

Absorption Rate of the Kerr-de Sitter Black Hole and the Kerr-Newman-de Sitter Black Hole

Hisao SUZUKI,^{1,*} Eiichi TAKASUGI^{2,**} and Hiroshi UMETSU^{2,***}

¹*Department of Physics, Hokkaido University, Sapporo 060-0810, Japan*

²*Department of Physics, Osaka University, Toyonaka 560-0043, Japan*

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By using an analytic solution of the Teukolsky equation in the Kerr-de Sitter and Kerr-Newman-de Sitter geometries, an analytic expression of the absorption rate formulae for these black holes is calculated.

§1. Introduction

In a series of papers,^{1),2)} we have developed a method to obtain an analytic solution of the Teukolsky equation³⁾ in Kerr geometries, the perturbation equation of massless particles in Kerr geometries. This method is applied to evaluate the gravitational wave from a binary of neutron stars.⁴⁾ Recently, we extended this method to Kerr-de Sitter and Kerr-Newman-de Sitter geometries and showed that an analytic solution can be similarly obtained.⁵⁾ It has been shown that the analytic solution obtained in Ref. 5) can be analytically continued to cover the entire physical region.⁶⁾ It should be noted that in Kerr-Newman-de Sitter geometries, electromagnetic and gravitational perturbations couple to each other, and thus these particles are excluded.

In this paper, we evaluate the absorption rate of the Kerr-de Sitter and the Kerr-Newman-de Sitter black holes by using the analytic solution. We construct the conserved current by evaluating the Wronskian, and we obtain an expression of the absorption rate. From this, we show explicitly that super-radiance occurs for the boson case similarly to the Kerr geometry case.²⁾

The paper is organized as follows. In §2 we summarize the construction of analytic solutions in order to define parameters involved in them. In §3 we choose the solution which satisfies the incoming boundary conditions at the outer horizon of the black hole and examine the asymptotic behavior at the de Sitter horizon. Then, we derive analytic expressions of the incident, the reflection and the transmission amplitudes. In §4 we derive the conserved current, from which we derive the absorption rate. A summary is given in §5.

*) E-mail: hsuzuki@particle.sci.hokudai.ac.jp

**) E-mail: takasugi@het.phys.sci.osaka-u.ac.jp

***) E-mail: umetsu@het.phys.sci.osaka-u.ac.jp

§2. The analytic solution

In the Boyer-Lindquist coordinates, the Kerr-Newman-de Sitter metric has the form

$$ds^2 = -\rho^2 \left(\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} \right) - \frac{\Delta_\theta \sin^2 \theta}{(1+\alpha)^2 \rho^2} [adt - (r^2 + a^2)d\varphi]^2 + \frac{\Delta_r}{(1+\alpha)^2 \rho^2} (dt - a \sin^2 \theta d\varphi)^2, \quad (2.1)$$

where $\alpha = \Lambda a^2/3$, $\rho^2 = \bar{\rho}\bar{\rho}^*$, $\bar{\rho} = r + ia \cos \theta$ and

$$\begin{aligned} \Delta_r &= (r^2 + a^2) \left(1 - \frac{\alpha}{a^2} r^2 \right) - 2Mr + Q^2 \\ &= -\frac{\alpha}{a^2} (r - r_+)(r - r_-)(r - r'_+)(r - r'_-), \\ \Delta_\theta &= 1 + \alpha \cos^2 \theta. \end{aligned} \quad (2.2)$$

Here Λ is the cosmological constant, M is the mass of the black hole, aM is its angular momentum, and Q is its charge.

Next, we deal with the Teukolsky equation. We assume that the time and the azimuthal angle dependence of the field are described by $e^{-i(\omega t - m\phi)}$. Then, the radial part of the equation with spin s and charge e is given by

$$\left\{ \Delta_r^{-s} \frac{d}{dr} \Delta_r^{s+1} \frac{d}{dr} + \frac{1}{\Delta_r} \left[(1+\alpha)^2 \left(K - \frac{eQr}{1+\alpha} \right)^2 - is(1+\alpha) \left(K - \frac{eQr}{1+\alpha} \right) \frac{d\Delta_r}{dr} \right] + 4is(1+\alpha)\omega r - \frac{2\alpha}{a^2} (s+1)(2s+1)r^2 + s(1-\alpha) - 2iseQ - \lambda_s \right\} R_s = 0, \quad (2.3)$$

with $K = \omega(r^2 + a^2) - am$. This equation has five regular singularities, at r_\pm, r'_\pm and ∞ , which are assigned such that $r_\pm \rightarrow M \pm \sqrt{M^2 - a^2 - Q^2} \equiv r_\pm^0$ and $r'_\pm \rightarrow \pm \frac{a}{\sqrt{\alpha}}$ in the limit $\alpha \rightarrow 0$ ($\Lambda \rightarrow 0$). In Ref. 5), it is shown that

$$\lambda_s = \lambda_{-s}. \quad (2.4)$$

Next, we define the variables x and z as

$$x = 1 - z = \frac{(r_- - r'_-)(r - r_+)}{(r_- - r_+)(r - r'_-)}. \quad (2.5)$$

This transformation maps the outer horizon r_+ , the inner horizon r_- , the de Sitter horizon r'_+ , and ∞ to 0, 1, x_r , and x_∞ , respectively:

$$\begin{aligned} x_r = 1 - z_r &= \frac{(r_- - r'_-)(r'_+ - r_+)}{(r_- - r_+)(r'_+ - r'_-)}, \\ x_\infty = 1 - z_\infty &= \frac{(r_- - r'_-)}{(r_- - r_+)}. \end{aligned} \quad (2.6)$$

Now, we give the solution of the Teukolsky equation which satisfies the incoming boundary condition on the outer horizon. Before doing so, we define parameters to specify the solution:

$$\begin{aligned} \sigma_+ &= 2a_2 - s + 1, & \sigma_- &= -2a_1 - 2a_3 + 1, \\ \gamma &= -2a_1 - s + 1, & \delta &= 2a_2 + s + 1, & \epsilon &= -2a_3 - s + 1, \\ \omega_H &\equiv \gamma + \delta - 1 = \sigma_+ + \sigma_- - \epsilon = -2a_1 + 2a_2 + 1. \end{aligned} \tag{2.7}$$

Here

$$\begin{aligned} a_1 &= i \frac{a^2(1+\alpha)}{\alpha} \frac{\left(\omega(r_+^2 + a^2) - am - \frac{eQr_+}{1+\alpha}\right)}{(r'_+ - r_+)(r'_- - r_+)(r_- - r_+)}, \\ a_2 &= i \frac{a^2(1+\alpha)}{\alpha} \frac{\left(\omega(r_-^2 + a^2) - am - \frac{eQr_-}{1+\alpha}\right)}{(r'_+ - r_-)(r'_- - r_-)(r_+ - r_-)}, \\ a_3 &= i \frac{a^2(1+\alpha)}{\alpha} \frac{\left(\omega(r_+^{\prime 2} + a^2) - am - \frac{eQr'_+}{1+\alpha}\right)}{(r_- - r'_+)(r'_- - r'_+)(r_+ - r'_+)}, \\ a_4 &= i \frac{a^2(1+\alpha)}{\alpha} \frac{\left(\omega(r_-^{\prime 2} + a^2) - am - \frac{eQr'_-}{1+\alpha}\right)}{(r_- - r'_-)(r'_+ - r'_-)(r_+ - r'_-)}, \end{aligned} \tag{2.8}$$

and the relation $a_1 + a_2 + a_3 + a_4 = 0$ is satisfied.

With these parameters, the solution can be expressed as

$$\begin{aligned} R_{\text{in};s}^\nu &= \tilde{A}_s R_{\text{in};\{0,1\};s}^\nu \\ &= \tilde{A}_s \left[K_\nu(s) R_{\{z_r, \infty\};s}^\nu(z) + K_{-\nu-1}(s) R_{\{z_r, \infty\};s}^{-\nu-1}(z) \right], \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} R_{\text{in};\{0,1\};s}^\nu &= (-x)^{-s-a_1} (1-x)^{a_2} \left(\frac{x-x_r}{1-x_r}\right)^{-s-a_3} \left(\frac{x-x_\infty}{1-x_\infty}\right)^{2s+1} \\ &\quad \times \sum_{n=-\infty}^{\infty} a_n^\nu(s) F\left(-n-\nu+\frac{\omega_H}{2}-\frac{1}{2}, n+\nu+\frac{\omega_H}{2}+\frac{1}{2}; \gamma; x\right), \\ R_{\{z_r, \infty\};s}^\nu(z) &= z^{a_2} (z-1)^{-s-a_1} \left(1-\frac{z}{z_r}\right)^{-s-a_3} \left(1-\frac{z}{z_\infty}\right)^{2s+1} \\ &\quad \times \sum_{n=-\infty}^{\infty} a_n^\nu(s) \frac{\Gamma(n+\nu+\sigma_+\frac{\omega_H}{2}+\frac{2}{3})\Gamma(n+\nu-\sigma_-\frac{\omega_H}{2}+\frac{2}{3})}{\Gamma(2a_2+2a_3+1)\Gamma(2n+2\nu+2)} \left(\frac{z}{z_r}\right)^{n+\nu+a_1-a_2} \\ &\quad \times F\left(n+\nu+\sigma_+\frac{\omega_H}{2}+\frac{1}{2}, n+\nu+\sigma_-\frac{\omega_H}{2}+\frac{1}{2}; 2n+2\nu+2; \frac{z}{z_r}\right), \end{aligned} \tag{2.10}$$

and the proportionality constant K_ν is determined by comparing the coefficients of $z^{r+\nu-\frac{\omega_H}{2}+\frac{1}{2}}$ as

$$\begin{aligned}
 K_\nu = & \frac{z_r^{r+\nu-\frac{\omega_H}{2}+\frac{1}{2}} \Gamma(\gamma) \Gamma(\sigma_+ - \epsilon + 1)}{\Gamma(r + \nu - \frac{\omega_H-3}{2}) \Gamma(r + \nu + \delta - \frac{\omega_H-1}{2}) \Gamma(r + \nu + \sigma_+ - \frac{\omega_H-1}{2}) \Gamma(r + \nu + \sigma_- - \frac{\omega_H-1}{2})} \\
 & \left[\sum_{n=r}^{\infty} a_n^\nu \frac{(-)^{n-r} \Gamma(n + \nu - \frac{\omega_H-3}{2}) \Gamma(n + \nu + \delta - \frac{\omega_H-1}{2}) \Gamma(r + n + 2\nu + 1)}{\Gamma(n + \nu + \frac{\omega_H+1}{2}) \Gamma(n + \nu + \gamma - \frac{\omega_H-1}{2}) (n-r)!} \right] \\
 & \left[\sum_{n=-\infty}^r \frac{a_n^\nu \Gamma(n + \nu - \sigma_+ + \frac{\omega_H+3}{2}) \Gamma(n + \nu - \sigma_- + \frac{\omega_H+3}{2})}{\Gamma(n + \nu + \sigma_+ - \frac{\omega_H-1}{2}) \Gamma(n + \nu + \sigma_- - \frac{\omega_H-1}{2}) \Gamma(r + n + 2\nu + 2) (r-n)!} \right]^{-1}, \tag{2.11}
 \end{aligned}$$

which should be independent of r , an integer value.

The coefficients are determined by solving the three-term recurrence relation

$$\alpha_n^\nu a_{n+1}^\nu + \beta_n^\nu a_n^\nu + \gamma_n^\nu a_{n-1}^\nu = 0, \tag{2.12}$$

where

$$\begin{aligned}
 \alpha_n^\nu = & - \frac{(n + \nu - \frac{\omega_H-3}{2})(n + \nu - \sigma_+ + \frac{\omega_H+3}{2})(n + \nu - \sigma_- + \frac{\omega_H+3}{2})(n + \nu + \delta - \frac{\omega_H-1}{2})}{2(n + \nu + 1)(2n + 2\nu + 3)}, \\
 \beta_n^\nu = & \frac{(1 - \omega_H)(\gamma - \delta)(\sigma_+ - \sigma_- + \epsilon - 1)(\sigma_+ - \sigma_- - \epsilon + 1)}{32(n + \nu)(n + \nu + 1)} + \left(\frac{1}{2} - x_r\right) (n + \nu)(n + \nu + 1) \\
 & + \frac{1}{4}[\epsilon(\gamma - \delta) + \delta(1 - \omega_H) + 2\sigma_+\sigma_-] + \frac{\omega_H^2 - 1}{4}x_r + \nu, \\
 \gamma_n^\nu = & - \frac{(n + \nu + \sigma_+ - \frac{\omega_H+1}{2})(n + \nu + \sigma_- - \frac{\omega_H+1}{2})(n + \nu + \gamma - \frac{\omega_H+1}{2})(n + \nu + \frac{\omega_H-1}{2})}{2(n + \nu)(2n + 2\nu - 1)}, \tag{2.13}
 \end{aligned}$$

with an appropriate initial condition. Here we set $a_0^\nu = a_0^{-\nu-1} = 1$. Then, we find that $a_{-n}^{-\nu-1} = a_n^\nu$ is satisfied.

The solution is characterized by the characteristic exponent ν (the shifted angular momentum), which is determined so that the coefficients are convergent as $n \rightarrow \pm\infty$. Since the solution is expressed by the sum of hypergeometric functions, the convergence of the series is examined. From the behavior of coefficients as $n \rightarrow \pm\infty$, we find that the solution $R_{\text{in};\{0,1\};s}^\nu$ converges for $r < r'_+$, and $R_{\{z_r,\infty\};s}^\nu$ converges for $r > r_+$, where r_+ and r'_+ ($r_+ \ll r'_+$) are the outer horizon and the de Sitter horizon, respectively. Therefore, the second expression of $R_{\text{in};s}^\nu$ is the analytic continuation of the first expression. By combining these two expressions, we can obtain a solution that covers the entire physical region.

We now summarize the properties of this solution. We showed that $\nu(s) = \nu(-s)$, which is crucial for solutions with spin s and $-s$ to satisfy the Teukolsky-Starobinsky

identity.^{7),8)} We showed explicitly that our solution satisfies the T-S identity, and as a consequence we can fix the relative normalization such that

$$\begin{aligned} \tilde{A}_s &= C_s^* \left[-\frac{a^2}{\alpha(r_+ - r_-)(r'_+ - r_-)(r'_- - r_-)} \right]^{2s} \\ &\times \frac{\Gamma(-2a_1 + s + 1)}{\Gamma(-2a_1 - s + 1)} \left| \frac{\Gamma(\nu + a_1 + a_2 - s + 1)}{\Gamma(\nu + a_1 + a_2 + s + 1)} \right|^2, \quad (s > 0) \end{aligned} \quad (2.14)$$

provided $\tilde{A}_{-s} = 1$ ($s > 0$). Here, C_s is the Starobinsky constant.⁹⁾

§3. Asymptotic behavior

In general, the asymptotic behavior of the solution is found by examining the Teukolsky equation to be¹⁰⁾

$$\begin{aligned} R_s &\longrightarrow R_s^{(\text{trans})} \Delta_r^{-s} e^{-ikr_*}, & (r \rightarrow r_+) \\ &\longrightarrow R_s^{(\text{inc})} \Delta_r^{-s} e^{-ipr_*} + R_s^{(\text{ref})} e^{ipr_*}, & (r \rightarrow r'_+) \end{aligned} \quad (3.1)$$

where k and p are defined by

$$\begin{aligned} k &= \frac{(1 + \alpha) \left[\omega(r_+^2 + a^2) - am - \frac{eQr_+}{1+\alpha} \right]}{r_+^2 + a^2}, \\ p &= \frac{(1 + \alpha) \left[\omega(r'_+{}^2 + a^2) - am - \frac{eQr'_+}{1+\alpha} \right]}{r'_+{}^2 + a^2}. \end{aligned} \quad (3.2)$$

Here r_* is defined by $\frac{dr_*}{dr} = \frac{r^2+a^2}{\Delta_r}$ and behaves asymptotically as

$$\begin{aligned} r_* &\longrightarrow \frac{-ia_1}{k} \ln(-x), & (r \rightarrow r_+) \\ &\longrightarrow \frac{-ia_3}{p} \ln \left(1 - \frac{z}{z_r} \right). & (r \rightarrow r'_+) \end{aligned} \quad (3.3)$$

Here, we give the explicit expressions of $R_s^{(\text{inc})}$, $R_s^{(\text{ref})}$ and $R_s^{(\text{trans})}$ by using our analytic solution $R_{\text{in};s}^\nu$.

From the behavior around $x = 1$, we find

$$\begin{aligned} R_s^{(\text{trans})} &= \tilde{A}_s \left[\frac{\alpha(r_+ - r_-)^2 (r_- - r'_-)^2 (r'_+ - r_-)}{a^2 (r_+ - r'_-)} \right]^s \left(\frac{-x_r}{1 - x_r} \right)^{-a_3} \left(\frac{-x_\infty}{1 - x_\infty} \right)^{-2s+1} \\ &\times \sum_{n=-\infty}^{\infty} a_n^\nu(s). \end{aligned} \quad (3.4)$$

Next, we consider the behavior near the cosmological horizon, i.e. as $z \rightarrow z_r$. First, we consider $R_{\{z_r, \infty\};s}^\nu$ and find

$$R_{\{z_r, \infty\};s}^\nu \longrightarrow A_{+;s}^\nu \Delta_r^{-s} e^{-ipr_*} + A_{-;s}^\nu e^{ipr_*}, \quad (3.5)$$

where

$$\begin{aligned}
 A_{+;s}^\nu &= \left[\frac{\alpha(r_+ - r_-)^2(r_- - r'_-)^2(r'_+ - r_-)}{a^2(r_+ - r'_-)} \right]^s z_r^{s+a_2} (z_r - 1)^{-a_1} \left(1 - \frac{z_r}{z_\infty} \right)^{-2s+1} \\
 &\quad \times \frac{\Gamma(2a_3 + s)}{\Gamma(2a_2 + 2a_3 + 1)} \sum_{n=-\infty}^\infty a_n^\nu(s), \\
 A_{-;s}^\nu &= z_r^{a_2} (z_r - 1)^{-s-a_1} \left(1 - \frac{z_r}{z_\infty} \right)^{2s+1} \frac{\Gamma(-2a_3 - s)}{\Gamma(2a_2 + 2a_3 + 1)} \\
 &\quad \times \sum_{n=-\infty}^\infty a_n^\nu(s) \frac{\Gamma(n + \nu + a_3 - a_4 + 1)\Gamma(n + \nu - a_1 - a_2 + s + 1)}{\Gamma(n + \nu - a_3 + a_4 + 1)\Gamma(n + \nu + a_1 + a_2 - s + 1)}. \quad (3.6)
 \end{aligned}$$

Since $R_{\text{in};s}^\nu$ is expressed by the linear combination of $R_{\{z_r, \infty\};s}^\nu$ and $R_{\{z_r, \infty\};s}^{-\nu-1}$, Eq. (2.9), we also need $A_{+;s}^{-\nu-1}$ and $A_{-;s}^{-\nu-1}$. These are given by using the explicit forms of $A_{+;s}^\nu$ and $A_{-;s}^\nu$ as

$$\begin{aligned}
 A_{+;s}^{-\nu-1} &= A_{+;s}^\nu, \\
 A_{-;s}^{-\nu-1} &= \frac{\sin \pi(\nu + a_3 - a_4) \sin \pi(\nu - a_1 - a_2 + s)}{\sin \pi(\nu - a_3 + a_4) \sin \pi(\nu + a_1 + a_2 - s)} A_{-;s}^\nu. \quad (3.7)
 \end{aligned}$$

Now, we obtain the asymptotic amplitudes $R_s^{(\text{inc})}$ and $R_s^{(\text{ref})}$ as

$$\begin{aligned}
 R_s^{(\text{inc})} &= \tilde{A}_s [K_\nu(s) + K_{-\nu-1}(s)] A_{+;s}^\nu, \\
 R_s^{(\text{ref})} &= \tilde{A}_s \left[K_\nu(s) + \frac{\sin \pi(\nu + a_3 - a_4) \sin \pi(\nu - a_1 - a_2 + s)}{\sin \pi(\nu - a_3 + a_4) \sin \pi(\nu + a_1 + a_2 - s)} K_{-\nu-1}(s) \right] A_{-;s}^\nu. \quad (3.8)
 \end{aligned}$$

Below, we give some useful relations to study the conserved current and absorption rate:

$$\begin{aligned}
 \frac{A_{+;s}^\nu}{A_{+;-s}^\nu} &= \left[\frac{\alpha}{a^2} (r'_+ - r_-)^2 (r_+ - r_-) (r'_+ - r'_-) \right]^{2s} \frac{\Gamma(2a_3 + s)}{\Gamma(2a_3 - s)} \frac{\sum_{n=-\infty}^\infty a_n^\nu(s)}{\sum_{n=-\infty}^\infty a_n^\nu(-s)}, \\
 \frac{A_{-;s}^\nu}{A_{-;-s}^\nu} &= \left[\frac{(r_+ - r_-)(r_- - r'_-)}{(r'_+ - r_+)(r'_+ - r'_-)} \right]^{2s} \frac{\Gamma(-2a_3 - s)}{\Gamma(-2a_3 + s)} \left| \frac{\Gamma(\nu + a_1 + a_2 + s + 1)}{\Gamma(\nu + a_1 + a_2 - s + 1)} \right|^2. \quad (3.9)
 \end{aligned}$$

These lead to the following relations for the asymptotic amplitudes:

$$\begin{aligned}
 \frac{R_s^{(\text{inc})}}{R_{-s}^{(\text{inc})}} &= \frac{1}{C_s} \left[\frac{\alpha}{a^2} (r'_+ - r_+)(r'_+ - r_-)(r'_+ - r'_-) \right]^{2s} \frac{\Gamma(2a_3 + s)}{\Gamma(2a_3 - s)}, \\
 \frac{R_s^{(\text{ref})}}{R_{-s}^{(\text{ref})}} &= C_s^* \left[\frac{\alpha}{a^2} (r'_+ - r_+)(r'_+ - r_-)(r'_+ - r'_-) \right]^{-2s} \frac{\Gamma(-2a_3 - s)}{\Gamma(-2a_3 + s)},
 \end{aligned}$$

$$\frac{R_s^{(\text{trans})}}{R_{-s}^{(\text{trans})}} = \frac{1}{C_s} \left[\frac{\alpha}{a^2} (r_+ - r_-)(r_+ - r'_+) (r_+ - r'_-) \right]^{2s} \frac{\Gamma(2a_1 + s)}{\Gamma(2a_1 - s)}. \quad (3.10)$$

Here we have used Eq. (4.29) in Ref. 6) which is the relation between the sums of the coefficients with spin weights s and $-s$.

§4. The conserved current and the absorption rate

The conserved current ⁷⁾ is obtained by examining the Wronskian between the incoming solution $R_{\text{in};s}^\nu$ and the outgoing solution $R_{\text{out};s}^\nu = (\Delta^{-s} R_{\text{in};-s}^\nu)^*$ on the outer horizon. We find

$$\begin{aligned} & \left[\Delta_r^{s+1} \left(R_{\text{in};s}^\nu \frac{d}{dr} (\Delta_r^{-s} R_{\text{in};-s}^\nu)^* - (\Delta_r^{-s} R_{\text{in};-s}^\nu)^* \frac{d}{dr} R_{\text{in};s}^\nu \right) \right]_{r=r_+} \\ &= \left[\Delta_r^{s+1} \left(R_{\text{in};s}^\nu \frac{d}{dr} (\Delta_r^{-s} R_{\text{in};-s}^\nu)^* - (\Delta_r^{-s} R_{\text{in};-s}^\nu)^* \frac{d}{dr} R_{\text{in};s}^\nu \right) \right]_{r=r'_+}. \end{aligned} \quad (4.1)$$

Then, by substituting the asymptotic behavior (3.1), we find

$$\begin{aligned} R_s^{(\text{inc})} (R_{-s}^{(\text{inc})})^* &= R_s^{(\text{ref})} (R_{-s}^{(\text{ref})})^* \\ &\quad - \frac{(r_+ - r_-)(r_+ - r'_-)(s + 2a_1)}{(r'_+ - r_-)(r'_+ - r'_-)(s + 2a_3)} R_s^{(\text{trans})} (R_{-s}^{(\text{trans})})^*. \end{aligned} \quad (4.2)$$

This relation can be rewritten by using the relations (3.10) as

$$\begin{aligned} |R_s^{(\text{inc})}|^2 &= \frac{1}{|C_s|^2} \left[\frac{\alpha}{a^2} (r'_+ - r_+)(r'_+ - r_-)(r'_+ - r'_-) \right]^{4s} \left| \frac{\Gamma(2a_3 + s)}{\Gamma(2a_3 - s)} \right|^2 |R_s^{(\text{ref})}|^2 \\ &\quad + \delta_s |R_s^{(\text{trans})}|^2, \end{aligned} \quad (4.3)$$

where

$$\delta_s = \left[-\frac{(r_+ - r_-)(r_+ - r'_-)}{(r'_+ - r_-)(r'_+ - r'_-)} \right]^{-2s+1} \frac{\Gamma(-2a_1 - s + 1)\Gamma(-2a_3 + s)}{\Gamma(-2a_1 + s)\Gamma(-2a_3 - s + 1)}. \quad (4.4)$$

For $s = 0, \frac{1}{2}, 1, \frac{3}{2}$ and 2, δ_s is explicitly given by

$$\begin{aligned} \delta_0 &= \frac{\omega(r_+^2 + a^2) - am - \frac{eQr_+}{1+\alpha}}{\omega(r'_+{}^2 + a^2) - am - \frac{eQr'_+}{1+\alpha}}, \\ \delta_{\frac{1}{2}} &= 1, \\ \delta_1 &= \frac{1}{\delta_0}, \\ \delta_{\frac{3}{2}} &= \left[-\frac{(r'_+ - r_-)(r'_+ - r'_-)}{(r_+ - r_-)(r_+ - r'_-)} \right]^2 \frac{\frac{1}{4} - 4a_3^2}{\frac{1}{4} - 4a_1^2}, \\ \delta_2 &= \left[-\frac{(r'_+ - r_-)(r'_+ - r'_-)}{(r_+ - r_-)(r_+ - r'_-)} \right]^2 \frac{(1 - 4a_3^2)}{(1 - 4a_1^2)} \frac{1}{\delta_0}. \end{aligned}$$

We recall that the above formulae are valid for all massless particles with spin 0, 1/2, 1, 3/2 and 2 in the Kerr-de Sitter geometry, but for the Kerr-Newman-de Sitter geometry, the photon and the graviton are exceptions.

The relation in Eq. (4.3) is interpreted as the energy conservation.⁷⁾ That is, the energy balances among the incident energy going into the black hole, the energy reflected by the black hole, and the energy absorbed by the black hole. It is evident that the δ_s are positive definite for fermions, because a_1 and a_3 are purely imaginary. For bosons, δ_s can be negative, and super-radiance occurs when

$$\frac{am + \frac{eQr'_+}{1+\alpha}}{r'^2_+ + a^2} < \omega < \frac{am + \frac{eQr_+}{1+\alpha}}{r_+^2 + a^2} \quad (4.5)$$

in the Kerr-de Sitter (for massless particles with any spin)⁹⁾ and the Kerr-Newman-de Sitter (excluding the photon and graviton) geometry.

We now give a formula for the absorption rate of the black hole in the Kerr-de Sitter geometry using our solution:

$$\begin{aligned} \Gamma_s &= 1 - \frac{1}{|C_s|^2} \left[\frac{\alpha}{a^2} (r'_+ - r_+) (r'_+ - r_-) (r'_+ - r'_-) \right]^{4s} \left| \frac{\Gamma(2a_3 + s)}{\Gamma(2a_3 - s)} \right|^2 \left| \frac{R_s^{(\text{ref})}}{R_s^{(\text{inc})}} \right|^2 \\ &= \delta_s \left| \frac{R_s^{(\text{trans})}}{R_s^{(\text{inc})}} \right|^2 \\ &= \delta_s \left| \frac{(r_+ - r_-)(r_+ - r'_-)}{(r'_+ - r_-)(r'_+ - r'_-)} \right|^{2s} \left(\frac{r'_+ - r'_-}{r_+ - r'_-} \right)^2 \left| \frac{\Gamma(2a_2 + 2a_3 + 1)}{\Gamma(2a_3 + s)} \right|^2 \\ &\quad \times |K_\nu(s) + K_{-\nu-1}(s)|^{-2} \\ &= \frac{1}{\pi^2 z_r^{2\nu+1}} \sin \pi(2a_1 + s) \sin \pi(2a_3 + s) D_s^\nu \left| 1 - \frac{p_s^\nu}{\pi^2 z_r^{2\nu+1} \sin^2 2\pi\nu} D_s^\nu \right|^{-2}, \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} p_s^\nu &= \sin \pi(\nu + a_1 - a_2) \sin \pi(\nu - a_3 + a_4) \\ &\quad \times \sin \pi(\nu + a_1 + a_2 + s) \sin \pi(\nu + a_1 + a_2 - s), \\ D_s^\nu &= |\Gamma(\nu + a_1 - a_2 + 1) \Gamma(\nu - a_3 + a_4 + 1) \\ &\quad \times \Gamma(\nu + a_1 + a_2 + s + 1) \Gamma(\nu + a_1 + a_2 - s + 1)|^2 d_s^\nu, \\ d_s^\nu &= \left| \sum_{n=-\infty}^0 \frac{a_n^\nu(s) \Gamma(n + \nu + a_3 - a_4 + 1) \Gamma(n + \nu - a_1 - a_2 + s + 1)}{\Gamma(n + \nu - a_3 + a_4 + 1) \Gamma(n + \nu + a_1 + a_2 - s + 1) \Gamma(n + 2\nu + 2) (-n)!} \right|^2 \\ &\quad \times \left| \sum_{n=0}^{\infty} a_n^\nu(s) \frac{(-)^n \Gamma(n + \nu + a_1 - a_2 + 1) \Gamma(n + \nu + a_1 + a_2 + s + 1) \Gamma(n + \nu + 1)}{\Gamma(n + \nu - a_1 + a_2 + 1) \Gamma(n + \nu - a_1 - a_2 - s + 1) n!} \right|^{-2}. \end{aligned} \quad (4.7)$$

Here we have set $r = 0$, which was an arbitrary integer, in $K_\nu(s)$ of (2.11). This quantity coincides with that in Ref. 2) in the Kerr-limit ($\Lambda \rightarrow 0$).

§5. Conclusions and discussion

We derived an analytic expression of the absorption rate. In particular, we found analytically that super-radiance occurs for bosons when the frequency satisfies the condition given in Eq. (4.5). In order to examine this property in detail, we must calculate coefficients by solving the three-term recurrence relations in Eq. (2.12). In general, we find

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}^\nu}{a_n^\nu} = \lim_{n \rightarrow -\infty} \frac{a_n^\nu}{a_{n+1}^\nu} = e^{-\xi r}, \quad (5.1)$$

where

$$e^{\xi r} = 1 - 2x_r + \sqrt{(1 - 2x_r)^2 - 1} > 1. \quad (x_r < 0) \quad (5.2)$$

Since x_r is negative and very large for very small Λ ,

$$x_r = \frac{(r_- - r'_-)(r'_+ - r_+)}{(r_- - r_+)(r'_+ - r'_-)} \simeq \frac{r'_-}{2(r_+ - r_-)}, \quad (5.3)$$

$e^{-\xi r}$ is very small. Thus, for larger n , the convergence of the series of coefficients is rapid. In practical cases where Λ is very small, we first expand α_n^ν , β_n^ν and γ_n^ν and also coefficients in terms of the small quantity $\alpha \equiv \Lambda a^2/3$, and then we expand in terms of $\epsilon \equiv 2M\omega$. In this way, we can obtain physical quantities as series in powers of α and ϵ .

At present, we do not know the physical meaning of the analysis presented here in comparison with the Kerr geometry case. However, we hope that our analysis may become important especially when we consider the early universe and also when we wish to obtain deeper insight regarding the correspondence between quantum gravity in anti-de Sitter space and conformal field theory defined on the boundary.^{11), 12)}

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