Absolutely singular dynamical foliations

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Introduction

Let A_2 be the automorphism of the 2-torus, $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$, given by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.
Let A_3 be the automorphism of the 3-torus $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$ given by $\begin{pmatrix} A_2 & 0 \\ 0 & 1 \end{pmatrix}$.
Let $\mathrm{Diff}^2_\mu(\mathbf{T}^3)$ be the set of C^2 diffeomorphisms of $\mathbf{T}^3$ that preserve Lebesgue-Haar measure μ.

In [SW1], M. Shub and A. Wilkinson prove the following theorem.

Theorem: *Arbitrarily close to A_3 there is a C^1-open set $U \subset \mathit{Diff}^2_\mu(\mathbf{T}^3)$ such that for each $g \in U$,*

1. *g is ergodic.*

2. *There is an equivariant fibration $\pi : \mathbf{T}^3 \to \mathbf{T}^2$ such that $\pi g = A_2\pi$ The fibers of π are the leaves of a foliation $\mathcal{W}^c_g$ of $\mathbf{T}^3$ by C^2 circles. In particular, the set of periodic leaves is dense in $\mathbf{T}^3$.*

3. *There exists $\lambda^c > 0$ such that, for μ-almost every $w \in \mathbf{T}^3$, if $v \in T_w\mathbf{T}^3$ is tangent to the leaf of $\mathcal{W}^c_g$ containing w, then*

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_w g^n v\| = \lambda^c.$$

4. *Consequently, there exists a set $S \subseteq \mathbf{T}^3$ of full μ-measure that meets every leaf of $\mathcal{W}^c_g$ in a set of leaf-measure 0. The foliation $\mathcal{W}^c_g$ is not absolutely continuous.*

Additionally, it is shown that the diffeomorphisms in U are nonuniformly hyperbolic and Bernoullian. In this note, we prove:

Theorem I: *Let g satisfy conclusions 1.–3. of the previous theorem. Then there exist $S \subseteq \mathbf{T}^3$ of full μ-measure and $k \in \mathbf{N}$ such that S meets every leaf of $\mathcal{W}_g^c$ in exactly k points. The foliation $\mathcal{W}_g^c$ is* absolutely singular.

Remark: Theorem I was also proved several years ago by Anatole Katok, as a first step in an attempt to show that examples such as those later constructed in [SW1] cannot exist (since the full argument turned out not to be valid, this work remains unpublished). We are indebted to Katok for useful conversations, and for pointing out the argument that shows that the number k in Theorem I might necessarily be greater than 1. We also thank Michael Shub for useful conversations.

In Katok's example of an absolutely singular foliation in [Mi], the leaves of the foliation meet the set of full measure in one point. In the [SW1] examples, the set S may necessarily meet leaves of $\mathcal{W}_g^c$ in more than one point, as the following argument of Katok's shows.

It follows from Theorem II in [SW2] that for $k \in \mathbf{Z}_+$ and for small $a, b > 0$, the map $g = j_{a,k} \circ h_b$ satisfies the hypotheses of Theorem I, where

$$h_b(x, y, z) = (2x + y, x + y, x + y + z + b \sin 2\pi y), \qquad \text{and}$$

$$j_{a,k}(x, y, z) = (x, y, z) + a \cos(2\pi k z) \cdot (1 + \sqrt{5}, 2, 0).$$

For $k \in \mathbf{N}$, let ρ_k be the vertical translation that sends (x, y, z) to $(x, y, z + \frac{1}{k})$. Note that $h_b \circ \rho_k = \rho_k \circ h_b$ and $j_{a,k} \circ \rho_k = \rho_k \circ j_{a,k}$. Thus $g \circ \rho_k = \rho_k \circ g$.

The fibration $\pi : \mathbf{T}^3 \to \mathbf{T}^2$ was obtained in [SW1] by using the persistence of normally hyperbolic submanifolds under perturbations. In the present case the symmetries ρ_k preserve the fibers of the trivial fibration $P : \mathbf{T}^3 \to \mathbf{T}^2$ from which one starts, and also the maps g. Therefore the fibers of $\pi : \mathbf{T}^3 \to \mathbf{T}^2$ (i.e., the leaves of center foliation $\mathcal{W}_g^c$) are invariant under the action of the finite group $< \rho_k >$.

Let S be the (full measure) set of points in $\mathbf{T}^3$ for which the center direction is a positive Lyapunov direction (i.e. for which conclusion 3 holds). Since $\rho_k(\mathcal{W}_g^c) = \mathcal{W}_g^c$, it follows that $\rho_k S = S$. If $p \in S \cap \mathcal{W}^c(p)$, then $\rho_k(p) \in \rho_k(S) \cap \rho_k(\mathcal{W}^c(p)) = S \cap \mathcal{W}^c(p)$; that is, $S \cap \mathcal{W}^c(p)$ contains at least k points.

Thus Theorem I is "sharp" in the sense that we cannot say more about the value of k in general. We see no reason why $k = 1$ should hold even for a residual set in U.

Theorem I has an interesting interpretation. Recall that a G-*extension* of a dynamical system $f : X \to X$ is a map $f_\varphi : X \times G \to X \times G$, where G is a compact group, of the form $(x, y) \mapsto (g(x), \varphi(x)y)$. If f preserves ν, and $\varphi : X \to G$ is measurable, then f_φ preserves the product of ν with Lebesgue-Haar measure on G. A $\mathbf{Z}/k\mathbf{Z}$-extension is also called a k-point extension.

Let λ be an invariant probability measure for a k-point extension of $f : X \to X$, and $\{\lambda_x\}$ the family of conditional measures associated with the partition $\{\{x\} \times G\}$. We remark that if λ is ergodic, then each atom of λ_x must have the same weight $1/k$ (up to a set of λ-measure 0).

Now take $g \in U$. Choose a coherent orientation on the leaves of $\{\pi^{-1}(x)\}_{x \in T^2}$. Take $h : \mathbf{T}^3 \to \mathbf{T}^2 \times \mathbf{T}$ to be any continuous change of coordinates such that h restricted to $\pi^{-1}(x)$ is smooth and orientation preserving to $\{x\} \times \mathbf{T}$. We may then write $F = h \circ g \circ h^{-1} : \mathbf{T}^2 \times \mathbf{T} \to \mathbf{T}^2 \times \mathbf{T}$ in the form

$$F(x, p) = (A_2 x, \varphi_x(p))$$

where $\varphi_x : \mathbf{T} \to \mathbf{T}$ is smooth and orientation preserving. If $P : \mathbf{T}^2 \times \mathbf{T} \to \mathbf{T}^2$ is the projection on the first factor of the product, we have $P \circ h = \pi$. Therefore, writing $\lambda = h^* \mu$, we have $P^* \lambda = \pi^* \mu$. Let $\{\lambda_x\}$ be the disintegration of the measure λ along the fibers $\{x\} \times \mathbf{T}$. By a further measurable change of coordinates, smooth along each $\{x\} \times \mathbf{T}$ fiber, we may assume that λ-almost everywhere, the atoms of λ_x are at l/k, for $l = 0, \ldots, k-1$. But then φ_x permutes the atoms cyclically, and we obtain the following corollary.

Corollary: *For every $g \in U$ there exists $k \in \mathbf{N}$ such that $(\mathbf{T}^3, \mu, g)$ is isomorphic to an (ergodic) k-point extension of $(\mathbf{T}^2, \pi^* \mu, A_2)$.*

M. Shub has observed that if $g = j_{a,k} \circ h_b$, then $\pi^* \mu$ is actually Lebesgue measure on $\mathbf{T}^2$.

1 Proof of Theorem I

The proof of Theorem I follows from a more general result about fibered diffeomorphisms. Before stating this result, we describe the underlying setup and assumptions.

Let (X, ν) be a probability space, and let $f : X \to X$ be invertible and ergodic with respect to ν. Let M be a closed Riemannian manifold and $\varphi : X \to \mathrm{Diff}^{1+\alpha}(M)$ a measurable map. Consider the skew-product transformation $F : X \times M \to X \times M$ given by

$$F(x, p) = (f(x), \varphi_x(p)).$$

Assume further that there is an F-invariant ergodic probability measure μ on $X \times M$ such that $\pi_* \mu = \nu$, where $\pi : X \times M \to X$ is the projection onto the first factor.

For $x \in X$, let $\varphi_x^{(0)}$ be the identity map on M and for $k \in \mathbf{Z}$, define $\varphi_x^{(k)}$ by

$$\varphi_x^{(k+1)} = \varphi_{f^k(x)} \circ \varphi_x^{(k)}.$$

Since the tangent bundle to M is measurably trivial, the derivative map of φ along the M direction gives a cocycle $D\varphi : X \times M \times \mathbf{Z} \to GL(n, \mathbf{R})$, where $n = \dim(M)$:

$$(x, p, k) \mapsto D_p \varphi_x^{(k)}.$$

Assume that $\log^+ \|D\varphi\|_\alpha \in L^1(X \times M, \mu)$, where $\| \cdot \|_\alpha$ is the α-Hölder norm. Let $\lambda_1 < \lambda_2 \cdots < \lambda_l$ be the Lyapunov exponents of this cocycle; they exist for μ-a.e. (x, p) by Oseledec's Theorem and are constant by ergodicity. We call these the *fiberwise exponents* of F. Under the assumptions just described, we have the following result.

Theorem II: *Suppose that $\lambda_l < 0$. Then there exists a set $S \subseteq X \times M$ and an integer $k \geq 1$ such that*

- $\mu(S) = 1$
- *For every $(x, p) \in S$, we have $\#(S \cap \{x\} \times M) = k$.*

This has the immediate corollary:

Corollary: *Let $f \in \mathit{Diff}^{1+\alpha}(M)$. If μ is an ergodic measure with all of its exponents negative, then it is concentrated on the orbit of a periodic sink.*

The corollary has a simple proof using regular neighborhoods. Our proof is a fibered version. Theorem I is also a corollary of Theorem II. For this, the argument is actually applied to the inverse of g, which has negative fiberwise exponents, rather than to g itself, whose fiberwise exponents are positive. As we described in the previous remarks, there is a measurable change of coordinates, smooth along the leaves of $\mathcal{W}^c_g$ in which g^{-1} is expressed as a skew product of $\mathbf{T}^2 \times \mathbf{T}$.

Remark: Without the assumption that f is invertible, Theorem II is false. An example is described by Y. Kifer [Ki], which we recall here. Let $f : \mathbf{T} \to \mathbf{T}$ be a $C^{1+\alpha}$ diffeomorphism with exactly two fixed points, one attracting and one repelling. Consider the following random diffeomophism of $\mathbf{T}$: with probability $p \in (0,1)$, apply f, and with probability $1-p$, rotate by an angle chosen randomly from the interval $[-\epsilon, \epsilon]$.

Let $X = (\{0,1\} \times \mathbf{T})^{\mathbf{N}}$. To generate a sequence of diffeomorphisms $f_0, f_1, \ldots$ according to the above rule, we first define $\varphi : X \to \mathrm{Diff}^{1+\alpha}(\mathbf{T})$ by

$$\varphi(\omega) = \begin{cases} f & \text{if } \omega(0) = (0,\theta), \\ R_\theta & \text{if } \omega(0) = (1,\theta), \end{cases}$$

where R_θ is rotation through angle θ. Next, we let ν_ϵ be the product of $p, 1-p$-measure on $\{0,1\}$ with the measure on $\mathbf{T}$ that is uniformly distributed on $[-\epsilon, \epsilon]$. Then corresponding to $\nu_\epsilon^{\mathbf{N}}$-almost every element $\omega \in X$ is the sequence $\{f_k = \varphi(\sigma^k(\omega))\}_{k=0}^{\infty}$, where $\sigma : X \to X$ is the one-sided shift $\sigma(\omega)(n) = \omega(n+1)$.

Put another way, the random diffeomorphism is generated by the (noninvertible) skew product $\tau : X \times \mathbf{T} \to X \times \mathbf{T}$, where $\tau(\omega, x) = (\sigma(\omega), \varphi(\omega)(x))$. An ergodic ν_ϵ-stationary measure for this random diffeomorphism is a measure μ_ϵ on $\mathbf{T}$ such that $\mu_\epsilon \times \nu_\epsilon^{\mathbf{N}}$ is τ-invariant and ergodic. Such measures always exist ([Ki], Lemma I.2.2), but, for this example, there is an ergodic stationary measure with additional special properties.

Specifically, for every $\epsilon > 0$, there exists an ergodic ν_ϵ-stationary measure μ_ϵ on $\mathbf{T}$ such that, as $\epsilon \to 0$, $\mu_\epsilon \to \delta_{x_0}$, in the weak topology, where δ_{x_0} is Dirac measure concentrated on the sink x_0 for f. From this, it follows that, as $\epsilon \to 0$, the fiberwise Lyapunov exponent for μ_ϵ approaches $\log |f'(x_0)| < 0$, which is the Lyapunov exponent of δ_{x_0}. Thus, for ϵ sufficiently small, the fiberwise exponent for τ with respect to μ_ϵ is negative. Nonetheless, it is easy to see that μ_ϵ for $\epsilon > 0$ cannot be uniformly distributed on k atoms; if μ_ϵ were atomic, then τ-invariance of $\mu_\epsilon \times \nu_\epsilon^{\mathbf{N}}$ would imply that, for every $x \in \mathbf{T}$,

$$\begin{aligned}\mu_\epsilon(\{x\}) &= p\mu_\epsilon(\{f^{-1}(x)\}) + (1-p)\int_{-\epsilon}^{\epsilon} \mu_\epsilon(\{R_\theta(x)\})d\theta \\ &= p\mu_\epsilon(\{f^{-1}(x)\}),\end{aligned}$$

which is impossible if μ_ϵ has finitely many atoms. In fact, μ_ϵ can be shown to be absolutely continuous with respect to Lebesgue measure (see [Ki], p. 173ff and the references cited therein). Hence invertibility is essential, and we indicate in the proof of Theorem II where it is used.

Proof of Theorem II: We first establish the existence of fiberwise "stable manifolds" for the skew product F. A general theory of stable manifolds for random dynamical systems is worked out in ([Ki], Theorem V.1.6; see also [BL]); since we are assuming that all of the fiberwise exponents for F are negative, we are faced with the simpler task of constructing fiberwise regular neighborhoods for F (see the Appendix by Katok and Mendoza in [KH]). We outline a proof, following closely [KH].

Theorem 1.1 *(Existence of Regular Neighborhoods) There exists a set $\Lambda_0 \subseteq X \times M$ of full measure such that for $\epsilon > 0$:*

- *There exists a measurable function $r : \Lambda_0 \to (0,1]$ and a collection of embeddings $\Psi_{(x,p)} : B(0, q(x,p)) \to M$ such that $\Psi_{(x,p)}(0) = p$ and $exp(-\epsilon) < r(F(x,p))/r(x,p) < exp(\epsilon)$.*
- *If $\varphi_{(x,p)} = \Psi^{-1}_{F(x,p)} \circ \varphi_x \circ \Psi_{(x,p)} : B(0, r(x,p)) \to \mathbf{R}^n$, then $D_0\varphi_{(x,p)}$ satisfies*

$$exp(\lambda_1 - \epsilon) \leq \|D_0\varphi^{-1}_{(x,p)}\|^{-1}, \|D_0\varphi_{(x,p)}\| \leq exp(\lambda_l + \epsilon).$$

- *The C^1 distance $d_{C^1}(\varphi_{(x,p)}, D_0\varphi_{(x,p)}) < \epsilon$ in $B(0, r(x,p))$.*
- *There exist a constant $K > 0$ and a measurable function $A : \Lambda_0 \to \mathbf{R}$ such that for $y, z \in B(0, r(x,p))$,*

$$K^{-1}d(\Psi_{(x,p)}(y), \Psi_{(x,p)}(z)) \leq \|y - z\| \leq A(x)d(\Psi_{(x,p)}(y), \Psi_{(x,p)}(z)),$$

with $exp(-\epsilon) < A(F(x,p))/A(x,p) < exp(\epsilon)$.

Proof: See the proof of Theorem S.3.1 in [KH]. □

Decompose μ into a system of fiberwise measures $d\mu(x,p) = d\mu_x(p)d\nu(x)$. Invariance of μ with respect to F implies that, for ν-a.e. $x \in X$,

$$\varphi_{x*}\mu_x = \mu_{f(x)}.$$

Corollary 1.2 *There exists a set $\Lambda \subseteq X \times M$, and real numbers $R > 0$, $C > 0$, and $c < 1$ such that*

(1) $\mu(\Lambda) > .5$, and, if $(x,p) \in \Lambda$, then $\mu_x(\Lambda_x) > .5$, where $\Lambda_x = \{p \in M \,|\, (x,p) \in \Lambda\}$,

(2) If $(x,p) \in \Lambda$ and $d_M(p,q) \le R$, then

$$d_M(\varphi_x^{(m)}(p), \varphi_x^{(m)}(q)) \le Cc^m d_M(p,q),$$

for all $m \ge 0$.

Proof: This follows in a standard way from the Mean Value Theorem and Lusin's Theorem.□

To prove Theorem II, it suffices to show that there is a positive ν-measure set $B \subseteq X$, such that for $x \in B$, the measure μ_x has an atom, as the following argument shows. For $x \in X$, let $d(x) = \sup_{p \in M} \mu_x(p)$. Clearly d is measurable, f-invariant, and positive on B. Ergodicity of f implies that $d(x) = d > 0$ is positive and constant for almost all $x \in X$. Let $S = \{(x,p) \in X \times M \,|\, \mu_x(p) \ge d\}$. Observe that S is F-invariant, has measure at least d, and hence has measure 1. The conclusions of Theorem II follow immediately.

Let Λ, $R > 0$, $C > 0$, and $c < 1$ be given by Corollary 1.2, and let $B = \pi(\Lambda)$. Let N be the number of $R/10$-balls needed to cover M. We now show that for ν-almost every $x \in B$, the measure μ_x has at least one atom.

For $x \in X$, let

$$m(x) = \inf \sum \text{diam } (U_j),$$

where the infimum is taken over all collections of closed balls $U_1, \ldots, U_k$ in M such that $k \leq N$ and $\mu_x(\bigcup_{j=1}^k U_j) \geq .5$. Let $m = \text{ess sup}_{x \in B} m(x)$.

We now show that $m = 0$. If $m > 0$, then there exists an integer J such that

$$C\Delta c^J N \;<\; m/2, \tag{1}$$

where Δ is the diameter of M. Let $\mathcal{U}$ be a cover of M by N closed balls of radius $R/10$. For $x \in B$, let $U_1(x), \ldots, U_{k(x)}(x)$ be those balls in $\mathcal{U}$ that meet Λ_x. Since these balls cover Λ_x, and $\mu_x(\Lambda_x) > .5$, it follows that $\mu_x(\bigcup_{j=1}^{k(x)} U_j(x)) \geq .5$. But ${\varphi_x^{(i)}}_* \mu_x = \mu_{f^i(x)}$, and so it's also true that

$$\mu_{f^i(x)}(\bigcup_{j=1}^{k(x)} \varphi_x^{(i)}(U_j(x))) \;\geq\; .5, \tag{2}$$

for all i.

We now use the fact that $\varphi_x^{(i)}$ contracts regular neighborhoods to derive a contradiction. The balls $U_j(x)$ meet Λ_x and have diameter less than $R/10$, and so by Corollary 1.2, (2), we have

$$\text{diam}\;(\varphi_x^{(i)}(U_j(x))) \;\leq\; C\Delta c^i. \tag{3}$$

Let $\tau : B \to \mathbf{N}$ be the first-return time of f^J to B, so that $f^{J\tau(x)}(x) \in B$, and $f^{Ji}(x) \notin B$, for $i \in \{1, \ldots, \tau(x) - 1\}$. Decompose the set B according to these first return times:

$$B = \bigcup_{i=1}^{\infty} B_i \quad (\text{mod } 0),$$

where $B_i = \tau^{-1}(i)$. *Because f is invertible and f^{-1} preserves measure*, we also have the mod 0 equivalence:

$$B' := \bigcup_{i=1}^{\infty} f^{Ji}(B_i) = B \quad (\text{mod } 0).$$

Let $y \in B'$. Then $y = f^{Ji}(x)$, where $x \in B_i \subseteq B$, for some $i \geq 1$. It follows from the definition of $m(y)$ and inequalities (2), (3) and (1) that

$$m(y) \;\leq\; \sum_{j=1}^{k(x)} \text{diam}\;(\varphi_x^{(Ji)}(U_j(x)))$$

$$
\begin{aligned}
&\leq Ck(x)\Delta c^{Ji} \\
&\leq CN\Delta c^{J} \\
&< m/2.
\end{aligned}
$$

But then

$$
\begin{aligned}
m &= \text{ess sup }_{x\in B} m(x) \\
&= \text{ess sup }_{y\in B'} m(y) \\
&< m/2,
\end{aligned}
$$

contradicting the assumption $m > 0$.

Thus $m = 0$, and, for ν-almost every $x \in B$, we have $m(x) = 0$. If $m(x) = 0$, then there is a sequence of closed balls $U^1(x), U^2(x), \cdots$ with $\lim_{i\to\infty} \text{diam } (U^i(x)) = 0$ and $\mu_x(U^i(x)) \geq .5/N$, for all i. Take $p_i \in U^i(x)$; any accumulation point of $\{p_i\}$ is an atom for μ_x. Since we have shown that μ_x has an atom, for ν-a.e. $x \in B$, the proof of Theorem II is complete.$\square$

References

[BL] J. Bahnmüller and P.-D. Liu. "Characterization of measures satisfying Pesin's entropy formula for random dynamical systems." J. Dynam. Diff. Equ. **10** (1998), no. 3, 425-448.

[KH] Katok, A. and B. Hasselblatt, "Introduction to the modern theory of dynamical systems." Cambridge, 1995.

[Ki] Kifer, Y. "Ergodic theory of random transformations," Birkhäuser Boston, 1986.

[Mi] Milnor, J. *Fubini foiled: Katok's paradoxical example in measure theory.* Math. Intelligencer, **19** (1997), no. 2, 30-32.

[SW1] Shub, M. and Wilkinson, A., *Pathological foliations and removable zero exponents*, Inv. Math., **139** (2000) 495–508.

[SW2] Shub, M. and Wilkinson, A., *A stably Bernoullian diffeomorphism that is not Anosov*, preprint.