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Abstract Cesàro Spaces. Optimal Range

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Abstract. Abstract Cesàro spaces are investigated from the optimal domain and optimal range point of view. There is a big difference between the cases on $[0,\infty)$ and on $[0,1]$, as we can see in Theorem 1. Moreover, we present an improvement of Hardy's inequality on $[0,1]$ which plays an important role in these considerations.

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1. Introduction and Basic Definitions

For a Banach ideal space X on $I=[0,1]$ or $I=[0,\infty)$ let us consider, as in [6], the abstract Cesàro space CX on I defined as $CX=\{f\in L^0(I): C|f|\in X\}$ with the norm given by

$$\|f\|_{CX}=\|C|f|\|_X,$$

where C is the Cesàro operator

$$Cf(x)=\frac{1}{x}\int_0^x f(t)\,dt,\ \ x\in I.$$

One may look at these spaces, on one hand, as on generalization of the well-known Cesàro spaces $Ces_p[0,1]$ and $Ces_p[0,\infty)$ which were investigated for example in [1]. On the other hand, CX is the optimal domain of C for X since, just by definition, $C: CX\to X$ is bounded and CX is the largest ideal space satisfying this relation. Consequently, the abstract Cesàro spaces may be considered also from the optimal domain point of view, as it was done in [3,9–11]. In this paper we discuss the Cesàro function spaces on $[0,\infty)$ and on $[0,1]$ from the point of view of optimal domain and optimal range of the Cesàro operator C. Such concept was already considered for $X=L^{p(\cdot)}$ on $[0,1]$ in [10,11] and for $X=L^{p(\cdot)}$ on $\mathbb{R}^n$ in [9], although the most interesting situation of CX on $[0,1]$ was omitted there. We develop and complete the

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discussion under some minimal assumptions. In this more interesting case of interval $[0,1]$ a very important role is played by the improvement of Hardy inequality presented in Theorem 2.

We present some basic definitions to understand further description of results. By $L^0 = L^0(I)$ we denote the space of Lebesgue measurable functions (in fact, respective equivalence classes with respect to equality almost everywhere) on $I = [0,1]$ or $I = [0,\infty)$. A Banach space $X \subset L^0$ is called a Banach ideal space on I if $g \in X, f \in L^0(I), |f| \le |g|$ a.e. on I implies $f \in X$ and $||f|| \le ||g||$. We will also assume that supp$X = I$, i.e. there exists $f \in X$ with $f(x) > 0$ for each $x \in I$.

For a given Banach ideal space X on I and a function $w \in L^0(I)$ such that $w(x) > 0$ a.e. on I, the weighted Banach ideal space $X(w)$ is defined as $X(w) = \{f \in L^0(I) : fw \in X\}$ with the norm

$$\|f\|_{X(w)} = \|fw\|_X.$$

In the whole paper only two concrete weights on $I = [0,1]$ will appear, namely v and $1/v$ where

$$v(x) = 1 - x. \tag{1.1}$$

We will need also a non-increasing majorant $\widetilde{f}$ of a given function f, which is just

$$\widetilde{f}(x) = \text{ess sup}_{t\in I,\, t\ge x} |f(t)|,\ x \in I.$$

Moreover, for a given Banach ideal space X on I, we define a new Banach ideal space $\widetilde{X} = \widetilde{X}(I)$ as $\widetilde{X} = \{f \in L^0(I) : \widetilde{f} \in X\}$ with the norm given by

$$\|f\|_{\widetilde{X}} = \|\widetilde{f}\|_X.$$

By a *symmetric function space* on I with the Lebesgue measure m (symmetric space in short), we mean a Banach ideal space $X = (X, \|\cdot\|_X)$ with the additional property that for any two equimeasurable functions $f \sim g, f, g \in L^0(I)$ (that is, they have the same distribution functions $d_f \equiv d_g$, where $d_f(\lambda) = m(\{x \in I : |f(x)| > \lambda\}), \lambda \ge 0$) and $f \in X$ we have $g \in X$ and $\|f\|_X = \|g\|_X$. In particular, $\|f\|_X = \|f^*\|_X$, where $f^*(t) = \inf\{\lambda > 0\colon\ d_f(\lambda) < t\},\ t \ge 0$.

The *dilation operators* σ_a $(a > 0)$ defined on $L^0(I)$ by

$$\sigma_a f(x) = f(x/a)\chi_I(x/a) = f(x/a)\chi_{[0,\,\min(1,\,a)]}(x),\ x \in I,$$

are bounded in any symmetric space X on I and $\|\sigma_a\|_{X\to X} \le \max(1,a)$ (see [2, p. 148] and [5, pp. 96–98]). They are also bounded in some Banach ideal spaces which are not necessarily symmetric spaces. Furthermore, recall that the Cesàro operator C, the Copson operator C^* and the Hardy–Littlewood maximal operator M are defined, respectively, by

$$Cf(x) = \frac{1}{x}\int_0^x f(t)dt,\ x \in I,\ C^*f(x) = \int_{I\cap[x,\infty)} \frac{f(t)}{t}dt,\ x \in I,$$

$$Mf(x) = \sup_{a,b\in I, 0\le a\le x\le b} \frac{1}{b-a}\int_a^b |f(t)|dt,\ x \in I.$$

We refer the reader to [6], where basic facts about the spaces CX and $\widetilde{X}$ were presented with more details. For more references on Banach ideal spaces and symmetric spaces we refer to [2,4,5,7,8].

2. Optimal Domain and Optimal Range

Let X and Y be two Banach ideal spaces on I and let $T : X \to Y$ be a bounded linear or sublinear operator. A Banach ideal space Z on I is called the *optimal domain* of T for Y within the class of Banach ideal spaces on I, if $T : Z \to Y$ is bounded and for each Banach ideal space W on I, $T : W \to Y$ is bounded implies that $W \subset Z$. The last implication may be formulated equivalently as: if Z and W are Banach ideal spaces on I and if $Z \subsetneq W$, then $T : W \not\to Y$. Of course in such a case $X \subset W$.

Similarly, we shall say that a Banach ideal space Z on I is the *optimal range* of T for X within the class of Banach ideal spaces on I, if $T : X \to Z$ is bounded and for each Banach ideal space W on I, $T : X \to W$ is bounded implies that $Z \subset W$. Once again, the last condition may be replaced by: $W \subsetneq Z$ implies $T : X \not\to W$. Such optimal range satisfies of course $Z \subset Y$.

The following theorem describes the optimal domain and optimal range problem for Cesàro operator within the class of Banach ideal spaces on I.

Theorem 1. *Let X be a Banach ideal space on I such that the maximal operator M is bounded on X.*

(i) *If $I = [0, \infty)$, then $C : CX \to \widetilde{X}$ is bounded. Moreover, the space CX is the optimal domain of C for X and for $\widetilde{X}$ (also for CX if the dilation operator σ_a is bounded on X for some $0 < a < 1$). The space $\widetilde{X}$ is the optimal range of C for CX, X and $\widetilde{X}$. In particular, $CX = C\widetilde{X}$.*

(ii) *If $I = [0, 1]$ and v is from (1.1), then $C : CX \to \widetilde{X(1/v)}(v)$ is bounded. The space CX is the optimal domain of C for X and also for $\widetilde{X(1/v)}(v)$. Moreover, if the maximal operator M is bounded on X', then the space $\widetilde{X(1/v)}(v)$ is the optimal range of C for CX and $X(v)$* (cf. Diagram 2). *In particular, $CX = C[\widetilde{X(1/v)}(v)]$.*

(iii) *If $I = [0, 1]$ and the dilation operator $\sigma_{1/2}$ is bounded on X, then $C : C\widetilde{X} \to \widetilde{X}$ is bounded. Moreover, the space $C\widetilde{X}$ is the optimal domain of C for $\widetilde{X}$ and the space $\widetilde{X}$ is the optimal range of C for $C\widetilde{X}$, X and $\widetilde{X}$. One also has $C\widetilde{X} = CX \cap L^1$.*

Before we prove the theorem, let us comment on the situation. Suppose that the corresponding assumptions in Theorem 1 are satisfied. Of course, boundedness of M on X implies also boundedness of C on X, therefore the support of CX is for sure the same as support of X (cf. [6]). Let $I = [0, \infty)$. Then the statement of (i) may be therefore pictured, putting the boundedness of C and respective embeddings, on Diagram 1.

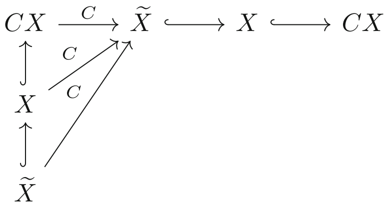

Diagram 1. The case of $I = [0, \infty)$

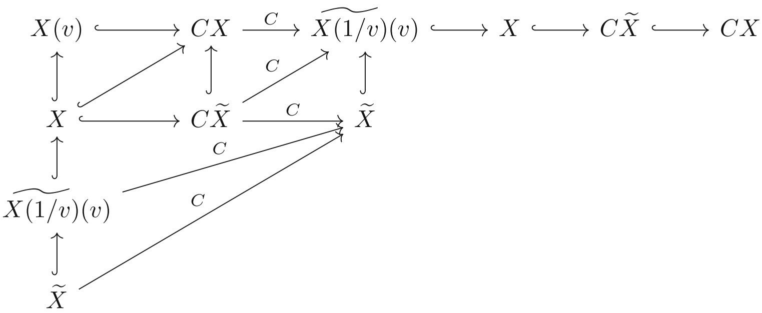

Diagram 2. The case of $I = [0, 1]$

Moreover, point (i) says that, in fact, CX is the optimal domain of C for $\widetilde{X}$, since $CX = C\widetilde{X}$. Even more can be said when the dilation operator σ_a is bounded on X for a certain $0 < a < 1$. Then CX is the optimal domain of C even for CX since, by Lemma 6 in [6], it follows that $CCX = CX$. On the other hand, we will see that $\widetilde{X}$ is the optimal range of C for $\widetilde{X}$, which by Diagram 1 means that also for X and for CX.

Much more interesting and delicate is the case of interval $[0, 1]$. Suppose that $C : X \to X$ is bounded and all assumptions of (ii) and (iii) are satisfied. Then $C : CX \to X$ is bounded, where CX is by definition the optimal domain of C for X. The case (ii) says that the optimal range of C for CX is then $\widetilde{X(1/v)}(v)$. It is however interesting that one may look at the situation also in another way. Let's start once again with $C : X \to X$ and find first the optimal range. It appears to be just $\widetilde{X}$ (cf. [10, Theorem 8.2], [11, Theorem 3.16] and [9, Theorem 4.1]) which is much smaller than $\widetilde{X(1/v)}(v)$. If we now find optimal domain of C for $\widetilde{X}$ it is then just $CX \cap L^1 = C(\widetilde{X})$. The diagram describing this dichotomy is now more complicated (see Diagram 2).

In general, there is no inclusion relation between $X(v)$ and $C\widetilde{X}$. For example, if X is a symmetric space on $I = [0, 1]$, we have for $f(x) := \frac{1}{1-x}$ that $f \in X(v)$ while $f \notin C\widetilde{X}$ because $Cf(x) \to \infty$ as $x \to 1^-$ and so $\widetilde{Cf}$ is not defined (or just ∞ everywhere). Therefore, $X(v) \not\subset C\widetilde{X}$. This means also that C does not act from $X(v)$ into $\widetilde{X}$. On the other hand, let $X = L^2$ and put $f(x) = |\frac{1}{2} - x|^{-1/2}$. Then $f \notin L^2$, but $Cf \in L^\infty$ and so $\widetilde{Cf} \in L^\infty \subset L^2$. This gives $C\widetilde{X} \not\subset X(v)$. For general symmetric space X on I such that $C: X \to X$ is bounded, one could take $f \in L^1$ in such a way that $f - f\chi_{[1/2-\epsilon, 1/2+\epsilon]} \in L^\infty$ for each $0 < \epsilon < 1/2$ but $f \notin X$, to achieve the same effect.

Proof of Theorem 1. (ii). Let $0 \leq f \in CX$. Suppose first that $0 \leq y \leq t \leq 2y \leq 1$. Then

$$Cf(t) = \frac{1}{t}\int_0^t f(s)ds \geq \frac{1}{2y}\int_0^y f(s)ds = \frac{1}{2}Cf(y). \tag{2.1}$$

If now $0 \leq x \leq y$ and $y \leq \frac{1}{2}$, then applying (2.1) one gets

$$\begin{aligned} MCf(x) &\geq \frac{1}{2y - x}\int_x^{2y} Cf(t)dt \geq \frac{1}{2y}\int_y^{2y} Cf(t)dt \\ &\geq \frac{1}{2y}\int_y^{2y} \frac{Cf(y)}{2}dt = \frac{1}{4}Cf(y) \geq \frac{1-y}{4(1-x)}Cf(y). \end{aligned}$$

Suppose now that $\frac{1}{2} \leq y \leq t \leq 1$. Then, similarly as in (2.1),

$$Cf(t) = \frac{1}{t}\int_0^t f(s)ds \geq \int_0^y f(s)ds \geq \frac{1}{2}Cf(y). \tag{2.2}$$

In consequence, when $0 \leq x \leq y$ and $\frac{1}{2} \leq y \leq 1$, applying (2.2) we obtain

$$\begin{aligned} MCf(x) &\geq \frac{1}{1-x}\int_x^1 Cf(t)dt \geq \frac{1}{1-x}\int_y^1 Cf(t)dt \\ &\geq \frac{1}{1-x}\int_y^1 \frac{Cf(y)}{2}dt = \frac{1-y}{2(1-x)}Cf(y). \end{aligned}$$

Consequently,

$$MCf(x) \geq \frac{1}{4(1-x)}\,\text{ess sup}_{0\leq x\leq y\leq 1}(1-y)Cf(y) = \frac{1}{4(1-x)}\widetilde{[vCf]}(x). \tag{2.3}$$

Since M is bounded on X, by our assumption, it follows that

$$\|Cf\|_{\widetilde{X(1/v)}(v)} = \|\widetilde{[vCf]}/v\|_X \leq 4\|M\|_{X\to X}\|Cf\|_X = 4\|M\|_{X\to X}\|f\|_{CX}.$$

This means that $C: CX \to \widetilde{X(1/v)}(v)$ is bounded and the first statement of (ii) is proved. It remains to show that the space $\widetilde{X(1/v)}(v)$ is the optimal range of C for CX (in fact, even for $X(v)$). Suppose that there is a Banach ideal space Z on I such that

$$Z \subsetneq Y \text{ but } C: CX \to Z \text{ is bounded.}$$

Let $0 \leq f \in Y \backslash Z$. Define

$$g(x) = \frac{1}{(1-x)}\widetilde{[vf]}(x), x \in I.$$

Then $f \leq g$ and $g \in \widetilde{X(1/v)}(v) \subset X$ because $\frac{1}{1-x}\widetilde{[vg]}(x) = \frac{1}{1-x}\widetilde{[vf]}(x)$. We have

$$C(g/v)(x) = \frac{1}{x}\int_0^x \frac{\widetilde{[vg]}(t)}{(1-t)^2}dt \geq \frac{\widetilde{[vf]}(x)}{x}\int_0^x \frac{1}{(1-t)^2}dt$$
$$= \frac{\widetilde{[vf]}(x)}{x}\frac{x}{(1-x)} \geq f(x),$$

which means that $C(g/v) \notin Z$. However, $g \in X$ and so $g/v \in X(v)$. Also, by Theorem 2 below, $X(v) \subset CX$ and therefore $g/v \in CX$ which means that $C : CX \not\to Z$. Note that we have already shown $C : X(v) \not\to Z$, which by inclusion $X(v) \subset CX$ means that $\widetilde{X(1/v)}(v)$ is the optimal range also for $X(v)$.

(iii). The argument is analogous to the one from statement (5.1) in [10]. However, we need to modify it because in [10] the maximal operator is defined on a larger interval than $[0, 1]$. Let $0 \leq f \in CX \cap L^1[0, 1]$. We shall understand that $f(x) = 0$ for $x > 1$. Of course, inequality from (2.1) remains true in this case, since $f \in CX$. Suppose that $0 < x \leq y \leq 1$ and consider two cases. If $y/2 \leq x$, then

$$M\sigma_{1/2}Cf(x) \geq \frac{2}{y}\int_{y/2}^y \sigma_{1/2}Cf(u)du.$$

If $x \leq y/2$, then

$$M\sigma_{1/2}Cf(x) \geq \frac{1}{y-x}\int_x^y \sigma_{1/2}Cf(u)du \geq \frac{1}{y}\int_{y/2}^y \sigma_{1/2}Cf(u)du.$$

Altogether we get

$$M\sigma_{1/2}Cf(x) \geq \frac{1}{y}\int_{y/2}^y \sigma_{1/2}Cf(u)du = \frac{1}{2y}\int_y^{2y} Cf(t)dt \geq \frac{1}{4}Cf(y).$$

Therefore, similarly as before,

$$M\sigma_{1/2}Cf(x) \geq \frac{1}{4}\,\text{ess sup}_{x\leq y}\, Cf(y) = \frac{1}{4}\widetilde{Cf}(x),$$

which gives

$$\|f\|_{C\widetilde{X}} = \|\widetilde{Cf}\|_X \leq 4\,\|M\sigma_{1/2}Cf\|_X \leq 4\,\|M\|_{X\to X}\|\sigma_{1/2}\|_{X\to X}\|Cf\|_X$$
$$= 4\,\|M\|_{X\to X}\|\sigma_{1/2}\|_{X\to X}\|f\|_{CX} \leq 4\,\|M\|_{X\to X}\|\sigma_{1/2}\|_{X\to X}\|f\|_{CX\cap L^1}.$$

On the other hand, if $0 \leq f \in C\widetilde{X}$, then

$$\|f\|_{L^1} = \int_0^1 f(t)dt\frac{\|\chi_{[0,1]}\|_X}{\|\chi_{[0,1]}\|_X} = \frac{\|(\int_0^1 f(t)dt)\chi_{[0,1]}\|_X}{\|\chi_{[0,1]}\|_X} \leq \frac{\|\widetilde{Cf}\|_X}{\|\chi_{[0,1]}\|_X}.$$

Thus also

$$\|f\|_{CX\cap L^1} \leq \max\{1, \frac{1}{\|\chi_{[0,1]}\|_X}\}\|\widetilde{Cf}\|_X,$$

which means that $C\widetilde{X} = CX \cap L^1$. For the sake of completeness we present the argument that $\widetilde{X}$ is the optimal range of C for $C\widetilde{X}$, although it works just like in [10, Theorem 8.2]. Let Z be a Banach ideal space on I and suppose that $0 \leq f \in \widetilde{X}\backslash Z$. Then also $\widetilde{f} \in \widetilde{X}\backslash Z$ and $C\widetilde{f} \geq \widetilde{f}$. However $\widetilde{f} \notin Z$, which means that $C\widetilde{f} \notin Z$ and $C : C\widetilde{X} \not\to Z$.

(i) This case is easier and may be deduced directly from [9]. Since for $0 < y$ also $2y \in I$ it is enough to follow (2.1) and after that to get for $y \geq x \geq 0$

$$MCf(x) \geq \frac{1}{2y - x} \int_x^{2y} Cf(t)dt \geq \frac{1}{4} Cf(y).$$

Then

$$\|Cf\|_{\widetilde{X}} = \|\widetilde{Cf}\|_X \leq 4\|MCf\|_X \leq 4\|M\|_{X\to X}\|Cf\|_X = 4\|M\|_{X\to X}\|f\|_{CX},$$

which means that $C : CX \to \widetilde{X}$ is bounded and $CX = C\widetilde{X}$. The optimal range of C for $\widetilde{X}, X, CX$ is once again $\widetilde{X}$ and the proof is the same as in (iii) (see also [10, Theorem 8.2], [11, Theorem 3.16] and [9, Theorem 4.1]). □

3. Hardy Inequality

We present an improvement of the Hardy inequality which appears for spaces on $I = [0, 1]$.

Theorem 2. *If C is bounded on a Banach ideal space X on $I = [0, 1]$ and the maximal operator M is bounded on X', then*

$$C : X(v) \to X$$

is also bounded, where v is from (1.1).

Proof. Let $0 \leq f \in X$. We have for $0 < x \leq \frac{1}{2}$

$$C(f/v)(x) = \frac{1}{x} \int_0^x \frac{f(s)}{1-s} ds \leq \frac{2}{x} \int_0^x f(s)ds$$

and for $\frac{1}{2} < x \leq 1$

$$C(f/v)(x) = \frac{1}{x} \int_0^x \frac{f(s)}{1-s} ds \leq 2 \int_0^x \frac{f(s)}{1-s} ds.$$

If we define the operator T as $Tf(x) = \int_0^x \frac{f(s)}{1-s} ds$, then

$$C(f/v) \leq 2(Cf + Tf).$$

Therefore, we need to show that T is bounded on X. Consider an involution operator $\tau : f(x) \mapsto f(1 - x)$. Then

$$Tf(x) = \int_0^x \frac{f(s)}{1-s} ds = \int_{1-x}^1 \frac{f(1-s)}{s} ds = \tau C^* \tau f(x). \tag{3.1}$$

Observe that the space

$$X^- = \{f : \tau f \in X\}$$

with its natural norm $\|f\|_{X^-} = \|\tau f\|_X$ is also a Banach ideal space on I and so $(X^-)^-$. Just by definition $\sigma : X \to X^-$, $\tau : X^- \to X$ are bounded and $\tau\tau = id$. Thus T is bounded on X if and only if C^* is bounded on X^-. We will prove the last equivalence. Notice that simply

$$Mf(1 - x) = \sup_{a\neq b, 0\leq a\leq 1-x\leq b\leq 1} \frac{1}{b - a} \int_a^b f(s)ds \tag{3.2}$$

$$= \sup_{a \neq b, 0 \leq 1-b \leq x \leq 1-a \leq 1} \frac{1}{b-a} \int_{1-b}^{1-a} f(1-s)ds = (M\tau f)(x) \tag{3.3}$$

and so $M\tau f = \tau M f$ which means that for any Banach ideal space Y, M is bounded on Y if and only if M is bounded on Y^-, which by our assumption gives that M is bounded on $(X')^-$. Thus also C is bounded on $(X')^-$ and by duality C^* is bounded on $[(X')^-]'$. However, it is evident that for any Banach ideal space Y there holds $(Y')^- = (Y^-)'$. Then $[(X')^-]' = (X'')^- = X^-$ and so C^* is bounded on X^-. □

Remark 1. If X is a symmetric space, then evidently $X = X^-$ and we get Lemma 10 from [6], whose proof was a generalization of the Astashkin–Maligranda result from [1]. Moreover, our Theorem 2 includes Theorem 9 in [6] for the weighted $L^p(x^\alpha)$ spaces when $1 \leq p < \infty$ and $-1/p < \alpha < 1-1/p$.

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