BULL. AUSTRAL. MATH. SOC. 03D45, 03D25, 03G10
VOL. 27 (1983), 461-464.

ABSTRACT DEPENDENCE, RECURSION THEORY, AND THE LATTICE OF RECURSIVELY ENUMERABLE FILTERS

R.G. DOWNEY

This thesis is divided into two sections. Both sections are devoted to the study of effectiveness in algebra, realized as analyses of substructures of recursive structures. Section One deals with closed subsets of a Steinitz closure system with recursive dependence as introduced by Metakides and Nerode. Initially we generalize many results proved by Metakides and Nerode and results concerning $L(V_\infty)$, the lattice of recursively enumerable subspaces, to considerably more general settings. To do this we introduce the notions of *semiregularity* and the *closure intersection property*, and show how they account for most of the observed phenomena in $L(V_\infty)$ and $L(\omega)$, the lattice of recursively enumerable sets. For example, we show that if (U, cl) has the closure intersection property and is semiregular then $\mathrm{Th}\big(L(U)\big)$ is undecidable. Similarly, we show that as (F_∞, cl) is regular, the Karp-Myhill theorem fails for F_∞ .

Shore defined *nowhere simplicity* in $L(\omega)$. We examine analogues of this notion in $L(U)$, and in particular $L(V_\infty)$. If (U, cl) has the closure intersection property then any recursively enumerable nondecidable closed subset can be decomposed into a pair of recursively enumerable nondecidable nowhere simple closed subsets. We use this for results concerning automorphisms of $L(V_\infty)$. We examine effective nowhere simplicity via the notion of a *maximal pair*. A recursively enumerable set A can be split into a maximal pair if and only if A is simple.

Received 22 November 1982. Thesis submitted to Monash University, August 1982. Degree approved November 1982. Supervisor: Professor J.N. Crossley.

We introduce a method of associating co-recursively enumerable subsets of recursive bases naturally with recursively enumerable closed sets, and thus answer a question of A. Retzlaff. If A and B are recursively enumerable subsets of a recursive basis R and A is a major subset of B in $L(R)$ then $\mathrm{cl}(A)$ is major in $\mathrm{cl}(B)$. This also holds more generally in *effective closure systems* (a question of J.B. Remmel).

This method also allows us to deduce many of the results in the literature; and to later examine complementation in $L(U)$ and $L(V_\infty)$.

THEOREM. *Every fully co-recursively enumerable (that is, generated by a co-recursively enumerable subset of a recursive basis) subspace has*

(i) a decidable complement and

(ii) a nondecidable nowhere simple complement.

THEOREM. *Every immune fully co-recursively enumerable subspace has*

(i) a supermaximal and

(ii) a k-thin complement.

Finally we examine automorphisms of supermaximal subspaces (a question of A. Nerode) via the notions of *superfluous* and *essential* bases. These bases are reflections of whether or not a supermaximal subspace has supermaximal subspaces.

The second part is an analysis of the lattice $L(Q)$ of recursively enumerable (proper) filters of the atomless free Boolean algebra Q. This study is intimately connected with the study of recursively bounded π_1^0 classes. Initially, we show that many of the results of Jockusch-Soare can be very easily deduced by studying $L(Q)$. This section basically reviews much of the existing folklore in this area. We then examine the degrees of *superthick* and *thick* filters. We say $F \in L(Q)$ is *super-thick* if for all $G \in L(Q)$ with $F \subset G$ there exists $\theta \in Q$ such that $\langle F, \theta\rangle = G$ (where $\langle A\rangle$ is the filter generated by A). Similarly F is *thick* if when $F \subset G$ there exists $\theta \in Q$ such that $\langle F, \theta\rangle = \langle G, \theta\rangle \neq Q$. A filter F is called *absolutely perfect* if $F = \langle P_i, \overline{P}_j \mid i \in A, j \in B\rangle$ where $A \cap B = \emptyset$ and $\{P_i\}_{i\in\omega}$ denotes a recursive set of free generators for Q.

THEOREMS. *(i) There exists a recursively enumerable absolutely perfect thick filter in each recursively enumerable nonzero degree. There exist recursively enumerable absolutely perfect superthick filters* F *such that*

(ii) $(F \equiv_T)$ $A \cup B$ *is incomplete,*

(iii) $A \cup B$ *is a (complete) maximal set,*

(iv) $A \cup B$ *is an incomplete maximal set,*

(v) $A \cup B$ *has low degree (so it is not even hyperhypersimple or dense simple).*

(vi) If F *is (super)thick then* $A \cup B$ *is hypersimple.*

We produce recursively enumerable (super)thick filters with complete decidable extensions (that is, contained in k recursive ultrafilters), and contained in exactly $2^k - 1$ decidable filters but contained in $2^{\aleph_0}$ or $\aleph_0$ nonrecursive ultrafilters. We produce recursive superthick filters contained in $\aleph_0$ recursive ultrafilters and exactly $k \geq 1$ or $\geq \aleph_0$ nonrecursive ones. These results give many "topological" generalizations of results by Jockusch and Soare on π_1^0 classes.

We also show how to embed the lattice of recursively enumerable sets as a filter in $L(Q)$. We give an analysis of properties of recursively enumerable axiomatizations (identifying filters with theories) of filters. Thus, for example, we produce a recursively enumerable theory T contained in a complete decidable theory T' such that T has a recursively enumerable set of axioms which can be extended to an axiomatization of T', but however no recursively enumerable set of axioms of T is contained in any recursively enumerable set of axioms of T'. We also generalize the concept of maximality to subfilters of a given recursively enumerable non-principal filter. Thus we show that any recursively enumerable non-principal filter has a "maximal" subfilter, but no recursively enumerable non-principal filter has a "supermaximal" subfilter. Finally, we show how we can definably embed $L(\omega)$ into the lattice of recursively enumerable filters.

Department of Mathematics,
National University of Singapore,
Kent Ridge 0511,
Singapore.

https://doi.org/10.1017/S0004972700025958 Published online by Cambridge University Press