Aust. J. Phys., 1979, **32**, 49–59

Absorption of Curvature Radiation

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Abstract

The reabsorption of curvature radiation, i.e. radiation from relativistic electrons moving along curved magnetic field lines, is discussed. The optical depth for the ray path is calculated by use of the Einstein coefficients. It is shown that the optical depth becomes negative (maser effect) if transitions between Landau levels are absent. However, maser action is ineffective if the energy density of the relativistic particles is less than that of the magnetic field. For pulsar radio emission the magnetic energy density is assumed to exceed the particle energy density, so the observed emission cannot be coherent curvature radiation.

1. Introduction

Curvature radiation from relativistic particles plays an important role in the theory of pulsars. Ruderman and Sutherland (1975) believe it to be responsible for the production of hard γ rays which decay into electron–positron pairs and populate the neutron star magnetosphere with charged particles. Goldreich *et al.* (1972) invoked curvature radiation to explain pulsar radio emission: they made use of the fact that the curvature of the magnetic field lines is sufficiently small ($R_B \gg r_B$, where r_B is the gyroradius) that the characteristic frequency of curvature radiation is much less than the synchrotron frequency of a given relativistic particle. This enabled them to propose a common explanation for the optical and radio emission from the Crab pulsar by means of synchrotron and curvature radiation from electrons near the light cylinder.

However, the extremely high brightness temperatures for pulsars (up to 10^{31} K) raise doubts concerning the relevance of any incoherent mechanism for the radio emission. This raises the question of what kind of coherent mechanism could enhance the curvature radiation to the observed level. Antenna mechanisms are unlikely, since these require the creation of electron bunches (with dimensions less than a wavelength) which must survive and emit coherently for a sufficiently long time. Consequently, it seems desirable to search for conditions under which a maser version of curvature radiation could be realized.

A maser corresponds to negative absorption, that is, amplification of radiation along the ray path. Blandford (1975) attempted to find such amplification for curvature radiation but his result remains uncertain. This problem has also been discussed recently by Melrose (1977), who concludes that amplification of curvature radiation is impossible. His conclusion, however, is not entirely convincing because his initial formulae for the amplification coefficient are rather intuitive. Therefore we feel that it may be in order to reconsider this problem.

2. Narrow-beam Pattern Approximation

The geometry of our problem is as follows (see Fig. 1). Relativistic particles ('electrons'*) move along the field lines of an inhomogeneous magnetic field $\boldsymbol{B}$, with a radius of curvature R_B of the field lines. The nonrelativistic gyrofrequency is $\omega_B = eB/m_0 c$ and the momentum distribution function is $f(\boldsymbol{p})$. It is known that the motion of charged particles along curved field lines is accompanied by a drift in the z direction (the z axis is perpendicular to the page in Fig. 1). The drift velocity v_z is determined by the condition that the Lorentz force be equal to the centrifugal force:

$$|ec^{-1}\boldsymbol{v}\times\boldsymbol{B}| = \mathscr{E}c^{-2}v_\phi^2/R_B, \tag{1}$$

where $\mathscr{E}$ is the particle energy and v_ϕ is the tangential velocity. According to this condition the momentum component p_z of the particle along z is given by

$$p_z = p_\phi^2/m_0\,\omega_B R_B. \tag{2}$$

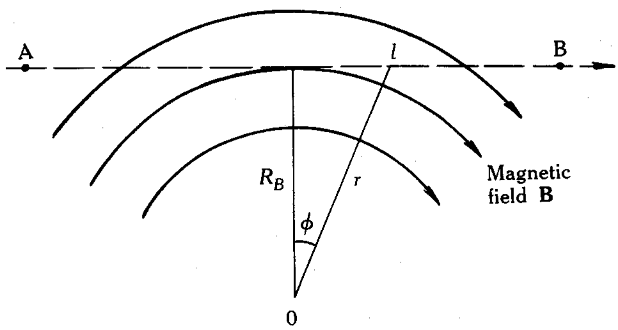

Fig. 1. Assumed field line geometry. Note that the magnetic field lines lie in the plane of the figure. The ray itself is at an angle θ to this plane and the line AB is the projection of the ray onto the plane. The z direction is perpendicular to the plane.

Our task is to calculate and analyse the expressions for the optical depth τ_j along the ray for a mode j:

$$\tau_j = \int_{l_2}^{l_1} \mu_j\,\mathrm{d}l. \tag{3}$$

Here μ_j is the absorption coefficient for curvature radiation and dl is an element of the ray path. By appealing to the Einstein coefficients the absorption coefficient is found to be (see e.g. Zheleznyakov 1977)

$$\mu_j = \{(2\pi)^3 c^2/n_j^2\,\omega^2\} \sum_{(m)\leftrightarrow(n)} A_m^n(N_n - N_m), \tag{4}$$

where n_j is the refractive index of the weakly anisotropic medium, A_m^n is the probability of spontaneous radiation (for curvature radiation in our case), N_m and N_n are the occupation numbers of the quantum states before and after the electron emits a photon, and the summation is over all possible transitions involving a radiation frequency ω.

* To be specific, we shall examine particles with charge $+e$ and rest mass m_0.

In the semiclassical approximation one has

$$\mathscr{P}_{\omega\Omega} = \hbar\omega A^n_m, \qquad N_m = f(\boldsymbol{p}_m), \qquad N_n = f(\boldsymbol{p}_n), \tag{5}$$

where $\mathscr{P}_{\omega\Omega}$ is the power radiated by a single particle (in an unspecified mode) per unit frequency ω and per unit solid angle Ω, and $\boldsymbol{p}_m$ and $\boldsymbol{p}_n$ are the particle momenta before and after emitting the photon with energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$. We then obtain

$$\tau_j = \frac{(2\pi)^3 c^2}{n_j^2 \hbar\omega^3} \int \mathscr{P}_{\omega\Omega}\{f(\boldsymbol{p}_n) - f(\boldsymbol{p}_m)\}\, \mathrm{d}^3\boldsymbol{p}\, \mathrm{d}l\,. \tag{6}$$

Let us find the expression for τ_j in the ultrarelativistic limit $\mathscr{E}/m_0 c^2 \to \infty$ when the width of the curvature radiation pattern may be assumed negligibly small (in other words, all the radiation is concentrated along the particle velocity direction*). In this limit we have

$$\mathscr{P}_{\omega\Omega} = P_\omega(R_B, p)\, \delta(\boldsymbol{\Omega} - \boldsymbol{\Omega}^k), \tag{7a}$$

with

$$\boldsymbol{p}_m = \boldsymbol{p}_n + \hbar\boldsymbol{k} \qquad (\boldsymbol{p} \parallel \boldsymbol{k}). \tag{7b}$$

In equation (7a) the delta function is defined by

$$\int \delta(\boldsymbol{\Omega} - \boldsymbol{\Omega}^k)\, \mathrm{d}\Omega = \int \delta(\boldsymbol{\Omega} - \boldsymbol{\Omega}^k)\, \mathrm{d}\Omega^k = 1\,, \tag{8}$$

where $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^k$ are unit vectors along $\boldsymbol{p}$ and $\boldsymbol{k}$, and $\mathrm{d}\Omega$ and $\mathrm{d}\Omega^k$ are elements of solid angle in $\boldsymbol{p}$ space and $\boldsymbol{k}$ space respectively.

Now consider a small variation of the distribution function $f(\boldsymbol{p})$ over the range $\Delta\boldsymbol{p} = \hbar\boldsymbol{k}$. The relations (7) imply

$$\begin{aligned} f(\boldsymbol{p}_n) - f(\boldsymbol{p}_m) &= f(\boldsymbol{p}_n) - f(\boldsymbol{p}_n + \hbar\boldsymbol{k}) \\ &\approx -(\mathrm{d}f/\mathrm{d}\boldsymbol{p})\,.\,\hbar\boldsymbol{k} = -(\mathrm{d}f/\mathrm{d}\boldsymbol{p})\,.\,\boldsymbol{\Omega}^k \hbar\omega c^{-1} n_j\,. \end{aligned} \tag{9}$$

Consequently,

$$\tau_j = -\frac{(2\pi)^3 c}{n_j \omega^2} \int P_\omega\, \delta(\boldsymbol{\Omega} - \boldsymbol{\Omega}^k) \frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{p}}\,.\,\boldsymbol{\Omega}^k p^2\, \mathrm{d}\Omega\, \mathrm{d}p\, \mathrm{d}l\,. \tag{10}$$

Integration over solid angle using the delta function then yields

$$\tau_j = -\frac{(2\pi)^3 c}{n_j \omega^2} \int P_\omega(R_B, p^k) \frac{\mathrm{d}f}{\mathrm{d}\boldsymbol{p}^k}\,.\,\boldsymbol{\Omega}^k (p^k)^2\, \mathrm{d}p^k\, \mathrm{d}l\,. \tag{11}$$

The classical distribution function $f(\boldsymbol{p})$ of drifting particles is assumed to be of the form

$$f(\boldsymbol{p}) = f(p_r, p_\phi, p_z) = \delta(p_r)\, \delta(p_z - p_\phi^2/m_0 \omega_B R_B)\, F(p_\phi)\,, \tag{12}$$

* This assumption is valid if the radiation beamwidth $\Delta\theta$ is much less than the angle $\psi \approx p_z/p_\phi$. From the formula (2) we obtain the condition $(p/m_0 c)^2 \gg \omega_B R_B/c$ (for $\theta \ll 1$ and the beamwidth $\Delta\theta \approx m_0 c^2/\mathscr{E}$).

with the normalization

$$\int f(p)\,\mathrm{d}^3p = \int F(p_\phi)\,\mathrm{d}p_\phi = N, \tag{13}$$

where N is a constant independent of the coordinates (r, ϕ, z).

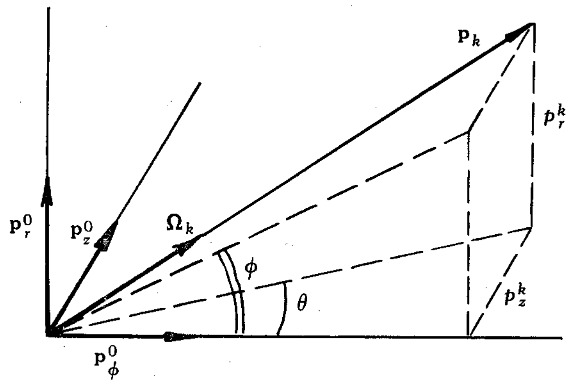

Fig. 2. Coordinate system used and relevant angles.

We now assume that the angles θ and ϕ shown in Fig. 2 are small, i.e.

$$\theta \ll 1, \qquad \phi \ll 1. \tag{14}$$

(The condition $\theta \ll 1$ implies $p_z \ll p_\phi$ or, equivalently, $p_\phi/m_0 c \ll \omega_B R_B/c$. The requirement $\phi \ll 1$ is fulfilled for $\Delta\phi \sim \Delta\theta \ll 1$; in this case the integration limits along the ray in the definition (3) may always be chosen to give $\phi \ll 1$ over the range which makes the important contribution to the optical depth τ_j.) With the conditions (14), we set

$$p^k = p_\phi^k, \qquad p_r^k = p_\phi^k \phi, \qquad p_z^k = p_\phi^k \theta, \qquad \mathrm{d}l = R_B\,\mathrm{d}\phi, \tag{15a}$$

$$\Omega_\phi^k = 1, \qquad \Omega_r^k = \phi, \qquad \Omega_z^k = \theta. \tag{15b}$$

Then the product $\mathbf{\Omega}^k\,\mathrm{d}f/\mathrm{d}p^k$ has the forms

$$\mathbf{\Omega}\,.\,\mathrm{d}f/\mathrm{d}p \approx (\partial f/\partial p_\phi) + (\partial f/\partial p_r)\phi + (\partial f/\partial p_z)\theta \tag{16a}$$

$$= \phi\,\delta'(p_\phi \phi)\,\delta(p_\phi \theta - bp_\phi^2)\,F(p_\phi) + (\theta - 2bp_\phi)\,\delta(p_\phi \phi)\,\delta'(p_\phi \theta - bp_\phi^2)\,F(p_\phi)$$
$$+ \delta(p_\phi \phi)\,\delta(p_\phi \theta - bp_\phi^2)\,\mathrm{d}F/\mathrm{d}p_\phi\,. \tag{16b}$$

(Here and below for simplicity of writing we omit the superscript k on $\boldsymbol{p}^k$, the vector with magnitude p directed along $\boldsymbol{k}$, and on the unit vector $\mathbf{\Omega}^k$.) In deriving the last equality (16b), we have assumed the form (12) for $f(\boldsymbol{p})$, taken the relations (15) into account and introduced the notation

$$b = (m_0\,\omega_B R_B)^{-1}.$$

Substituting equation (16b) into (11) and integrating over the delta functions δ and their derivatives δ', we obtain finally

$$\tau_j = \frac{(2\pi)^3 c}{n_j \omega^2} m_0 \omega_B R_B^2 \frac{F(p_\phi)}{p_\phi^2} \frac{\mathrm{d}(p_\phi^2 P_\omega)}{\mathrm{d}p_\phi}\bigg|_{p_\phi = \theta m_0 \omega_B R_B} . \tag{17}$$

The momentum $p_\phi = \theta m_0 \omega_B R_B$ is that for those drifting electrons which move at an angle θ to the plane of the magnetic field lines. It is natural that in the narrow-beam approximation it is these particles which determine the optical depth for radiation along a ray inclined to the given plane by an angle θ.

According to equation (17) the optical depth τ_j is negative (i.e. amplification takes place) under the condition

$$\mathrm{d}(p_\phi^2 P_\omega)/\mathrm{d}p_\phi < 0 . \tag{18}$$

The expression for the specific power $P_\omega(p_\phi)$ of the curvature radiation may be obtained from the similar expression for the synchrotron radiation by the replacement $\omega_B \to p_\phi/m_0 R_B$ (see e.g. Zheleznyakov 1977). This means that, in a weakly anisotropic rarefied plasma with refractive index close to unity ($1 - n_j^2 \approx \omega_L^2/\omega^2 \ll 1$), the product $p_\phi^2 P_\omega$ is given by (for electrons with $p_z \ll p_\phi$)

$$p_\phi^2 P_\omega \approx \frac{1}{2\sqrt{3}\pi} \frac{e^2 \omega (m_0 c)^2}{c} \left\{1 + \left(1 - n_j^2\right)\left(\frac{p_\phi}{m_0 c}\right)^2\right\}$$
$$\times \left\{\int_{\omega/\omega_c}^{\infty} \mathrm{K}_{5/3}(x)\,\mathrm{d}x + \frac{1 - K_j^2}{1 + K_j^2} \mathrm{K}_{2/3}\left(\frac{\omega}{\omega_c}\right)\right\}, \tag{19}$$

where

$$\omega_c = \frac{3c}{2R_B}\left(\frac{p_\phi}{m_0 c}\right)^3 \left\{1 + \left(1 - n_j^2\right)\left(\frac{p_\phi}{m_0 c}\right)^2\right\}^{-3/2} \tag{20}$$

and K_j is the polarization coefficient for the extraordinary ($j = 1$) and the ordinary ($j = 2$) modes. It is not difficult to see that equation (19) implies $\mathrm{d}(p_\phi^2 P_\omega)/\mathrm{d}p_\phi > 0$. Thus the condition (18) cannot be satisfied and amplification of curvature radiation is impossible in the approximation considered.

3. General Expression for Optical Depth in Finite Radiation Pattern Approximation

Let us consider a specific configuration of the magnetic field, assuming it to be created by a linear current I flowing along the z axis. The vector potential A of such a field is given by

$$A = -z^0 a \ln r , \tag{21}$$

where z^0 is the unit vector along the z axis, a is a constant proportional to the current I, and r is the distance between the z axis and the given point. The magnetic field lines in the case (21) are circular with centres on the z axis. An electron moves in a circular orbit of radius R_B along the field line in the plane $z =$ const. and drifts in the direction z^0. The components of the electron momentum along the field line (p_ϕ) and along the current (p_z) are related to each other as before through equation (2). The above trajectory of the particle will be called 'the drift orbit'. In the semiclassical approximation the electron motion is quantized according to the Bohr–Sommerfeld law

$$\oint p_\phi \, \mathrm{d}\xi = 2\pi n\hbar;$$

we have $\mathrm{d}\xi = R_B \mathrm{d}\phi$ and consequently

$$p_\phi R_B = n\hbar, \tag{22}$$

where n is a positive integer.

Let an electron make a transition from one drift orbit m to another orbit n and radiate a photon with energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$ at an angle α to the z axis. In this case the conservation laws for energy and for the z component of momentum are

$$\mathscr{E}_m - \mathscr{E}_n \equiv (m_0^2 c^4 + p_{zm}^2 c^2 + p_{\phi m}^2 c^2)^{\frac{1}{2}} - (m_0^2 c^4 + p_{zn}^2 c^2 + p_{\phi n}^2 c^2)^{\frac{1}{2}} = \hbar\omega, \tag{23a}$$

$$(\boldsymbol{p} + ec^{-1}\boldsymbol{A})_{zm} - (\boldsymbol{p} + ec^{-1}\boldsymbol{A})_{zn} = \hbar\omega c^{-1} n_j \cos\alpha. \tag{23b}$$

Here one must bear in mind that the generalized momentum of an electron in a magnetic field is $\boldsymbol{p} + ec^{-1}\boldsymbol{A}$. From equation (21) we have

$$A_{zm} - A_{zn} = -B\Delta R_B. \tag{24}$$

In the semiclassical limit $\hbar \to 0$ we obtain from equations (23)

$$\hbar\omega = c^2(p_\phi \Delta p_\phi + m_0 \omega_B p_z \Delta R_B)/\mathscr{E}(1 - \beta_z n_j \cos\alpha), \tag{25}$$

with $\beta_z = v_z/c$. (For simplicity, the index m is omitted on p_z, p_ϕ, R_B, β_z and $\mathscr{E}$.)

From equation (22) it follows that, for an electron transition from the quantum state m to the state n, the values $\Delta p_\phi \equiv p_{\phi m} - p_{\phi n}$ and $\Delta R_B \equiv R_{Bm} - R_{Bn}$ are related by

$$R_B \Delta p_\phi + p_\phi \Delta R_B = s\hbar \qquad (s \equiv m - n). \tag{26}$$

Using the relations (2) and (26), we obtain from equation (25) a set of frequencies for curvature radiation in a magnetic field with circular field lines:

$$\omega = s\Omega/(1 - \beta_z n_j \cos\alpha), \qquad \Omega = v_\phi/R_B. \tag{27}$$

Then from equations (2) and (23) it follows that

$$\Delta p_z = 2p_\phi \Delta p_\phi / m_0 \omega_B R_B, \qquad p_z \Delta p_z + p_\phi \Delta p_\phi = \mathscr{E} c^{-2} \hbar\omega.$$

Combining these relations we obtain

$$\Delta p_z = 2v_z \hbar\omega/(c^2 + v_z^2), \qquad \Delta p_\phi = v_\phi \hbar\omega/(c^2 + v_z^2). \tag{28}$$

The optical depth τ_j along the path in this approximation has, as before, the form (6). Now, however, the change in the momentum $\boldsymbol{p}$ due to emission of a photon differs from that given by equation (7b), and instead we have

$$f(\boldsymbol{p}_m) - f(\boldsymbol{p}_n) \approx (\partial f/\partial p_\phi)\Delta p_\phi + (\partial f/\partial p_z)\Delta p_z, \tag{29}$$

where Δp_ϕ and Δp_z are determined by the expressions (28). The distribution function $f(\boldsymbol{p})$ still has the form (12). The quantity $F(p_\phi)$ is assumed to be different from zero only for values of $p_\phi \gg m_0 c$ (ultrarelativistic electrons) and $\beta_z^2 \ll 1$ (slow enough drift along the z axis). Integration of equation (6) over $\mathrm{d}p_z$, $\mathrm{d}p_r$ and $\mathrm{d}l \approx R_B \,\mathrm{d}\phi$ then gives

$$\tau_j = \frac{(2\pi)^4 c R_B}{n_j^2 \omega^2} \int \frac{F(p_\phi)}{p_\phi^2} \frac{\mathrm{d}(p_\phi^2 P_{\omega\Omega})}{\mathrm{d}p_\phi} \mathrm{d}p_\phi, \tag{30}$$

where

$$P_{\omega\Omega} \equiv P_{\omega\Omega}\big(R_B, \omega, p_\phi, \psi(p_\phi), \alpha\big) = (2\pi)^{-1} \int_0^{2\pi} \mathscr{P}_{\omega\Omega} \,\mathrm{d}\phi, \tag{31}$$

with $\psi \approx p_z/p_\phi$ being the angle between $\boldsymbol{p}$ and the plane of the magnetic field. After integration by parts the expression (30) becomes

$$\tau_j = -\frac{(2\pi)^4 c R_B}{n_j^2 \omega^2} \int \frac{\mathrm{d}}{\mathrm{d}p_\phi}\left(\frac{F(p_\phi)}{p_\phi^2}\right) p_\phi^2 P_{\omega\Omega} \,\mathrm{d}p_\phi. \tag{32}$$

The formulae for the angular distribution of synchrotron radiation from an electron moving along a helical orbit are well known. By making the replacement $\omega_B \to p_\phi/m_0 R_B$ in these formulae we obtain the following expression for the power in curvature radiation as a function of angle α:

$$P_{\omega\Omega} = \frac{n_j e^2 \gamma_j^2 \omega^2 R_B}{6\pi^3 c^2 p_\phi^2}\left\{p_\phi x \mathrm{K}_{2/3}(q) + \left(\frac{g_j p_\phi(1 - n_j \beta_z \cos\alpha)}{n_j \sin\alpha} + h_j p_z\right) x^{\frac{1}{2}} \mathrm{K}_{1/3}(q)\right\}^2, \tag{33}$$

where

$$q = \tfrac{1}{3} s x^{3/2}, \qquad x = 1 - \frac{\chi^2}{s^2}, \qquad \chi = \frac{s n_j \beta_\phi \sin\alpha}{1 - n_j \beta_z \cos\alpha}, \tag{34a}$$

$$g_j = T_j \sin\alpha + K_j \cos\alpha, \qquad h_j = T_j \cos\alpha - K_j \sin\alpha, \tag{34b}$$

$$\gamma_j^2 = (1 + K_j^2)^{-1}. \tag{34c}$$

Here the index j denotes the extraordinary ($j = 1$) and ordinary ($j = 2$) modes and s is the harmonic number; it follows from the relation (27) that we have $s \gg 1$. For the relations (34b, c), in a rarefied plasma we have $T_j = 0$ and, for quasi-linear (QL) propagation, $K_1 = 1$ and $K_2 = -1$ while, for quasi-transverse (QT) propagation, $K_1 = 0$ and $K_2 = \infty$. When differentiating over p_ϕ in equation (30) one should bear in mind that p_z and β_z in (33) depend on p_ϕ (see equation 2).

4. Transition to Narrow-beam Pattern Approximation

The expression (32) for τ_j simplifies in the narrow-radiation pattern approximation for $\Delta\alpha \ll \psi \approx p_z/p_\phi$. By virtue of equation (2) one has $\psi \approx p_\phi/m_0 \omega_B R_B$ and the integration over p_ϕ in equation (32) may be replaced by that over ψ. For $\Delta\alpha \ll \psi$ the function $P_{\omega\Omega}$ is sharp with a maximum near $\psi = \theta \equiv \frac{1}{2}\pi - \alpha$. By evaluating the smooth function multiplying $P_{\omega\Omega}$ in equation (32) at the point $p_\phi/m_0 \omega_B R_B = \theta$ and taking it outside the integral, we obtain

$$\tau_j \approx -\frac{(2\pi)^3 m_0 c \omega_B R_B^2}{n_j^2 \omega^2} \frac{\mathrm{d}}{\mathrm{d}p_\phi}\left(\frac{F(p_\phi)}{p_\phi^2}\right) p_\phi^2 P_\omega \bigg|_{p_\phi = \theta m_0 \omega_B R_B}. \tag{35}$$

Here it has been taken into account that, in the case of a narrow-beam pattern for $\psi \ll 1$ and $\theta \ll 1$, one has

$$2\pi \int P_{\omega\Omega}\, d\psi \approx 2\pi \int P_{\omega\Omega}\, d\theta = P_\omega,$$

which is just the total power in curvature radiation in a single mode. One can easily see that the optical depth τ_j as given by equation (35) differs radically from the result (17) obtained in Section 2 for the narrow-beam pattern. Here, from (35), negative values of τ_j occur in the directions $\theta = p_\phi/mR_B\omega_B$ whenever the derivative of $p_\phi^{-2}F(p_\phi)$ is positive.

To resolve the dilemma as to which of the expressions (17) or (35) is correct for the narrow pattern, it is necessary for us to analyse the range of validity of the formula (3). This formula is obtained under the assumption that photon emission is accompanied by an electron transition from one drift orbit to another. However, the motion of an electron in a smoothly inhomogeneous magnetic field involves both drift motion and rotation about a field line. In a homogeneous magnetic field this rotation (with momentum $p_\perp$) is quantized. It corresponds to Landau levels spaced by

$$\Delta p_\perp^2 = 2s'm_0\omega_B, \tag{36}$$

where s' is an integer. In a smoothly inhomogeneous magnetic field these levels are quasi-stationary, with the approximate separation between them still defined by equation (36). When the photon is emitted, the Landau levels are not excited if the change in the drift component squared satisfies $\Delta p_z^2 \ll \Delta p_\perp^2$. According to equations (2), (28) and (36) this inequality, which is the condition of validity of the expression (3) for the optical depth, takes the form (we have $v_z \ll c$ and $\mathscr{E} \approx p_\phi c$)

$$\psi^3 \approx p_z^3/p_\phi^3 \ll c/2\omega R_B. \tag{37}$$

It is not difficult to establish the order-of-magnitude relation $c/2\omega R_B \sim (\Delta\theta)^3$, where $\Delta\theta$ is the beamwidth of the curvature radiation. Indeed in a vacuum, where we have $\omega \sim \omega_{\max} \approx (c/R_B)(p_\phi/m_0c)^3$, the beamwidth is $\Delta\theta \approx m_0c/p_\phi \sim (c/R_B\omega)^{\frac{1}{3}}$. At lower frequencies ($\omega \ll \omega_{\max}$) we have $\Delta\theta \sim (c/R_B\omega)^{\frac{1}{3}}$; at high frequencies ($\omega \gg \omega_{\max}$) both P_ω and τ_j are exponentially small and this region is of no interest. In the case when the medium exerts a dominant influence upon curvature radiation (i.e. when $\omega \ll \omega_L p_\phi/m_0c$), the beamwidth is $\Delta\theta \sim \omega_L/\omega$ for $\omega \sim (\omega_L^3 R_B/c)^{\frac{1}{2}}$, that is, $\Delta\theta \sim (c/R_B\omega)^{\frac{1}{3}}$. This approximation for $\Delta\theta$ remains valid for $\omega \gg (\omega_L^3 R_B/c)^{\frac{1}{2}}$ as well; for $\omega \ll (\omega_L^3 R_B/c)^{\frac{1}{2}}$ the absorption again becomes exponentially small. It follows that, for curvature radiation and absorption, the Landau levels are not excited if the beamwidth $\Delta\theta$ is sufficiently great,

$$(\Delta\theta)^3 \gg \psi^3. \tag{38}$$

Only in this case, where transitions between Landau levels can be neglected, is the formula (30) for the optical depth τ_j relevant to curvature radiation.

(From a classical point of view the meaning of the criterion (38) is quite evident. Indeed, this inequality implies that the number of electron revolutions around the field line is $\omega_B(m_0c/p_\phi)\Delta t \gg 1$, where $\Delta t = (R_B/c)\Delta\theta$ is the characteristic emission time along the given ray path for an electron moving along the field line with a radius of curvature R_B. It is clear that in this case one may average the particle motion over the fast rotation with frequency $\omega_B m_0 c/p_\phi$ and disregard this rotation altogether when studying curvature radiation and absorption.)

The narrow-pattern approximation corresponds to the opposite inequality to (38), namely $\Delta\theta \ll \psi$. The formula (35) for τ_j obtained in this approximation is incorrect because it ignores possible transitions between Landau levels. On the other hand, in the momentum conservation law (7b), $\Delta \boldsymbol{p} = \hbar \boldsymbol{k}$, which is the basis for the formula (17), transitions between Landau levels are automatically taken into account. (When the photon is emitted, the electron, according to equation (7b), transits from one drift orbit to a state which corresponds to a different drift orbit with a different value of the transverse momentum p_ϕ.)

Thus, in the narrow-pattern approximation, we can conclude that the correct expression for τ_j is just the formula (17), according to which τ_j is strictly positive, and negative curvature absorption does not occur.

5. Wide-beam Pattern Approximation

Let us now investigate curvature absorption for a wide-beam pattern; that is, under the condition (38) when transitions between Landau levels do not occur. In this case the optical depth is described by the formulae (30) and (33), from which it follows that

$$\tau_j = \frac{4\Omega_L^2 m_0 c}{3\Omega_c^2 N(1+K_j^2)} \int \mathrm{d}p_\phi \frac{F(p_\phi)}{p_\phi}\Big(h\,\mathrm{K}_{2/3}(q) + K_j(\theta-\psi)h^{\frac{1}{2}}\mathrm{K}_{1/3}(q)\Big)$$
$$\times\Big(h\,\mathrm{K}_{2/3}(q) + 3\{(m_0 c/p_\phi)^2 + (\theta-\psi)\psi\}q\,\mathrm{K}_{1/3}(q)$$
$$+K_j h^{-\frac{1}{2}}[3(\theta-\psi)\{(m_0 c/p_\phi)^2+(\theta-\psi)\psi\}q\,\mathrm{K}_{2/3}(q)$$
$$+(\theta-2\psi)h\,\mathrm{K}_{1/3}(q)]\Big). \tag{39}$$

Here

$$\Omega_L^2 = 4\pi e^2 N/m_0, \qquad \Omega_c = c/R_B, \tag{40a}$$

$$q = (\omega/3\Omega_c)h^{3/2}, \qquad h = (m_0 c/p_\phi)^2 + (\theta-\psi)^2. \tag{40b}$$

For the ordinary mode ($j = 2$) in the QT approximation, $K_j = \infty$ (linear polarization with the electric vector $\boldsymbol{E}$ along the z axis, which is the electron drift direction), and in this case

$$\tau_2 = \frac{4\Omega_L^2 m_0 c}{3\Omega_c^2 N}\int \mathrm{d}p_\phi \frac{F(p_\phi)}{p_\phi}(\theta-\psi)\mathrm{K}_{1/3}(q)\,[3(\theta-\psi)\{(m_0 c/p_\phi)^2+(\theta-\psi)\psi\}q\,\mathrm{K}_{2/3}(q)$$
$$+(\theta-2\psi)h\,\mathrm{K}_{1/3}(q)]. \tag{41}$$

At frequencies $\omega \sim \Omega_c(p_\phi/m_0 c)^3$ corresponding to the maximum power of the curvature radiation, the beamwidth is $\Delta\theta \approx m_0 c/p_\phi$ and the condition (38) reduces to the inequality $\psi \ll m_0 c/p_\phi$. Then for the directions $|\theta| \lesssim m_0 c/p_\phi$ in equation (41) one may put

$$(m_0 c/p_\phi)^2 + (\theta-\psi)\psi \approx (m_0 c/p_\phi)^2. \tag{42}$$

From this result we can conclude that the absorption is positive ($\tau_2 > 0$) for rays with $\theta > 2\psi$ and $\theta < \psi$ (the functions $K_{1/3}(q)$ and $K_{2/3}(q)$ are positive for all values of their arguments). However, for $\psi < \theta < 2\psi$ the optical depth τ_2 may be negative. Indeed, within this latter interval, we have $h \approx (m_0 c/p_\phi)^2$ and $q \approx (\omega/3\Omega_c)(m_0 c/p_\phi)^3$, and therefore

$$\tau_2 \approx \frac{4\Omega_L^2 m_0 c}{3\Omega_c^2 N} \int \mathrm{d}p_\phi \frac{F(p_\phi)}{p_\phi}\left(\frac{m_0 c}{p_\phi}\right)^2 (\theta-\psi)K_{1/3}(q)$$
$$\times \{3(\theta-\psi)q\,K_{2/3}(q) + (\theta-2\psi)K_{1/3}(q)\}. \qquad (43)$$

Now assume that the distribution function $F(p_\phi)$ has a maximum at the point p_ϕ^* and differs from zero only within the interval $\Delta p_\phi \ll p_\phi^*$. Then the integrand in the formula (43), except for $F(p_\phi)$, may be evaluated at $p_\phi = p_\phi^*$ and taken outside the integral sign, to yield

$$\tau_2 \approx \frac{4\Omega_L^2}{3\Omega_c^2}\left(\frac{m_0 c}{p_\phi^*}\right)^3 (\theta-\psi^*)K_{1/3}(q^*)\{3(\theta-\psi^*)q^* K_{2/3}(q^*) + (\theta-2\psi^*)K_{1/3}(q^*)\}, \qquad (44)$$

where the asterisk denotes values of ψ and q taken at $p_\phi = p_\phi^*$. The range of the values of θ/ψ^* within which curvature absorption becomes negative is shown in Fig. 3 as a function of the parameter q^*.

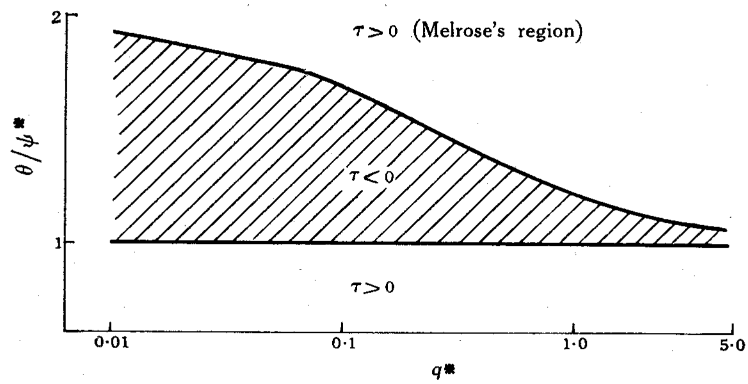

Fig. 3. Range of parameters for which the optical depth τ is negative in the case of ordinary mode QT propagation.

We have not found negative absorption here for QT propagation of the extraordinary mode or for QL propagation of either mode. However, it should be emphasized that the present investigation of τ_j in the wide-beam approximation is based on the formula (39) in which the influence of the medium is ignored ($n_j = 1$). In fact, the polarization of the modes does depend on the properties of the medium, which may be a system of low energy (but relativistic) electrons or even a system of radiating (and absorbing) particles itself. The characteristics of the polarization of modes propagating almost along a magnetic field in such relativistic plasmas need further investigation.

Although we have found above that τ_2 may be negative in the wide-beam approximation, the reversal of the sign of the absorption coefficient alone is not sufficient for a maser mechanism to be effective. For this to occur a second criterion

$|\tau_j| \gg 1$ must also be satisfied and, for curvature absorption, this means that we cannot have

$$D \equiv 8\pi \mathscr{E} N/B^2 \lesssim 1\,. \tag{45}$$

To see this, rewrite equation (44) as

$$\tau_2 = -\tfrac{2}{3} D(m_0\, c/p_\phi^*)^2\, \Phi(\theta/\psi^*, q^*)\,, \tag{46}$$

where

$$\Phi(\theta/\psi^*, q^*) = (\theta/\psi^* - 1)\mathrm{K}_{1/2}(q^*)\{-3(\theta/\psi^* - 1)q^*\, \mathrm{K}_{2/3}(q^*) + (2-\theta/\psi^*)\mathrm{K}_{1/3}(q^*)\}\,. \tag{47}$$

Now consider the possible range of $|\tau_2|$ from equation (46) for variation in q^*:
(i) For $q^* \ll 1$, in the region where τ_2 is negative we have

$$\Phi(\theta/\psi^*, q^*) \approx 2{\cdot}5(p_\phi^*/m_0\, c)^2(3\Omega_c/\omega)^{\frac{2}{3}}(\theta/\psi^* - 1)(2-\theta/\psi^*)$$

and, after substituting this expression into equation (46), we find that, under the condition (45), $|\tau_2|$ is much less than unity (since curvature radiation occurs at $\omega \gg \Omega_c$).
(ii) For $q^* \sim 1$, in the negative τ_2 region we have

$$0 < \Phi(\theta/\psi^*, q^*) \lesssim 1$$

and, since $m_0\, c/p_\phi^* \ll 1$, it follows from the formula (46) that under the condition (45) we again have $|\tau_2| \ll 1$.
(iii) Finally, for $q^* \gg 1$ the function $\Phi(\theta/\psi^*, q^*)$ and hence the optical depth τ_2 are exponentially small.

Thus we arrive at the conclusion that, although a maser mechanism of curvature radiation is possible, in order to be effective it requires the energy density of the relativistic particles to exceed that of the magnetic field ($D \gg 1$). In our problem we have $D \ll 1$, since it is only in this case that the motion of the charged particles is directed by the magnetic field and that curvature radiation occurs in the form usually discussed in pulsar theory. It is not inconceivable that in neutron star magnetospheres there are regions with $D \gg 1$. However, in such regions the character of particle motion (and hence the radiation and absorption) will differ fundamentally from that considered in the present report.

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Manuscript received 17 July 1978