Delta F_\lambda(x) \subset [0,1] \backslash \Gamma_p$. Let $F_\lambda^{-1}(z)$ be the inverse diffeomorphism, and let $N = N(F)$ be the number of iterations of the initial map $g_\lambda : x \mapsto \lambda x(1-x) \bmod 1$ corresponding to F_λ (i.e., $F_\lambda = g_\lambda^N$). Then*

$$\left| \frac{\partial}{\partial \lambda} \frac{\partial F_\lambda^{-1}}{\partial z} \right| < \frac{\lambda^{2sp}}{8\lambda} N \left| \frac{\partial F_\lambda^{-1}}{\partial z} \right| \sum_{i=0}^{n} \lambda^{-si}. \tag{11.5}$$

Proof. We proceed by induction. Assuming Lemma 5 holds for $k \leq n$, we need to prove the corresponding estimates for $n+1$. We begin by estimating $\dfrac{\partial}{\partial \lambda} \dfrac{\partial \phi^{-1}}{\partial z}$ for a composition of maps. Let

$$\phi(\lambda, x) = \varphi_n \circ \varphi_{n-1} \circ \ldots \circ \varphi_1(\lambda, x),$$

where our notation is similar to that in the calculations for (9.1). Several applications of the chain rule give

$$\frac{\partial}{\partial \lambda}(\partial \phi^{-1}/\partial z) = [\partial \phi^{-1}/\partial z] \sum_{i=1}^{n} \left[\frac{\partial/\partial\lambda(\partial \varphi_i^{-1}/\partial z)}{\partial \varphi_i^{-1}/\partial z} - \frac{\partial^2 \varphi_i/\partial x^2}{(\partial \varphi_i/\partial x)^2} \frac{\partial(\varphi_{i+1}^{-1} \circ \ldots \circ \varphi_n^{-1})}{\partial \lambda} \right], \tag{11.6}$$

where as before the arguments of φ_i and its derivatives are λ and $\varphi_{i-1} \circ \ldots \circ \varphi_1(x)$ while those of φ_i^{-1} are λ and $\varphi_{i+1}^{-1} \circ \ldots \circ \varphi_n^{-1}(z)$, $z = \phi(\lambda, x)$ (for $i = n$, $\varphi_n^{-1} = \varphi_n^{-1}(\lambda, z)$, and there is no second term in the brackets).

Let $F_1 = f_{n-1} \circ \ldots \circ f_1 \circ g$, $\Delta F_1 \subset [0,1] \backslash \Gamma_p$, and let N_i denote the number of iterations corresponding to f_i. The expression $\partial/\partial\lambda(\varphi_{i+1}^{-1} \circ \ldots \circ \varphi_n^{-1})$ in (11.6) equals $\partial x_{i+1}/\partial \lambda$, where $x_{i+1}(\lambda)$ satisfies

$$\varphi_n \circ \ldots \circ \varphi_{i+1}(\lambda, x_{i+1}(\lambda)) = z.$$

In our case $\varphi_1 = g$, $\varphi_{i+1} = f_i$ $1 \leq i \leq n-1$ and the estimates of Sect. 9 give

$$|\partial/\partial\lambda(f_i^{-1} \circ \ldots \circ f_{n-1}^{-1})(\lambda, z)| < \frac{1 + \varepsilon_1}{8\lambda} \lambda^{si}.$$

For $g = \lambda x(1-x)$ we have

$$\frac{\partial}{\partial \lambda} \frac{\partial g^{-1}}{\partial z} = \frac{1}{2\lambda^2(x - \frac{1}{2})}, \qquad \frac{-\partial^2 g/\partial x^2}{(\partial g/\partial x)^2} = \frac{1}{2\lambda(x - \frac{1}{2})^2}.$$

For f_i we have by estimates b_{2n} of Sect. 6 that

$$\frac{\partial^2 f_i / \partial x^2}{(\partial f_i / \partial x)^2} < 1 + \varepsilon_2 .$$

Thus, (11.5) and (11.6) give

$$\left| \frac{\partial}{\partial \lambda} \frac{\partial F_1^{-1}}{\partial z} \right| \leqq \left| \frac{\partial F_1^{-1}}{\partial z} \right| \cdot \left[\left(\frac{\lambda^{sp}}{2\lambda^2} + \frac{\lambda^{2sp}}{2\lambda}(1 + \varepsilon_1) \frac{\lambda^s}{8\lambda} \right) + \left(\frac{\lambda^{2s}}{8\lambda} N_1 + \frac{\lambda^{2s}}{8\lambda}(1 + \varepsilon_1)(1 + \varepsilon_2) \right) \right.$$
$$\left. + \ldots + \left(\frac{\lambda^{2s(n-2)}}{8\lambda} N_{n-2} + \frac{\lambda^{s(n-1)}}{8\lambda}(1 + \varepsilon_1)(1 + \varepsilon_2) \right) + \frac{\lambda^{2s(n-1)}}{8\lambda} N_{n-1} \right] \quad (11.7)$$

Let $F_2 = G \circ F_1$, where $G = G_n : \Delta G \to \delta_n$, $\Delta G \subset [0,1] \setminus \delta_n$, and N_G is the number of iterates for G. The estimates of Sect. 9 imply $\left| \dfrac{\partial G^{-1}(z, \lambda)}{\partial \lambda} \right| < \dfrac{\lambda^{sn}(1 + \varepsilon_1)}{8\lambda}$.

Because $\dfrac{|\partial^2 F_1 / \partial x^2|}{(\partial F_1 / \partial x)^2} < v(F_1, \Delta(F_1))$ for $x \in \Delta F_1$, we obtain using (4.7), a_{2n} and b_{2n} that

$$\frac{|\partial^2 F_1 / \partial x^2|}{(\partial F_1 / \partial x)^2} < (1 + O(\lambda^{-\gamma})) \left(1 + \frac{\lambda^{2sp}}{\lambda^{s/2}(2\lambda^{co})^{n+1}} \right)$$
$$+ \frac{\lambda^{2sp}}{\lambda^{s/2}(2\lambda^{co})^{n+1}} = \frac{\lambda^{2sp}(1 + \varepsilon_3)}{\lambda^{s/2}(2\lambda^{co})^{n+1}} + (1 + \varepsilon_3).$$

Using (11.6) for $F_2 = G \circ F_1$ we have

$$\left| \frac{\partial}{\partial \lambda} \frac{\partial F_2^{-1}}{\partial z} \right| < \left[\left| \frac{\partial}{\partial \lambda} \frac{\partial F_1^{-1}}{\partial z} \right| / \left| \frac{\partial F_1^{-1}}{\partial z} \right| + \left(\frac{\lambda^{2sp}(1 + \varepsilon_3)}{\lambda^{s/2} \cdot (2\lambda^{co})^{n+1}} + (1 + \varepsilon_3) \right) \right.$$
$$\left. \cdot \frac{\lambda^{sn}(1 + \varepsilon_1)}{8\lambda} + \frac{\lambda^{2sn}}{8\lambda} \cdot N_G \right] \cdot \left| \frac{\partial F_2^{-1}}{\partial z} \right|. \quad (11.8)$$

Substituting (11.7) in (11.8) we obtain (11.5) for $G'_n = G_n \circ F_{n-1} \circ g$ constructed at the beginning of step $n+1$ (we have besides an additional factor less than λ^{-co} in the right part of (11.5)). The proof for $G'_n = G_n \circ \hat{f}_n$, $\Delta \hat{f}_n \subset [0,1] \setminus \Gamma_p$, is analogous. Considering $p = n + 1$ in (11.7) we obtain the assertion of Lemma 5 for $f_{n+1 \, 1}$. Then we consider the compositions $f_{n+1 \, k} = f_{n+1 \, 1} \circ \tilde{G}_{n_k} \circ \ldots \circ \tilde{G}_{n_1}$. The induction hypotheses and the previous estimates give

$$\left| \frac{\partial}{\partial \lambda} \frac{\partial G'^{-1}_{n_i}}{\partial z} \right| < \frac{\lambda^{2s(n+1)}}{8\lambda} \cdot N_i \left| \frac{\partial G^{-1}_{n_i}}{\partial z} \right| \cdot \sum_{i=0}^{n} \lambda^{-si} .$$

The estimates of Sect. 9 give

$$\left| \frac{\partial (\tilde{G}^{-1}_{n_i} \circ \ldots \circ \tilde{G}^{-1}_{n_k} \circ f^{-1}_{n+1 \, 1})}{\partial \lambda} \right| < \frac{\lambda^{s(n+1)}}{8\lambda}(1 + \varepsilon_1).$$

Taking into account

$$\left| \frac{\partial^2 G_{n_i}}{\partial x^2} \middle/ \left(\frac{\partial G_{n_i}}{\partial x} \right)^2 \right| < (1 + \varepsilon_2) \lambda^{-\alpha n},$$

(11.6) implies

$$\left|\frac{\partial}{\partial\lambda}\frac{\partial f_{n+1\,k}^{-1}}{\partial z}\right| < \left[\sum_{i=1}^{k}\left(\frac{\lambda^{2s(n+1)}}{8\lambda}\cdot N_i\cdot\sum_{i=0}^{n}\lambda^{-si} + \frac{(1+\varepsilon_2)(1+\varepsilon_1)\cdot\lambda^{s(n+1)}}{\lambda^{\alpha n}\cdot 8\lambda}\right)\right.$$
$$\left. + \frac{\lambda^{2s(n+1)}}{8\lambda}\cdot N(f_{n+1\,1})\right]\left|\frac{\partial f_{n+1\,k}^{-1}}{\partial z}\right|.$$

This proves Lemma 5 for f_{n+1}, and the sum in round brackets gives the desired estimate for G_{n+1}. The proof for $\hat{f}_{n+1} = f_{n+1}\circ G_n$ is similar. $\quad\square$

   b)   Consider $\mathscr{I}(\delta_{q(n+1)}) = \{\lambda: F_n\circ h(\lambda,\tfrac{1}{2})\in\delta_{q(n+1)}(\lambda)\} = [\lambda_0,\lambda_1]$.   (11.4)   gives $|\mathscr{I}(\delta_{q(n+1)})| < \dfrac{(1+\varepsilon_{11.4})\cdot 16\cdot\lambda_0^{1+s}}{(2\cdot\lambda_0^{c_0+s-2\alpha})^{n+1}}$. Let $\varDelta(\lambda)$ be any interval in $L_{n+1}\bigcap\delta_{q(n+1)}$. The comparison of velocities (11.1) and (11.2) shows that the time it takes for $F_n\circ h(\lambda,\tfrac{1}{2})$ to traverse $\varDelta(\lambda)$ equals $\dfrac{|\varDelta(\lambda)|}{v_n(\lambda)}\cdot(1+o(\lambda^{-c_0n}))$, where λ is any moment of passing by. We want to reduce all these moments (for different $\varDelta(\lambda)$) to the same one, namely to λ_0, and then use the relation (11.3) for λ_0. This can be done for given $\varDelta(\lambda)$ if for any $\lambda\in\mathscr{I}(\delta_{q(n+1)})$,

$$\frac{|\varDelta(\lambda)|}{|\varDelta(\lambda_0)|} > (1-\alpha_{n+1}), \quad \alpha_{n+1} = o(\lambda_0^{-t(n+1)}).$$

   Let $N = N(\varDelta f_k) = N(f_k)$. If $N < \lambda_0^{s(n+1)}$, Lemma 5 and the estimate of $\mathscr{I}(\delta_{q(n+1)})$ imply

$$|\varDelta(\lambda_0)| - |\varDelta(\lambda)| < \frac{\lambda_0^{2s(n+1)}}{8\lambda_0}\cdot\lambda_0^{s(n+1)}\cdot\frac{16\cdot\lambda_0^{1+s}\cdot(1+\varepsilon_{11.9})}{(2\lambda_0^{c_0+s-2\alpha})^{n+1}}\cdot|\varDelta(\lambda_0)|. \qquad (11.9)$$

Thus for such $\varDelta$, $\alpha_{n+1} = O(\lambda_0^{[c_0-2(s+\alpha)](n+1)})$.

   Lemma 7 of Sect. 12 gives the following relation between $N(\varDelta)$ and $|\varDelta|$ for $\varDelta\in[0,1]\setminus\delta_n$:

$$N < \frac{\sqrt{n}\cdot 2s}{c_0}\,|\log_{\lambda_0}|\varDelta||.$$

Thus $N < \lambda_0^{s(n+1)}$, if $\dfrac{2s}{c_0}\sqrt{n+1}\,|\log_{\lambda_0}|\varDelta|| < \lambda_0^{s(n+1)}$. Lemma 7 also gives the following estimate for a domain $\varDelta(F_{n-1}\circ h)$ of the central branch $F_{n-1}\circ h(\lambda,x)$. If $\Gamma_p\supset\varDelta(F_{n-1}\circ h)\supset\Gamma_{p+1}$, and $N = N(F_{n-1}\circ h) = N(F_{n+1})+1$, then

$$N < \frac{4s}{c_0}\sqrt{n}\cdot p.$$

   When constructing L_{n+1} in Sect. 10, we had $L_{n+1}\bigcap(\delta_n\setminus\delta_{n+1})\subset\bigcup\varDelta f_{n+1\,1}$. Using this fact one can check inductively following the proofs of Lemmas 2 and 4 that the following construction gives a set $\mathscr{X}'_{n+1}\subset\mathscr{X}_{n+1}$ with

$$\text{mes}(L_{n+1}^{-k}\bigcap\mathscr{X}'_{n+1}) > (1-\varepsilon'_0)\lambda^{2\alpha(n+1)}\text{mes}\,\delta_{n+1}^{-k}$$

for every δ_{n+1}^{-k}.

We begin by constructing at step 2 the maps $f_1 \circ g$ and the holes $g^{-1}\delta_1$. Then at step $n+1$, $n \geq 2$, we reproduce on each interval inside δ_n the structure obtained after step n on $[0,1]\backslash\delta_n$, and on each hole δ_i^{-k} [$1 \leq i \leq n-1$ and $k \leq k_0(n)$ here, contrary to $i=n$ and $1 \leq k < \infty$ in the construction of Sect. 3] we reproduce the structure of $\delta_i \backslash \delta_n$ obtained after step n. Ignoring $N(F_{n-1} \circ h)$ this construction gives for $N(\mathcal{X}_n') = \max\{N(\Delta f_k), \Delta f_k \in \mathcal{X}_n'\}$ the upper estimate 2^n. Taking into account $N(F_{n-1} \circ h)$ estimated above, we obtain

$$N(\mathcal{X}_n') < n \cdot 2^n.$$

This implies the following

Lemma 6.

$$\frac{\mathrm{mes}\{\Delta(f_k) \in \hat{L}_{n+1}(\lambda): N(f_k) < n \cdot 2^n\}}{\mathrm{mes}\,\hat{\delta}_{n+1}(\lambda)} > (1-\varepsilon)\lambda^{\alpha(n+1)}.$$

Lemma 6 implies the predominance of Δ satisfying (11.9) in $\hat{L}_{n+1}$. Thus (11.3) implies

$$\frac{\mathrm{mes}\{\lambda \in \mathcal{J}(\delta_{q(n+1)}), F_n \circ h(\lambda, \tfrac{1}{2}) \in \hat{L}_{n+1}(\lambda)\}}{\mathrm{mes}\{\lambda \in \mathcal{J}(\delta_{q(n+1)}), F_n \circ h(\lambda, \tfrac{1}{2}) \in \hat{\delta}_{n+1}(\lambda)\}} > (1-\varepsilon_{11.10})\lambda_0^{\alpha(n+1)}. \qquad (11.10)$$

c) Let $\mathcal{J} = \mathcal{J}(\delta_{q(n+1)}^{-k}) = [\lambda_2, \lambda_3] = \{\lambda: F_n \circ h(\lambda, \tfrac{1}{2}) \in \delta_{q(n+1)}^{-k}(\lambda)\}$. (11.4) and Lemma 5 imply that for any $\lambda \in \mathcal{J}$

$$\left\| \left| \frac{\partial G_{n+1\,\lambda}^{-1}}{\partial z} \right|_{z=1/2} - \frac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z} \right|_{z=1/2} \right\|$$

$$< \frac{\left(\left. \frac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z} \right|_{z=1/2} \right)^2 \cdot N(G_{n+1}) \cdot 2\lambda_2^s(1+\varepsilon_{11.11})}{(2\lambda_2^{c_0-(s+2\alpha)})^{n+1}}. \qquad (11.11)$$

(11.11) and the estimate b_{4n+1} of $\mu(G_{n+1})$ give for any $\Delta(\lambda) \subset (\hat{L}_{n+1} \cap \mathcal{X}_{n+1}')^{-k}$

$$\frac{\mathrm{mes}\,G_{n+1\,\lambda}^{-1}\Delta(\lambda)}{\mathrm{mes}\,G_{n+1\,\lambda_2}^{-1}\Delta(\lambda_2)} > (1-\alpha_{n+1})\left(1 - \frac{\left| \frac{\partial G_{n+1\,\lambda}^{-1}}{\partial z} - \frac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z} \right|_{z=1/2}}{\left| \frac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z} \right|_{z=1/2}} \right)$$

$$\cdot \exp(\mu(G_{n+1})) > \left(1 - \frac{N(G_{n+1}) \cdot \left| \frac{\partial G_{n+1\,\lambda_2}^{-1}}{\partial z} \right| 2\lambda_2^s(1+\varepsilon_{11.12})}{(2\lambda_2^{c_0-(s+2\alpha)})^{(n+1)}} \right)$$

$$\cdot (1-\alpha_{n+1})(1 - O(\lambda^{-\alpha(n+1)})). \qquad (11.12)$$

As $N(G) \cdot \left| \frac{\partial G^1}{\partial z} \right| = o(1)$, we obtain from (11.12) and (11.3)

$$\frac{\mathrm{mes}\{\lambda \in \mathcal{J}(\delta_{q(n+1)}^{-k}), F_n \circ h(\lambda, \tfrac{1}{2}) \in \hat{L}_{n+1}^{-k}\}}{\mathrm{mes}\{\lambda \in \mathcal{J}(\delta_{q(n+1)}^{-k}), F_n \circ h(\lambda, \tfrac{1}{2}) \in \hat{\delta}_{n+1}^{-k}\}} > (1-\varepsilon_{11.13}) \cdot \lambda_2^{\alpha(n+1)} \qquad (11.13)$$

Using $\hat{L}_{n+1}^{-k}(\lambda) \cap \hat{L}_{n+1}^{-\ell}(\lambda) = \emptyset$, if $\hat{\delta}_{n+1}^{-k} \neq \hat{\delta}_{n+1}^{-\ell}$, we obtain from (11.10) and (11.13)

Proposition 2. *Let* $\mathscr{I}_n = [\lambda_{0n}, \lambda_{1n}] \subset \mathscr{M}_n$ *be any interval on the* λ-*axis constructed at Step n. Then*

$$\frac{\mathrm{mes}\left\{\lambda \in \mathscr{I}_n, F_n \circ h(\lambda, \tfrac{1}{2}) \in \bigcup_{k=0}^{\infty} \bigcup \hat{\delta}_{n+1}^{-k}(\lambda)\right\}}{\mathrm{mes}\,\mathscr{I}_n} < \frac{1+\varepsilon_{11}}{\lambda_{0n}^{\alpha(n+1)}},$$

where $\varepsilon_{11} < \lambda^{-t}$.

We define

$$\mathscr{M}_{n+1} \cap \mathscr{I}_n = \left\{\lambda : F_n \circ h(\lambda, \tfrac{1}{2}) \in [0,1] \Big\backslash \bigcup_{k=0}^{\infty} \bigcup \hat{\delta}_{n+1}^{-k}(\lambda)\right\},$$

and obtain

$$\mathrm{mes}(\mathscr{M}_{n+1} \cap \mathscr{I}_n) > 1 - \frac{1+\varepsilon_{11}}{\lambda_{0n}^{\alpha(n+1)}}$$

and consequently

$$\mathrm{mes}\,\mathscr{M}_{n+1} > \left(1 - \frac{1+\varepsilon_{11}}{N_0^{\alpha(n+1)}}\right) \mathrm{mes}\,\mathscr{M}_n.$$

Remark XI/1. Any λ such that $F_n \circ h(\lambda, \tfrac{1}{2}) \in [0,1] \backslash \bigcup_{k=0}^{\infty} \bigcup \hat{\delta}_{n+1}^{-k}(\lambda)$ lies in one of the intervals $\mathscr{I}_{n+1}(\Delta_k)$ corresponding to the relation $F_n \circ h(\lambda, \tfrac{1}{2}) \in \Delta_k(\lambda)$, or is a limit point of such intervals. One can apparently prove that

$$\mathrm{mes}\{\lambda : F_n \circ h(\lambda, \tfrac{1}{2}) \in \bigcup \Delta_k(\lambda)\} > \left(1 - \frac{1+\varepsilon'_{11}}{\lambda_{0n}^{t(n+1)}}\right) \mathrm{mes}\,\mathscr{I}_n,$$

but there is no reason to avoid λ lying in the limit set. They are even better in some sense (see Remark VIII/3).

12. Transition from T_λ-Invariant Measure to f_λ-Invariant Measure

The previous relations between $\mathrm{mes}\,\mathscr{M}_{n+1}$ and $\mathrm{mes}\,\mathscr{M}_n$, and the choice of the position of the top

$$F_{n-1} \circ h(\lambda, \tfrac{1}{2}) \in [0,1] - \left(\tfrac{1}{2} - \frac{1}{\lambda^{s/2}}, \tfrac{1}{2} + \frac{1}{\lambda^{s/2}}\right)$$

within the first steps $1, 2, \ldots, n_0$, imply that there exists a set $\mathscr{M} = \bigcap_{n=1}^{\infty} \mathscr{M}_n$ on the λ-axis with measure

$$\mathrm{mes}\,\mathscr{M} > 4\left[\prod_{n=1}^{n_0}\left(1 - \frac{2(1+\varepsilon)}{N_0^{s/2}}\right)\right] \cdot \left[\prod_{n=n_0+1}^{\infty}\left(1 - \frac{1+\varepsilon}{N_0^{\alpha n}}\right)\right]$$

such that for any $\lambda \in \mathscr{M}$ the partition ξ_λ of Sect. 1 exists.

Conditions i–iv of Sect. 1 imply that for T_λ defined by $T_\lambda|\Delta_i(\lambda) = f_\lambda^{n_i}$ there exists a unique T_λ – invariant probabilistic measure $\nu_\lambda < dx$ with a density $\varrho_\lambda(x) \in C_{[0,1]}$, $\varrho_\lambda > c > 0$. The endomorphism T_λ of the Lebesgue space $([0,1], \nu_\lambda)$ is exact and its natural extension is isomorphic to a Bernouli shift (see [12, 13]).

In order to finish the proof of Theorem A for the family $f_\lambda : x \to \lambda x(1-x)$ (mod 1) we have to construct an invariant measure $\mu_\lambda < dx$.

Let $f^{-k}(A)$ be the full preimage of $A \subset [0,1]$ under f^k, $f^{-k}A = \{x : f^k x \in A\}$.

Suppose $\sum_{\Delta_i \in \xi_\lambda} n_i \nu_\lambda(\Delta_i) < \infty$. Then the measure defined for any dx-measurable set A by

$$\mu_\lambda(A) = \sum_{\Delta_i \in \xi_\lambda} \sum_{0 \leq j < n_i} \nu_\lambda(f^{-j}A \cap \Delta_i) \tag{12.1}$$

is absolutely continuous with respect to dx, by a theorem on integrability of a series of positive functions (see for example [15] Sect. 14).

We show μ_λ is f-invariant.

By definition

$$\mu_\lambda(f^{-1}A) = \sum_{\Delta_i \in \xi_\lambda} \sum_{0 \leq j < n_i} \nu_\lambda(f^{-j} \circ f^{-1}A \cap \Delta_i). \tag{12.2}$$

If $j < n_i - 1$, every term $f^{-j} \circ f^{-1}A \cap \Delta_i$ in (12.2) coincides with $f^{-(j+1)}A \cap \Delta_i$ in (12.1). After excluding these terms, there remain in (12.1) terms with $j = 0$, which give $\sum_{\Delta_i \in \xi_\lambda} \nu_\lambda(A \cap \Delta_i) = \nu_\lambda(A)$, and in (12.2) terms with $j = n_i - 1$, which give

$$\sum_{\Delta_i \in \xi} \nu_\lambda(f^{-n_i}A \cap \Delta_i) = \sum_{\Delta_i \in \xi} \nu_\lambda(T_\lambda^{-1}A \cap \Delta_i) = \nu_\lambda T_\lambda^{-1}A.$$

Thus (12.1) equals (12.2) because of the T_λ-invariance of ν_λ.

Let $\beta = \frac{2}{5}$. The following proposition implies $\sum n(\Delta_i)|\Delta_i| < \infty$.

Proposition 3. $\displaystyle\sum_{\Delta_i \in \delta_n \setminus \delta_{n-1}} n(\Delta_i)|\Delta_i| < \dfrac{n^{3/2}}{\lambda^{s(1-\beta)n}}$.

Proof. a) Consider step n of the induction construction of Sect. 3. If Φ_n is one of f_n, $\hat{f}_n$, G_n, F_{n-1} obtained with N successive iterates of f_λ, we use an upper index so that $\Phi_n^N = f_\lambda \circ f_\lambda \circ \ldots \circ f_\lambda$, and $\Phi_n^{-N} = (\Phi_n^N)^{-1}|\mathrm{Im}\,\Phi_n^N$.

Lemma 7. $|Df_n^N| > \lambda^{\frac{c_0 N}{2\sqrt{n}} + s}$

Let

$$\Gamma_\ell = \begin{cases} \delta_\ell & \text{if } \ell \leq n \\ [\frac{1}{2} - \lambda^{-s\ell}, \frac{1}{2} + \lambda^{-s\ell}] & \text{if } \ell \geq n+1. \end{cases}$$

We prove Lemma 7 by induction and assume that for $k = 1, \ldots, n$ Lemma 7 holds together with the following properties:

i) Let $\delta_n^{-N} = G^{-N}\delta_n \subset [0,1] \setminus \Gamma_\ell$, and let $r = \max(1, n)$. Then $|DG^N| \, \delta_n^{-N}| > \lambda^{\frac{c_0 N}{2\sqrt{r}} + s}$.

ii) Let $\Delta \hat{f}_n^N \subset [0,1] \setminus \Gamma_\ell$. Then $|D\hat{f}_n^N| > \lambda^{\frac{c_0 N}{2\sqrt{\ell}}}$.

Consider $k = n + 1$. Notice that if $x \in [0, 1] \setminus \Gamma_2$, then $|Df_\lambda| > \lambda^{1 - 2s} > \lambda^{\frac{c_0}{2} + s}$. If $x \in \Gamma_2$, then $\ell, r \geq 3$ in i), ii) and $\frac{c_0}{2\sqrt{3}} + s < \frac{c_0}{2}$. As $|Df_\lambda| > \lambda^{\frac{c_0}{2}}$ on any hole δ_1^{-1} we obtain i) for $n = 1$. ii) for $n = 1$ holds because of i) and $|Df_1| > \lambda^{c_0} > \lambda^{\frac{c_0}{2} + s}$.

Let $F_{n-1} \circ h(\lambda, x)$ be the central branch, $F_{n-1} = f_{i_{n-1}} \circ \ldots \circ f_{i_1}$, $1 \leq i_k \leq k$, $N(f_{i_k}) = N_k$ the number of iterations of f_λ corresponding to f_{i_k}, $k \in [1, n-1]$, $\delta = G^{-N_n} \delta_n$, $M = 1 + \sum_{k=1}^{n} N_k$. Then $(F_{n-1} \circ h)^{-1} \delta = \delta_n^{-M}$. (In the notation of Sect. 3, $G^M : \delta_n^{-M} \to \delta_n$ is one of the G'_n constructed at the beginning of step n.)

Let D_{n-1} be the domain of $F_{n-1} \circ h$, and let $p_0 = \min\{p | \delta_n^{-M} \subset [0, 1] \setminus \Gamma_p\}$. Then (see (6.12))

$$p_0 > \frac{c_0}{2s} n. \tag{12.3}$$

According to the construction of Sect. 3,

$$\mathrm{dist}(\delta, F_{n-1} \circ h(\lambda, \tfrac{1}{2})) > \frac{\delta}{2} \lambda^{\alpha n}(1 - \varepsilon),$$

which implies (see (6.4)) that

$$|DG^M| > \frac{\sqrt{\lambda} \left[|DG^{N_n}| \prod_{k=1}^{n-1} |Df_{i_k}| \right]^{1/2}}{\lambda^{(s - \alpha)n/2}}, \tag{12.4}$$

where DG^M is evaluated on δ_n^{-M} and DG^{N_n} on δ. By the induction hypotheses we have

$$|Df_{ik}| > \lambda^{[c_0 N_k / 2\sqrt{k}] + s} \quad 1 \leq k \leq n - 1$$

$$|DG^{N_n}| > \lambda^{[c_0 N_n / 2\sqrt{n}] + s}.$$

Hence, on δ_n^{-M},

$$|DG^M| > \lambda^\theta,$$

where

$$\theta = \tfrac{1}{2} + \frac{s(n - 1)}{2} + \left(\sum_{i=1}^{n} N_i \right) \frac{c_0}{4\sqrt{n}} - \frac{(s - \alpha)}{2} n. \tag{12.5}$$

We have to prove

$$|DG^M| > \lambda^{[c_0 M / 2\sqrt{p_0}] + s}. \tag{12.5a}$$

Now, $M = \left(\sum_{i=1}^{n} N_i \right) - 1$,

$$\frac{1 - s + n\alpha}{2} > s + \frac{c_0}{2\sqrt{p_0}}$$

and $c_0/2s > 4$ imply

$$2\sqrt{n} < \sqrt{c_0 n/2s} < \sqrt{p_0}$$

and (12.5a) follows from (12.5).

So i) is proved for the holes $\delta_n^{-M} = (F_{n-1} \circ h)^{-1} \delta_n^{-N}$. Any branch $f_{i_{n-1}} \circ \ldots \circ f_{i_1} \circ g$ is some composition of the form $f_{i_{n-1}} \circ \ldots \circ f_{i_k} \circ (f_{i_{k-1}} \circ \ldots \circ f_{i_1} \circ h)$, where $f_{i_{k-1}} \circ \ldots \circ f_{i_1} \circ h$ is a central branch of some previous step. Thus the same arguments prove i) for $\delta_n^{-M} = (F_{n-1} \circ g)^{-1} \delta_n^{-N}$ (the estimates are better in this case). If $\delta_n^{-M} = \hat{f}_n^{-1} \delta_n^{-N}$, i) follows from i) and ii) of Step $n-1$.

Let $G^{-N} \delta_n = \delta_n^{-N} \subset [0,1] \backslash \delta_{n+1}$. Then we have $\max(\ell, n) = n+1$. Now i) follows for $G_{n+1}^M : \delta_{n+1}^{-M} \to \delta_{n+1}$ with $\delta_{n+1}^{-M} \subset [0,1] \backslash \delta_{n+1}$ because they are compositions of maps satisfying i) with $r \leq n+1$. Similarly for $G_{n+1}^M : \delta_{n+1}^{-M} \to \delta_{n+1}$, $\delta_{n+1}^{-M} \subset [0,1] \backslash \Gamma_\ell$, $\ell > n+1$. This proves i_{n+1}.

Let $f_{n+1\,1} = f_{i_n} \circ f_{i_{n-1}} \circ \ldots \circ f_{i_1} \circ g_\lambda | [0,1] \backslash \delta_{n+1}$. The induction conditions on $|Df_{i_k}|$ imply that $|Df_{n+1\,1}| = \prod_{k=1}^{n} |Df_{i_k}| \cdot 2\lambda |x - \tfrac{1}{2}|$ satisfies Lemma 7. The same is true for $f_{n+1\,1} = f_{i_n} \circ \hat{f}_n$, because of ii). Taking into account i_{n+1}, we obtain Lemma 7 for $f_{n+1\,k}$ with $k > 1$. Finally ii) at Step $n+1$ follows from i) and the assertion of Lemma 7 for f_{n+1}. $\square$

b) We shall use the following estimates for compositions of maps.

Let $g : B \to J$ be given by $g(x) = ax^2$, where $B = [x_{min}, x_{max}]$ and $J = \bigcup \Delta$, where $\mathrm{int}\,\Delta_1 \cap \mathrm{int}\,\Delta_2 = \emptyset$ if $\Delta_1 \neq \Delta_2$. Let $\Delta = [h_\Delta, h_\Delta + |\Delta|]$ and denote by $n(\Delta)$ the number of iterations corresponding to Δ. Then $B = \bigcup g^{-1} \Delta$, where

$$|g^{-1}\Delta| = \frac{1}{\sqrt{a}} \left(\sqrt{h_\Delta + |\Delta|} - \sqrt{h_\Delta} = \frac{|\Delta|}{\sqrt{a}(\sqrt{h_\Delta + |\Delta|} + \sqrt{h_\Delta})} \right)$$

and

$$n(g^{-1}\Delta) = 1 + n(\Delta).$$

Hence

$$\sum n(g^{-1}\Delta)|g^{-1}\Delta| = \frac{1}{\sqrt{a}} \sum \frac{(1 + n(\Delta))|\Delta|}{\sqrt{h_\Delta + |\Delta|} + \sqrt{h_\Delta}}$$

$$= \frac{1}{\sqrt{a}} \sum \frac{|\Delta|}{\sqrt{h_\Delta + |\Delta|} + \sqrt{h_\Delta}} + \frac{1}{\sqrt{a}} \sum \frac{n(\Delta)|\Delta|}{\sqrt{h_\Delta + |\Delta|} + \sqrt{h_\Delta}}. \quad (12.6)$$

Let us now consider $\{\Delta', f', n'\}$, where $\mathrm{int}\,\Delta_1' \cap \mathrm{int}\,\Delta_2' = \emptyset$, $n' = n'(\Delta') = n'(f')$. Suppose every f' maps its domain onto the same interval, $f' : \Delta' \to J$, and $\mu(f', \Delta') < c$. Let $\{\Delta, f, n\}$ be so that $\Delta \subset J$, $\mathrm{int}\,\Delta_1 \cap \mathrm{int}\,\Delta_2 = \emptyset$, $n = n(\Delta) = n(f)$. Then

$$\sum_{\Delta, \Delta'} n(f'^{-1}(\Delta))|f'^{-1}\Delta| < \left(\sum_{\Delta, \Delta'} (n + n')|\Delta||\Delta'| \right) \frac{\exp(c)}{|J|}$$

$$= [(\sum n |\Delta|)(\sum |\Delta'|) + (\sum n'|\Delta'|)(\sum |\Delta|)] \frac{\exp(c)}{|J|}.$$

c) When estimating $\sum n(\Delta)|\Delta|$ after step n of the induction construction we shall attribute to any preimage δ_n^{-N} mapped onto δ_n by G_n^N the number of iterations N,

ignoring the structure inside δ_n. But when considering δ_n itself, we take into account this structure. This gives the estimate of $\sum n(\Delta)|\Delta|$ on any domain inside δ_n. Then according to the construction of Sect. 3 we introduce at step $n+1$ the structure from $\delta_n \backslash \delta_{n+1}$ inside every domain $(\delta_n \backslash \delta_{n+1})^{-N}$.

Before formulating the induction hypotheses of Proposition 3 we introduce a new notation. Let $\Delta_0 = \Delta f \subset [0,1] \backslash \delta_n$ be a domain of some f, constructed after Step n. We define a "block" $B(\Delta_0)$ as a maximal interval containing Δ_0, which doesn't contain any hole δ_n^{-k}. If $B(\Delta_0) \cap \delta_n = \emptyset$, then $B(\Delta_0) = \bigcup \Delta_i$, where any $\Delta_i = [a_i, a_{i+1}]$, $i \in \mathbb{Z}$, is a domain of some Δf_ℓ, $\ell \leq n$. If $B(\Delta_0) \cap \delta_n \neq \emptyset$, then a part of the Δ_i are as above and the others are $\Delta(F_{n-1} \circ g)$ or $\Delta \hat{f}_n$.

After Step 2 we obtain two exceptional one-side blocks B_0^*, which contains 0, and B_1^*, containing 1, and for any $\tilde{B} \neq B_0^*$, B_1^*, $\tilde{B} = \tilde{B}_1 \bigcup \tilde{B}_2$, where $\tilde{B}_1 = g^{-n} B_0^*$, $\tilde{B}_2 = g^{-n} B_1^*$.

The structure of B_0^* is: $B_0^* = \bigcup B_{0i}$, $i = 1, 2, \ldots$, where $B_{0i} = \bigcup \Delta_{ik}$, $k \in [1, n_0]$, $n_0 = \text{card}\{\Delta f_1 \subset [0, \frac{1}{2}]\}$, $\Delta_{1k} = \Delta f_1$, $\Delta_{ik} = \Delta f_{2i-1}$ for $i \geq 2$, and the corresponding number of iterations $N(\Delta_{ik}) = i$. The structure of B_1^* is similar.

Let $\tilde{B}$ be some block of step $n+1$. Then either $\tilde{B} = \tilde{B}_1 \bigcup \tilde{B}_2$, where $\tilde{B}_1 \subset [a_1, a]$, $\tilde{B}_2 \subset [a, a_2]$, and $[a_1, a]$, $[a, a_2]$ are two adjacent intervals constructed at step n, $\tilde{B}_1 \cap \tilde{B}_2 = a$, and both $\tilde{B}_1$, $\tilde{B}_2$ are preimages of B_0^* or $\tilde{B}_1^*$, or $\tilde{B}$ is some preimage of such blocks constructed at previous steps.

When constructing δ_{n+1} we shall take the precaution to choose two adjacent intervals $\Delta' \subset [0,1] \backslash \delta_{n+1}$ and $\Delta'' \subset \delta_{n+1}$ which are the preimages of Δ_{ik} with the same i. This can be done by moving if necessary the point ξ_{1n} of Sect. 10 a distance less than $(2\lambda^{c_1})^{-(n+1)}$ and still having (10.10) true.

Let $B_+(\Delta_0) = \bigcup (\Delta_i \subset B(\Delta_0), i > 0)$, $B_-(\Delta_0) = \bigcup (\Delta_i \subset B(\Delta_0), i < 0)$. Then the preceding implies

$$\min(\operatorname{mes} B_+(\Delta_0), \operatorname{mes} B_-(\Delta_0)) > \frac{1 - \lambda^{-t}}{2} |\Delta_0|. \tag{12.8}$$

(12.8) together with (4.6) imply the following

Property. Let $\Delta_0 = \Delta f_k \subset B(\Delta_0) \subset \operatorname{Im} F_{n-1} \circ h(\lambda, x)$ be so that $F_{n-1} \circ h(\lambda, \frac{1}{2}) \notin B(\Delta_0)$. Then

$$\mu(F_{n-1} \circ h(\lambda, x), \Delta_0) < 3. \tag{12.9}$$

d) Let $\mathscr{D}_n = \Delta(F_{n-1} \circ h)$, $F_{n-1} = f_{i_{n-1}} \circ \ldots \circ f_{i_1}$, and let $\Delta_0^{(n)} = \Delta f_{i_n}$ be so that $F_{n-1} \circ h(\lambda, \frac{1}{2}) \in \Delta_0^{(n)}$. Then $\mathscr{D}_{n+1} = \Delta(f_{i_n} \circ F_{n-1} \circ h)$. Let $B_n = B(\Delta_0^{(n)})$ be the block of $\Delta_0^{(n)}$, $\mathscr{U}_n = (F_{n-1} \circ h)^{-1} B(\Delta_0^{(n)})$. Notice that $\mathscr{D}_{n+1}$ may be equal to $\mathscr{D}_n$ (it is, if $\operatorname{im} F_{n-1} \circ h(\lambda, x) \subset \Delta(0)$ – the first interval Δf_1 on $[0, 1]$ (or $\subset \Delta(1)$ – the last one)), but always $\mathscr{U}_{n+1} \subsetneqq \mathscr{U}_n$.

We now formulate the induction hypotheses for the proof of Proposition 3. Let $R(n) = \max\{R : \mathscr{U}_{n-1} \subset \Gamma_R\}$ where $\Gamma_R = (\frac{1}{2} - \lambda^{-sR}, \frac{1}{2} + \lambda^{-sR})$. Let $\sum_n^k = \sum N(\Delta)|\Delta|$ after step n, where $\Delta \subset \delta_k \backslash \delta_{k+1}$ if $k < n$, or $\Delta \subset \Gamma_k \backslash \Gamma_{k+1}$ if $k \geq n$, are either intervals Δf_i, $i \leq n$, $\Delta F_{n-1} \circ g$, $\Delta \hat{f}_n$ or holes δ_n^{-M}. $(N(\delta_n^{-M}) = M$ for holes.) Then for $k \leq R(n) - 1$,

i) $\sum_n^k < \dfrac{k^{3/2}}{\lambda^{s(1-\beta)k}} \sum_{i=0}^{n-1} \lambda^{-it}$.

Consider any x_0, x_1, $x_2 \in \delta_{k-2} \backslash \delta_k$ (respectively $\Gamma_{k-2} \backslash \Gamma_k$), so that $x_1 \in [x_0, x_2]$. Then for $k \leqq R(n)$

ii) $\displaystyle\sum_{\Delta \subset [x_0, x_2]} N(\Delta) |\Delta| / |x_0 - x_2| < \lambda^{s \beta k}$

$$\cdot \left(\sum_{\Delta \subset [x_0, x_1]} N(\Delta) |\Delta| / |x_0 - x_1| \right) \left(\sum_{i=0}^{n-1} \lambda^{-it} \right).$$

We have to prove (i) and (ii) for $n+1$ and $k \leqq R(n+1) - 1$ (respectively $k \leqq R(n+1)$), where $R(n+1) = \max\{R | \mathcal{U}_n \subset \Gamma_R\}$.

We shall assume that the boundary points of δ_n, δ_{n+1}, $\mathcal{D}_n$ and $\mathcal{U}_n$ lie in $\{\lambda^{-sm}\}$, that is, $\delta_k = (\frac{1}{2} - \lambda^{-sk}, \frac{1}{2} + \lambda^{-sk})$ for $k = n$, $n+1$, and for some r, $p \in \mathbb{Z}$,

$$\mathcal{D}_n = (\tfrac{1}{2} - \lambda^{-sr}, \tfrac{1}{2} + \lambda^{-sr})$$
$$\mathcal{U}_n = (\tfrac{1}{2} - \lambda^{-s(r+p)}, \tfrac{1}{2} + \lambda^{-s(r+p)}).$$

In addition we suppose $\dfrac{c_0}{2s}$ and $\dfrac{\alpha}{s} n$ to be integers. The reader can check there is no loss of generality here.

Let $F_{n-1} \circ h(\lambda, \frac{1}{2}) \in \delta_{q-1} \backslash \delta_q$. According to the main construction $q \leqq \left(1 - \dfrac{\alpha}{s}\right) n$. Let $N(f_{i_k}) = N_k$, $k \in [1, n-1]$. Lemma 7 implies

$$|Df_{i_k}| > \lambda^{\frac{N_k c_0}{2\sqrt{k}} + s}.$$

As $\frac{1}{2} - \lambda^{-sr}$ is a root of the equation

$$F_{n-1} \circ [\lambda x(1-x)] = F_{n-1} \circ h(\lambda, \tfrac{1}{2}) \quad (\text{mod } 1)$$

we have

$$\frac{1}{\lambda^{sr}} < \frac{1}{\left[\lambda \prod\limits_{k=1}^{n-1} |Df_{ik}| \right]^{1/2}} < \exp\left[-\frac{1}{2} \left(1 + \sum_{k=1}^{n-1} \left(\frac{N_k c_0}{2\sqrt{k}} + s \right) \right) \ell n \lambda \right].$$

Hence

$$sr > \frac{1}{2} \left[1 + (n-1)s + \frac{c_0}{2} \sum_{k=1}^{n-1} N_k / \sqrt{k} \right]$$

$$> \frac{1}{2} \left[1 + (n-1)s + \frac{c_0}{2\sqrt{n-1}} \sum_{k=1}^{n-1} N_k \right].$$

This implies that the number of iterations $N(F_{n-1} \circ h) = 1 + \sum\limits_{k=1}^{n-1} N_k$ satisfies

$$N(F_{n-1} \circ h) < \frac{4s}{c_0} r \sqrt{n-1} - \frac{2s}{c_0} (n-1)^{3/2}.$$

Taking into account that $r > c_0 n / 2s$, we obtain

$$N(F_{n-1} \circ h) < 2 \left(\frac{2sr}{c_0} \right)^{3/2} - \frac{2s}{c_0} (n-1)^{3/2}. \tag{12.10}$$

We shall denote $\Delta_0^{(n)}$ by Δ_0 and B_n by B below. As $F_{n-1} \circ h(\lambda, \frac{1}{2}) \in \Delta_0$, $\operatorname{im} F_{n-1} \circ h(\lambda, x)$ contains either $B_+(\Delta_0)$, or $B_-(\Delta_0)$. Suppose the former. The number of iterations $N(\Delta_i)$ are either increasing, or they decrease till some $N_{\min}$, and then increase up to infinity. Let $\mathscr{S} = \operatorname{im} F_{n-1} \circ h(\lambda, x) \cap B$. The properties of blocks are so that in the second case $|\mathscr{S}| = 2n_0 \cdot |\Delta_{\min}|(1 + \varepsilon)$, where $\Delta_{\min}$ is any interval corresponding to $N_{\min}$, $n_0 = \operatorname{card}\{\Delta f_1 \subseteq [0, \frac{1}{2}] \backslash \delta_1\} < \lambda$, $\varepsilon < \lambda^{-t}$. In the first case more than $1 - \varepsilon$ of $|\mathscr{S}|$ consists of intervals with $N(\Delta_i) = N(\Delta_0)$, and $N(\Delta_i) = N(\Delta_0) + 1$ (the distribution depends on the number of first Δ_i with $N(\Delta_i) = N(\Delta_0)$). In both cases we have

$$\sum_{\Delta \subset \mathscr{S}} N(\Delta)|\Delta| < (1 + \varepsilon) \cdot N(\mathscr{S}) \cdot |\mathscr{S}| \tag{12.11}$$

and

$$|\mathscr{S}| < 2 \cdot \lambda \cdot (1 + \varepsilon)|D f_k^{-(N(\mathscr{S}) - 1)}|, \tag{12.12}$$

where $N(\mathscr{S}) = N_{\min}$ in the second case, $N(\mathscr{S}) = N(\Delta_0) + 1$ in the first case. Taking into account that Δ_0, $\Delta_{\min} \subseteq B \subseteq [0, 1] \backslash \delta \left(1 - \frac{\alpha}{s}\right) n$ and thus $f_k^{N_{\min}} : \Delta_{\min} \to [0, 1]$ (correspondingly $f_k^{N(\Delta_0)}$) satisfies Lemma 7 with $\left(1 - \frac{\alpha}{s}\right) n$, and proceeding as above when deriving (12.10), we obtain for $\mathscr{U}_n = (F_{n-1} \circ h(\lambda, x))^{-1} \mathscr{S} = \Gamma_{r+p}$

$$N(F_{n-1} \circ h) + N(\mathscr{S}) < 2 \left(\frac{2s}{c_0}(r + p)\right)^{3/2} - \frac{2s}{c_0}(n - 1)^{3/2}. \tag{12.13}$$

e) As $\Gamma_r = \mathscr{D}_n = \Delta(F_{n-1} \circ h)$ consists of a unique Δ, after step n we have $\sum_n^k = N(F_{n-1} \circ h) \cdot 2(\lambda^{-sk} - \lambda^{-s(k+1)})$ for $k \geq r$.

Let us estimate $\sum n(\Delta)|\Delta|$ after taking the first compositions $f_i \circ F_{n-1} \circ h$ on every $\Gamma_k \backslash \Gamma_{k+1} \overset{\text{def}}{=} \tilde{\Gamma}_k$, $r \leq k < r + p$. We shall denote this sum by $\sum_{n+11}^k$.

Let $\mathscr{S}_0 = \mathscr{S}$, and let $\mathscr{S}_i$ be the λ^{2si}-enlargement of $\mathscr{S}_0$ with center $F_{n-1} \circ h(\lambda, \frac{1}{2})$. Then $\Gamma_{r+p} = (F_{n-1} \circ h)^{-1} \mathscr{S}_0$, $\tilde{\Gamma}_{r+p-i} = (F_{n-1} \circ h)^{-1}(\mathscr{S}_i \backslash \mathscr{S}_{i-1})$, $i = 1, 2, \ldots, p$. Applying (12.7) to $\{\Delta \subseteq \mathscr{S}_i \backslash \mathscr{S}_{i-1}\}$ and $\Delta' = \Delta F_{n-1}$ we obtain using b_{2n}

$$\sum_{\Delta \subset \mathscr{S}_i \backslash \mathscr{S}_{i-1}} (N(F_{n-1}) + n(\Delta))|F_{n-1}^{-1} \Delta| < (1 + O(\lambda^{-\gamma}))|\Delta F_{n-1}|$$

$$\left(N(F_{n-1})|\mathscr{S}_i \backslash \mathscr{S}_{i-1}| + \sum_{\Delta \subset \mathscr{S}_i \backslash \mathscr{S}_{i-1}} n(\Delta)|\Delta|\right). \tag{12.14}$$

Applying (12.6) to $F_{n-1}^{-1}(\mathscr{S}_i \backslash \mathscr{S}_{i-1})$ we have $h_\Delta > \lambda^{1 - 2s(r+p-(i-1))}$ and consequently

$$\sum_{n+11}^{r+p-i} = \sum_{(F_{n-1} \circ h)^{-1} \Delta} (N(F_{n-1}) + n(\Delta) + 1)|(F_{n-1} \circ h)^{-1} \Delta|$$

$$< 2^{-1} \lambda^{-1 + s(r+p-(i-1))} (1 + O(\lambda^{-\gamma})) \Delta F_{n-1}$$

$$\cdot \left(|\mathscr{S}_i \backslash \mathscr{S}_{i-1}| + N(F_{n-1})|\mathscr{S}_i \backslash \mathscr{S}_{i-1}| + \sum_{\Delta \subset \mathscr{S}_i \backslash \mathscr{S}_{i-1}} n(\Delta)|\Delta|\right). \tag{12.15}$$

We shall assume $\operatorname{im} F_{n-1} \circ h(\lambda, x) \subseteq [0, \frac{1}{2}]$, and leave to the reader the modifications corresponding to another position of $\operatorname{im} F_{n-1} \circ h(\lambda, x)$ in $[0, 1]$. Let

$\ell = \max\{i : \mathscr{S}_i \subseteq \delta_{q-2}\backslash\delta_q\}$. Then for $i \leq \ell$ we can apply (ii). Together with (12.11) this gives

$$\sum_{\Delta \subset \mathscr{S}_i\backslash\mathscr{S}_{i-1}} n(\Delta)|\Delta| < \sum_{\Delta \subset \mathscr{S}_i} n(\Delta)|\Delta| < \lambda^{s\beta\left(1-\frac{\alpha}{s}\right)n} \cdot N(\mathscr{S}_0) \cdot |\mathscr{S}_0| \cdot \lambda^{2si} \cdot (1+\varepsilon)$$

$$= N(\mathscr{S}_0) \cdot |\mathscr{S}_i\backslash\mathscr{S}_{i-1}| \cdot (1 - \lambda^{-2s})^{-1} \cdot (1+\varepsilon) \cdot \lambda^{\beta(s-\alpha)n}.$$

Substituting this estimate in (12.15), we obtain

$$\sum_{n+1\,1}^{r+p-1} < \frac{(1+O(\lambda^{-\gamma})) \cdot |\Delta F_{n-1}| \cdot |\mathscr{S}_i\backslash\mathscr{S}_{i-1}| (1 + N(F_{n-1}) + N(\mathscr{S}_0)(1+\varepsilon_1) \cdot \lambda^{\beta(s-\alpha)n})}{2\lambda^{1-s(r+p-(i-1))}}.$$

We have by definition $\mathscr{S}_k\backslash\mathscr{S}_{k-1} = F_{n-1} \circ h(\tilde{\Gamma}_{r+p-k})$ and using b_{2n}, this implies

$$|\Delta F_{n-1}| \cdot |\mathscr{S}_k\backslash\mathscr{S}_{k-1}| = \lambda^{1-2s(r+p-k)} \cdot (1+O(\lambda^{-\gamma}))(1-\lambda^{-2s}). \qquad (12.16)$$

Thus

$$\sum_{n+1\,1}^{r+p-i} < \frac{(1+\varepsilon_2) \cdot \lambda^s \cdot (1 + N(F_{n-1}) + N(\mathscr{S}_0)\lambda^{\beta(s-\alpha)n})}{2\lambda^{s(r+p-i)}}.$$

Consequently, by (12.13) this implies

$$\sum_{n+1\,1}^{r+p-i} < \frac{(1+\varepsilon_{12.17}) \cdot \lambda^s \cdot \left(\frac{2s}{c_0}(r+p)\right)^{3/2} \lambda^{\beta(s-\alpha)n}}{\lambda^{s(r+p-i)}}$$

$$< (1+\varepsilon_{12.17})\left(\frac{2s}{c_0}\right)^{3/2} \frac{(r+p-i)^{3/2}}{\lambda^{s(1-\beta)(r+p-i)}} \cdot \frac{\lambda^{s\beta\left(1-\frac{\alpha}{s}\right)n+s+2\beta}}{\lambda^{s\beta(r+p-i)}}. \qquad (12.17)$$

Thus for $1 \leq i \leq \ell$ we have on $\tilde{\Gamma}_{r+p-i}$ the analogue of assumption (i) but with an additional factor less than $\left(\text{we use } n < \dfrac{2s}{c_0}r\right)$

$$\frac{\lambda^{s+2\beta}}{\lambda^{s\beta\left(r+p-i-\left(1-\frac{\alpha}{s}\right)n\right)}} < \frac{\lambda^{s+2\beta}}{\left[\lambda^{s\beta\left(p-i+r\left(1-\frac{2(s-\alpha)}{c_0}\right)\right)}\right]}.$$

In a general case we have $\ell < p$ (this is not so only if $F_{n-1} \circ h(\lambda, \frac{1}{2}) \in [0,1]\backslash\delta_3$), and we have also to estimate $\sum_{n+1\,1}^{r+j}$, $0 \leq j < p - \ell$. Let us consider

$$\mathscr{S}_{\ell+1} = \lambda^{2s(\ell+1)}\mathscr{S}_0, \ldots, \mathscr{S}_p = \lambda^{2sp} \cdot \mathscr{S}_0 = \operatorname{im} F_{n-1} \circ h(\lambda, x).$$

We have $\mathscr{S}_p\backslash\mathscr{S}_{p-1} = [0, a_{p-1}]$, where $\frac{1}{2} - \lambda^{-2s}/2 \approx a_{p-1} \in \delta_2\backslash\delta_3$,

$$\mathscr{S}_{p-1}\backslash\mathscr{S}_{p-2} = [a_{p-1}, a_{p-2}], \quad \frac{1}{2} - \frac{1}{2\lambda^{4s}} \approx a_{p-2} \in \delta_4\backslash\delta_5 \ldots,$$

$$\mathscr{S}_{\ell+1}\backslash\mathscr{S}_\ell = [a_{\ell+1}, a_\ell],$$

$$\frac{1}{2} - \frac{1}{2}\lambda^{-2s(p-\ell-1)} \approx a_{\ell+1} \in \delta_{2(p-\ell-1)}\backslash\delta_{2(p-\ell-1)+1},$$

$$\frac{1}{2} - \frac{1}{2}\lambda^{-2s(p-\ell)} \approx a_\ell \in \delta_{2(p-\ell)}\backslash\delta_{2(p-\ell)+1}.$$

By construction $\mathscr{S}_\ell = [a_\ell, F_{n-1} \circ h(\lambda, \frac{1}{2})]$ is the last enlargement of $\mathscr{S}_0$ which lies in $\delta_{q-2} \backslash \delta_q$. Hence either $a_\ell \in \delta_{q-1} \backslash \delta_q$, or $a_\ell \in \delta_{q-2} \backslash \delta_{q-1}$. For definiteness let $a_\ell \in \delta_{q-1} \backslash \delta_q$. Then $q = 2(p-\ell) + 1$, $\ell = p - \frac{q-1}{2}$,

$$\mathscr{S}_{\ell+j} \backslash \mathscr{S}_{\ell+j-1} \subseteqq \delta_{2(p-\ell-j)} \backslash \delta_{2(p-\ell-j)+3}, \quad j \in [1, p-\ell],$$

where $\delta_0 = [0, \frac{1}{2}]$.

(i) for δ_k with $k = 2(p-\ell-j)$, $k+3 \le q \le \left(1 - \frac{\alpha}{s}\right) n$ implies

$$\sum_n^k + \sum_n^{k+1} + \sum_n^{k+2} < \frac{k^{3/2}}{\lambda^{s(1-\beta)k}} (1+\varepsilon_3).$$

By construction

$$\frac{1-\varepsilon_4}{2} \lambda^{-2s(p-\ell-j)} < |\mathscr{S}_{\ell+j} \backslash \mathscr{S}_{\ell+j-1}| < \lambda^{-2s(p-\ell-j)}.$$

Hence using (12.7) we obtain similarly to (12.14) for $j \in [1, p-\ell]$

$$\begin{aligned}
\sum_{A \subset \mathscr{S}_{\ell+j} \backslash \mathscr{S}_{\ell+j-1}} &(N(F_{n-1}) + n(\Delta)) |F_{n-1}^{-1}(\Delta)| \\
&< (1 + O(\lambda^{-\gamma})) |\Delta F_{n-1}| \{N(F_{n-2}) |\mathscr{S}_{\ell+j} \backslash \mathscr{S}_{\ell+j-1}| \\
&\quad + [2(p-\ell-j)]^{3/2} \lambda^{-s(1-\beta)2(p-\ell-j)}\} \\
&< (1 + \varepsilon_{12.19}) |\Delta F_{n-1}| |\mathscr{S}_{\ell+j} \backslash \mathscr{S}_{\ell+j-1}| \\
&\quad \cdot [N(F_{n-1}) + 2(2(p-\ell-j))^{3/2} \lambda^{2s\beta(p-\ell-j)}].
\end{aligned} \qquad (12.19)$$

By construction

$$\lambda^{2s} \cdot \mathrm{dist}(a_\ell, F_{n-1} \circ h(\lambda, \tfrac{1}{2})) > \frac{1 - \lambda^{-s}}{\lambda^{s(q-2)}}.$$

This implies

$$h_\Delta > (1 - \lambda^{-s}) |\Delta F_{n-1}| \lambda^{-s(2(p-\ell)+1)}$$

on $F_{n-1}^{-1}(\mathscr{S}_{\ell+1} \backslash \mathscr{S}_\ell)$ and

$$h_\Delta > \frac{(1 - \lambda^{-s}) |\Delta F_{n-1}|}{\lambda^{s(2(p-\ell-j)+3)}}$$

on $F_{n-1}^{-1}(\mathscr{S}_{\ell+j} \backslash \mathscr{S}_{\ell+j-1})$. Applying (12.6) we obtain from (12.19) that

$$\begin{aligned}
\sum_{n+1}^{r+p-(\ell+j)} &< (1 + \varepsilon_{12.20}) |\Delta F_{n-1}|^{1/2} |\mathscr{S}_{\ell+j} \backslash \mathscr{S}_{\ell+j-1}| [N(F_{n-1}) + 1 \\
&\quad + (2(p-\ell-j))^{3/2} 2 \lambda^{2s\beta(p-\ell-j)}] \lambda^{-1/2} \lambda^{-s(p-\ell-j+3/2)}.
\end{aligned} \qquad (12.20)$$

Now, (12.16) with $k = \ell + j$ implies

$$|\Delta F_{n-1}| |\mathscr{S}_{\ell+j} \backslash \mathscr{S}_{\ell+j-1}| < \frac{\sqrt{\lambda} \lambda^{-s(p-\ell-j)} (1+\varepsilon_5)}{\lambda^{s(r+p-\ell-j)}}.$$

Substituting this into (12.20) we obtain

$$\sum_{n+1\,1}^{r+p-(\ell+j)} < (1+\varepsilon_{12.21})\lambda^{3s/2}\,\frac{N(F_{n-1})+1+(2(p-\ell-j))^{3/2}\lambda^{2s\beta(p-\ell-j)}}{\lambda^{s(r+p-\ell-j)}}.$$

$$(12.21)$$

According to (12.10), $N(F_{n-1})+1 < 2(2sr/c_0)^{3/2}$. Because

$$2(p-\ell) < \left(1-\frac{\alpha}{s}\right)n < \left(1-\frac{\alpha}{s}\right)\frac{2sr}{c_0},$$

we can rewrite (12.21) as

$$\sum_{n+1\,1}^{r+p-\ell-j} < (1+\varepsilon_{12.22})\left(\frac{2s}{c_0}\right)^{3/2}\frac{r^{3/2}}{\lambda^{s(1-\beta)(r+p-\ell-j)}}$$

$$\cdot\,\frac{2\lambda^{\frac{3s}{2}}}{\lambda^{s\beta(r+p-\ell-j)\left(1-\frac{2s}{c_0}\right)}}.$$

$$(12.22)$$

Thus for $0\le k < p-\ell$ we have on $\tilde{\Gamma}_{r+k}$ an additional exponentially small factor compared with the assumption (i), as well as for $p-\ell\le k < p$ (see (12.18)).

f) In order to estimate the contribution of terms in $(\delta_n-\delta_{n+1})^{-M}$ we first do it in $\delta_n\backslash\delta_{n+1}$.

Step $n+1$ on $\delta_n\backslash\delta_{n+1}$ divides into substeps $\ell=1,2,\ldots$ corresponding to the construction of $f_{n+1\ell}$ (see Sect. 3).

We use the following notation: Δ is any interval $\Delta F_{n-1}\circ g$, $\Delta\hat{f}_n$, $\Delta f_{n+1\ell}$; δ is any hole δ_n^{-M}, $\delta_{n+1}^{-N}\subseteqq\delta_n\backslash\delta_{n+1}$; $n(\Delta)$, $n(\delta)$ are corresponding numbers of iterations.

Let $i_{n\ell}=\sum|\Delta|$ after substep ℓ of step $n+1$ and with the same meaning of indices n, ℓ

$$h_{n\ell}=\sum|\delta|;\qquad x_{n\ell}=\sum n(\Delta)|\Delta|;\qquad y_{n\ell}=\sum n(\delta)|\delta|.$$

We consider also the corresponding sums on $[0,1]\backslash\delta_n$ namely

$$I_n=\sum|\Delta|,\qquad \Delta\subseteqq[0,1]\backslash\delta_n,$$

after step n:

$$H_n=\sum|\delta|;\qquad X_n=\sum n(\Delta)|\Delta|;\qquad Y_n=\sum n(\delta)|\delta|.$$

Then $I_n+H_n+|\delta_n|=1$, $i_{n\ell}+h_{n\ell}+|\delta_{n+1}|=|\delta_n|$ for all ℓ. Besides, let $\ell=0$ correspond to i,h,x,y constructed after step n, and $\ell=\infty$ after step $n+1$, so that (n,∞) equals $(n+1,0)$.

We may assume all the compositions to be linear (see Remark XII/1 below) and thus using (12.7) we obtain

$$\left.\begin{aligned}
i_{n1} &= i_{n0}(1-H_n-|\delta_n|)\\
h_{n1} &= h_{n0}+i_{n0}(H_n+|\delta_n|)\\
x_{n1} &= x_{n0}I_n+X_n\cdot i_{n0}\\
y_{n1} &= y_{n0}+x_{n0}(H_n+|\delta_n|)+Y_n\cdot i_{n0}.
\end{aligned}\right\}$$

$$(12.23)$$

The holes and intervals of subsequent substeps $1 = 2, 3, \ldots$ are obtained using compositions of maps $\tilde{G}_n : \delta_n^{-M} \to \delta_n$, so that after substep ℓ the remaining preimages of δ_n are of the form $\tilde{G}_{n_\ell}^{-1} \circ \ldots \circ \tilde{G}_{n_1}^{-1} \delta_n$ and preimages of δ_{n+1} are $\tilde{G}_{n_i}^{-1} \circ \ldots \circ \tilde{G}_{n_1}^{-1} \delta_{n+1}$, $i = 1, 2, \ldots, \ell - 1$ (compare with (3.6), (3.7)). Let $\tilde{h}_{n\ell}$ and $\tilde{y}_{n\ell}$ correspond to preimages of δ_n and $\tilde{\tilde{h}}_{n\ell}, \tilde{\tilde{y}}_{n\ell}$ to preimages of δ_{n+1}. With this notation we have

$$\tilde{y}_{n1} = \tilde{\tilde{h}}_{n1} = 0, \qquad h_{n\ell} = \tilde{h}_{n\ell} + \tilde{\tilde{h}}_{n\ell}, \qquad y_{n\ell} = \tilde{y}_{n\ell} + \tilde{\tilde{y}}_{n\ell},$$

and for $\ell \geq 2$

$$\left.\begin{aligned}
&i_{n\ell} = i_{n\ell-1} + i_{n\ell-1} \cdot \tilde{h}_{n\ell-1} \cdot |\delta_n|^{-1}\\
&\tilde{h}_{n\ell} = \tilde{h}_{n\ell-1} \cdot \tilde{h}_{n\ell-1} \cdot |\delta_n|^{-1};\\
&\tilde{\tilde{h}}_{n\ell} = \tilde{\tilde{h}}_{n\ell-1} + \tilde{h}_{n\ell-1} \cdot \frac{\tilde{\tilde{h}}_{n\ell-1} + |\delta_{n+1}|}{|\delta_n|}\\
&x_{n\ell} = x_{n\ell-1} + (\tilde{y}_{n\ell-1} \cdot i_{n\ell-1} + x_{n\ell-1} \cdot \tilde{h}_{n\ell-1}) \cdot |\delta_n|^{-1}\\
&\tilde{y}_{n\ell} = 2\tilde{y}_{n\ell-1} \cdot \tilde{h}_{n\ell-1} \cdot |\delta_n|^{-1};\\
&\tilde{\tilde{y}}_{n\ell} = \tilde{\tilde{y}}_{n\ell-1} + \frac{\tilde{y}_{n\ell-1} \cdot \tilde{\tilde{h}}_{n\ell-1} + \tilde{\tilde{y}}_{n\ell-1} \cdot \tilde{h}_{n\ell-1} + \tilde{y}_{n\ell-1} |\delta_{n+1}|}{|\delta_n|}.
\end{aligned}\right\} \quad (12.24)$$

According to Sect. 10, to any hole $\delta_n^{-M} = G_n^{-M}(\delta_n)$ there corresponds uniquely a set $L_n^{-M} = G_n^{-M}(L)$. As for any interval $\Delta \subseteq L_n^{-M}$, $n(\Delta) > n(\delta_n^{-M})$; this implies

$$\tilde{h}_{n1} < (|\delta_n| - |\delta_{n+1}|)(1 + \varepsilon) \cdot \lambda^{-2\alpha n}.$$

Using $\dfrac{|\delta_{n+1}|}{|\delta_n|} < (1 + o(\lambda^{-s(n+1)}))\lambda^{-s}$, the recurrent formulas (12.24) give

$$x_{n\infty} = x_{n+10} < (x_{n1} + y_{n1})(1 + O(\lambda^{-2\alpha n}));$$

$$y_{n\infty} = y_{n+10} < y_{n0} \cdot \frac{1 + o(\lambda^{-tn})}{\lambda^s}.$$

The induction hypotheses imply

$$X_n < 1 - \frac{2}{\lambda^s} + \left(\sum_{k=1}^{n} \frac{k^{3/2}}{\lambda^{s(1-\beta)k}}\right)\left(\sum_{i=0}^{n-1} \lambda^{-ti}\right) < 1 + \varepsilon_1;$$

$$x_{n0} < \frac{n^{3/2}}{\lambda^{s(1-\beta)n}} \sum_{i=0}^{n-1} \lambda^{-ti};$$

$$i_{n0} < 2(1 + o(\lambda^{-sn})) \cdot (1 - \lambda^{-s}) \cdot \lambda^{-sn};$$

$$I_n < 1 - 2\lambda^{-sn};$$

$$H_n + |\delta_n| < [\lambda^s \cdot (1 + \varepsilon_2)]^{-n}.$$

By the above reasons

$$y_{n0} < x_{n0} \cdot \lambda^{-2\alpha n} \cdot (1 + \varepsilon); \qquad Y_n < X_n \cdot \lambda^{-2\alpha n} \cdot (1 + \varepsilon).$$

Using (12.23) we obtain

$$\sum_{n+1}^{n} = x_{n+1\,0} + y_{n+1\,0} < \left(x_{n0} + \frac{2}{\lambda^{sn}}\right)(1 + O(\lambda^{-2\alpha n})) < \frac{n^{3/2}}{\lambda^{s(1-\beta)n}} \sum_{i=0}^{n} \lambda^{-ti} \tag{12.25}$$

which proves i_{n+1} for $\delta_n \backslash \delta_{n+1}$. The proof is similar for $\delta_k \backslash \delta_{k+1}$, $1 \leq k < n$.

Now we can estimate the contribution of $\sum n(\Delta)|\Delta|$ in every hole $(\delta_n \backslash \delta_{n+1})^{-M}$ on $\tilde{\Gamma}_{r+i} = \Gamma_{r+i} \backslash \Gamma_{r+i+1}$, $i \in [0, p-1]$.

Though we cannot correspond the $\lambda^{2\alpha n}$-enlargement to any δ_n^{-M}, we can consider its $\lambda^{\alpha n/2}$-enlargement. The construction of Sect. 10 gives, as above,

$$\sum_{\delta \subset \tilde{\Gamma}_{r+i}} |\delta| < \frac{|\tilde{\Gamma}_{r+i}|(1+\varepsilon)}{\lambda^{\alpha n/2}}; \qquad \sum_{\delta \subset \tilde{\Gamma}_{r+i}} n(\delta)|\delta| < \left(\sum_{\Delta \subset \tilde{\Gamma}_{r+i}} n(\Delta)|\Delta|\right) \frac{1+\varepsilon}{\lambda^{\alpha n/2}}.$$

Using (12.7) with $\mu = O(\lambda^{-\alpha n/2})$ we obtain after step $(n+1)$

$$\sum_{n+1}^{r+i} < \sum_{n+1\,1}^{r+i} + \left(\sum_{n+1\,1}^{r+i} \cdot (1 - \lambda^{-s})|\delta_n| \cdot \lambda^{-\alpha n/2} + \sum_{n+1}^{n} \cdot |\tilde{\Gamma}_{r+i}| \cdot \lambda^{-\alpha n/2}\right) \frac{1 + O(\lambda^{-\alpha n/2})}{|\delta_n|}$$

$$< (1 + \varepsilon_{12.26}) \sum_{n+1\,1}^{r+i} + \frac{n^{3/2}}{\lambda^{s(r+i)}} \cdot \frac{\lambda^{s\beta n}}{\lambda^{\alpha n/2}}. \tag{12.26}$$

Thus we still have for $\sum_{n+1}^{r+i}$ an exponentially better estimate than that required by (i).

This proves (i_{n+1}) for $k \in [r, R(n+1)-1]$. Now $\tilde{\Gamma}_k$ $k \in [R(n), r-1]$ are contained in the union of preimages $(F_{n-2} \circ h)^{-1} \Delta_i^{(n-1)}$, where $\Delta_i^{(n-1)} \subset B_{n-1}(\Delta_0^{(n-1)})$. One obtains (i_{n+1}) for such $\tilde{\Gamma}_k$ in a similar way, using the construction of block B_{n-1} (the estimates are better in this case).

In order to obtain (i_{n+1}) for $n+1 \leq k < R(n)-1$, we notice that at step $m(k)$ corresponding to the first consideration of $\Gamma_k \backslash \Gamma_{k+1}$, we have on $\Gamma_k \backslash \Gamma_{k+1}$ an exponential reserve by comparison with $(i_{m(k)})$. (12.7) and Property (12.9) imply that the nonlinearity at Step $(m(k)+1)$ gives an additional factor less than 3. Any of the subsequent steps implies the diminishing of the maximal interval $\Delta \subset \Gamma_k \backslash \Gamma_{k+1}$ at least $3\lambda^{c_0}$ times (because of taking compositions), and we obtain the following:

Remark XII/1. The total non-linear effect of steps $m(k)+1$, $m(k)+2$, ... on $\Gamma_k \backslash \Gamma_{k+1}$ is less than

$$\exp\left(3 \cdot \sum_{n \subset 0}^{\infty} \lambda^{-c_0 n}\right).$$

In particular this shows that when proving (i) for $\Gamma_k \backslash \Gamma_{k+1}$ it suffices to consider only step $m(k)$.

g) In order to prove (ii_{n+1}) of Proposition 3 we consider three points x_0, x_1, $x_2 \in \Gamma_{r+i-2} \backslash \Gamma_{r+i}$ and their images under $F_{n-1} \circ h(\lambda, x)$. We may suppose x_2 to be closer to $\frac{1}{2}$ than x_0, (otherwise h_Δ for $\Delta \subset [x_1, x_2]$ is larger than for $\Delta \subset [x_0, x_1]$ and an estimate for x_0, x_1, x_2 is better than for their images).

Let $Q_1 = F_{n-1} \circ h[x_0, x_1]$, $Q_2 = F_{n-1} \circ h[x_0, x_2]$. Using (12.7) and (12.6) with $h_\Delta \geq h(x_2) \geq \lambda^{1-2s(r+i)}$ we obtain

$$\sum_{\Delta \subset [x_0, x_2]} n(\Delta)|\Delta| < \frac{1 + O(\lambda^{-\gamma}))|\Delta F_{n-1}| \left((N(F_{n-1})+1)|Q_2| + \sum_{\Delta \subset Q_2} n(\Delta)|\Delta|\right)}{2\sqrt{\lambda} \lambda^{-s(r+i)}}. \tag{12.27}$$

For $\Delta \subset [x_0, x_1]$ $h_\Delta \leqq h(x_0) \leqq \lambda^{1-2s(r+i)+4s}$. Hence

$$\sum_{\Delta \subset [x_0, x_1]} n(\Delta)|\Delta| > \frac{(1-O(\lambda^{-\gamma}))|\Delta F_{n-1}|\left((N(F_{n-1})+1)|Q_1| + \sum_{\Delta \subset Q_1} n(\Delta)|\Delta|\right)}{2\sqrt{\lambda}\lambda^{-s(r+i)+2s}}. \qquad (12.28)$$

As $\dfrac{|h[x_0, x_2]|}{|h[x_0, x_1]|} < \dfrac{|x_0 - x_2|}{|x_0 - x_1|}$, we have $\dfrac{|Q_2|}{|Q_1|} < \dfrac{|x_0 - x_2|}{|x_0 - x_1|}$ $(1 + O(\lambda^{-\gamma}))$. First let $Q_2 \subset \delta_{q-2} \backslash \delta_q$. Then we can use (ii_n) for $F_{n-1} \circ h(x_0, x_1, x_2) \subset [0, 1] \backslash \delta_{(1-\alpha/s)n}$. Applying (12.27), (12.28), we obtain

$$\frac{\displaystyle\sum_{\Delta \subset [x_0, x_2]} n(\Delta)|\Delta|}{\displaystyle\sum_{\Delta \subset [x_0, x_1]} n(\Delta)|\Delta|} < \frac{|x_0 - x_2|}{|x_0 - x_1|}\lambda^{(s-\alpha)\beta n + 2s}.$$

If $F_{n-1} \circ h[x_0, x_2]$ is not contained in $\delta_{q-2}\backslash\delta_q$, we have $Q_2 = Q_2' \cup Q_2''$ where $Q_2' \subset \delta_{q-2}\backslash\delta_q$, $Q_2'' \subset [0, 1]\backslash\delta_{q-2}$. We estimate $\displaystyle\sum_{\Delta \subset Q_2'} n(\Delta)|\Delta|$ as above, and $\displaystyle\sum_{\Delta \subset Q_2''} n(\Delta)|\Delta|$ using (i_n) similarly to (12.19)–(12.21), and obtain

$$\frac{\displaystyle\sum_{\Delta \subset [x_0, x_2]} n(\Delta)|\Delta|}{\displaystyle\sum_{\Delta \subset [x_0, x_1]} n(\Delta)|\Delta|} < \frac{|x_0 - x_2|}{|x_0 - x_1|}\sqrt{n}\lambda^{(s-\alpha)\beta n + 2s}. \qquad (12.29)$$

For large λ, $\sqrt{n} \ll \lambda^{\alpha\beta n}$. Comparing with the requirement (ii_{n+1}) for $k = r+i$, we obtain a sufficient condition on r

$$s\beta n + 2s \leqq s\beta r.$$

As $r > \dfrac{c_0}{2s}n$, it suffices to have

$$n \geqq \frac{2}{5}\left(\frac{c_0}{2s} - 1\right)^{-1} \qquad (12.30)$$

which holds for $s \leqq \frac{1}{13}$, $\beta = \frac{2}{5}$, $n \geqq 1$.

The account of $\Delta \subset \delta_n^{-M}$ gives an additional factor $(1 + O(\lambda^{-\alpha n/2}))$ and one finishes the proof of (ii_{n+1}) as above (i_{n+1}). $\quad\square$

Remark XII/2. One can check that for $n \leqq n_0$, when

$$F_{n-1} \circ h(\lambda, \tfrac{1}{2}) \in [0, 1]\backslash[\tfrac{1}{2} - \lambda^{-s/2}, \tfrac{1}{2} + \lambda^{-s/2}],$$

(ii) is satisfied with $\beta = 0$ [$\lambda^{s\beta k}$ on the right side of (ii) can be replaced by a constant]. From Remark VI/5 and (12.30) it follows that one can take β arbitrarily small. It seems that more careful estimates should give Proposition 3 with $\beta = 0$ and $k^{1+\varepsilon}$ ($\varepsilon > 0$ small) instead of $k^{3/2}$.

Remark XII/3. Lemma 7 implies that for any $\lambda \in \mathcal{M}$ and for $\Delta f_k \in \xi_\lambda$ so that $f_k : \Delta f_k \to [0, 1]$, $f_k = f_\lambda^N|\Delta f_k$,

$$|Df_k| > \lambda^{c_0\sqrt{N}/2}.$$

Collet and Eckmann [10] proved for a particular smooth family f_δ that the Liapunov exponent is positive on the trajectory of $\frac{1}{2}$ for a set of λ of positive measure.

13. Theorem A for a General Family. The Reduction of Theorem B to Theorem A

a) Let $f(x):[0,1]\to[0,1]$, $f(0)=f(1)=0$, be a C^3-map, c a single critical point of f. Consider a family $f_\lambda(x):x\to\lambda\cdot f(x)$ (mod 1). We take λ sufficiently large and imitate the construction used for $\lambda x(1-x)$.

We take $T_0=(f(c))^{-1}$ so as to make $\lambda\cdot f(c)$ traverse $[0,1]$ when λ crosses $[L,L+T_0]$.

Then we choose a small $\varepsilon>0$ and consider an ε-neighbourhood U of the critical point c. Using the Hadamard lemma we represent $f(x)$ and its derivatives in the form

$$f(x)=f(c)-a(x-c)^2(1+(x-c)\theta_1(x))$$
$$f'(x)=-2a(x-c)(1+(x-c)\theta_2(x)) \tag{13.1}$$
$$f''(x)=-2a(1+(x-c)\theta_3(x)),$$

where $-2a=f''(c)<0$, $|\theta_i(x)|<c_1$. Using (13.1), one can check that (4.6) with $\frac{|\Delta|}{H}$ instead of $\frac{|\Delta|}{2H}$, and (4.7) with $\frac{|\Delta|}{ax^2}$ instead of $\frac{|\Delta|}{2ax^2}$ are still true in U.

Remark XIII/1. Notice that the condition $f''(c)\neq0$ is not necessary, $f^{(n)}(c)\neq0$ for some $n\geq2$ will do as well.

Then we consider

$$Df_\lambda=\lambda f'(x), \qquad \frac{D^2f_\lambda}{(Df_\lambda)^2}=\frac{1}{\lambda}\frac{f''(x)}{(f'(x))^2},$$

$$\frac{\partial f/\partial\lambda}{\partial f/\partial x}=\frac{1}{\lambda}\frac{f(x)}{f'(x)}, \qquad \frac{\dfrac{\partial}{\partial\lambda}\dfrac{\partial f_\lambda^{-1}}{\partial z}}{\dfrac{\partial f_\lambda^{-1}}{\partial z}}=-\frac{1}{\lambda}\left(1-\frac{f''\cdot f}{(f')^2}\right).$$

Let

$$A=\max_{x\in[0,1]\setminus U}\left\{\frac{1}{|f''(x)|},\ 1+\frac{|f''(x)|}{(f'(x))^2}\right\}.$$

We take s from Sect. 2, and we take a λ as a parameter. We choose λ so large that

$$\lambda^s>\max\left\{\frac{2}{\varepsilon a^s},2Aa^{1-s}\right\}.$$

Then we choose $\delta_1\approx(c-(\lambda a)^{-s},\ c+(\lambda a)^{-s})$ as in Sect. 2, and define $f_1(\lambda,x)$ so that $\Delta f_1\subseteq[0,1]\setminus\delta_1$. One can check that the branches f_1 and their derivatives satisfy the conditions of Step 1 with $\max_{x\in[0,1]}f(x)$ instead of $\frac{1}{4}=\max_{x\in[0,1]}x(1-x)$. Then

for $a\lambda > N_0$ of Sect. 2, the inductive construction may be used, and we obtain Theorem A for the family $\lambda \cdot f(x)$.

Remark XIII/2. Theorem A holds also in the case of a family $\lambda \cdot f(x)$, $f(0)=0=f(1)$, $f'(0) \neq 0$, when $f(x)$ has several extremal points $c^{(1)}$, $c^{(2)}$, ..., $c^{(k)}$. Then the construction can be generalized in the following manner. During step n we construct intervals $\delta_n^{(i)} \approx (c^{(i)} - (\lambda a_i)^{-sn}, c^{(i)} + (\lambda a_i)^{-sn})$, $1 \leq i \leq k$, their preimages $(\delta_n^{(i)})^{-m}$, and enlarged preimages $(\hat{\delta}_n^{(i)})^{-m}$; the constants a_i are defined according to the map f. The condition

$$F_{n-1}^{(i)} \circ h^{(i)}(\lambda, c^{(i)}) \in [0, 1] \setminus \bigcup_{j=1}^{k} \bigcup_{m=0}^{\infty} (\hat{\delta}_n^{(j)})^{-m}$$

defines on step n the set of admissible values of the parameter $\mathcal{M}_n^{(i)}$, the set $\mathcal{M}$ is defined as $\mathcal{M} = \bigcap_{i=1}^{k} \bigcap_{n=1}^{\infty} \mathcal{M}_n^{(i)}$.

b) We reduce the proof of Theorem B to the proof of Theorem A using the induced map studied in [5]. Let $f_\lambda(x) = \lambda x(1-x)$, $0 < \lambda \leq 4$, and $t_\lambda = 1 - 1/\lambda$ its fixed point. We consider for $\lambda \in [4 - \varepsilon, 4]$ the induced map T_λ on the interval $I_\lambda = [1/\lambda, 1 - 1/\lambda]$. T_λ has $2p$ monotone branches $T_{i\lambda}$, $i = \pm 1, \ldots, \pm p$ ($p = p(\lambda)$) and one middle branch S_λ. Furthermore, $T_{i\lambda} = f_\lambda^{i+1}$ on $\Delta T_{i\lambda}$ and $S_\lambda = f_\lambda^{p+2}$ on ΔS_λ. The interval $[4 - \varepsilon, 4]$ is divided into a countable number of intervals $[\lambda_p, \lambda_{p+1}]$ such that for $\lambda \in [\lambda_p, \lambda_{p+1}]$ the number $p(\lambda)$ defined above is constant and as λ passes λ_p, the old parabolic branch S_λ breaks up into two branches T_λ, a new branch S_λ is born, and $p(\lambda)$ grows from p to $p+1$.

For some constants $c_1, c_2 > 0$ we have

$$2^i c_2 < |\partial T_i / \partial x| < 2^i c_1 \quad 1 \leq i \leq p - 1$$
$$4^{p+1} c_2 |x - \tfrac{1}{2}| < |\partial T_p / \partial x|, |\partial S / \partial x| < 4^{p+1} c_1 |x - \tfrac{1}{2}|. \tag{13.2}$$

Applying (9.1) to $T_{i\lambda}$ we obtain

$$\left| \frac{\partial T_i / \partial \lambda}{\partial T_i / \partial x} \right| < 2^i c_3 \quad 1 \leq i \leq p - 1,$$

$$\left| \frac{\partial T_p / \partial \lambda}{\partial T_p / \partial x} \right|, \left| \frac{\partial S / \partial x}{\partial S / \partial x} \right| < \frac{c_3}{|x - \tfrac{1}{2}|}. \tag{13.3}$$

The estimate for the velocity of the top is

$$v_p(\lambda) = -4^p (1 + O(\lambda^{-p})). \tag{13.4}$$

We have

$$\frac{|D^2 T_{i\lambda}|}{|D T_{i\lambda}|^2} < c_4 \quad 1 \leq i \leq p - 1$$

$$\frac{|D^2 T_{p\lambda}|}{|D T_{p\lambda}|^2}, \frac{|D^2 S_\lambda|}{|D S_\lambda|^2} < c_4 \left(1 + \frac{1}{4^p (x - \tfrac{1}{2})^2} \right). \tag{13.5}$$

Using (11.6) we obtain for all i and $z = T_i(\lambda, x)$, $x \in \Delta T_i$

$$\left| \frac{\frac{\partial}{\partial \lambda} \partial T_{i\lambda}^{-1}/\partial z}{\partial T_{i\lambda}^{-1}/\partial z} \right| < c_5 \left(i + \frac{1}{(x - \frac{1}{2})^2} \right). \tag{13.6}$$

Now we use the following property of T_4 (see [5]). There exists $d > 1$ and a positive integer q so that

$$|DT_4^q| > d. \tag{13.7}$$

Remark XIII/3. Apparently $q = 1$ but it is not essential for our purpose.

For any fixed i, $T_i(\lambda, x)$ and its derivatives uniformly converge to $T_i(4, x)$ when $\lambda \to 4$. Thus for $i_s \in [0, i_0]$ and for λ sufficiently close to 4 we still have

$$|D(T_{\lambda i_1} \circ T_{\lambda i_2} \circ \ldots \circ T_{\lambda i_q})| > d. \tag{13.8}$$

Choose a very large k and some $p \gg k$, and consider $\lambda \in [\lambda_p, \lambda_{p+1}]$. Let

$$n = \left[\frac{k}{\log_2 d} + 1 \right].$$

Let us consider consecutive compositions of the form

$$T_{\lambda\tau_r} = T_{\lambda i_1} \circ \ldots \circ T_{\lambda i_r}, \qquad i_s \in [1, k]$$

until we have on the domain of $T_{\lambda\tau}$

$$|DT_{\lambda\tau}| > 2^k.$$

Because of (13.8), for any $T_{\lambda\tau_p}$, $r \leq qn$ (really (13.2) implies $r \ll qn$ for many $T_{\lambda\tau_r}$). Let

$$\delta_1(\lambda) = \Delta S_\lambda \cup \left(\bigcup_{i > k} \Delta T_{\lambda i} \right), \qquad J_\lambda = I_\lambda \backslash \delta_1(\lambda).$$

Then we obtain the following partition of J_λ.

$$J_\lambda = \left(\bigcup \Delta T_{\lambda\tau} \right) \cup \left(\bigcup_{m \leq qn} \delta_1^{-m}(\lambda) \right). \tag{13.9}$$

(13.5), (13.8) and a modification of Lemma 1 imply

$$\frac{|D^2 T_{\lambda\tau_r}|}{|DT_{\lambda\tau_r}|^2} < c_6$$

independent of k. Hence we obtain

$$\text{mes} \bigcup \delta_1^{-m}(\lambda) < 1 - \left(1 - \frac{c_7}{2^k} \right)^{qn} < c_8 \frac{k}{2^k}. \tag{13.10}$$

Using (13.8), (9.1), and (11.6) we obtain

$$\left| \frac{\partial T_{\lambda\tau}/\partial \lambda}{\partial T_{\lambda\tau}/\partial x} \right| < 2^k c_9,$$

$$\left| \frac{\frac{\partial}{\partial \lambda} \partial T_{\lambda\tau}^{-1}/\partial z}{\partial T_{\lambda\tau}^{-1}/\partial z} \right| < c_{10}(k + 4^k). \tag{13.11}$$

Although the estimates (13.11) grow with k, we can choose p so large that the time that the top $S_\lambda(\frac{1}{2})$ spends inside the union of the enlarged domains $\bigcup \hat{\delta}_1^{-m}(\lambda)$ will still be proportional to its measure.

Now we are able to begin the inductive construction, with branches $T_{\lambda\tau}$ instead of f_1 and $\bigcup_{m \lneqq ck} \hat{\delta}_1^{-m}(\lambda)$ instead of $\hat{\delta}_1(\lambda)$. In particular, the intervals δ_n have the form $\delta_n \approx 2^{-sk(n-1)}\delta_1$. The estimates (13.2)–(13.10) allow the induction to continue, and if we denote by $\mathscr{M}_p$ the set of $\lambda \in [\lambda_p, \lambda_{p+1}]$ obtained by using an inductive construction similar to that in Sect. 3, we obtain that the induced map $T_\lambda : I_\lambda \to I_\lambda$ has a measure $\tilde{\mu}_\lambda$ absolutely continuous with respect to dx. Besides, for some constants c, $\alpha > 0$ independent of k and p we have

$$\frac{\text{mes}\, \mathscr{M}_p}{|\lambda_p - \lambda_{p+1}|} > 1 - c\frac{k}{(2^k)^\alpha}. \tag{13.12}$$

The measure $\tilde{\mu}_\lambda$ induces an f_λ-invariant measure on $[0,1]$ supported on $[f_\lambda^2(\frac{1}{2})$, $f_\lambda(\frac{1}{2})]$. Since the time of return to I_λ is finite for all $x \in I_\lambda$, μ_λ is certainly finite.

Let $\Lambda_1 = \bigcup_{p=p_0}^\infty \mathscr{M}_p$. We take $k \to \infty$ together with p, and obtain from (13.12) that $\lambda = 4$ is a Lebesgue point (from one side) of Λ_1. This proves Theorem B and the Remark of the introduction for $f_\lambda(x) = \lambda x(1-x)$ $0 < \lambda \leqq 4$.

Remark XIII/4. The measures μ_λ certainly are ergodic, because the v_λ are. It follows from the recent results by Ledrappier [16] that the natural extensions of (f_λ, μ_λ) are Bernoulli.

Remark XIII/5. One may conjecture that the densities μ_λ converge in L_1 to $\varrho_4(x)$ $= (\pi\sqrt{x(1-x)})^{-1}$, when $\lambda \to 4$. Notice that the construction always gives measures supported on the maximal possible interval $[f_\lambda^2(\frac{1}{2}), f_\lambda(\frac{1}{2})]$ and thus avoids λ corresponding to measures supported by pairwise disjoint intervals permuted by f_λ.

   c) Consider any $f(x):[0,1] \to [0,1]$, $f(0) = f(1) = 0$, $f'(c) = 0$, lying in a sufficiently small C^3-neighbourhood of $x(1-x)$. Then for a family $\lambda \cdot f(x)$ there exists some λ_0 close to 4 so that $\lambda_0 f(c) = 1$. Considering for $\lambda \in [\lambda_0 - \varepsilon, \lambda_0]$ the corresponding induced map $T_{f\lambda} : I_{f\lambda} \supset$, we obtain that $T_{f\lambda}$ has on $I_{f\lambda}$ a structure similar to the one described above for $T_\lambda = T_{x(1-x)\lambda}$ and (13.7) still holds for $T_{f\lambda}$. This implies Theorem B for $f_\lambda = \lambda \cdot f(x)$.

   Now, if for some $\lambda_0 \neq 4$, $f_{\lambda_0} = \lambda_0 x(1-x)$ or its iteration on some interval admits the induced map described above, the construction still goes and we obtain absolutely continuous measures invariant under f or under some iteration of f for a set of $\lambda \in [\lambda_0 - \varepsilon, \lambda_0]$ of positive measure.

   One can check this is so for a countable set $\{\lambda_{0n} : f\lambda_{0n}(\frac{1}{2})$ falls into a periodic unstable orbit$\}$ and for a set $\Phi = \{\lambda : f_\lambda(\frac{1}{2}) \in K_\lambda = $ an invariant unstable Cantor set$\}$, card $\Phi = $ continuum (see [5–7]), thus all these λ are Lebesgue density points of $\mathscr{M}_1$.

Remark XIII/5. As Misiurewicz pointed out, for a family $f_\lambda = \lambda f(x)$ with unimodal $f(x):[0,1] \to [0,1]$, $f(0) = f(1) = 0$, having negative Schwarzian derivative, and for λ_0 such that $f_{\lambda_0}(c)$ falls into an unstable periodic orbit or an invariant unstable Cantor set, the corresponding incuded map also satisfies (13.7). Thus the same

construction implies that for a set of λ of positive measure f_λ admits an absolutely continuous invariant measure and λ_0 is a Lebesgue density point of this set.

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