Filomat 26:5 (2012), 1005–1022
DOI 10.2298/FIL1205005M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Abstract convexity of extended real valued increasing and radiant functions

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Abstract. In this paper, we first investigate abstract convexity of non-negative increasing and radiant (IR) functions over a topological vector space X. We also characterize the essential results of abstract convexity such as support set, subdifferential set and polarity of this class of functions. Finally, we examine abstract convexity, polarity and subdifferential of extended real valued increasing and radiant functions.

1. Introduction

It is well-known that every proper and lower semi-continuous convex function can be expressed as a point-wise supremum of a family of affine functions majorized by it (see [13]). It is natural to see what happens if we replace affine functions by a certain class of functions which is so-called elementary functions. This gave rise to the subject of Abstract Convexity (for more details see [12, 14, 15]). It is well-known that some classes of increasing functions are abstract convex, for example, the class of increasing and positively homogeneous (IPH) functions and the class of increasing and convex-along-rays (ICAR) functions are abstract convex. The first studies of these functions were carried out over the cones in topological vector spaces (see [2, 3]). Some suitable extensions for these functions defined over the whole of a topological vector space were obtained in [7–9].

The class of increasing and co-radiant (ICR) functions is another class of increasing functions which is abstract convex. The theory of ICR functions can be applied in mathematical economics (see, e.g., [5]), where quasi-concave ICR functions have been studied. The first characterization of these functions has been shown in [13] over the cone $\mathbb{R}^n_+$. This was generalized in [4], where ICR functions defined over cones in a topological vector space. A generalization of ICR functions defined on the whole of a topological vector space has been given in [1].

Recently, abstract convexity of lower semi-continuous and radiant functions have been characterized in [17]. In [18] it has given a characterization for abstract convexity of evenly radiant functions defined on real normed linear spaces. Also, abstract convexity of non-positive increasing and radiant functions defined on a topological vector space has been characterized in [6]. In this paper, we are going to extend the results obtained in [6] for extended real valued IR functions defined on a topological vector space.

2010 *Mathematics Subject Classification*. Primary 26B25; Secondary 26A48, 26A51, 52A30
Keywords. Monotonic analysis, increasing and radiant function, radiant set, co-radiant set, abstract convexity
Received: 28 January 2012; Accepted: 27 March 2012
Communicated by Qamrul Hasan Ansari and Ljubiša D.R. Kočinac
This research was supported partially by Kerman Graduate University of Technology, Mahani Mathematical Research Center and Linear Algebra and Optimization Central of Excellence
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The layout of the paper is as follows. In Section 2, we collect definitions, notations and preliminary results related to IR functions and abstract convexity. In Sections 3 and 4, we obtain some results of abstract convexity of non-negative IR functions and characterize their subdifferential and support sets. We study polarity of non-negative IR functions in Section 5. The relation between non-negative IR and DPH (see Definition 2.2) functions will be given in Section 6. Finally, characterizations of extended real valued IR functions and abstract convexity of this class of functions are given in section 7.

2. Preliminaries

Let X be a topological vector space. We assume that X is equipped with a closed convex pointed cone S (the latter means that $S \cap (-S) = \{0\}$). We say $x \leq y$ or $y \geq x$ if and only if $y - x \in S$.

A function $f : X \longrightarrow [-\infty, +\infty]$ is called radiant if $f(\lambda x) \leq \lambda f(x)$ for all $x \in X$ and all $\lambda \in (0, 1]$. It is easy to see that f is radiant if $f(\lambda x) \geq \lambda f(x)$ for all $x \in X$ and all $\lambda \geq 1$. The function f is called increasing if $x \geq y \implies f(x) \geq f(y)$. A function $f : X \longrightarrow [-\infty, +\infty]$ is called positively homogeneous if $f(\lambda x) = \lambda f(x)$ for all $x \in X$ and all $\lambda > 0$.

Definition 2.1. A function $f : X \to [-\infty, +\infty]$ is called IR if f is increasing and radiant.

Definition 2.2. A function $f : X \to [-\infty, +\infty]$ is called DPH if f is decreasing and positively homogeneous.

Definition 2.3. A function $f : X \to [-\infty, +\infty]$ is called convex-along-rays, if for each $x \in X$ the function $f_x(\alpha) := f(\alpha x)$ $(\alpha \in (0, +\infty))$ is convex.

In this paper, we study IR (increasing and radiant) functions f such that

$$0 \in dom f := \{ x \in X : -\infty < f(x) < +\infty \}.$$

Remark 2.1. Let $f : X \longrightarrow [0, +\infty]$ be an IR function. Then it is clear that $f(x) = 0$ for all $x \in -S$.

The following definitions are well-known (see [14]).

i) A non-empty subset A of X is called downward, if $x \in A$, $x' \in X$ and $x' \leq x$ imply $x' \in A$.

ii) A non-empty subset B of X is called upward, if $x \in B$, $x' \in X$ and $x \leq x'$ imply $x' \in B$.

iii) A non-empty subset A of X is called radiant, if $x \in A$ and $0 < \lambda \leq 1$ imply $\lambda x \in A$. Also, a subset B of X is called co-radiant, if $x \in B$ and $\lambda \geq 1$ imply $\lambda x \in B$.

Now, let X be a set and Δ be a non-empty set of extended real valued functions $l : X \longrightarrow [-\infty, +\infty]$ defined on X. Recall (see [14]) that a function $f : X \longrightarrow [-\infty, +\infty]$ is called abstract convex with respect to Δ (or Δ-convex) if

$$f(x) = \sup\{l(x) : l \in \text{supp}\,(f, \Delta)\},\ \forall\ x \in X,$$

where

$$\text{supp}\,(f, \Delta) := \{l \in \Delta : l \leq f\}$$

is called the support set of the function f, and $l \leq f$ if and only if $l(x) \leq f(x)$ for all $x \in X$. The set Δ will be referred to as a set of elementary functions.

Also, the Δ-subdifferential of a function $f : X \longrightarrow [-\infty, +\infty]$ at a point $x_0 \in X$ is defined by:

$$\partial_\Delta f(x_0) := \{l \in \Delta\ :\ f(x) - f(x_0) \geq l(x) - l(x_0)\ \forall\ x \in X\}.$$

Now, consider the function $u : X \times X \times (-\infty, 0) \longrightarrow [-\infty, 0]$ defined by:

$$u(x, y, \beta) := \sup\{\lambda \leq \beta : \lambda y \geq -x\}, \quad (x,\ y \in X;\ \beta \in (-\infty, 0)), \tag{1}$$

(we use the convention $\sup \emptyset = -\infty$).

This function was introduced and examined in [6]. The following properties of the function u have been proved in ([6], Proposition 3.1). In fact, for every $x,\ y,\ x',\ y' \in X;\ \gamma \in (0,1];\ \mu,\ \beta,\ \beta' \in (-\infty, 0)$, one has

$$u(-\mu x, y, \beta) = -\mu u(x, y, \frac{\beta}{-\mu}), \tag{2}$$

$$u(x, -\mu y, \beta) = \frac{1}{-\mu} u(x, y, -\mu\beta), \tag{3}$$

$$x \leq x' \implies u(x, y, \beta) \leq u(x', y, \beta), \tag{4}$$

$$y \leq y' \implies u(x, y', \beta) \leq u(x, y, \beta), \tag{5}$$

$$\beta \leq \beta' \implies u(x, y, \beta) \leq u(x, y, \beta'), \tag{6}$$

$$u(\gamma x, y, \beta) \leq \gamma u(x, y, \beta), \tag{7}$$

$$u(x, \gamma y, \beta) \geq \frac{1}{\gamma} u(x, y, \beta), \tag{8}$$

$$u(x, y, \beta) = \beta \iff \beta y \geq -x. \tag{9}$$

The following results for non-positive IR functions which will be used later were given in [6]. For each $(y, \beta) \in X \times (-\infty, 0)$, define the function $u_{(y,\beta)} : X \longrightarrow [-\infty, 0]$ by $u_{(y,\beta)}(x) := u(x, y, \beta)$ for all $x \in X$. Let

$$L := \{u_{(y,\beta)} : y \in X,\ \ \beta \in (-\infty, 0)\}.$$

Note that each $u_{(y,\beta)} \in L$ is a non-positive IR function. L is called the set of elementary functions defined on X.

Theorem 2.1. ([6, Theorem 3.1]) *Let $f : X \to [-\infty, 0]$ be a function. Then the following assertions are equivalent:*
(i) f is IR.
(ii) $f(x) \geq -\lambda f(y)$ for all $x,\ y \in X$ and all $\lambda \leq -1$ such that $\lambda y \geq -x$.
(iii) $u(x, y, \beta) f(-\beta y) \geq \beta f(x)$ for all $x,\ y \in X$ and all $\beta \in (-\infty, 0)$, with the convention $0 \times (-\infty) = +\infty$.

Theorem 2.2. ([6, Theorem 3.2]) *Let $f : X \to [-\infty, 0]$ be a function. Then f is an IR function if and only if there exists a set $U \subseteq L$ such that*

$$f(x) = \sup_{u_{(y,\beta)} \in U} u_{(y,\beta)}(x), \quad (x \in X).$$

In this case, one can take $U := \{u_{(y,\beta)} \in L : f(-\beta y) \geq \beta\}$. Hence, f is IR if and only if f is an L-convex function.

3. Abstract convexity of non-negative IR functions

In this section, we discuss on abstract convexity of non-negative IR functions with respect to a certain class of non-negative IR functions. We also investigate subdifferential and a special kind of polarity of these functions.

Consider the function $k : X \times X \times \mathbb{R}_{++} \longrightarrow [0, +\infty]$ defined by:

$$k(x, y, \alpha) := \sup\{\lambda \geq \alpha : \lambda y \leq x\}, \quad (x,\ y \in X;\ \alpha \in \mathbb{R}_{++}), \tag{10}$$

(we use the convention $\sup \emptyset = 0$).

In the following, we give some properties of the function k.

Proposition 3.1. *For every* $x,\ y,\ x',\ y' \in X;\ \gamma \in (0,1];\ \mu,\ \alpha,\ \alpha' \in \mathbb{R}_{++}$, *one has*

$$k(\mu x, y, \alpha) = \mu k(x, y, \frac{\alpha}{\mu}), \tag{11}$$

$$k(x, \mu y, \alpha) = \frac{1}{\mu} k(x, y, \mu\alpha), \tag{12}$$

$$x \le x' \implies k(x, y, \alpha) \le k(x', y, \alpha), \tag{13}$$

$$y \le y' \implies k(x, y', \alpha) \le k(x, y, \alpha), \tag{14}$$

$$\alpha \le \alpha' \implies k(x, y, \alpha') \le k(x, y, \alpha), \tag{15}$$

$$k(\gamma x, y, \alpha) \le \gamma k(x, y, \alpha), \tag{16}$$

$$k(x, \gamma y, \alpha) \ge \frac{1}{\gamma} k(x, y, \alpha), \tag{17}$$

$$k(x, x, 1) = 1 \iff x \notin -S, \tag{18}$$

$$x \in S,\ y \in -S \implies k(x, y, \alpha) = +\infty, \forall \alpha \in \mathbb{R}_{++}, \tag{19}$$

$$k(x, y, \alpha) = +\infty \implies y \in -S. \tag{20}$$

Proof. We only prove parts (11) and (20). For (11) we have:

$$\begin{aligned} k(\mu x, y, \alpha) &= \sup\{\lambda \ge \alpha : \lambda y \le \mu x\} \\ &= \sup\{\lambda \ge \alpha : \frac{\lambda}{\mu} y \le x\} \\ &= \sup\{\mu\tilde{\lambda} \ge \alpha : \tilde{\lambda} y \le x\} \\ &= \mu k(x, y, \frac{\alpha}{\mu}). \end{aligned}$$

To prove (20), let $k(x, y, \alpha) = +\infty$, then by (10), there exists a sequence $\{\lambda_n\}_{n\ge1}$ such that $\lambda_n \longrightarrow +\infty$ and $y \le \frac{1}{\lambda_n} x$ for all $n \ge 1$. Since S is closed, we get $y \in -S$. This proves (20). □

Example 3.1. Let $X = \mathbb{R}^n$ and S be the cone $\mathbb{R}^n_+$ of all vectors in $\mathbb{R}^n$ with non-negative coordinates. Let $I = \{1, 2, ..., n\}$. Each vector $x \in \mathbb{R}^n$ generates the following sets of indices:

$$I_+(x) = \{i \in I : x_i > 0\},\ \ I_0(x) = \{i \in I : x_i = 0\},\ \ I_-(x) = \{i \in I : x_i < 0\}.$$

Let $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Denote by $\frac{c}{x}$ the vector with coordinates

$$(\frac{c}{x})_i := \begin{cases} \frac{c}{x_i}, & i \notin I_0(x), \\ 0, & i \in I_0(x). \end{cases}$$

Then, for each $x,\ y \in \mathbb{R}^n$, we have

$$k(x, y, \alpha) = \begin{cases} \max\{\min_{i\in I_+(y)} \frac{x_i}{y_i}\ , 0\}, & x \in K^+_{y,\alpha}, \\ 0, & x \notin K^+_{y,\alpha}, \end{cases}$$

where $\inf\emptyset = +\infty$, $\sup\emptyset = 0$, and

$$K^+_{y,\alpha} := \{x \in \mathbb{R}^n :\ \forall\ i \in I_-(y) \cup I_0(y),\ \ x_i \le 0;\ \max_{i\in I_-(y)} \frac{x_i}{y_i} \le \min_{i\in I_+(y)} \frac{x_i}{y_i}, \textit{and } \alpha \le \min_{i\in I_+(y)} \frac{x_i}{y_i}\}.$$

We can also introduce the function $t : X \times X \times \mathbb{R}_{++} \longrightarrow [0, +\infty]$ defined by

$$t(x, y, \beta) := \inf\{0 \le \lambda \le \beta : \lambda y \ge x\}, \quad (x,\ y \in X;\ \beta \in \mathbb{R}_{++}), \tag{21}$$

(with the convention $\inf\emptyset = +\infty$).

The following proposition gives us some properties of the function t.

Proposition 3.2. *For every $x,\ y,\ x',\ y' \in X$; $\gamma \in (0,1]$; $\mu,\ \beta,\ \beta' \in \mathbb{R}_{++}$, one has*

$$t(\mu x, y, \beta) = \mu t(x, y, \frac{\beta}{\mu}), \tag{22}$$

$$t(x, \mu y, \beta) = \frac{1}{\mu} t(x, y, \mu\beta), \tag{23}$$

$$x \le x' \implies t(x, y, \beta) \le t(x', y, \beta), \tag{24}$$

$$y \le y' \implies t(x, y', \beta) \le t(x, y, \beta), \tag{25}$$

$$\beta \le \beta' \implies t(x, y, \beta') \le t(x, y, \beta), \tag{26}$$

$$t(\gamma x, y, \beta) \le \gamma t(x, y, \beta), \tag{27}$$

$$t(x, \gamma y, \beta) \ge \frac{1}{\gamma} t(x, y, \beta), \tag{28}$$

$$t(x, x, 1) = 1,\ \forall x \in X, \tag{29}$$

$$t(x, y, \beta) = 0 \iff x \in -S. \tag{30}$$

In the following proposition we give the relation between the functions k and t.

Proposition 3.3. *Let k and t be as the above. Then, for all $x,\ y \in X$ and all $\mu > 0$, we have:*

$$k(x, y, \mu) t(y, x, \frac{1}{\mu}) = 1, \tag{31}$$

(*with the convention* $0 \times (+\infty) = (+\infty) \times 0 = 1$).

Proof. This is an immediate consequence of the definition k and t. □

Theorem 3.1. *Let $f : X \to [0, +\infty]$ be a function. Then the following assertions are equivalent:*
(i) f is IR.
(ii) $\lambda f(y) \le f(x)$ for all $x,\ y \in X$ and all $\lambda \ge 1$ such that $\lambda y \le x$.
(iii) $k(x, y, \alpha) f(\alpha y) \le \alpha f(x)$ for all $x,\ y \in X$ and all $\alpha \in \mathbb{R}_{++}$ with the convention $0 \times (+\infty) = 0$.
(iv) $t(x, y, \beta) f(\beta y) \ge \beta f(x)$ for all $x,\ y \in X$ and all $\beta \in \mathbb{R}_{++}$ with the convention $0 \times (+\infty) = +\infty$.

Proof. $(i) \Rightarrow (ii)$. It is obvious.
$(ii) \Rightarrow (iii)$. Let $\alpha > 0$ and $x,\ y \in X$ be arbitrary. Notice first that due to (20) $k(x, y, \alpha) = +\infty$ implies that $y \in -S$, and so $f(\alpha y) = 0$. Then, by the convention $0 \times (+\infty) = 0$, we have $\alpha f(x) \ge k(x, y, \alpha) f(\alpha y)$. If $k(x, y, \alpha) = 0$, then, since f is non-negative and the convention $0 \times (+\infty) = 0$ holds, one has $\alpha f(x) \ge k(x, y, \alpha) f(\alpha y)$. Finally, let $0 < k(x, y, \alpha) < +\infty$. Then in view of (10) and closedness of S, we have $k(x, y, \alpha) y \le x$, and also $1 \le \frac{k(x,y,\alpha)}{\alpha} < +\infty$. Thus, by the hypothesis and the fact that $\frac{k(x,y,\alpha)}{\alpha}(\alpha y) \le x$, we conclude that $\frac{k(x,y,\alpha)}{\alpha} f(\alpha y) \le f(x)$. Therefore, (iii) holds.
$(iii) \Rightarrow (i)$. Now, let $y \le x$. Then, by (10), $k(x, y, 1) \ge 1$. Then (iii) yields that $f(y) \le f(y) k(x, y, 1) \le f(x)$. So, f is increasing. Moreover, according to (19), we have $\lambda k(x, x, 1) \le k(\lambda x, x, 1)$ for all $\lambda \ge 1$ and all $x \in X$, Thus

$$\lambda f(x) \le \lambda k(x, x, 1) f(x) \le k(\lambda x, x, 1) f(x) \le f(\lambda x).$$

Hence, f is IR.
$(i) \Rightarrow (iv)$. Let $t(x, y, \beta) = 0$. It follows from (30) that $x \in -S$, and so $f(x) = 0$. Thus, $\beta f(x) = 0 \le t(x, y, \beta) f(\beta y)$. We now assume that $0 < t(x, y, \beta) < +\infty$. Then, in view of (21) and closedness of S, we get $t(x, y, \beta) y \ge x$, and also $0 < \frac{t(x,y,\beta)}{\beta} \le 1$. Therefore, $\frac{t(x,y,\beta)}{\beta}(\beta y) \ge x$. Since f is an IR function, it follows that

$$\frac{t(x, y, \beta)}{\beta} f(\beta y) \ge f(\frac{t(x, y, \beta)}{\beta}(\beta y)) \ge f(x).$$

Also, (iv) holds if $t(x, y, \beta) = +\infty$ because of the convention $0 \times (+\infty) = +\infty$ holds.
Finally, the proof of the implication $(iv) \Rightarrow (i)$ can be done in a similar manner as the proof of the implication $(iii) \Rightarrow (i)$. □

Now, we are going to show that each non-negative IR function is supremally generated by a certain class of IR functions.

Assume that $y \in X$ and $\alpha \in \mathbb{R}_{++}$ are arbitrary. Consider the function $k_{(y,\alpha)} : X \to [0, +\infty]$ defined by $k_{(y,\alpha)}(x) = k(x, y, \alpha)$ for all $x \in X$. Let

$$\tilde{L} := \{k_{(y,\alpha)} : y \in X,\ \alpha \in \mathbb{R}_{++}\}. \tag{32}$$

$\tilde{L}$ is called the set of elementary functions defined on X.

Remark 3.1. By (13) and (16), the function $k_{(y,\alpha)}$ is an IR function.

Proposition 3.4. *Let $f : X \to [0, +\infty]$ be an IR function and $x_0 \in X$ be such that $f(x_0) \neq 0, +\infty$. Then, $y := \frac{x_0}{f(x_0)} \notin -S$.*

Proof. Assume if possible that $y \in -S$. Since $0 < f(x_0) < +\infty$ and S is a cone, it follows that $x_0 \in -S$. So, since f is an IR function, in view of Remark 2.1 we have $f(x_0) \leq f(0) = 0$. This is a contradiction. □

Theorem 3.2. *Let $f : X \to [0, +\infty]$ be a function. Then, f is IR if and only if there exists a set $A \subseteq \tilde{L}$ such that*

$$f(x) = \sup_{k_{(y,\alpha)} \in A} k_{(y,\alpha)}(x), \quad (x \in X).$$

In this case, one can take $A := \{k_{(y,\alpha)} \in \tilde{L} : f(\alpha y) \geq \alpha\}$. Hence, f is IR if and only if f is $\tilde{L}$-convex.

Proof. We only prove that each IR function $f : X \to [0, +\infty]$ satisfies

$$f(x) = \sup_{k_{(y,\alpha)} \in A} k_{(y,\alpha)}(x), \quad (x \in X).$$

For this end, according to Theorem 3.1 (the implication $(i) \Rightarrow (iii)$), we have $k_{(y,\alpha)}(x) f(\alpha y) \leq \alpha f(x)$ for all $x,\ y \in X$ and all $\alpha \in \mathbb{R}_{++}$. So, if $x \in X$ and $k_{(y,\alpha)} \in A$ are arbitrary, then $k_{(y,\alpha)}(x) \leq f(x)$. Let $0 < f(x) < +\infty$. Therefore, one has $k_{(\frac{x}{f(x)}, f(x))} \in A$, and in view of Proposition 3.4 we have $\frac{x}{f(x)} \notin -S$. This implies that $x \notin -S$. Thus, by (11) and (18) we have $k_{(\frac{x}{f(x)}, f(x))}(x) = f(x) k(x, x, 1) = f(x)$, it follows that

$$f(x) = \max_{k_{(y,\alpha)} \in A} k_{(y,\alpha)}(x).$$

Now, suppose that $f(x) = 0$. Assume if possible that there exists $k_{(y,\alpha)} \in A$ such that $k_{(y,\alpha)}(x) \neq 0$. According to Theorem 3.1 (the implication $(i) \Rightarrow (iii)$), we have $k_{(y,\alpha)}(x) f(\alpha y) \leq \alpha f(x) = 0$, which implies that $f(\alpha y) = 0$. But, $0 = f(\alpha y) \geq \alpha$. This is a contradiction. So, $k_{(y,\alpha)}(x) = 0$ for all $k_{(y,\alpha)} \in A$. Hence,

$$f(x) = 0 = \sup_{k_{(y,\alpha)} \in A} k_{(y,\alpha)}(x).$$

Finally, assume that $f(x) = +\infty$. For each $\alpha > 1$, put $y_\alpha := \frac{x}{\alpha}$. Trivially, we get $f(\alpha y_\alpha) = f(x) = +\infty \geq \alpha$, and thus $k_{(y_\alpha, \alpha)} \in A$ for all $\alpha > 1$. Also, in view of (12) one has $k_{(y_\alpha, \alpha)}(x) = \alpha k(x, x, 1) \geq \alpha$ for all $\alpha > 1$. Therefore, we have

$$f(x) = +\infty = \sup_{\alpha > 1} k_{(y_\alpha, \alpha)}(x) \leq \sup_{k_{(y,\alpha)} \in A} k_{(y,\alpha)}(x) \leq f(x).$$

Hence, the proof is complete. □

As the above, we can also show that each non-negative IR function $f : X \to [0, +\infty]$ is infimally generated by a certain class of IR functions. Let $y \in X$ and $\beta \in \mathbb{R}_{++}$ be arbitrary. Consider the function $t_{(y,\beta)} : X \to [0, +\infty]$ defined by $t_{(y,\beta)}(x) = t(x, y, \beta)$ for all $x \in X$. Also, let

$$T := \{t_{(y,\beta)} : y \in X,\ \beta \in \mathbb{R}_{++}\}. \tag{33}$$

Then, T is called the set of elementary functions defined on X.

Remark 3.2. By (24) and (27), the function $t_{(y,\beta)}$ is an IR function.

The proof of the following theorem is similar to the one of Theorem 3.2, and therefore we omit it.

Theorem 3.3. *Let $f : X \to [0, +\infty]$ be a function. Then, f is IR if and only if there exists a set $B \subseteq T$ such that*

$$f(x) = \inf_{t_{(y,\beta)}\in B} t_{(y,\beta)}(x), \quad (x \in X).$$

In this case, one can take $B := \{t_{(y,\beta)} \in T : f(\beta y) \le \beta\}$. Hence, f is IR if and only if f is T-concave.

Recall that a function f is inf-abstract-convex if $f(x) = \inf_\alpha f_\alpha(x)$ such that each f_α is abstract-convex.

Corollary 3.1. *If $f : X \to [0, +\infty]$ is an IR function, then f is inf-abstract-convex.*

Proof. This is an immediate consequence of Theorem 3.2 and Theorem 3.3. □

4. Subdifferential and support set of non-negative IR functions

In this section, we present a description of support set and $\tilde{L}$-subdifferential of a non-negative IR function f defined on a topological vector space X (for properties and definition of support set and subdifferential see [10, 11]), and we also investigate some properties of support set. Let $\tilde{L}$ and T be as defined by (32) and (33), respectively.

The proof of the following proposition is similar to the one of Proposition 4.1 in [6], and therefore we omit it.

Proposition 4.1. *Let $f : X \to [0, +\infty]$ be an IR function. Then,*

$$supp(f, \tilde{L}) = \{k_{(y,\alpha)} \in \tilde{L} \ : \ f(\alpha y) \ge \alpha\}.$$

Proposition 4.2. *Let $f : X \to [0, +\infty]$ be an IR function and $x_0 \in X$ be such that $f(x_0) \ne 0, +\infty$. Then*

$$\{k_{(y,\alpha)} \in \tilde{L} \ : \ f(\alpha y) \ge \alpha, \ k_{(y,\alpha)}(x_0) = f(x_0)\} \subset \partial_{\tilde{L}} f(x_0).$$

Proof. This is an immediate consequence of Proposition 4.1 . □

Corollary 4.1. *Let $f : X \to [0, +\infty]$ be an IR function and $x_0 \in X$ be such that $f(x_0) \ne 0, +\infty$. Let $y := \frac{x_0}{f(x_0)}$, $\alpha := f(x_0)$. Then, $k_{(y,\alpha)} \in \partial_{\tilde{L}} f(x_0)$, and hence $\partial_{\tilde{L}} f(x_0)$ is non-empty.*

Proof. It follows from Proposition 3.4 and (18) that $k(x_0, x_0, 1) = 1$. Therefore, $k_{(y,\alpha)}(x_0) = f(x_0)$ and $f(\alpha y) = \alpha$. Hence, in view of Proposition 4.2, we conclude that $k_{(y,\alpha)} \in \partial_{\tilde{L}} f(x_0)$. □

Theorem 4.1. *Let $f : X \to [0, +\infty]$ be an IR function and $x_0 \in X$ be such that $f(x_0) \ne +\infty$. Then*

$$\{k_{(y,\alpha)} \in \tilde{L} : k_{(y,\alpha)}(x_0) \le f(x_0), \ \alpha - k_{(y,\alpha)}(x_0) \le f(\alpha y) - f(x_0)\} \subset \partial_{\tilde{L}} f(x_0).$$

Proof. Let $D := \{k_{(y,\alpha)} \in \tilde{L} : k_{(y,\alpha)}(x_0) \le f(x_0), \ \alpha - k_{(y,\alpha)}(x_0) \le f(\alpha y) - f(x_0)\}$ and $k_{(y,\alpha)} \in D$ be arbitrary. Since $\frac{k_{(y,\alpha)}(x)}{\alpha} \ge 1$ and $0 \le f(x_0) - k_{(y,\alpha)}(x_0)$, it follows that

$$\frac{k_{(y,\alpha)}(x)}{\alpha}(f(\alpha y) - \alpha) \ge \frac{k_{(y,\alpha)}(x)}{\alpha}(f(x_0) - k_{(y,\alpha)}(x_0)) \ge f(x_0) - k_{(y,\alpha)}(x_0), \tag{34}$$

for all $x \in X$. According to Theorem 3.1 (the implication $(i) \Rightarrow (iii)$) one has $\frac{k_{(y,\alpha)}(x)}{\alpha} f(\alpha y) \le f(x)$ for all $x \in X$. This, together with (34) implies that

$$f(x) - k_{(y,\alpha)}(x) \ge \frac{k_{(y,\alpha)}(x)}{\alpha}(f(\alpha y) - \alpha) \ge f(x_0) - k_{(y,\alpha)}(x_0),$$

for all $x \in X$. Hence, $k_{(y,\alpha)} \in \partial_{\tilde{L}} f(x_0)$. □

In the sequel, we introduce $X \times \mathbb{R}_{++}$-support sets for a non-negative IR function which are essential to characterize polar sets.

Let $f : X \to [0, +\infty]$ be a function. The lower $(X \times \mathbb{R}_{++})$-support set of f (with respect to $\tilde{L}$), $supp_l(f, X \times \mathbb{R}_{++})$, is defined by:

$$\text{supp}_{\,l}(f, X \times \mathbb{R}_{++}) := \{(y, \alpha) \in X \times \mathbb{R}_{++} \ : \ k_{(\frac{y}{\alpha}, \alpha)} \le f\}. \tag{35}$$

Also, we define the upper $(X \times \mathbb{R}_{++})$-support set of f (with respect to T), $\text{supp}_{\,u}(f, X \times \mathbb{R}_{++})$, by:

$$\text{supp}_{\,u}(f, X \times \mathbb{R}_{++}) := \{(y, \beta) \in X \times \mathbb{R}_{++} \ : \ t_{(\frac{y}{\beta}, \beta)} \ge f\}. \tag{36}$$

Let $W \subseteq X \times \mathbb{R}_{++}$; recall that the α-section of W (W^α) is defined by $W^\alpha := \{y \in X \ : \ (y, \alpha) \in W\}$. Also, the y-section of W (W_y) is defined by $W_y := \{\alpha \in \mathbb{R}_{++} \ : \ (y, \alpha) \in W\}$.

Remark 4.1. Let $f : X \to [0, +\infty]$ be a function. According to (12), (14), (15) and (35), $\text{supp}_{\,l}(f, X \times \mathbb{R}_{++})$ is a co-radiant set and has upward α-section, also the y-section of $\text{supp}_{\,l}(f, X \times \mathbb{R}_{++})$ is a downward and closed set in $\mathbb{R}_{++}$.

Remark 4.2. *Let $f : X \to [0, +\infty]$ be a function. According to (24), (26) and (36),* $\text{supp}_{\,u}(f, X \times \mathbb{R}_{++})$ *is a radiant set and has downward β-section, also the y-section of* $\text{supp}_{\,u}(f, X \times \mathbb{R}_{++})$ *is an upward and closed set in $\mathbb{R}_{++}$.*

The proof of the following proposition is similar to the one of Propositions 4.3 in [6], and therefore we omit it.

Proposition 4.3. *Let $W \subseteq X \times \mathbb{R}_{++}$ and $W \neq \emptyset$. Then the following assertions are equivalent:*
(*i*) *W is co-radiant, the α-section W^α is upward for all $\alpha > 0$, and for all $y \in X$ the y-section W_y is downward and closed in $\mathbb{R}_{++}$.*
(*ii*) *There is a unique IR function $f : X \to [0, +\infty]$ such that* $\text{supp}_{\,l}(f, X \times \mathbb{R}_{++}) = W$.
(*iii*) *There is a unique function $f : X \to [0, +\infty]$ such that* $\text{supp}_{\,l}(f, X \times \mathbb{R}_{++}) = W$.
Furthermore, the function f of (*ii*) *is defined by $f(y) := \sup\{\alpha > 0 : \ (y, \alpha) \in W\}$ for all $y \in X$* (*with the convention* $\sup \emptyset = 0$).

The proof of the following proposition is similar to the one of Proposition 4.3, and therefore we omit it.

Proposition 4.4. *Let $U \subseteq X \times \mathbb{R}_{++}$ and $U \neq \emptyset$. Then the following assertions are equivalent:*
(*i*) *U is radiant, the β-section U^β is downward for all $\beta > 0$, and for all $y \in X$ the y-section U_y is upward and closed in $\mathbb{R}_{++}$.*
(*ii*) *There is a unique IR function $f : X \to [0, +\infty]$ such that* $\text{supp}_{\,u}(f, X \times \mathbb{R}_{++}) = U$.
(*iii*) *There is a unique function $f : X \to [0, +\infty]$ such that* $\text{supp}_{\,u}(f, X \times \mathbb{R}_{++}) = U$.
Furthermore, the function f of (*ii*) *is defined by $f(x) := \inf\{\beta > 0 : \ (x, \beta) \in U\}$ for all $x \in X$* (*with the convention* $\inf \emptyset = 0$).

5. Polarity of non-negative IR functions and radiant sets

In the this section, we introduce the polarity of non-negative IR functions and some radiant sets. Also, we present a separation theorem for these sets. Throughout this section, we assume that $\tilde{L}$ and T are as defined by (32) and (33), respectively.

Definition 5.1. The lower polar function of $f : X \to [0, +\infty]$ is the function $f_l^0 : \tilde{L} \to [0, +\infty]$ defined by

$$f_l^0(k_{(y,\alpha)}) := \sup_{x \in X} \frac{k_{(y,\alpha)}(x)}{f(x)} \quad \forall\, k_{(y,\alpha)} \in \tilde{L}, \tag{37}$$

(with the conventions $\frac{0}{0} = \frac{+\infty}{+\infty} = 0$).

Proposition 5.1. *Let $f : X \to [0, +\infty]$ be a function. Then*

$$f_l^0(k_{(y,\alpha)}) \geq \frac{\alpha}{f(\alpha y)} \quad \forall\, k_{(y,\alpha)} \in \tilde{L}.$$

Moreover, f is an IR function if and only if

$$f_l^0(k_{(y,\alpha)}) = \frac{\alpha}{f(\alpha y)} \quad \forall\, k_{(y,\alpha)} \in \tilde{L}. \tag{38}$$

Proof. By (37), it follows that $f_l^0(k_{(y,\alpha)}) \geq \frac{k_{(y,\alpha)}(x)}{f(x)}$ for all $x \in X$ and all $k_{(y,\alpha)} \in \tilde{L}$. This, together with (11) implies that $\frac{\alpha}{f(\alpha y)} \leq \frac{k_{(y,\alpha)}(\alpha y)}{f(\alpha y)} \leq f_l^0(k_{(y,\alpha)})$. Now, suppose that f is an IR function and $x, y \in X$, $\alpha > 0$ are arbitrary. According to Theorem 3.1 (the implication $(i) \Rightarrow (iii)$), we have $k_{(y,\alpha)}(x) f(\alpha y) \leq \alpha f(x)$ for all $x \in X$. This, together with the convention $\frac{0}{0} = \frac{+\infty}{+\infty} = 0$ implies that

$$f_l^0(k_{(y,\alpha)}) \leq \frac{\alpha}{f(\alpha y)} \quad \forall\, k_{(y,\alpha)} \in \tilde{L}.$$

Hence, we get (38). Now, assume that (38) holds. Then, in view of (37) and Theorem 3.1 (the implication $(iii) \Rightarrow (i)$), one has f is an IR function. □

Corollary 5.1. *Let $f : X \to [0, +\infty]$ be an IR function. Then*

$$\text{supp}\,(f, \tilde{L}) = \{k_{y,\alpha)} \in \tilde{L} \ : \ f_l^0(k_{(y,\alpha)}) \leq 1\ \}.$$

Proof. This is an immediate consequence of (38). □

We can also define the upper polar functions which are defined by the elementary functions $t_{(y,\beta)} \in T$.

Definition 5.2. The upper polar function of $f : X \to [0, +\infty]$ is the function $f_u^0 : T \to [0, +\infty]$ defined by

$$f_u^0(t_{(y,\beta)}) := \inf_{x \in X} \frac{t_{(y,\beta)}(x)}{f(x)} \quad \forall\, t_{(y,\beta)} \in T,$$

(with the conventions $\frac{0}{0} = \frac{+\infty}{+\infty} = +\infty$).

The proof of the following proposition is similar to the one of Proposition 5.1, and therefore we omit it.

Proposition 5.2. *Let $f : X \to [0, +\infty]$ be a function. Then*

$$f_u^0(t_{(y,\beta)}) \leq \frac{\beta}{f(\beta y)} \quad \forall\, t_{(y,\beta)} \in T.$$

Moreover, f is an IR function if and only if

$$f_u^0(t_{(y,\beta)}) = \frac{\beta}{f(\beta y)} \quad \forall\, t_{(y,\beta)} \in T.$$

Definition 5.3. *Let $W \subseteq X \times \mathbb{R}_{++}$. The left polar set of W (W^l) is defined by:*

$$W^l := \{(x, \beta) \in X \times \mathbb{R}_{++} \ : \ k_{(\frac{y}{\alpha},\alpha)}(x) \leq \beta,\ \forall\, (y, \alpha) \in W\}. \tag{39}$$

Proposition 5.3. *Let $W \subseteq X \times \mathbb{R}_{++}$. Then*

$$W^l = supp_u(h_W, X \times \mathbb{R}_{++}),$$

where the function $h_W : X \to [0, +\infty]$ is defined by:

$$h_W(y) := sup\{\alpha > 0 \ : \ (y, \alpha) \in W\}, \quad \forall\, y \in X, \tag{40}$$

(*with the convention* $\sup \emptyset = 0$).

Proof. By (39), (22), (23), and (31), we conclude that

$$\begin{aligned} W^l &= \{(x,\beta) \in X \times \mathbb{R}_{++} \ : \ k_{(\frac{y}{\alpha},\alpha)}(x) \leq \beta, \ \forall \ (y,\alpha) \in W\} \\ &= \{(x,\beta) \in X \times \mathbb{R}_{++} \ : \ \beta t_{(x,\frac{1}{\alpha})}(\frac{y}{\alpha}) \geq 1, \ \forall \ (y,\alpha) \in W\} \\ &= \{(x,\beta) \in X \times \mathbb{R}_{++} \ : \ t_{(\frac{x}{\beta},\beta)}(y) \geq \alpha, \ \forall \ (y,\alpha) \in W\} \\ &= \{(x,\beta) \in X \times \mathbb{R}_{++} \ : \ t_{(\frac{x}{\beta},\beta)}(y) \geq h(y), \ \forall \ y \in X\} \\ &= supp_u(h_W, X \times \mathbb{R}_{++}). \end{aligned}$$

□

Definition 5.4. Let $W \subseteq X \times \mathbb{R}_{++}$. The right polar set of W (W^r) is defined by:

$$W^r := \{(y,\alpha) \in X \times \mathbb{R}_{++} \ : \ k_{(\frac{y}{\alpha},\alpha)}(x) \leq \beta, \ \forall \ (x,\beta) \in W\}. \tag{41}$$

The proof of the following proposition is similar to the one of Proposition 5.3, and therefore we omit it.

Proposition 5.4. *Let $W \subseteq X \times \mathbb{R}_{++}$. Then*

$$W^r = supp_l(e_W, X \times \mathbb{R}_{++}),$$

where the function $e_W : X \to [0,+\infty]$ is defined by:

$$e_W(x) := \inf\{\beta > 0 \ : \ (x,\beta) \in W\}, \quad \forall \ x \in X, \tag{42}$$

(*with the convention* $\inf \emptyset = +\infty$).

Remark 5.1. Let $W \subseteq X \times \mathbb{R}_{++}$ and $W \neq \emptyset$. According to (14), (15) and (41), we have W^r is a co-radiant set, the α-section $(W^r)^\alpha$ is upward for all $\alpha > 0$, and for all $y \in X$ the y-section $(W^r)_y$ is a downward and closed set in $\mathbb{R}_{++}$.
Also, by (13), (16) and (39), we have W^l is a radiant set, the β-section $(W^l)^\beta$ is downward for all $\beta > 0$, and for all $x \in X$ the x-section $(W^l)_x$ is an upward and closed set in $\mathbb{R}_{++}$.

The sets which are closed under the closure operators $W \to W^{rl}$ and $W \to W^{lr}$ are identified in the following.

Theorem 5.1. *Let $W \subseteq X \times \mathbb{R}_{++}$. Then the following assertions are true:*
(*i*) *One has $W = W^{rl}$ if and only if W is radiant and has the downward β-section and closed upward x-section for all $\beta > 0$ and all $x \in X$.*
(*ii*) *One has $W = W^{lr}$ if and only if W is co-radiant and has the upward α-section and closed downward y-section for all $\alpha > 0$ and all $y \in X$.*

Proof. We only prove the part (*i*) and the proof of the part (*ii*) is similar. Let $W = W^{rl}$. By Remark 5.1, one has W is a radiant set and has the downward β-section and closed upward x-section.

Conversely, let W be a radiant set and has the downward β-section and closed upward x-section. Then, by Proposition 4.4, there exists a unique IR function $f : X \longrightarrow [0,+\infty]$ such that $W = \text{supp}_u(f, X \times \mathbb{R}_{++})$. In view of Proposition 4.4(*iii*) and (42) we conclude that $f = e_W$. Moreover, Proposition 5.4 and the fact that $f = e_W$ implies that $W^r = \text{supp}_l(f, X \times \mathbb{R}_{++})$. Also, according to Remark 5.1, we have W^r is a co-radiant set and has the upward α-section and closed downward y-section. Thus, by Proposition 4.3 there exists a unique function $g : X \longrightarrow [0,+\infty]$ such that $\text{supp}_l(g, X \times \mathbb{R}_{++}) = W^r$. By (40) and Proposition 4.3(*iii*) we obtain $g = h_{W^r}$. Since g is unique and $\text{supp}_l(g, X \times \mathbb{R}_{++}) = W^r = \text{supp}_l(f, X \times \mathbb{R}_{++})$, it follows that $f = h_{W^r}$. Now, by Proposition 5.3, we conclude that

$$W^{rl} = \text{supp}_u(h_{W^r}, X \times \mathbb{R}_{++}) = \text{supp}_u(f, X \times \mathbb{R}_{++}) = W,$$

which completes the proof. □

Many applications of convexity are based on the separation properties. Some notions of separability of radiant and co-radiant sets has been introduced and studied in [16]. In the following theorem, we give a kind of separation property for a certain class of radiant sets by an elementary IR function.

Theorem 5.2. *Let $W \subseteq X \times \mathbb{R}_{++}$. Then the following assertions are equivalent:*
(i) *W is a radiant set and has the downward β-section and closed upward x-section for all $\beta > 0$ and all $x \in X$.*
(ii) *For each $(x_0, \beta_0) \notin W$, there exists $(y, \alpha) \in X \times \mathbb{R}_{++}$ such that*

$$\frac{1}{\beta} k_{(y,\alpha)}(x) \leq 1 < \frac{1}{\beta_0} k_{(y,\alpha)}(x_0) \ \ \forall\ (x, \beta) \in W. \tag{43}$$

Proof. (i) $\Rightarrow$ (ii). Let $(x_0, \beta_0) \notin W$. It follows from Theorem 5.1(i) that $(x_0, \beta_0) \notin W^{rl}$. This, together with the definition of W^l implies that there exists $(\tilde{y}, \alpha) \in W^r$ such that $k_{(\frac{\tilde{y}}{\alpha},\alpha)}(x_0) > \beta_0$ and $k_{(\frac{\tilde{y}}{\alpha},\alpha)}(x) \leq \beta$ for all $(x, \beta) \in W$. Let $y := \frac{\tilde{y}}{\alpha}$. Thus, $k_{(y,\alpha)}$ satisfies (43).
(ii) $\Rightarrow$ (i). According to Theorem 5.1(i), we only show that $W^{rl} \subseteq W$. For this end, assume that $(x_0, \beta_0) \in W^{rl}$ and $(x_0, \beta_0) \notin W$, so by the hypothesis there exists $(y, \alpha) \in X \times \mathbb{R}_{++}$ such that

$$\frac{1}{\beta} k_{(y,\alpha)}(x) \leq 1 < \frac{1}{\beta_0} k_{(y,\alpha)}(x_0) \ \ \forall\ (x, \beta) \in W. \tag{44}$$

The left inequality in (44) shows that $(\alpha y, \alpha) \in W^r$. Then, from $(x_0, \beta_0) \in W^{rl}$ and $(\alpha y, \alpha) \in W^r$ we conclude that $k_{(y,\alpha)}(x_0) \leq \beta_0$, and this contradicts the right inequality in (44). □

In the following, we present a kind of separation property for a certain class of co-radiant sets by an elementary IR function. By a similar argument as in the proof of Theorem 5.2 and by using Theorem 5.1(ii) we have the following result.

Theorem 5.3. *Let $W \subseteq X \times \mathbb{R}_{++}$. Then the following assertions are equivalent:*
(i) *W is a co-radiant set and has the upward α-section and closed downward y-section for all $\alpha > 0$ and all $y \in X$.*
(ii) *For each $(y_0, \alpha_0) \notin W$, there exists $(x, \beta) \in X \times \mathbb{R}_{++}$ such that*

$$\frac{1}{\alpha_0} t_{(x,\beta)}(y_0) < 1 \leq \frac{1}{\alpha} t_{(x,\beta)}(y) \ \ \forall\ (y, \alpha) \in W.$$

6. Non-negative IR functions and DPH functions

Abstract concavity of DPH (decreasing and positively homogeneous) functions on topological vector spaces has been studied in [9]. In this section, we study non-negative IR functions by means of DPH functions, which are simpler. For this end, we need the following definition:

Let $f : X \to [0, +\infty]$ be a function. The positively homogeneous extension function $\hat{f}$ of f is defined by $\hat{f} : X \times \mathbb{R}_{++} \cup \{(0, 0)\} \to [0, +\infty]$

$$\hat{f}(x, \lambda) := \lambda f(\frac{-x}{\lambda}),\ (x \in X,\ \lambda > 0),\ \hat{f}(0, 0) = 0.$$

For a function $f : X \longrightarrow [-\infty, +\infty]$, in a manner analogous to the case of Δ-subdifferential, we now define the $\tilde{\Delta}$-supperdifferential (see [10, 11]) of f at $x_0 \in X$ as follows:

$$\partial^+_{\tilde{\Delta}} f(x_0) := \{l \in \tilde{\Delta} :\ l(x) - l(x_0) \geq f(x) - f(x_0) \quad \forall\ x \in X\},$$

where $\tilde{\Delta}$ is a set of elementary functions defined on X.

We consider the natural order relation with respect to the convex cone $S \times \mathbb{R}_{++}$ on the space $X \times \mathbb{R}_{++}$ by:

$$(x_1, c_1) \leq (x_2, c_2) \Leftrightarrow x_2 - x_1 \in S,\ c_1 \leq c_2.$$

Now, we have the following result.

Theorem 6.1. *A function $f : X \to [0, +\infty]$ is IR if and only if its positively homogeneous extension $\hat{f}$ is decreasing.*

Proof. Suppose that f is an IR function. Let $(x_1, \lambda_1), (x_2, \lambda_2) \in X \times (0, +\infty)$ with $(x_1, \lambda_1) \le (x_2, \lambda_2)$. It follows from the definition that $x_1 \le x_2$ and $\lambda_1 \le \lambda_2$. Then one has

$$\hat{f}(x_1, \lambda_1) = \lambda_1 f(\frac{-x_1}{\lambda_1}) \ge \lambda_1 f(\frac{-x_2}{\lambda_1}) = \lambda_1 f((\frac{\lambda_2}{\lambda_1})\frac{-x_2}{\lambda_2}) \ge \lambda_2 f(\frac{-x_2}{\lambda_2}) = \hat{f}(x_2, \lambda_2),$$

and hence $\hat{f}$ is a decreasing function. Conversely, assume that $\hat{f}$ is decreasing. Let $x_1, x_2 \in X$ with $x_1 \le x_2$. Then we have

$$f(x_1) = \hat{f}(-x_1, 1) \le \hat{f}(-x_2, 1) = f(x_2).$$

That is f is increasing. Now, let $0 < \lambda \le 1$ and $x \in X$ be arbitrary. Then, $(-\lambda x, \lambda) \le (-\lambda x, 1)$. So, one has

$$\lambda f(x) = \hat{f}(-\lambda x, \lambda) \ge \hat{f}(-\lambda x, 1) = f(\lambda x).$$

Therefore, f is radiant, and hence f is an IR function. □

Remark 6.1. Let $f : X \to [0, +\infty]$ be an IR function.Then, in view of Theorem 6.1 we conclude that the positively homogeneous extension $\hat{f}$ of f is a DPH function.

The following results have been proved in [7].

Theorem 6.2. ([7, Theorem 3.2]) *Let $p : X \to [-\infty, 0]$ be a function. Then, p is IPH if and only if p is Ω-convex, where $\Omega := \{k_y \;:\; y \in X\}$ and*

$$k_y(x) := \max\{\lambda \le 0 : \lambda y \ge -x\}, \quad (x, y \in X). \tag{45}$$

Theorem 6.3. ([7, Theorem 3.4]) *Let $p : X \to [-\infty, 0]$ be an IPH function and $x \in X$ be such that $p(x) \ne -\infty, 0$. Then*

$$\partial_\Omega p(x) = \{k_y \in \Omega \;:\; k_y(x) = p(x),\;\; p(y) = -1\}.$$

Remark 6.2. Let $q : X \to [0, +\infty]$ be a DPH function and let $p : X \to [-\infty, 0]$ be defined by $p(x) := -q(x)$ for all $x \in X$. It is clear that p is an IPH function. Therefore, in view of Theorem 6.2, one has q is DPH if and only if q is $\tilde{\Omega}$-concave, where $\tilde{\Omega} := \{-k_y : k_y \in \Omega,\; y \in X\}$. Also, by a similar argument as in the proof of Theorem 6.3, we have if $q : X \to [0, +\infty]$ is a DPH function such that $q(x) \ne 0, +\infty$, then we deduce that

$$\partial^+_{\tilde{\Omega}} q(x) = \{-k_y \in \tilde{\Omega} : -k_y(x) = q(x), q(y) = 1\}.$$

Now, for each $(y, \beta) \in X \times (0, +\infty)$, define the function $k'_{(y,\beta)} : X \times (0, +\infty) \to [-\infty, 0]$ as in (45).

Remark 6.3. It is not difficult to show that for each $(y, \beta) \in X \times (0, +\infty)$, one has

$$-k'_{(y,\beta)}(x, c) = t_{(-y, \frac{c}{\beta})}(-x), \quad \forall\, (x, c) \in X \times (0, +\infty),$$

where $t_{(y,\beta)}$ defined by (21).

In the sequel, let

$$\bar{L} := \{-k'_{(y,\beta)} : y \in X\,,\ \beta \in (0, +\infty)\}.$$

Remark 6.4. Let $f : X \to [0, +\infty]$ be an IR function. Then, $\hat{f}$ is an $\bar{L}$-concave function. In this case, $\tilde{\Omega}$ in Theorem 6.2 is exactly $\bar{L}$. Indeed, let

$$\triangle := \{-k'_{(-y,\frac{1}{\beta})} : y \in X,\ \beta \in (0, +\infty)\} \subset \bar{L}.$$

Then, in view of Theorem 3.3 and Remark 6.3 we conclude that

$$\begin{aligned}\hat{f}(x, c) &= cf(\frac{-x}{c}) \\ &= c \inf_U \{t_{(y,\beta)}(\frac{-x}{c})\} \\ &= \inf_U \{ct_{(y,\beta)}(\frac{-x}{c})\} \\ &= \inf_U \{t_{(y,\beta c)}(-x)\} \\ &= \inf_{\triangle} \{-k'_{(-y,\frac{1}{\beta})}(x, c)\}, \quad \forall\ (x, c) \in X \times (0, +\infty).\end{aligned}$$

This implies that $\hat{f}$ is an $\bar{L}$-concave function.

Now, we give a description of supperdifferential $\partial^+_{\bar{L}} \hat{f}$ of $\hat{f}$, where $f : X \to [0, +\infty]$ is an IR function. Note that $f(x) = \hat{f}(-x, 1)$ for all $x \in X$.

Theorem 6.4. *Let $f : X \to [0, +\infty]$ be an IR function, and $x_0 \in X$ be such that $f(x_0) \neq 0, +\infty$. Then*

$$\partial^+_{\bar{L}} \hat{f}(x_0) = \{-k'_{(y,\beta)} \in \bar{L}\ :\ f(x_0) = t_{(y,\beta)}(x_0),\ f(\beta y) = \beta\}.$$

Proof. According to Remark 6.1, Remark 6.2 and Remark 6.4, we have

$$\partial^+_{\bar{L}} \hat{f}(-x_0, 1) = \{-k'_{(y,\beta)} \in \bar{L}\ :\ \hat{f}(-x_0, 1) = -k'_{(-y,\frac{1}{\beta})}(-x_0, 1),\ \hat{f}(-y, \frac{1}{\beta}) = 1\}.$$

Therefore, the result follows by Remark 6.3 and the definition of the positively homogeneous extension function $\hat{f}$ of f. □

Example 6.1. Let $X = \mathbb{R}^n$ and S be the cone $\mathbb{R}^n_+$ of all vectors in $\mathbb{R}^n$ with non-negative coordinates. Let $I = \{1, 2, ..., n\}$. Each vector $x \in \mathbb{R}^n$ generates the following sets of indices:

$$I_+(x) = \{i \in I : x_i > 0\},\ \ I_0(x) = \{i \in I : x_i = 0\},\ \ I_-(x) = \{i \in I : x_i < 0\}.$$

Let $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Denote by $\frac{c}{x}$ the vector with coordinates

$$(\frac{c}{x})_i := \begin{cases} \frac{c}{x_i}, & i \notin I_0(x), \\ 0, & i \in I_0(x). \end{cases}$$

Then, for each $x,\ y \in \mathbb{R}^n$, in view of (3.12) we have

$$t(x, y, \beta) = \begin{cases} \max\{\max_{i \in I_+(y)} \frac{x_i}{y_i}\ ,\ 0\}, & x \in K^+_{y,\beta}, \\ +\infty, & x \notin K^+_{y,\beta}, \end{cases}$$

where

$$K^+_{y,\beta} := \{x \in \mathbb{R}^n\ :\ \forall\ i \in I_-(y) \cup I_0(y),\ \ x_i \leq 0;\ \max_{i \in I_+(y)} \frac{x_i}{y_i} \leq \min_{i \in I_-(y)} \frac{x_i}{y_i}, and\ \beta \geq \max_{i \in I_+(y)} \frac{x_i}{y_i}\}.$$

(Note that we use the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = 0$). Now, assume that $f : \mathbb{R}^n \to [0, +\infty]$ be an IR function and $x_0 \in \mathbb{R}^n$ be such that $f(x_0) \neq 0, +\infty$. Then

$$\partial^+_{\bar{L}} \hat{f}(x_0) = \{-k'_{(y,\beta)}\ :\ f(x_0) = \max\{\max_{i \in I_+(y)} \frac{(x_0)_i}{y_i}\ ,\ 0\},\ f(\beta y) = \beta\},$$

where

$$-k'_{(y,\beta)}(x,c) = t_{(-y,\frac{c}{\beta})}(-x) = \begin{cases} \max\{\max_{i \in I_+(y)} \frac{x_i}{y_i}, 0\}, & x \in K^+_{y,\frac{c}{\beta}}, \\ +\infty, & x \notin K^+_{y,\frac{c}{\beta}}, \end{cases}$$

for all $x \in \mathbb{R}^n$ and all $c > 0$.

7. Abstract convexity of extended real valued IR functions

In this section, we are going to extend the results obtained in Sections 3 and 4 for a function $f : X \to [-\infty, +\infty]$, where f is an IR function. On the other hand, in this section we assume that $f(0) = 0$ and the sets $\{x \in X : f(x) \leq 0\}$ and $\{x \in X : f(x) \geq 0\}$ are not equal to the whole of the space X.

First, consider two functions $f^+ : X \to [0, +\infty]$ and $f^- : X \to [-\infty, 0]$ defined as follows

$$f^+(x) := \max\{f(x), 0\}, \quad (x \in X),$$

and

$$f^-(x) := \min\{f(x), 0\}, \quad (x \in X).$$

It is easy to see that f is an IR function if and only if f^+ and f^- are IR functions.

Now, consider the function $T : X \times X \times X \times R_{++} \times (-\infty, 0) \to [-\infty, +\infty]$ defined by

$$T(x, y, y', \alpha, \alpha') := \begin{cases} k_{(y,\alpha)}(x), & x \in S, \\ u_{(y',\alpha')}(x), & x \notin S, \end{cases} \tag{46}$$

for all $x,\ y,\ y' \in X$, all $\alpha \in R_{++}$ and all $\alpha' \in (-\infty, 0)$. We also introduce a class of elementary functions such that the extended real valued IR functions are supremally generated. Let $(y, y', \alpha, \alpha') \in X \times X \times R_{++} \times (-\infty, 0)$ be arbitrary. Define the function $T_{(y,y',\alpha,\alpha')} : X \to [-\infty, +\infty]$ by

$$T_{(y,y',\alpha,\alpha')}(x) := T(x, y, y', \alpha, \alpha'), \quad \forall\, x \in X. \tag{47}$$

Proposition 7.1. *Let $(y, y', \alpha, \alpha') \in X \times X \times R_{++} \times (-\infty, 0)$ be arbitrary. Then the function $T_{(y,y',\alpha,\alpha')} : X \to [-\infty, +\infty]$ defined by (47) is an IR function.*

Proof. Since S is a convex cone in X, it follows that S is also an upward conic subset of X, and hence in view of (46) and (47) the result follows. □

In the sequel, define the set H by

$$H := \{T_{(y,y',\alpha,\alpha')} : y \in S \setminus \{0\},\ \ y' \in X,\ \alpha \in R_{++},\ \alpha' \in (-\infty, 0)\}. \tag{48}$$

The set H is called a set of elementary functions defined by (47). In view of Proposition 7.1, we have each element of H is an IR function.

In the following, we show that each extended real valued IR function defined on X_0 is an H_0-convex function, where $X_0 := S \cup (-S)$ and

$$H_0 := \{T_{(y,y',\alpha,\alpha')}|_{X_0} : T_{(y,y',\alpha,\alpha')} \in H\}. \tag{49}$$

It is clear that X_0 is a closed cone in X. Also, we mean by $T_{(y,y',\alpha,\alpha')}|_{X_0}$ the restriction of the function $T_{(y,y',\alpha,\alpha')}$ to X_0. Notice that each element of H_0 is an IR function defined on X_0.

Theorem 7.1. *Let $f : X_0 \to [-\infty, +\infty]$ be a function and H_0 be the set described by (49). Then, f is an IR function if and only if there exists a set $\triangle_0 \subseteq H_0$ such that*

$$f(x) = \sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x), \quad (x \in X_0).$$

In this case, one can take

$$\triangle_0 := \{T_{(y,y',\alpha,\alpha')} \in H_0 : f^+(\alpha y) \geq \alpha, \;\; f^-(-\alpha' y') \geq \alpha'\}.$$

Hence, f is an IR function if and only if it is an H_0-convex function.

Proof. We only show that every extended real valued IR function $f : X_0 \to [-\infty, +\infty]$ satisfies

$$f(x) = \sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x), \quad (x \in X_0). \tag{50}$$

For this end, let $x \in X_0$ be fixed and $T_{(y,y',\alpha,\alpha')} \in \triangle_0$ be arbitrary. Assume that $x \in S$. Then, $f(x) \geq 0$, and so $f^+(x) = f(x)$. Since $f^+(\alpha y) \geq \alpha$, it follows from (46) and Theorem 3.1(*iii*) that

$$T_{(y,y',\alpha,\alpha')}(x) = k_{(y,\alpha)}(x) \leq f^+(x) = f(x). \tag{51}$$

Suppose that $x \notin S$. Then, we conclude from (46) that $T_{(y,y',\alpha,\alpha')}(x) = u_{(y',\alpha')}(x)$. On the other hand, one has $f^-(x) \leq f(x)$. Therefore, since $\alpha' \leq f^-(-\alpha' y')$, by Theorem 2.1(*iii*), we obtain

$$T_{(y,y',\alpha,\alpha')}(x) = u_{(y',\alpha')}(x) \leq f^-(x) \leq f(x). \tag{52}$$

Hence, in view of (51) and (52) we get

$$T_{(y,y',\alpha,\alpha')}(x) \leq f(x), \quad \forall\, T_{(y,y',\alpha,\alpha')} \in \triangle_0, \;\; (x \in X_0). \tag{53}$$

Now, consider six possible cases.

Case (*i*). If $x \notin S$ and $-\infty < f(x) < 0$. Thus, one has $f^-(x) = f(x)$. Now, let $y' := \frac{x}{-f(x)}$, $\alpha' := f(x)$ and $y \in S \setminus \{0\}, \alpha \in R_{++}$ be such that $f^+(\alpha y) \geq \alpha$. This implies that $f^-(-\alpha' y') = \alpha'$, and hence $T_{(y,y',\alpha,\alpha')} \in \triangle_0$. Therefore, in view of (3) and (46) we deduce that

$$T_{(y,y',\alpha,\alpha')}(x) = u_{(\frac{x}{-f(x)}, f(x))}(x) = f(x). \tag{54}$$

This, together with (53) implies (50).

Case (*ii*). If $x \in S$ and $0 < f(x) < +\infty$. Then, we have $f^+(x) = f(x)$, and moreover; $x \notin -S$. By putting $y := \frac{x}{f(x)}$, $\alpha := f(x)$ and $y' \in X, \alpha' \in (-\infty, 0)$ with $f^-(-\alpha' y') \geq \alpha'$, one has $y \in S \setminus \{0\}$ and $f^+(\alpha y) = \alpha$, and hence $T_{(y,y',\alpha,\alpha')} \in \triangle_0$. Thus, in view of (12) and (46) we get

$$T_{(y,y',\alpha,\alpha')}(x) = k_{(\frac{x}{f(x)}, f(x))}(x) = f(x). \tag{55}$$

This, together with (53) implies (50).

Case (*iii*). If $x \notin S$ and $f(x) = -\infty$, then $f^-(x) = f(x) = -\infty$. Now, let $T_{(y,y',\alpha,\alpha')} \in \triangle_0$ be arbitrary. Since $f^-(-\alpha' y') \geq \alpha'$, it follows from Theorem 2.1(*iii*) that $u_{(y',\alpha')}(x) = -\infty$ for all $T_{(y,y',\alpha,\alpha')} \in \triangle_0$. Therefore, since $x \notin S$, in view of (46) one has

$$\sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x) = \sup_{\triangle_0} u_{(y',\alpha')}(x) = -\infty = f(x). \tag{56}$$

Case (*iv*). If $x \notin S$ and $f(x) = 0$. This implies that $f^-(x) = f(x) = 0$. Let $T_{(y,y',\alpha,\alpha')} \in \triangle_0$ be arbitrary. Since $x \notin S$, in view of (46) we conclude that

$$T_{(y,y',\alpha,\alpha')}(x) = u_{(y',\alpha')}(x) \leq 0, \;\; \forall\, T_{(y,y',\alpha,\alpha')} \in \triangle_0,$$

and hence

$$\sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x) \leq 0. \tag{57}$$

On the other hand, we have $f^-(\lambda(\frac{1}{\lambda}x)) = f^-(x) = 0 > -\lambda$, for all $\lambda > 0$. Therefore, $T_{(y,\frac{1}{\lambda}x,\alpha,-\lambda)} \in \triangle_0$ for all $\lambda > 0$. So, since $x \notin S$, in view of (3) and (46) one has

$$T_{(y,\frac{1}{\lambda}x,\alpha,-\lambda)}(x) = u_{(\frac{1}{\lambda}x,-\lambda)}(x) = -\lambda, \ \ \forall\, \lambda > 0.$$

Hence

$$\sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x) \geq T_{(y,\frac{1}{\lambda}x,\alpha,-\lambda)}(x) = -\lambda, \quad \forall\, \lambda > 0.$$

This implies that

$$\sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x) \geq 0. \tag{58}$$

Thus, in view of (57) and (58) we get

$$\sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x) = 0 = f(x). \tag{59}$$

Case (v). If $x \in S$ and $f(x) = 0$. Then, $f^+(x) = f(x) = 0$. Now, let $T_{(y,y',\alpha,\alpha')} \in \triangle_0$ be arbitrary. Since $f^+(\alpha y) \geq \alpha$, in view of Theorem 3.1(*iii*) we obtain $k_{(y,\alpha)}(x) = 0$ for all $T_{(y,y',\alpha,\alpha')} \in \triangle_0$. Therefore, since $x \in S$, we conclude from (46) that

$$\sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x) = \sup_{\triangle_0} k_{(y,\alpha)}(x) = 0 = f(x). \tag{60}$$

Case (vi). Finally, assume that $x \in S$, $f(x) = +\infty$ and $T_{(y,y',\alpha,\alpha')} \in \triangle_0$ is arbitrary. Thus, one has $f^+(x) = f(x) = +\infty$. This implies that $x \notin -S$. Also, we have $f^+(\lambda(\frac{1}{\lambda}x)) = f^+(x) = +\infty > \lambda$ for all $\lambda > 0$. So, $T_{(\frac{1}{\lambda}x,y',\lambda,\alpha')} \in \triangle_0$ for all $\lambda > 0$. Hence, since $x \in S$ and $x \notin -S$, it follows from (12) and (46) that

$$T_{(\frac{1}{\lambda}x,y',\lambda,\alpha')}(x) = k_{(\frac{1}{\lambda}x,\lambda)}(x) = \lambda, \ \ \forall\, \lambda > 0 \ (\textit{note that } \frac{1}{\lambda}x \in S \setminus \{0\}).$$

This implies that

$$\sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x) \geq T_{(\frac{1}{\lambda}x,y',\lambda,\alpha')}(x) = \lambda, \quad \forall\, \lambda > 0,$$

and hence

$$\sup_{\triangle_0} T_{(y,y',\alpha,\alpha')}(x) = +\infty = f(x). \tag{61}$$

This completes the proof. □

Recall that the support set of an abstract convex function accumulates an essential part of global information about this function in terms of elementary functions. In the following, we characterize the support set and the H-subdifferential of an extended real valued IR function.

Theorem 7.2. *Let $f : X \to [-\infty, +\infty]$ be an IR function. Then*

$$\text{supp}\,(f, H) = \triangle,$$

where $\triangle$ is defined as follows

$$\triangle := \{T_{(y,y',\alpha,\alpha')} \in H : f^+(\alpha y) \geq \alpha, \ \alpha' \leq f^-(-\alpha' y')\}.$$

Notice that the set H defined by (48).

Proof. By the definition of the support set for a function, we have

$$\text{supp}\,(f,H) := \{T_{(y,y',\alpha,\alpha')} \in H : T_{(y,y',\alpha,\alpha')}(t) \le f(t),\ \forall\, t \in X\}. \tag{62}$$

Now, let $T_{(y,y',\alpha,\alpha')} \in \triangle$ be fixed and $t \in X$ be arbitrary. Assume that $t \in S$. Then, $f(t) \ge 0$, and so $f^+(t) = f(t)$. Since $f^+(\alpha y) \ge \alpha$, it follows from (46) and Theorem 3.1(*iii*) that

$$T_{(y,y',\alpha,\alpha')}(t) = k_{(y,\alpha)}(t) \le f^+(t) = f(t). \tag{63}$$

Suppose that $t \notin S$. Then, we conclude from (46) that $T_{(y,y',\alpha,\alpha')}(t) = u_{(y',\alpha')}(t)$. On the other hand, one has $f^-(t) \le f(t)$. Therefore, since $\alpha' \le f^-(-\alpha' y')$, by Theorem 2.1(*iii*), we obtain

$$T_{(y,y',\alpha,\alpha')}(t) = u_{(y',\alpha')}(t) \le f^-(t) \le f(t). \tag{64}$$

Hence, in view of (63) and (64) we get

$$T_{(y,y',\alpha,\alpha')}(t) \le f(t),\ \ \forall\, t \in X,$$

and so $\triangle \subseteq \text{supp}\,(f,H)$. Conversely, we show that $\text{supp}\,(f,H) \subseteq \triangle$. Let $T_{(y,y',\alpha,\alpha')} \in \text{supp}\,(f,H)$ be arbitrary. Then we have $T_{(y,y',\alpha,\alpha')} \in H$ and

$$T_{(y,y',\alpha,\alpha')}(t) \le f(t),\ \ \forall\, t \in X. \tag{65}$$

Since, by the definition of H, one has $y \in S \setminus \{0\}$, it follows that $f(\alpha y) \ge 0$, and so $f^+(\alpha y) = f(\alpha y)$. Moreover, one has $y \notin -S$. Therefore, by (11) and (46) we have $T_{(y,y',\alpha,\alpha')}(\alpha y) = k_{(y,\alpha)}(\alpha y) = \alpha$. Thus, in view of (65) we conclude that

$$f^+(\alpha y) = f(\alpha y) \ge T_{(y,y',\alpha,\alpha')}(\alpha y) = \alpha. \tag{66}$$

On the other hand, if $f(-\alpha' y') \ge 0$, then one has

$$f^-(-\alpha' y') = \min\{f(-\alpha' y'), 0\} = 0 > \alpha'. \tag{67}$$

Now, suppose that $f(-\alpha' y') < 0$. This implies that $f^-(-\alpha' y') = f(-\alpha' y')$ and $-\alpha' y' \notin S$. Then, by (2), (46) and (65) one has

$$\alpha' = u_{(y',\alpha')}(-\alpha' y') = T_{(y,y',\alpha,\alpha')}(-\alpha' y') \le f(-\alpha' y') = f^-(-\alpha' y'). \tag{68}$$

Therefore, it follows from (66), (67) and (68) that $T_{(y,y',\alpha,\alpha')} \in \triangle$, which completes the proof. □

In the final part of this section, we present a characterization for the H-subdifferential of an extended real valued IR function. Notice that, by definition, the H-subdifferential of an extended real valued IR function f at a point $x_0 \in X$ is defined as follows

$$\partial_H f(x_0) := \{T_{(y,y',\alpha,\alpha')} \in H : T_{(y,y',\alpha,\alpha')}(x_0) \in \mathbb{R},$$
$$f(t) - f(x_0) \ge T_{(y,y',\alpha,\alpha')}(t) - T_{(y,y',\alpha,\alpha')}(x_0),\ \ \forall\, t \in X\}.$$

Theorem 7.3. *Let $f : X \to [-\infty, +\infty]$ be an IR function and $x \in X$ be such that $f(x) \ne -\infty, 0, +\infty$. Then*

$$\begin{cases} \{T_{(y,y',\alpha,\alpha')} \in H : f^+(\alpha y) \ge \alpha,\ \alpha' \le f^-(-\alpha' y'),\ k_{(y,\alpha)}(x) = f(x)\} \subset \partial_H f(x), & \text{if } x \in S \\ \{T_{(y,y',\alpha,\alpha')} \in H : f^+(\alpha y) \ge \alpha,\ \alpha' \le f^-(-\alpha' y'),\ u_{(y',\alpha')}(x) = f(x)\} \subset \partial_H f(x), & \text{if } x \notin S. \end{cases}$$

Proof. It is easy to see that

$$\{T_{(y,y',\alpha,\alpha')} \in \text{supp}\,(f,H) : f(x) = T_{(y,y',\alpha,\alpha')}(x)\} \subset \partial_H f(x),\ (x \in X). \tag{69}$$

First, suppose that $x \in S$. Then, in view of (46) one has

$$T_{(y,y',\alpha,\alpha')}(x) = k_{(y,\alpha)}(x),\ \forall\, T_{(y,y',\alpha,\alpha')} \in H. \tag{70}$$

Now, consider the set A defined as follows

$$A := \{T_{(y,y',\alpha,\alpha')} \in H : f^+(\alpha y) \geq \alpha,\ \alpha' \leq f^-(-\alpha' y'),\ k_{(y,\alpha)}(x) = f(x)\}.$$

According to (69), (70) and Theorem 7.2, we conclude that $A \subseteq \partial_H f(x)$. Now, assume that $x \notin S$. Thus, in view of (46) we have

$$T_{(y,y',\alpha,\alpha')}(x) = u_{(y',\alpha')}(x),\ \forall\ T_{(y,y',\alpha,\alpha')} \in H. \tag{71}$$

Consider the set B defined as follows

$$B := \{T_{(y,y',\alpha,\alpha')} \in H : f^+(\alpha y) \geq \alpha,\ \alpha' \leq f^-(-\alpha' y'),\ u_{(y',\alpha')}(x) = f(x)\}.$$

According to (69), (71) and Theorem 7.2, we conclude that $B \subseteq \partial_H f(x)$. $\square$

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