ABSOLUTELY SUMMING OPERATORS AND LOCAL UNCONDITIONAL STRUCTURES

BY

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1. Introduction

In his remarkable paper [8] Grothendieck defined a one absolutely summing operator between two Banach spaces, to be an operator which maps every unconditionally convergent series to an absolutely convergent series (see definition below). It is well known that a one absolutely summing operator factors through an $L_\infty(\mu)$-space and for every $p\,(1 \leqslant p < \infty)$ also through a certain subspace of $L_p(\mu)$. It was asked in [8] problem 2, p. 72 whether every one absolutely summing operator can be factored through an $L_1(\mu)$-space, and other equivalent formulations of the problem were presented. We establish here the negative answer to this question and related results as well.

The literature on one absolutely summing maps, and more generally p-absolutely summing maps introduced by Pietsch [22], is very extensive and varied. Some results of Grothendieck are by now classical, such as the facts that every operator from an $L_1(\mu)$-space to a Hilbert space is one absolutely summing, and every operator from $L_\infty(\mu)$ to $L_1(\mu)$ is 2-absolutely summing [8], [18]. However, we shall generally make use here only of the definitions and basic results on these spaces. The class of p-absolutely summing operators forms only a single example in the classes of Banach ideals of operators. Equally important, and related by duality, are the Banach ideals of p-integral operators, and L_p-factorizable operators which we mention later in this section.

Our approach to the problem mentioned is to consider various inclusion maps I_n: $E_n \to F_n$ $(n = 1, 2, \ldots)$ between certain sequences of finite-dimensional Banach spaces and carefully evaluate the ratios $\gamma_1(I_n)/\pi_1(I_n)$ between their L_1-factorizable norms and

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one absolutely summing norms, and to show that for the suitable examples chosen in section 2, the ratios increase to infinity with the dimensions of the spaces involved. This, among other things, provides the counter example in section 4 which states that the inclusion map, whose domain is the Banach space of operators from l_∞ to l_1, and whose range is the space of Hilbert-Schmidt operators on l_2, is one absolutely summing and cannot be factored through any L_1-space.

The unbounded sequence of norm ratios has bearing on another problem considered in section 3. It is shown that $\gamma_1(I_n)/\pi_1(I_n)$ is less than or equal to the unconditional basis constant $\mathfrak{X}(E_n)$ of the domain space E_n, and thus we obtain the first example of a sequence E_n $(n=1, 2, \ldots)$ of finite-dimensional Banach spaces whose unconditional basis constants tend to infinity. This answers the well known question which may be found in [6], [19], [11] or [12], and provides a method for computing the unconditional basis constant of a given finite-dimensional space. In fact a stronger implication is that the local unconditional constants introduced in section 3, $\mathfrak{X}_u(E_n)$, tend to infinity. The local unconditional constant of a given Banach space E, in one formulation, measures how well the identity operator of every finite-dimensional subspace of E may be represented as some unconditionally convergent sum (in the norm of operators) of rank one operators whose ranges lie in the entire space E.

The infinite-dimensional version of these results says that many of the common spaces of linear operators considered in section 3, do not have local unconditional structure and are therefore not isomorphic to complemented subspaces of spaces with unconditional bases; moreover it implies also that these spaces cannot have sufficiently many Boolean algebras of projections, in the terminology of Lindenstrauss and Zippin [19], thus answering the question raised in their paper as to whether there exist such spaces.

We pass to some specific examples. The space $c_p(H)$ $(1 \leqslant p \leqslant \infty)$ is the Banach space of compact operators T defined on a Hilbert space H and equipped with the norm $c_p(T) = [\text{trace}\,(T^*T)^{p/2}]^{1/p}$ for $p<\infty$, and $c_\infty(T) = \|T\|$. A systematic study of the c_p spaces may be found in McCarthy [20] where, among other things, it is shown that for $1<p<\infty$ c_p is uniformly convex, and the classical result $c_p(H)' = c_q(H)$, $1/p+1/q=1$. Additional recent results on c_p-spaces are included in [15] and [25]. We prove in section 5 that for $p \neq 2$ and infinite-dimensional H, $c_p(H)$ does not have local unconditional structure, and therefore does not have an unconditional basis. This result answers Problem 2 [15] of Kwapien and Pelczynski, who have shown that $c_\infty(H)$, $c_1(H)$ and in general the spaces of all compact operators from l_p to l_q (for $p \geqslant q$) are not isomorphic to subspaces of spaces with unconditional bases. We do not know whether for $1<p<\infty, p \neq 2, c_p(H)$ is isomorphic to a subspace of a space with an unconditional basis. We show that for finite-dimensional spaces H and any fixed value $p(1 \leqslant p \leqslant \infty)$, both $\mathfrak{X}_u(c_p(H))$ and $\mathfrak{X}(c_p(H))$ are asymptotically equivalent to $(\dim H)^{|1/p-1/2|}$, and complement the above mentioned results of [15] by proving that if $p > 1$ and $q < \infty$, the space of compact operators from l_p to l_q does not have local unconditional structure, hence is not isomorphic to a complemented subspace of a space with an unconditional basis, but for $1 < p < q < \infty$ it is still unknown whether these spaces embed isomorphically in spaces with unconditional bases.

The results on c_p are closely related to a general result proved in section 5 which essentially says that the Banach ideals of operators on l_2, except those ideals which are "close" to being Hilbert spaces themselves, lack local unconditional structures. We conjecture this to be true for all Banach ideals of operators on l_2, which are not isomorphic to Hilbert spaces. The rest of the section is concerned with obtaining estimates on the projection constants of $c_p(H)$, for finite-dimensional H, and their distances from the subspace of L_1. The results confirm a conjecture of [20].

Let us now introduce some definitions. All Banach spaces E are over the same scalar field, either real or complex, with E' the dual space of E. In the proofs only real spaces are considered, as similar arguments are possible in the complex case. The space of all continuous linear operators from E into F is written $L(E, F)$.

By a *Banach ideal of operators* $[A, a]$, [23], we mean a method which associates with each pair (E, F) of Banach spaces an algebraic subspace $A(E, F)$ of $L(E, F)$ together with a norm α on $A(E, F)$ in such a way that the following requirements are fulfilled:

(a) $A(E, F)$ contains all the finite rank operators from E into F, and $\alpha(x' \otimes y) = \|x'\| \, \|y\|$ (here $x' \otimes y$ is the rank-one operator defined by $x' \otimes y(x) = \langle x, x' \rangle y$;

(b) if $u \in L(X, E)$, $v \in A(E, F)$ and $w \in L(F, Y)$, then $wvu \in A(X, Y)$ and $\alpha(wvu) \leqslant \|w\| \alpha(v) \|u\|$; and lastly

(c) $A(E, F)$ is complete under α.

Given a Banach ideal of operators $[A, \alpha]$ α is referred to as a *Banach ideal norm*, and $\alpha(u)$ is the *α-norm* of u. It is convenient to consider α as defined for all elements of $L(E, F)$ and we write $\alpha(u) < \infty$ iff $u \in A(E, F)$. The α-norm of the identity operator on E is written $\alpha(E)$. For u a finite rank operator on E with representation $u = \sum_{i \leqslant n} x_i' \otimes x_i$, the trace of u is $\operatorname{tr}(u) = \sum_{i \leqslant n} \langle x_i, x_i' \rangle$.

The following ideals are used throughout this paper.

For $1 \leqslant p \leqslant \infty$ the ideal $[I_p, i_p]$ of *p-integral operators* [21] is defined as follows: $u \in I_p(E, F)$ iff there is a probability measure μ and operators $v \in L(E, L_\infty(\mu))$, $w \in L(L_p(\mu), F'')$ such that $iu = w\varphi v$ where i is the natural embedding of F into F'' and φ is the inclusion of $L_\infty(\mu)$ into $L_p(\mu)$. The *p-integral norm* of u is $i_p(u)=\inf \|v\|\,\|w\|$, where the infimum is taken over all possible factorizations.

For $1\leqslant p<\infty$ the ideal $[\Pi_p, \pi_p]$ of *p-absolutely summing* operators [22] is defined as follows: $u\in\Pi_p(E, F)$ iff there is a constant $\lambda>0$ with

$$(\sum_{i\leqslant n}\|u(x_i)\|^p)^{1/p}\leqslant \lambda \sup_{\|x'\|\leqslant 1}(\sum_{i\leqslant n}|\langle x_i, x'\rangle|^p)^{1/p}$$

for all finite sets $(x_i)_{i\leqslant n}\subset E$. The *p-absolutely summing norm* $\pi_p(u)$ is the smallest such constant λ.

The ideal $[\Gamma_p, \gamma_p]$ of *L_p-factorizable operators* [7], [16]: $u\in\Gamma_p(E, F)$ iff there is a measure μ and operators $v\in L(E, L_p(\mu))$, $w\in L(L_p(\mu), F'')$ such that $iu=wv$, where i is again the canonical embedding of F into F''. The *γ_p-norm* of u is $\gamma_p(u)=\inf\|v\|\,\|w\|$, with the infimum taken over all possible factorizations.

The *adjoint ideal*, $[A^*, \alpha^*]$, of $[A, \alpha]$ is defined in the following manner [7], [23]: $u\in A^*(E, F)$ if and only if there is a constant $\lambda>0$ such that for any finite-dimensional spaces X and Y, and any $v\in L(X, E)$, $w\in L(F, Y)$ and $t\in A(Y, X)$, $|\operatorname{tr}(twuv)|\leqslant\lambda\|v\|\,\|w\|\,\alpha(t)$. The α^*-norm of u is the smallest such constant λ. We shall frequently use the elementary fact that if E or F has the metric approximation property and $u\in A^*(E, F)$, then $\alpha^*(u)$ is equal to the smallest constant C for which $|\operatorname{trace}(Lu)|\leqslant C\alpha(L)$ whenever $L\in L(F, E)$ has finite rank [7], [23].

It is immediate that $[A^*, \alpha^*]$ is also a Banach ideal of operators and it is known that $i_1^*=\|\;\|$ and $\pi_p^*=i_{p'}$, where $1/p+1/p'=1$ with the usual convention about $p=1$ and $p=\infty$ ([23]). The ideal $[A, \alpha]$ is called *perfect* if $\alpha^{**}=\alpha$. The ideals π_p, i_p and $\gamma_p(1\leqslant p\leqslant\infty)$ are all perfect (cf. [7]). In addition to these general ideals of operators we consider the classes $c_p(H_1, H_2)$ of operators between Hilbert spaces H_1 and H_2 (cf. [15], [20]). Given $1\leqslant p<\infty$, $u\in c_p(H_1, H_2)$ if and only if u is compact and $(u^*u)^{p/2}\in I_1(H_1, H_2)$ in which case $c_p(u)=[\operatorname{tr}(u^*u)^{p/2}]^{1/p}$. $c_\infty(H_1, H_2)$ will denote the space of all compact operators with the usual operator norm. Use will be made of the well known fact that $c_2(H_1, H_2)=\Pi_2(H_1, H_2)$ with equality of norms.

It will be convenient to adopt the notation of tensor products. An elementary tensor $u\in E\otimes F$ will be regarded, when convenient, as an operator from E' to F. The *least $\otimes$-norm* of u is defined by

$$|u|_\vee=\sup\{|\langle u, x'\otimes y'\rangle|;\, x'\in E',\, y'\in F',\, \|x'\|=\|y'\|=1\}$$

and is eqnal to $\|u\|$ where u is regarded as an element of $L(E', F)$.

The *greatest $\otimes$-norm* of u is defined by

$$|u|_{\wedge}=\inf\left\{\sum_{i=1}^{n}\|e_i\|\,\|f_i\|;\ u=\sum_{i=1}^{n}e_i\otimes f_i\right\}$$

and is equal to $\sup\{\langle u, v\rangle;\ v\in L(F, E'),\ \|v\|\leqslant 1\}$ where the action marks $\langle .,.\rangle$ represent the trace of the composition. The completion of $E\otimes F$ under $\alpha=\wedge$ or $\vee$ is written $E\overset{\alpha}{\otimes}F$. In particular, $L(l_{p'}^n, l_q^n)=l_p^n\overset{\vee}{\otimes}l_q^n$, $I_1(l_{p'}^n, l_q^n)=l_p^n\overset{\wedge}{\otimes}l_q^n$. For $u\in L(E, G)$, $v\in L(F, H)$ and $\alpha=\vee$ or $\wedge$, there is always the operator of norm $\leqslant\|u\|\,\|v\|$ from $E\overset{\alpha}{\otimes}F$ into $G\overset{\alpha}{\otimes}H$, denoted by $u\otimes v$, which maps $x\otimes y$ to $u(x)\otimes v(y)$.

2. The basic inequalities

The first lemma is an immediate consequence of the definition of the adjoint ideal, and was used in [6], [7].

LEMMA 2.1. *For E and F finite-dimensional spaces and α a Banach ideal norm, $A(E, F)'=A^*(F, E)$ naturally and isometrically, where $\langle u, v\rangle=\operatorname{trace}(uv)$, $u\in A(E, F)$, $v\in A^*(F, E)$.*

Given a locally compact space M and a positive measure μ, it was shown in [8] Théorème 3, p. 21, and [10] Théorème 2, p. 59 that the natural map of $E\otimes L_1(\mu)$ to $L_1(\mu, E)$ (=the space of μ-integrable vector valued functions) given by $e\otimes f\to f(\cdot)e$ extends to an isometry of $E\overset{\wedge}{\otimes}L_1(\mu)$ onto $L_1(\mu, E)$. It follows ([8] Corollaire 2 p. 20, or [10] Proposition 9 p. 64) that $u\in L(E, L_1(\mu))$ is 1-integral if and only if the image of the unit sphere of E by u is lattice bounded, and that $i_1(u)=\int\sup_{\|x\|\leqslant 1}|u(x)(t)|\,\mu(dt)$. This fact will be used in the following theorem.

THEOREM 2.2. (a) *The inclusion map of $l_1^n\overset{\vee}{\otimes}l_1^n$ into $c_2(l_2^n, l_2^n)$ has π_1-norm at most* 3.

(b) *The inclusion map of $l_2^n\overset{\wedge}{\otimes}l_2^n$ into $c_2(l_2^n, l_2^n)$ has π_1-norm at most $3\sqrt{n}$.*

Proof. (a) Let M be the subset of $(l_1^n\overset{\vee}{\otimes}l_1^n)'=l_\infty^n\overset{\wedge}{\otimes}l_\infty^n$ defined by

$$M=\{\varepsilon\otimes\delta;\ \varepsilon, \delta=(\pm 1, \pm 1, \ldots, \pm 1)\},$$

and let μ be the probability measure in $C(M)'$ given by

$$\mu(f)=2^{-2n}\sum_{\varepsilon,\delta}f(\varepsilon\otimes\delta),\ f\in C(M).$$

For $u\in l_1^n\overset{\vee}{\otimes}l_1^n$ we have by the well known Khinchin's inequality that

$$\mu(|\langle u,\cdot\rangle|)=2^{-n}\sum_{\varepsilon}2^{-n}\sum|\langle u(\varepsilon),\delta\rangle|\geqslant 3^{-\frac{1}{2}}2^{-n}\sum_{\varepsilon}\|u(\varepsilon)\|_2.$$

Now let $K=\{\varepsilon;\ \varepsilon=(\pm 1, \pm 1, \ldots, \pm 1)\}$, and consider the probability measure ν on

$C(K)$ given by $\nu(f)=2^{-n}\sum_\varepsilon f(\varepsilon)$, $f\in C(K)$, and the operator w from l_2^n to $L_1(\nu)$ given by $w(x)(\varepsilon)=\langle x,\varepsilon\rangle$. It again follows from Khinchin's inequality that w is an isomorphic embedding with $\|w^{-1}\|\leqslant 3^{\frac12}$. Now regard u as an operator on l_2^n, then by the remark above

$$2^{-n}\sum_\varepsilon \|u(\varepsilon)\|_2 = i_1(wu^*)\geqslant \pi_1(wu^*)\geqslant 3^{-\frac12}\pi_1(u^*)\geqslant 3^{-\frac12}\pi_2(u^*)=3^{-\frac12}c_2(u),$$

so that $c_2(u)\leqslant 3\mu(|\langle u,\cdot\rangle|)$.

We note again that M is a subset of the unit sphere of $(l_1^n\overset{\backslash}{\otimes} l_1^n)'$, it then follows that for any finite subset $\{u_j\}_{j=1}^m\subset l_1^n\overset{\vee}{\otimes} l_1^n$

$$\sum_{j=1}^m c_2(u_j)\leqslant 3\sum_{j=1}^m \mu(|\langle u_j,\cdot\rangle|)\leqslant 3\max_{\varepsilon,\delta}\sum_{j=1}^m|\langle u_j,\varepsilon\otimes\delta\rangle|$$

$$=3\max_{\varepsilon,\delta,\pm}\langle\sum_{j=1}^m\pm u_j,\varepsilon\otimes\delta\rangle\leqslant 3\max_{\pm}\left|\sum_{j=1}^m\pm u_j\right|_\vee,$$

hence the inclusion map considered in (a) has π_1-norm $\leqslant 3$.

Proof of (b): Let G be the compact group of orthogonal transformations on l_2^n, and dg the unique normalized Haar measure on G. Consider G as a subset of the unit sphere of $(l_2^n\overset{\wedge}{\otimes} l_2^n)'=l_2^n\overset{\vee}{\otimes} l_2^n$, then concluding as in part (a) it will suffice to prove the inequality

$$c_2(u)\leqslant 3n^{\frac12}\int_G|\langle u,g\rangle|\,dg,\quad\text{for all } u\in l_2^n\overset{\wedge}{\otimes} l_2^n. \tag{1}$$

Any given u can be written as $u=\sum_{i=1}^n\lambda_i e_i\otimes b_i$, where (e_i) and (b_i) are orthonormal bases and (λ_i) is some sequence of non-negative reals. Choose $h\in G$ so that $h(b_i)=e_i$. Then $c_2(hu)=c_2(u)$, so by the invariance of dg it will suffice to prove (1) for a diagonal multiplication operator $u(e_i)=\lambda_i e_i$ with respect to some fixed orthonormal basis (e_i).

For $g\in G$ set $g_{ik}=\langle g(e_i),e_k\rangle$, let $S=\{x\in l_2^n;\ \|x\|_2=1\}$ be the unit sphere and dm be the $(n-1)$-dimensional, normalized, rotational-invariant measure on S. From [5] we have

$$\int_G g_{ii}^4\,dg=\int_S\langle x,e_i\rangle^4\,dm(x)=\frac{3}{n(n+2)},$$

and from the orthogonality of the function g_{ik}, also

$$\int_G\langle u,g\rangle^2\,dg=n^{-1}c_2(u)^2. \tag{2}$$

In addition we need the following inequalities for $1\leqslant i,k,s,t\leqslant n$,

$$\int_G g_{ii}g_{kk}g_{ss}g_{tt}\,dg\begin{cases}=\dfrac{3}{n(n+2)}; & \text{if } i=k=s=t\\[2mm] \leqslant\dfrac{3}{n(n+2)}; & \text{if } i=k\neq s=t\\[2mm] =0\quad; & \text{otherwise.}\end{cases} \tag{3}$$

The first equality is given above, and the second inequality follows from the first by the Cauchy-Schwarz inequality. The last equality follows by considering the various cases, for example, if $h \in G$ is such that $he_2 = e_2$ and $he_1 = -e_1$, then by the multiplication invariance of dg

$$\int_G g_{11}^3 g_{22}\, dg = \int_G \langle ge_1, e_1\rangle^3 \langle ge_2, e_2\rangle\, dg$$
$$= -\int_G \langle ge_1, he_1\rangle^3 \langle ge_2, he_2\rangle\, dg = -\int_G g_{11}^3\, g_{22}\, dg$$

so $\int_G g_{11}^3 g_{22} dg = 0$. Now by (3)

$$\int_G \langle u, g\rangle^4 dg = \sum_{i \leq n} \lambda_i^4 \int_G g_{ii}^4 dg + 6 \sum_{1 \leq i < k \leq n} \lambda_i^2 \lambda_k^2 \int_G g_{ii}^2 g_{kk}^2 dg$$
$$\leq \frac{3}{n(n+2)} (3\|\lambda\|_2^4 - 2\|\lambda\|_4^4) \leq 9n^{-2} c_2(u)^4,$$

and from (2) and Hölder's inequality

$$n^{-1} c_2(u)^2 = \int_G |\langle u, g\rangle|^{\frac{4}{3}} |\langle u, g\rangle|^{\frac{2}{3}} dg \leq \left(\int_G \langle u, g\rangle^4 dg\right)^{\frac{1}{3}} \left(\int_G |\langle u, g\rangle|\, dg\right)^{\frac{2}{3}}$$

so the desired inequality follows.

Remark. Professor H. P. Rosenthal drew our attention to the fact that another form of inequality (1) appears in [2] Lemma 1, and indeed seems to originate even farther back. We included its proof for the sake of completeness.

THEOREM 2.3. (a) *The inclusion J_n of $l_2^n \hat{\otimes} l_2^n$ into $c_2(l_2^n, l_2^n)$ satisfies the inequalities: $n^{\frac{1}{2}} \leq \pi_1(J_n) \leq 3n^{\frac{1}{2}}$, and $n/3 \leq \gamma_1(J_n) \leq n$.*

(b) *The inclution I_n of $l_2^n \check{\otimes} l_2^n$ into $c_2(l_2^n, l_2^n)$ satisfies the inequalities: $n \leq \pi_1(I_n) \leq 3n$, and $n^{\frac{3}{2}}/3 \leq \gamma_1(I_n) \leq n^{\frac{3}{2}}$.*

Proof. The estimate $\pi_1(J_n) \leq 3n^{\frac{1}{2}}$ is given in Theorem 2.2 (b). Fix $e \in l_2^n$, $\|e\|_2 = 1$, and set $Q(x) = e \otimes x$. Clearly

$$n^{\frac{1}{2}} \leq \pi_1(l_2^n) = \pi_1(Q) \leq \pi_1(J_n),$$

the first inequality is by [4].

To estimate $\pi_1(I_n)$ from above consider the factorization of I_n given by

$$l_2^n \check{\otimes} l_2^n \xrightarrow{A} l_1^n \check{\otimes} l_1^n \xrightarrow{B} c_2(l_2^n, l_2^n)$$

where A and B are the formal identities. Then by Theorem 2.2 (a) $\pi_1(I_n) \leqslant 3\|A\| \leqslant 3n$. For the lower estimate observe that since $\pi_2 \leqslant \pi_1$, and $\pi_2(E) = \sqrt{\dim E}$ for any space E [4], we have

$$n = \pi_2(l_2^{n^2}) \leqslant \pi_1(l_2^{n^2}) = \pi_1(I_n I_n^{-1}) \leqslant \pi_1(I_n).$$

From [4], or [14], it is known that the projection constant of a space is at most the square root of its dimension, so

$$\gamma_1(c_2(l_2^n, l_2^n)) = \gamma_\infty(l_2^{n^2}) \leqslant n$$

and thus

$$\gamma_1(J_n) \leqslant \|J_n\| \gamma_1(c_2(l_2^n, l_2^n)) \leqslant n.$$

For the lower bound on $\gamma_1(J_n)$ observe that $J_n' = I_n^{-1}$.

Since $\gamma_\infty = \pi_1^*$, Lemma 2.1 gives that

$$n^2 = \text{trace } (I_n I_n^{-1}) \leqslant \gamma_\infty(J_n')\pi_1(I_n) \leqslant 3n\gamma_1(J_n).$$

Finally,

$$\gamma_1(I_n) \leqslant \|I_n\| \gamma_1(c_2(l_2^n, l_2^n)) \leqslant n^{\frac{3}{2}},$$

last inequality as above. For the lower estimate,

$$\gamma_1(I_n) \geqslant n^2 \pi_1(J_n)^{-1} \geqslant n^{3/2}/3.$$

Remarks. We do not know the exact values of the norms estimated in Theorem 2.3, although the given values may be slightly improved. The somewhat better estimate $\pi_1(J_n) \leqslant (3n)^{\frac{1}{2}}$ may be obtained from the proof of Theorem 2.2 by using the equality

$$\int_G g_{ii}^2 g_{kk}^2 = \frac{n+1}{(n-1)\, n(n+2)}, \quad i \neq k,$$

in equation (3). Similarly the proof of Theorem 4.2 will show $\pi_1(I_n) \leqslant (\pi/2)n$. In addition, the constant $\sqrt{3}$ appearing in Khinchin's inequality can be replaced by $\sqrt{e}$ [25], though the exact value is unknown yet.

Given a finite-dimensional Banach space E and *a* compact topological group G, *a* (G, E)*-representation* is a continuous homomorphism $g \to a_g^E$ of G into the group of isometries of E. Say that $T \in L(E, F)$ is invariant under the (G, E) and (G, F)-representations if $Ta_g^E = a_g^F T$ for every $g \in G$. The following result was proved in [7]:

LEMMA 2.4. *Let E, F be n-dimensional and $T \in L(E, F)$ be invertible. Suppose that the only operators in $L(E, F)$ which are invariant under the (G, E) and (G, F)-representations are the scalar multiples of T. Then for every ideal norm α, $\alpha(T)\alpha^*(T^{-1}) = n$.*

We then obtain,

THEOREM 2.5. *Let* $1 \leqslant p, q, r, s \leqslant \infty$ *and* α, β, γ *be any ideal norms. Let* J_n *be the natural inclusion of* $L((l_p^n, l_q^n), \alpha)$ *into* $(L(l_r^n, l_s^n), \beta)$. *Then* $\gamma(J_n)\gamma^*(J_n^{-1}) = n^2$.

Proof. Let e_i, $1 \leqslant i \leqslant n$, denote the usual ith unit vector of the n-dimensional vector space R^n. For each vector $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ $\varepsilon_i = \pm 1$, define the linear operator g_ε: $R^n \to R^n$ by: $g_\varepsilon(e_i) = \varepsilon_i e_i$, $1 \leqslant i \leqslant n$. For each permutation σ of $\{1, 2, ..., n\}$ define the operator h_σ: $R^n \to R^n$ by: $h_\sigma(e_i) = e_{\sigma(i)}$, $1 \leqslant i \leqslant n$. Let G be the group of operators on R^n generated by all products of g_ε and h_σ. We claim that the only operators T: $R^n \otimes R^n \to R^n \otimes R^n$ which commute with all operators of the set $\{a \otimes b;\, a, b \in G\}$ are the scalar multiples of the identity I on $R^n \otimes R^n$. Indeed if T is a commuting operator, and has the representation $T(e_i \otimes e_j) = \sum_{r,s \leqslant n} t_{rs}^{ij} e_r \otimes e_s$, then

$$T(g_\varepsilon \otimes g_\theta)(e_i \otimes e_j) = \sum_{k,l \leqslant n} t_{kl}^{ij} \varepsilon_i \theta_j e_k \otimes e_l,$$

and

$$(g_\varepsilon \otimes g_\theta) T(e_i \otimes e_j) = \sum_{k,l \leqslant n} t_{kl}^{ij} \varepsilon_k \theta_l e_k \otimes e_l.$$

Therefore, $\varepsilon_i \theta_j t_{kl}^{ij} = \varepsilon_k \theta_l t_{kl}^{ij}$ for all choices of vectors ε, θ and indices k, l, i, j. This implies that $t_{kl}^{ij} = t_{ij} \delta_{ik} \delta_{jl}$ (where $\delta_{ik} = 1$ if $i = k$, 0 otherwise). Similarly

$$T(h_\tau \otimes h_\sigma)(e_i \otimes e_j) = t_{\tau(i)\sigma(j)} e_{\tau(i)} \otimes e_{\sigma(j)}$$

and

$$(h_\tau \otimes h_\sigma) T(e_i \otimes e_j) = t_{ij} e_{\tau(i)} \otimes e_{\sigma(j)}.$$

Consequently, $t_{ij} = t_{\sigma(i)\tau(j)}$ for all permutations τ, σ and indices i, j, hence $t_{ij} = t$, where t is a constant, so $T = tI$. The set $\{a \otimes b;\, a, b \in G\}$ forms in a natural way a group of isometries for $(L(l_p^n, l_q^n), \alpha)$ and also for $(L(l_r^n, l_s^n), \beta)$, and by Lemma 2.4 this implies that $\gamma(J_n)\,\gamma^*(J_n^{-1}) = n^2$.

COROLLARY 2.6. *Let* I_n *and* J_n *be as in Theorem* 2.3. *Then* $\pi_1(J_n)\gamma_1(I_n) = n^2$ and $\pi_1(I_n)\gamma_1(J_n) = n^2$.

3. Unconditional structures

The *unconditional basis constant* $\mathfrak{X}(E)$ of a given Banach space E is the least constant λ having the following property: There exists a basis $\{e_i\}_{i \in I}$ for E which $\|\sum_{i \in I} \varepsilon_i x_i e_i\| \leqslant \lambda$ whenever $\sum_{i \in I} x_i e_i \in E$ has norm one and $\varepsilon_i = \pm 1$ $(i \in I)$, with $\varepsilon_i = 1$ for all but finitely many i. If no such λ exist, set $\mathfrak{X}(E) = \infty$. We do not exclude the case where the index set I is uncountable, in which case all vectors $\sum_{i \in I} x_i e_i$ have $x_i = 0$ for all but countably many indices i.

More generally define the *local unconditional constant* of E, $\mathfrak{X}_u(E)$, to be the infimum of all scalars λ having the following property: Given any finite-dimensional subspace $F \subseteq E$, there exists a space U and operators $\alpha \in L(F, U)$ $\beta \in L(U, E)$, such that $\beta\alpha$ is the identity on F and $\|\alpha\|\|\beta\|\mathcal{X}(U)\leqslant\lambda$. If no such λ exist, set $x_u(E)=\infty$. In case $\mathcal{X}_u(E)<\infty$, we say that E has *local unconditional structure*. Of course, if E is finite-dimensional $\mathcal{X}_u(E)=\mathcal{X}_u(E')$.

We introduce the following definition of [19]: *A* set $\mathcal{B}$ of commuting projections, that is idempotent bounded linear operators, on a Banach space E is called a *Boolean algebra of projections* on E if whenever $P, Q\in\mathcal{B}$ also $PQ(=QP)$, $P+Q$ and $I-P$ are in $\mathcal{B}$, and $\|\mathcal{B}\|=\sup\{\|P\|; P\in\mathcal{B}\}<\infty$. E is said to have *sufficiently many Boolean algebras of projections* if there is a constant λ with the following property: For every finite-dimensional subspace F of E there is a Boolean algebra of projections $\mathcal{B}$ on E with $\|\mathcal{B}\|\leqslant\lambda$ and an $e\in E$ such that F is contained in the closed linear space of $\{Pe; P\in\mathcal{B}\}$. The least such λ will be denoted by $b(E)$. When no such λ exists set $b(E)=\infty$. The relations between the three constants introduced are as follows.

LEMMA 3.1. *For any Banach space* E, $\mathcal{X}_u(E)\leqslant 2b(E)\leqslant 2\mathcal{X}(E)$.

Proof. The inequality $b(E)\leqslant\mathcal{X}(E)$ is obvious. It follows from [19] Proposition 1 that for any $\lambda>b(E)$ and finite-dimensional subspace $F\subseteq E$ there is a Boolean algebra of projections $\mathcal{B}$ on E with $\|\mathcal{B}\|\leqslant\lambda$, disjoint $\{P_i\}_{i=1}^n$ in $\mathcal{B}$ and $e_i\in P_iE$ such that $F\subseteq\operatorname{span}\{e_i\}_1^n$.

Define a new norm $|||\cdot|||$ on span $\{e_i\}_1^n$ by

$$|||\sum_1^n\lambda_i e_i||| = \max_{\pm}\left\|\sum_1^n \pm\lambda_i e_i\right\|$$

and denote the space thus obtained by U. Each $e\in F$ can be written as $e=\sum_1^n\lambda_i e_i$, so $P_ie=\lambda_ie_i$ and hence

$$|||e||| = \max_{\pm}\left\|\sum_1^n \pm P_i e\right\| \leqslant 2\lambda\|e\|,$$

therefore the inclusion map α of F into U has norm $\leqslant 2\lambda$. Of course $|||\cdot|||\geqslant\|\ \|$, so the inclusion map β of U into E has norm $\leqslant 1$; $\beta\alpha$ is the identity on F and $\mathcal{X}(U)=1$, therefore $\|\alpha\|\,\|\beta\|\,\mathcal{X}(U)\leqslant 2\lambda$. This concludes the proof.

Remarks. Clearly $\mathcal{X}_u(E)\leqslant\mathcal{X}(E)$ and there are spaces with local unconditional structure which have no unconditional bases; simple examples are furnished by $C[0, 1]$ and $L_1[0, 1]$. Moreover Enflo and Rosenthal [2] have shown that for every $1<p<\infty$, $p\neq 2$, and a finite measure μ with dim $(L_p(\mu))\geqslant\aleph_\omega$, $L_p(\mu)$ can have no unconditional basis. On the other hand every L_p-space has sufficiently many Boolean algebras of projections. We do not know of an example in which $b(E)=\infty$ and $\mathcal{X}_u(E)<\infty$.

It is easily seen that if E is isomorphic to a complemented subspace of a space with an unconditional basis then E has local unconditional structure. This fact is also a consequence of the following easily proved lemma.

Lemma 3.2. *Let X and E be Banach spaces and μ a scalar, and suppose for any finite-dimensional subspace $Y \subseteq X$ there are operators $A \in L(Y, E)$, $B \in L(E, X)$ such that BA is the identity operator on Y and $\|B\|\,\|A\| \leqslant \mu$. Then $\mathfrak{X}_u(X) \leqslant \mu \mathfrak{X}_u(E)$.*

Recall that the *Banach-Mazur distance* between isomorphic Banach spaces E and F is defined to be $d(E, F) = \inf \|T\|\,\|T^{-1}\|$, where the infimum is taken over all isomorphisms T mapping E onto F. It follows from Lemma 3.2 that $\mathfrak{X}_u(F) \leqslant \mathfrak{X}_u(E)d(E, F)$.

Lemma 3.3. *If $A \in \Pi_1(E, M)$, then $\gamma_1(A) \leqslant \mathfrak{X}_u(E)\pi_1(A)$.*

Proof. Let $\lambda > \mathfrak{X}_u(E)$ and $F \subseteq E$ be any finite-dimensional subspace. Choose α, β, U as in the definition, $\mu > \mathfrak{X}(U)$ and $\{u_i\}_{i \in I}$ to be an unconditional basis for U such that $\|\Sigma_{i \in I} \pm t_i u_i\| \leqslant \mu \|\Sigma_{i \in I} t_i u_i\|$ for every vector $\Sigma_{i \in I} t_i u_i \in U$ and every choice of $\pm$signs. Then

$$\sum_{i \in I} |t_i|\,\|A\beta u_i\| \leqslant \pi_1(A\beta) \sup_{\|u'\| \leqslant 1} \sum |\langle t_i u_i\, u'\rangle| \leqslant \|\beta\| \pi_1(A)\,\mu \Big\|\sum_{i \in I} t_i u_i\Big\|.$$

Define $C\colon U \to l_1(I)$ and $D\colon l_1(I) \subset M$ by: $C(\Sigma_{i \in I} t_i u_i) = (t_i \|A\beta u_i\|)_{i \in I}$, and $D((\xi_i)_{i \in I}) = \Sigma_{i \in I} \xi_i \|A\beta_i u_i\|^{-1} A\beta u_i$, where the last sum is on all indices i for which $A\beta u_i \neq 0$. Clearly $\|D\| \leqslant 1$, $\|C\| \leqslant \mu \|\beta\| \pi_1(A)$ and $DC = A\beta$, so $DC\alpha = A \mid F$, hence,

$$\gamma_1(A \mid F) \leqslant \|D\|\,\|C\alpha\| \leqslant \|\alpha\|\,\|\beta\|\,\mu\pi_1(A).$$

This inequality implies that $\gamma_1(A \mid F) \leqslant \|\alpha\|\,\|\beta\|\,\mathfrak{X}(U)\pi_1(A) \leqslant \lambda\pi_1(A)$. The norm γ_1 is perfect ([7], [16]), so

$$\gamma_1(A) = \sup \{\gamma_1(A \mid F);\ F \subseteq E,\ \dim F < \infty\} \leqslant \lambda \pi_1(A),$$

letting $\lambda \to \mathfrak{X}_u(E)$ completes the proof.

Recall the following definition of [24]. A Banach space E is termed *sufficiently Euclidean* if there is a constant $b_E > 0$ and sequences $S_n \in L(l_2^n, E)$, $T_n \in L(E, l_2^n)$ such that $T_n S_n$ is the identity and $\|S_n\|\,\|T_n\| \leqslant b_E$, $n = 1, 2, \ldots$

Theorem 3.4. *If both E and F are sufficiently Euclidean, then $E \overset{\vee}{\otimes} F$ and $E \overset{\wedge}{\otimes} F$ their duals, biduals, etc., do not have local unconditional structure.*

Proof. Choose b_E, b_F and sequences $s_n \in L(l_2^n, E)$, $T_n \in L(E, l_2^n)$, $A_n \in L(l_2^n, F)$ and $B_n \in L(F, l_2^n)$ to meet the requirements of the definition. First consider the least $\otimes$-norm. Clearly $(T_n \otimes B_n) \circ (S_n \otimes A_n)$ is the identity on $l_2^n \otimes l_2^n$ and $\|T_n \otimes B_n\|\,\|S_n \otimes A_n\| \leqslant b_E b_F$. By Lemma 3.2

$$\mathfrak{X}_u(l_2^n \overset{\vee}{\otimes} l_2^n) \leqslant b_E b_F \mathfrak{X}_u(E \overset{\vee}{\otimes} F),$$

and by Theorem 2.3 (b) and Lemma 3.3

$$n^{\frac{1}{2}}/9 \leqslant \mathfrak{X}_u(l_2^n \overset{\vee}{\otimes} l_2^n).$$

Thus $\mathfrak{X}_u(E \overset{\vee}{\otimes} F) = \infty$. The greatest $\otimes$-norm may be dealt with in the same manner using the inclusion J_n of Theorem 2.3, and the remaining assertions follow by considering the adjoints, biadjoints, etc., of the sequences $T_n \otimes B_n$ and $S_n \otimes A_n$.

Remarks. (1) It is proved in [24] that every $\mathcal{L}_p$-space, $1 < p < \infty$, is sufficiently Euclidean, so Theorem 3.4 applies to $\otimes$-products of such spaces.

(2) The proof of Theorem 3.4 gives that $\mathfrak{X}(l_2^n \overset{\alpha}{\otimes} l_2^n) \geqslant n^{\frac{1}{2}}/9$, for $\alpha = \vee$ or $\wedge$. This solves the problem of finding a sequence of finite dimensional spaces whose unconditional basis constants tend to infinity [6], [11], [12], [19].

(3) It is well-known that it is possible to embed l_2^n as a complemented subspace of $l_n^{2^n}$, $1 < p < \infty$, in such a way that neither the norm of the embedding nor the norm of the projection depend on n. Thus the proof of Theorem 3.4 gives $\mathfrak{X}_u(l_p^{2^n} \overset{\alpha}{\otimes} l_q^{2^n}) \geqslant c_{pq} n^{\frac{1}{2}}$ for $1 < p$, $q < \infty$ and $\alpha = \wedge$ or $\vee$, where c_{pq} is a contsant independent of n. Again by Lemma 3.2 $\mathfrak{X}_u(l_q^n \overset{\alpha}{\otimes} l_q^n \geqslant c_{pq} (\log n)^{\frac{1}{2}}$. We now wish to find more precise lower bounds for the parameters.

For positive functions f and g defined on the natural numers the notation $f(n) \lesssim g(n)$ means $\sup_n f(n)/g(n) < \infty$, and $f(n) \sim g(n)$ means $f(n) \lesssim g(n)$ and $g(n) \lesssim f(n)$.

THEOREM 3.5. *Let* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$.

$$\mathfrak{X}_u(l_{q'}^n \overset{\vee}{\otimes} l_{p'}^n) = \mathfrak{X}_u(l_q^n \overset{\wedge}{\otimes} l_p^n) \gtrsim \begin{cases} n^{1/2} & , \text{ if } 2 \leqslant p, q \leqslant \infty \\ n^{1/q'} & , \text{ if } 1 \leqslant q \leqslant 2 \leqslant p \\ n^{1/p'} & , \text{ if } 1 \leqslant p \leqslant 2 \leqslant q \\ n^{3/2 - 1/p - 1/q}, & \text{ if } 1 \leqslant p, q \leqslant 2. \end{cases}$$

Proof. For the greatest $\otimes$-norm we wish to apply Lemma 3.3 with R_n the inclusion of $l_p^n \overset{\wedge}{\otimes} l_q^n$ into $c_2(l_2^n, l_2^n)$. Consider the factorization of the identity on $c_2(l_2^n, l_2^n)$ given by

$$c_2(l_2^n, l_2^n) \xrightarrow[R_n']{} l_{p'}^n \overset{\vee}{\otimes} l_{q'}^n \xrightarrow[A]{} l_1^n \overset{\vee}{\otimes} l_1^n \xrightarrow[B]{} c_2(l_2^n, l_2^n)$$

where A and B are the inclusions. Using the identity $\pi_1^* = \gamma_\infty$ and Theorem 2.2 (a)

$$n^2 = \operatorname{tr}(BAR_n') \leqslant \pi_1(B) \|A\| \gamma_\infty(R_n') = 3n^{1/p + 1/q} \gamma_1(R_n).$$

To bound $\pi_1(R_n)$ above, let C_n and D_n be the inclusions of l_p^n into l_2^n and l_q^n into l_2^n respectively, and factor R_n as

$$l_p^n \overset{\wedge}{\otimes} l_q^n \xrightarrow[C_n \otimes D_n]{} l_2^n \overset{\wedge}{\otimes} l_2^n \xrightarrow[J_n]{} c_2(l_2^n, l_2^n),$$

so that by Theorem 2.3 (a)

$$\pi_1(R_n) \leqslant \|C_n\| \, \|D_n\| \, 3n^{\frac{1}{2}}.$$

Combining inequalities with Lemma 3.3 yields

$$9^{-1} n^{3/2-1/p-1/q} \leqslant \|C_n\| \, \|D_n\| \, \mathfrak{X}_u(l_p^n \hat{\otimes} l_q^n).$$

The estimates now follow by considering cases. For the least $\otimes$-norm apply the same proof with $R_n^{-1'}$, or use Theorem 2.4 to get $\pi_1(R_n^{-1'})\gamma_1(R_n) = n^2$ and $\gamma_1(R_n^{-1'})\pi_1(R_n) = n^2$.

As in Theorem 3.4 we now have

COROLLARY 3.6. *For* $1 < p, q \leqslant \infty$ *neither* $l_p \hat{\otimes} l_q$ *nor* $l_{p'} \otimes l_{q'}$, *their duals, biduals, etc., have local unconditional structure.*

Remark. It is proved in [15] that if $1/p + 1/q \geqslant 1$ then $l_p \check{\otimes} l_q$ is not isomorphic to a subspace of a space with an unconditional basis. We do not know if this stronger result is true for $p, q < \infty$ and $1/p + 1/q < 1$.

COROLLARY 3.7. *Let* $1 \leqslant r \leqslant 2 < q \leqslant \infty$ *and* $p < q$. *Then* $I_r(l_p, l_q)$ *and* $\Pi_{r'}(l_q, l_p)$ *have no local unconditional structure.*

Proof. By [18] there is a constant K such that for any $u \in l_1^n \check{\otimes} l_p^n = L(l_\infty^n, l_p^n)$ and $p \leqslant 2$, $\|u\| \leqslant \pi_{r'}(u) \leqslant K\|u\|$.

Applying Lemma 3.2

$$n^{\frac{1}{2}} \lesssim \mathfrak{X}_u(l_1^n \check{\otimes} l_p^n) \leqslant K \mathfrak{X}_u(\Pi_{r'}(l_\infty^n, l_p^n)),$$

the first inequality by Theorem 3.5. The distance from $\Pi_{r'}(l_\infty^n, l_p^n)$ to $\Pi_{r'}(l_q^n, l_p^n)$ is at most $n^{1/q}$ (consider the norm of the natural inclusion and its inverse), hence by Lemma 3.2 $n^{1/2-1/q} \lesssim \mathfrak{X}_u(\Pi_{r'}(l_q^n, l_p^n))$, and again the lemma implies that $\Pi_{r'}(l_q, l_p)$ has no local unconditional structure.

Now, if $q > p > 2$, $d(l_p^n, l_{p'}^n) \sim n^{1/2-1/p}$ [11], so the distance of $\Pi_{r'}(l_q^n, l_p^n)$ from $\Pi_{r'}(l_q^n, l_{p'}^n)$ is at most, asymptotically, $n^{1/2-1/p}$. By Lemma 3.2

$$n^{1/2-1/q} \lesssim \mathfrak{X}_u(\Pi_{r'}(l_q^n, l_{p'}^n)) \leqslant n^{1/2-1/p} \mathfrak{X}_u(\Pi_{r'}(l_q^n, l_p^n)),$$

so that $\Pi_{r'}(l_q, l_p)$ has no local unconditional structure.

A dual argument, using the identity of $\Pi_{r'}^*(l_p^n, l_q^n) = I_r(l_p^n, l_q^n)$ (Lemma 2.1, and $\pi_{r'}^* = i_r$), yields the other assertion.

Remark. We can show that if $1 \leqslant r' \leqslant p \leqslant 2 \leqslant q \leqslant r$, $\Pi_{r'}(l_q, l_p)$ has local unconditional structure, moreover, the sequence $\mathfrak{X}(\Pi_{r'}(l_q^n, l_p^n))$ $n = 1, 2, 3, \ldots$ is bounded and $\Pi_{r'}(l_q^n, l_p^n)$ embeds isometrically in $L_{r'}(\mu)$ for some finite measure μ.

4. Factorizations of absolutey summing maps

We now give an example which answers [8] problem 2 negatively.

THEOREM 4.1. *The natural inclusion of $l_1 \check{\otimes} l_1$ into $c_2(l_2, l_2)$ is absolutely summing yet does not have the lifting property, that is does not factor through any L_1-space.*

Proof. Write R for the inclusion and let P_n be the projection of l_1 onto the span of the first n unit vectors. Then $P_n \otimes P_n$ is a sequence of norm one projections which converges simply to the identity on $l_1 \check{\otimes} l_1$, and whose range are the natura images of $l_1^n \check{\otimes} l_1^n$. Thus by Theorem 2.2(a) $\pi_1(R) \leq \sup_n \pi_1(R \circ (P_n \otimes P_n)) \leq 3$. Consider the factorization of I_n given by

$$l_2^n \check{\otimes} l_2^n \xrightarrow{A} l_1^n \check{\otimes} l_1^n \xrightarrow{R|l_1^n \check{\otimes} l_1^n} c_2(l_2^n, l_2^n)$$

where A is just the identity. By Theorem 2.3 (b)

$$3^{-1} n^{\frac{3}{2}} \leq \|A\| \gamma_1 (R | l_1^n \check{\otimes} l_1^n) \leq n \gamma_1 (R)$$

so that $\gamma_1(R) = \infty$.

Remarks: (1) By taking adjoints it is easily seen that the injection of $c_2(l_2, l_2)$ into $(l_1 \check{\otimes} l_1)' = I_1(l_1, l_\infty)$ has absolutely summing adjoint, yet does not have the extension property.

(2) Problem 2 of [8] was possibly motivated by the following considerations (see [8] problem 5). The identity operator on $L = L_1(\mu)$ induces a continuous mapping from $l_1 \check{\otimes} L$ into $l_2 \hat{\otimes} L$ of norm at most $\sqrt{3}$([8], Théorème 5). Thus if $u \in \Gamma_1(E, F)$ then $1 \otimes u$ gives rise to a continuous mapping of $l_1 \check{\otimes} E$ into $l_2 \hat{\otimes} F$ of norm at most $\sqrt{3}\, \gamma_1(u)$. The converse is false since an absolutely summing operator $u \in \Pi_1(E, F)$ gives rise to a continuous mapping $1 \otimes u$ of $l_1 \check{\otimes} E$ into $l_2 \hat{\otimes} F$ and we saw that u need not factor through an L_1-space. Yet it is unknown whether the identity operator on E must factor through an L_1-space if the identity map $l_1 \check{\otimes} E \to l_2 \hat{\otimes} E$ is continuous (see [8] problem 5, [9] Proposition 9 and subsequent discussion, [18] problem 2).

(3) Using the closed graph theorem and Theorem 2.3 it follows that, for E and F sufficiently Euclidean, there are absolutely summing maps from $E \hat{\otimes} F$ (or $E \check{\otimes} F$) into Hilbert spaces which do not factor through $L_1(\mu)$-spaces.

The *right injective envelope* of $[A, \alpha]$, denoted by $[A\backslash, \alpha\backslash]$, is defined as follows: $u \in A\backslash(E, F)$ if and only if there is an isometric embedding w of F into a $C(K)$-space such that $wu \in A(E, C(K))$, and $\alpha\backslash$-norm of u is $\alpha\backslash(u) = \alpha(wu)$. The *left injective envelope,* $[/A, /\alpha]$, may be defined by $u \in /A(E, F)$ iff $u' \in A\backslash(F', E')$, with $/\alpha(u) = \alpha\backslash(u')$.

Lemma 4.2. *For $u \in \Pi_1(E, F)$, the inequalities $\gamma_1(u) \leqslant \gamma_1\backslash(E)\pi_1(u)$ and $\gamma_1(u) \leqslant /\gamma_\infty(E)\pi_1(u)$ hold.*

Proof. In case $\gamma_1\backslash(E) < \infty$, let $\varepsilon > 0$ and find a subspace $L \subset L_1(\mu)$ and an isomorphism $s: L \to E$ such that $\|s\|\,\|s^{-1}\| \leqslant (1+\varepsilon)\gamma_1\backslash(E)$. Then $\pi_1(us) \leqslant \|s\|\pi_1(u)$ so us has a factorization

$$L \xrightarrow{v} L_\infty(\nu) \xrightarrow{w} F$$

with $wv = us$ and $\|v\|\,\|w\| \leqslant \pi_1(us)$. Since $L_\infty(\nu)$ is injective there is a $\tilde{v} \in L(L_1(\mu), L_\infty(\nu))$ with $\|\tilde{v}\| = \|v\|$ and $\tilde{v}|L = v$. Then $w\tilde{v}s^{-1} = u$ and

$$\gamma_1(u) \leqslant \|s^{-1}\|\,\|\tilde{v}\|\,\|w\| \leqslant (1+\varepsilon)\gamma_1\backslash(E)\pi_1(u).$$

In case $/\gamma_\infty(E) < \infty$ let $i: E \to E''$ be the canonical embedding and factor $i = vw$, where Q is a quotient of a $C(K)$-space, $w \in L(E, Q)$ and $v \in L(Q, E'')$. Let η be the quotient map from $C(K)$ onto Q. Then $u''v\eta$ is absolutely summing on a $C(K)$-space, and so by [21] $u''v\eta$ is integral and $i_1(u''v\eta) = \pi_1(u''v\eta) \leqslant \|v\|\pi_1(u)$. But then

$$\pi_1(v'u''') \leqslant i_1((u''v\eta)') = i_1(u''v\eta) \leqslant \|v\|\pi_1(u),$$

and so $\pi_1(i'u''') \leqslant \|v\|\,\|w\|\pi_1(u)$. Then as above

$$\gamma_1(u) = \gamma_\infty(u') \leqslant \pi_1(u') \leqslant \pi_1(i'u'''\,|\,F'),$$

so that $\gamma_1(u) \leqslant \|v\|\,\|w\|\pi_1(u)$. Taking the infimum over all such factorizations gives the inequality.

Theorem 4.3. *There are spaces E and F, and non-integral operators $u \in L(E, F)$ with the following property: if G is isomorphic to a subspace of an L_1-space or to a quotient of a $C(K)$-space, or if G has local unconditional structure, then $1 \otimes u$ extends to a continuous linear map from $G \overset{\vee}{\otimes} E$ into $G \overset{\wedge}{\otimes} F$.*

Proof. Let $E = l_1 \overset{\vee}{\otimes} l_1$ and v be the inclusion of $l_1 \overset{\vee}{\otimes} l_1$ into $c_2(l_2, l_2)$. Suppose for any Banach space F and any $w \in L(c_2(l_2, l_2), F)$ with absolutely summing adjoint, that wv is integral. Then from [7] Corollary 2.21 it would follow that $\gamma_\infty(v') = \gamma_1(v) < \infty$, that is, v factors through an L_1-space, contradicting Theorem 4.1. Thus $u = wv$ is non-integral for some w with w' absolutely summing.

Now let $\lambda = \min\{\gamma_1\backslash(G),\ /\gamma_\infty(G),\ \mathfrak{X}_u(G)\}$. For $t \in L(G, F')$ $u't$ must be integral; in fact it follows from Lemmas 3.3 and 4.2 that $\gamma_1(w't) \leqslant \lambda\pi_1(w't)$, so that

$$i_1(u't) = i_1(t'w''v'') \leqslant \gamma_\infty((w't)')\pi_1(v'') \leqslant 3\lambda\pi_1(w')\,\|t\|.$$

Thus setting $\varphi(t)=u't$ gives a continus linear operator from $L(G, F')$ into $I_1(G, E')$. By checking elementary tensors it is easy to see that the diagram

$$\begin{array}{ccc} I_1(G,E')' & \xrightarrow{\varphi'} & L(G,F')' \\ \uparrow & & \uparrow \\ G\otimes E & \xrightarrow[1\otimes u]{} & G\otimes F \end{array}$$

commutes, where the unmarked arrows are the natural embeddings. But then $1\otimes u=\varphi'\,|\,G\otimes E$ is continuous with the inductive topology on $G\otimes E$, and the projective topology on $G\otimes F$, and hence has an extension.

Remarks. (1) The interest in Theorem 4.3 is that if the conclusion holds for *all* G then u must be integral (essentially the same proof as above shows that integral operators must satisfy the conclusion of the theorem). In fact, taking $G=F'$, $u'=(1_{F'}\otimes u)\,1_{F'}$ is an element of $(F'\overset{\vee}{\otimes} E)'=I_1(F', E')$.

(2) An operator T is in $\Gamma_p^*(E, F)$ if and only if the map $1\otimes T$ from $l_q\overset{\vee}{\otimes} E$ to $l_q\overset{\wedge}{\otimes} F$ is continuous $(1/p+1/q=1)$, and then $\gamma_p^*(T)=\|1\otimes T\|$, [1], [16], [17]. Thus by setting $G=l_q$ in Theorem 4.3 it follows that there is a non-integral operator T which is of type γ_p^* for every p, $1\leqslant p\leqslant\infty$. This solves a problem raised by the second named author at the Louisiana State University conference on $\mathcal{L}_p$ spaces in 1971.

(3) The construction in the proof above yields a non-integral operator u of the form wv where both v and w' are 1-absolutely summing. The question whether there exists a non-integral operator u of this form was observed by Grothendieck [8] (remarks on p. 39) to be equivalent to the question whether there exists an absolutely summing operator not factorizable through any L_1-space.

5. Spaces of operators on l_2^n

We begin by considering the classes $c_p(H)=c_p(H, H)$ of operators on a Hilbert space H.

THEOREM 5.1. *For* $1\leqslant p\leqslant\infty\, \mathcal{X}_u(c_p(l_2^n))\sim n^{|1/p-1/2|}$. *For* $p\neq 2$ *and* H *an infinite dimensional Hilbert space,* $c_p(H)$ *has no local unconditional structure.*

Proof. Theorem 3.5 gives $\mathcal{X}_u(c_p(l_2^n))\gtrsim n^{\frac{1}{2}}$ *for* $p=1$ *and* $p=\infty$. *Given* $1\leqslant p\leqslant 2$ *it follows easily that* $c_p(u)\leqslant c_1(u)\leqslant n^{1/q'}c_p(u)$ so that by Lemma 3.2

$$n^{\frac{1}{2}}\lesssim \mathcal{X}_u(c_1(l_2^n))\leqslant n^{1/p'}\mathcal{X}_u(c_p(l_2^n))$$

For $2\leqslant p\leqslant\infty$ we may compare c_p to c_∞ and obtain in either case that

$$n^{|1/p-1/2|} \lesssim \mathfrak{X}_u(c_p(l_2^n)).$$

But the distance from $c_p(l_2^n)$ to $c_2(l_2^n)$ is always at most $n^{|1/p-1/2|}$ so that by Lemma 3.2

$$\mathfrak{X}_u(c_p(l_2^n)) \lesssim n^{|1/p-1/2|}.$$

Remark. Theorem 5.1 solves problem 2 of [15] by showing that $c_p(H)$ has no unconditional basis for $p \neq 2$. Also observe that the proof gives $\mathfrak{X}(c_p(l_2^n)) \sim n^{|1/p-1/2|}$.

The unconditional structures in sequences of spaces of the form $A(l_2^n, l_2^n)$, $[A, \alpha]$ a Banach ideal norm, seem to depend largely on the behaviour of $\alpha(l_2^n)$ and on the best constants relating the α-norm with the Hilbert-Schmidt. The following two theorems of this section are indicative of this fact.

THEOREM 5.2. *Let $[A, \alpha]$ be a Banach ideal. Then*

$$\mathfrak{X}_u(A(l_2^n, l_2^n)) \geqslant (2/3\pi) \max\{n^{-\frac{1}{2}}\alpha(l_n^2),\ n^{\frac{1}{2}}\alpha(l_2^n)^{-1}\}.$$

Proof. We are going to show that

$$(2/3\pi)\, \alpha(l_2^n)\, n^{-\frac{1}{2}} \leqslant \mathfrak{X}_u(A(l_2^n, l_2^n)).$$

Let R_n be the inclusion of $A(l_2^n, l_2^n)$ into $c_2(l_2^n, l_2^n)$. We first estimate $\pi_1(R_n)$. Let S be the unit sphere of l_2^n, dm the normalized $(n-1)$-dimensional, rotational invariant measure on S and

$$K = \{x \otimes y;\ \|x\|_2 = \|y\|_2 = 1\}.$$

Then K is a compact subset of $A(l_2^n, l_2^n)' = A^*(l_2^n, l_2^n)$. Define $\nu \in C(K)'$ a probability measure by

$$\nu(f) = \int_S \int_S f(x \otimes y)\, dm(x)\, dm(y),\ f \in C(K).$$

For every $u \in A(l_2^n, l_2^n)$ we have by [5]

$$\nu(|\langle u, \cdot \rangle|) = \int_S \int_S |\langle ux, y \rangle|\, dm(y)\, dm(x) = \pi_1(l_2^n)^{-1} \int_S \|ux\|_2\, dm(x).$$

Consider the isometric embedding φ of l_2^n into $L_1(S)$ given by $\varphi(x) = \pi_1(l_2^n)\langle x, \cdot \rangle$. Then as in Theorem 2.2 (a)

$$i_1(\varphi u^*) = \pi_1(l_2^n) \int_S \|ux\|_2\, dm(x)$$

and
$$c_2(u) = \pi_2(u) = \pi_2(u^*) \leqslant \pi_1(u^*) \leqslant i_1(\varphi u^*),$$

so
$$c_2(u) \leqslant \pi_1(l_2^n)^2 \nu(|\langle u, \cdot \rangle|).$$

Thus, $\pi_1(R_n) \leqslant \pi_1(l_2^n)^2 \leqslant \pi n/2$, the last inequality by [5].

To estimate $\pi_1(R_n^{-1'})$, recall that from the proof of Theorem 2.2 (b)

$$c_2(u) \leqslant 3\,n^{\frac{1}{2}}\,\alpha^*(l_2^n)\int_G |\langle u, \alpha^*(l_2^n)^{-1}g\rangle|\,dg.$$

Since $\alpha^*(g) = \alpha^*(l_2^n)$ for each isometry g,

$$\pi_1(R_n^{-1'}) \leqslant 3n^{\frac{1}{2}}\alpha^*(l_2^n) = 3n^{3/2}\alpha(l_2^n)^{-1},$$

the last equality is by Lemma 2.4. By Theorem 2.5, $n^2 = \gamma_1(R_n)\pi_1(R_n^{-1'})$, so that

$$\gamma_1(R_n) \geqslant \tfrac{1}{3}\,n^{\frac{1}{2}}\,\alpha(l_2^n).$$

Applying Lemma 3.3 gives the inequality

$$(2/3\pi)\,\alpha(l_2^n)\,n^{-\frac{1}{2}} \leqslant \mathfrak{X}_u(A(l_2^n, l_2^n)).$$

Consideration of the operator $R^{-1'}$ gives in the same manner the analogeous inequalities

$$\pi_1(R_n^{-1'}) \leqslant \pi n/2 \quad \text{and} \quad \gamma_1(R_n^{-1'}) \geqslant \tfrac{1}{3}\,n^{\frac{1}{2}}\,\alpha^*(l_2^n)$$

and by Theorem 2.5, $n^2 = \gamma_1(R_n)\pi_1(R_n^{-1'}) = \pi_1(R_n)\,\gamma_1(R_n^{-1'})$, so that applying Lemma 3.3 again with the equality $\alpha(l_2^n)\,\alpha^*(l_2^n) = n$, gives that

$$(2/3\pi)\,n^{\frac{1}{2}}\alpha(l_2^n)^{-1} \leqslant \mathfrak{X}_u(A(l_2^n, l_2^n)).$$

Remark. It of course follows that if in Theorem 5.2 $\lim\sup_n \{n^{-\frac{1}{2}}\alpha(l_2^n), n^{\frac{1}{2}}\alpha(l_2^n)^{-1}\} = \infty$ and both E and F are sufficiently Euclidean, then $A(E, F)$ has no local unconditional structure. This is true in particular for $\Gamma_p(E, F)$ $1 < p < \infty$ and $\Gamma_p^*(E, F)$.

THEOREM 5.3. *Let α be an ideal norm for which $ak^{1/p} \leqslant \alpha(l_2^k) \leqslant bk^{1/p}$, $k = 1, 2, \ldots, n$. Then for $u \in A(l_2^n, l_2^n)$*

$$a(ln(en))^{-1/p}c_p(u) \leqslant \alpha(u) \leqslant b(ln(en))^{1/p'}c_p(u).$$

Proof. For $u \in A(l_2^n, l_2^n)$ choose orthonormal bases $(e_i)_{i\leqslant n}$ and $(b_i)_{i\leqslant n}$, and a decreasing sequence of non-negative scalars λ_i so that $u = \sum_{i\leqslant n}\lambda_i e_i \otimes b_i$. Let g be the isometry $g(b_i) = e_i$ and, for each $k = 1, 2, \ldots, n$, let v_k be the orthogonal projection onto $[b_i]_{i\leqslant k}$. For each $k = 1, 2, \ldots, n$

$$\textstyle\sum_{i\leqslant k}\lambda_i = \operatorname{tr}(ugv_k) \leqslant \alpha(u)\,\alpha^*(gv_k) \leqslant \alpha(u)\,\alpha^*(l_2^k).$$

But since $\alpha^*(l_2^k)\,\alpha(l_2^k) = k$, [6] (or Lemma 2.4),

$$\lambda_k \leqslant k^{-1}\textstyle\sum_{i\leqslant k}\lambda_i \leqslant a^{-1}\alpha(u)\,k^{-1/p}$$

and hence $$c_p(u) = (\Sigma_{k\leqslant n}\lambda_k^p)^{1/p} \leqslant a^{-1}\alpha(u)\,(\Sigma_{k\leqslant n}k^{-1})^{1/p}$$

which gives the first inequality. In a similar manner $c_{p'}(u) \leqslant b(ln(en))^{1/p'}\alpha^*(u)$ and hence by duality, using the relation $c_p(l_2^n)' = c_{p'}(l_2^n)$ [20], the second inequality follows.

COROLLARY 5.4. *If α is an ideal norm and $d(A(l_2^n, l_2^n), l_2^{n^2})/ln(n)$ is not bounded then $\mathcal{X}_u(A(l_2^n, l_2^n))$ is not bounded. In particular, if E and F are sufficiently Euclidean $A(E, F)$ has no local unconditional structure.*

Proof. We claim that $\alpha(l_2^n) \nsim n^{\frac{1}{2}}$; if not, Theorem 5.3 with $p=2$ gives a contradiction. But since $\alpha(l_2^n) \nsim n^{\frac{1}{2}}$, Theorem 5.2 yields the result. The last statement follows by Lemma 3.2.

Remarks. Under the assumptions of Corollary 5.4 it follows from Lemma 4.2 that $\gamma_1 \diagdown (A(l_2^n, l_2^n)) \to \infty$. Hence $A(E, F)$ is neither isomorphic to a subspace of L_1, nor to a quotient of L_∞.

We conjecture that the assertion of Corollary 5.4 is true also in the case when $d(A(l_2^n, l_2^n), l_2^{n^2}) \xrightarrow[n\to\infty]{} \infty$.

Given a Banach space E, $s(E)$ will denote the least number λ for which there is a multiplicative group of isomorphisms on E, G, all of norms at most λ, which has the property that an operator on E which commutes with each element of G must be a scalar multiple of the identity. E is said to have *enough symmetries* if $s(E)=1$ (cf. [4]).

LEMMA 5.5. *For E, F finite-dimensional spaces and α an ideal norm, $s(A(E, F)) \leqslant s(E)s(F)$.*

Proof. Regard $A(E, F)$ as $E' \otimes F$, algebraically. Let G and H be groups of isometries on E' and F, respectively, such that only the scalar multiples of the identity commute with each group, and with $\|g\| \leqslant \lambda$, $\|h\| \leqslant \mu$, for all $g \in G$ and $h \in H$. Let M be the group of all isomorphisms on $E' \otimes F$ of form $g \otimes h$, $g \in G$ and $h \in H$. Then each element of M has norm $\leqslant \lambda\mu$. Let T be an operator on $E' \otimes F$ which commutes with each element of M. For $y \in F$, $y' \in F'$, define S on E' by $\langle x, S(x')\rangle = \langle T(x' \otimes y), x \otimes y'\rangle$. Then for $g \in G$

$$\begin{aligned}\langle x, g^{-1}Sg(x')\rangle &= \langle T\circ(g\otimes 1)(x'\otimes y), (g^{-1}\otimes 1)'(x\otimes y')\rangle \\ &= \langle T(x'\otimes y), (g\otimes 1)'(g^{-1}\otimes 1)'(x\otimes y')\rangle \\ &= \langle x, S(x')\rangle,\end{aligned}$$

so that $S = \lambda(y, y')1_{E'}$ for some scalar $\lambda(y, y')$, and hence the equality

$$\langle T(x'\otimes y), x\otimes y'\rangle = \lambda(y, y')\langle x, x'\rangle$$

always holds. Chose $x_0 \in E$ and $x_0' \in E'$ with $\langle x_0, x_0'\rangle = 1$, and define R on F by $\langle Ry, y'\rangle =$

$\langle T(x_0'\otimes y),\ x_0\otimes y'\rangle$. Repeating the same argument gives that $R=t\,1_F$ for some scalar t, so that

$$\langle T(x'\otimes y),\ x\otimes y'\rangle = t\langle x, x'\rangle\langle y, y'\rangle = \langle t(x'\otimes y),\ x\otimes y'\rangle$$

always holds. This gives T as t times the identity, so the lemma is established.

THEOREM 5.6. *Let E and F be finite-dimensional spaces with enough symmetries. Then*

(a) $\pi_1(E\overset{\vee}{\otimes}F)=\pi_1(E)\,\pi_1(F)$.

(b) $\gamma_\infty(E\overset{\vee}{\otimes}F)=\gamma_\infty(E)\,\gamma_\infty(F)$.

(c) $\gamma_1(E\overset{\wedge}{\otimes}F)=\gamma_1(E)\,\gamma_1(F)$.

(d) $\gamma_\infty(c_p(l_2^n))\ \sim n,\ 1\leqslant p\leqslant\infty$.

(e) $\gamma_1\backslash(c_p(l_2^n)),/\gamma_\infty(c_p(l_2^n))\sim n^{|1/p-1/2|},\ 1\leqslant p\leqslant\infty$.

Proof. Let K_1, and K_2 be the closed unit balls of E' and F', respectively, with $\mu\in C(K_1)'$ and $\nu\in C(K_2)'$ probability measures such that $\|x\|\leqslant\pi_1(E)\mu(|\langle x,\cdot\rangle|)$, $x\in E$, and $\|y\|\leqslant\pi_1(F)\nu(|\langle y,\cdot\rangle|)$, $y\in F$. Let $u\in E\overset{\vee}{\otimes}F=L(E',F)$ and choose $y_0'\in F'$, $\|y_0'\|=1$, so that $\|u\|=\|u'(y_0')\|$. If $\mu\otimes\nu\in C(K_1\times K_2)'$ is the product of μ and ν then

$$\begin{aligned}\mu\otimes\nu(|\langle u,\cdot\rangle|)&=\int_{K_1}\int_{K_2}|\langle u(x'),y'\rangle|\,\mu(dx')\,\nu(dy')\\&\geqslant\pi_1(F)^{-1}\int_{K_2}\|u(x')\|\,\mu(dx')\\&\geqslant\pi_1(F)^{-1}\int_{K_2}|\langle x',u'(y_0')\rangle|\,\mu(dx')\\&\geqslant\pi_1(F)^{-1}\pi_1(E)^{-1}\|u'(y_0')\|,\end{aligned}$$

so $\|u\|\leqslant\pi_1(E)\,\pi_1(F)\,(\mu\otimes\nu)\,(|\langle u,\cdot\rangle|)$. Thus $\pi_1(E\overset{\vee}{\otimes}F)\leqslant\pi_1(E)\,\pi_1(F)$.

Now let $1_E=uv$ and $1_F=st$ be arbitrary factorizations through $C(K)$ and $C(M)$, respectively. Since $(u\otimes s)\circ(v\otimes t)$ is the identity on $E\overset{\vee}{\otimes}F$ and $C(K)\overset{\vee}{\otimes}C(M)=C(K\times M)$, $\gamma_\infty(E\overset{\vee}{\otimes}F)\leqslant\|u\|\,\|s\|\,\|v\|\,\|t\|$, so that $\gamma_\infty(E\overset{\vee}{\otimes}F)\leqslant\gamma_\infty(E)\gamma_\infty(F)$. Let Z be one of the spaces E, F or $E\overset{\vee}{\otimes}F$. Since Z has enough symmetries $\gamma_\infty(Z)\pi_1(Z)=\dim Z$, so (a) and (b) follow by combining inequalities. For (c) $\gamma_1(E\overset{\wedge}{\otimes}F)=\gamma_1((E'\overset{\vee}{\otimes}F')')=\gamma_\infty(E'\overset{\vee}{\otimes}F')$.

To show (e) consider the factorization of J_n given by

$$c_1(l_2^n)\xrightarrow[A]{}c_p(l_2^n)\xrightarrow[B]{}c_1(l_2^n)\xrightarrow[J_n]{}c_2(l_2^n),\tag{1}$$

with A, B the identities. By Theorem 2.3 and Lemma 4.2,

$$n/3 \leqslant \gamma_1(J_n BA) \leqslant \|A\| \pi_1(J_n B)\, \gamma_1\backslash(c_p(l_2^n)) \leqslant \|A\|\,\|B\|\,\pi_1(J_n)\, \gamma_1\backslash(c_p(l_2^n))$$
$$\leqslant 3n^{1/2+1/p'} \gamma_1\backslash(c_p(l_2^n)).$$

Factoring the operator I_n of Theorem 2.3 in a similar manner gives $n^{1/2-1/p} \lesssim \gamma_1\backslash(c_p(l_2^n))$, so the lower estimate holds. But $c_2(l_2^n)$ is isometric to a subspace of $L_1[0, 1]$ so that $\gamma_1\backslash(c_p(l_2^n)) \leqslant d(c_p, c_2) \leqslant n^{|1/p-1/2|}$. The second part of (e) follows from $/\gamma_\infty(c_p(l_2^n)) = \gamma_1\backslash(c_p(l_2^n)')$ and $c_{p'}(l_2^n) = c_p(l_2^n)'$.

To prove (d) first suppose that $1 \leqslant p \leqslant 2$. In the sequence (1) let $R = J_n B$. Then

$$i_1(J_n B) \leqslant \pi_1(J_n B)\, \gamma_\infty(c_p(l_2^n)) \leqslant \pi_1(J_n BA)\, \|A^{-1}\|\, \gamma_\infty(c_p(l_2^n)) \leqslant 3n^{1/2} \|A^{-1}\|\, \gamma_\infty(c_p(l_2^n))$$

by Theorem 2.3. But also $n^2 \leqslant \|R^{-1}\|\, i_1(R)$ so that $n^2 \leqslant 3n^{1/2} \|A^{-1}\|\, \|R^{-1}\| \gamma_\infty(c_p)$. But $\|A^{-1}\| \leqslant n^{1/p'}$ and $\|R^{-1}\| \leqslant n^{1/p-1/2}$ since $1 \leqslant p \leqslant 2$, and thus $n/3 \leqslant \gamma_\infty(c_p(l_2^n))$. But the projection constant is always at most the square root of the dimension. For $2 \leqslant p \leqslant \infty$, a similar argument may be applied with I_n.

Remark. The estimate given in (e) partially verifies a conjecture of [20] by showing that the best distance from c_p to a subspace of L_p behaves like $n^{|1/p-1/2|}$ for $1 \leqslant p \leqslant 2$. This is the case since by [13], L_p is isometric to a subspace of L_1.

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