

Abstract reduction and topology

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Abstract Reduction and Topology

by

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Abstract

If $\mathcal{A} = \langle A, \rightarrow \rangle$ is an abstract reduction system, we use the reduction \rightarrow to introduce a closure operator on the set $\mathcal{P}(A)$ of parts of A . We describe the topologies obtained in this way, and establish a bijective correspondence between these topologies and the category of abstract reduction systems with transitive reductions. Hence, we can view and formulate Church-Rosser (and the converse), Strong and Weak Normalization in topological terms. Also, we give topological proofs of some results on abstract reduction systems.

Chapter 1

Reduction topology

1.1 Abstract reduction systems

We use the notations and definitions of [Klo90]. Here, for the reader's convenience, we recall the ones more frequently used.

Definition 1.1.1.

An abstract reduction system (ARS) is a pair $\mathcal{A} = \langle A, (\rightarrow_\alpha)_{\alpha \in I} \rangle$ such that:

- *A is a set*
- *For all $\alpha \in I$, \rightarrow_α is a subset of $A \times A$, i.e. $\rightarrow_\alpha \subseteq A \times A$. These subsets (or binary relations) are called reduction relations or simply reductions.*

If $(c, d) \in \rightarrow_\alpha$ we write $c \rightarrow_\alpha d$ and call d a one step α -reduct of c .

Example 1.1.2. In this paper the main examples of ARS's will be based on $\mathcal{A} = \langle \Lambda, \rightarrow_\beta \rangle$ with Λ the set of λ -terms and $\rightarrow_\beta = \{((\lambda x.d)e, d[x := e])\}$ as defined in [Bar85].

Definition 1.1.3. *Suppose that $\mathcal{A} = \langle A, \rightarrow_\alpha \rangle$ and $\mathcal{B} = \langle B, \rightarrow_\beta \rangle$ are two ARS's and that $f : A \rightarrow B$ is a function between the base sets. We say that f preserves the reduction if $\forall a, a' \in A, a \rightarrow_\alpha a' \Rightarrow f(a) \rightarrow_\beta f(a')$. In this case we say that f is a morphism of ARS's.*

Observe that in the definition above f preserves the reduction if and only if $\rightarrow_\alpha \subseteq (f \times f)^{-1}(\rightarrow_\beta)$.

We use the symbol \equiv to indicate identity of elements in A .

Definition 1.1.4.

- a) We call \rightarrow_α the relation in A that is the transitive, reflexive closure of \rightarrow_α . In other words $a \rightarrow_\alpha b$ iff there exists a finite sequence a_1, a_2, \dots, a_n of elements of A such that

$$a \equiv a_1 \rightarrow_\alpha a_2 \rightarrow_\alpha \dots \rightarrow_\alpha a_n \equiv b$$

- b) We call $=_\alpha$ the equivalence relation generated by \rightarrow_α . In other words $a =_\alpha b$ if there exists a finite sequence a_1, a_2, \dots, a_n of elements of A such that

$$a \xrightarrow{\alpha} a_1 \rightarrow_\alpha a_2 \xleftarrow{\alpha} \dots \rightarrow_\alpha a_{n-1} \xleftarrow{\alpha} a_n \rightarrow_\alpha b$$

- c) We call \rightarrow^{-1} or \leftarrow the converse relation of \rightarrow .

Definition 1.1.5. Let $\mathcal{A} = \langle A, \rightarrow_\alpha \rangle$ and $\mathcal{B} = \langle B, \rightarrow_\beta \rangle$ be two ARS's. We say that \mathcal{B} is a sub-ARS of \mathcal{A} - and write $\mathcal{B} \subseteq \mathcal{A}$ - if the following two conditions hold:

- $B \subseteq A$
- \rightarrow_β is the restriction of \rightarrow_α , i.e. $\forall a, a' \in B (a \rightarrow_\beta a' \Leftrightarrow a \rightarrow_\alpha a')$

\mathcal{A} is also called an extension of \mathcal{B} .

Definition 1.1.6.

Let $\mathcal{A}_i = \langle A_i, \rightarrow_{\alpha_i} \rangle$ with $i \in I$ be a family of ARS's. The product reduction is denoted as $\rightarrow_{\prod_{i \in I} \alpha_i}$ or $\prod_{i \in I} \rightarrow_{\alpha_i}$ and defined on $\prod_{i \in I} A_i$ as follows:

$$\{a_i\}_{i \in I} \rightarrow_{\prod_{i \in I} \alpha_i} \{b_i\}_{i \in I} \text{ if and only if } a_i \rightarrow_{\alpha_i} b_i \text{ for all } i \in I$$

We say that $\langle \prod_{i \in I} A_i, \rightarrow_{\prod_{i \in I} \alpha_i} \rangle$ is the product of the \mathcal{A}_i 's and is denoted as $\prod_{i \in I} \mathcal{A}_i$

Property 1.1.7.

- a) If $\mathcal{A}_i = \langle A_i, \rightarrow_{\alpha_i} \rangle$ with $i \in I$ is a family of ARS's, then the canonical projections $\pi_i : \prod_{i \in I} A_i \rightarrow A_i$ are morphisms of ARS's, from $\prod_{i \in I} \mathcal{A}_i$ into \mathcal{A}_i .
- b) If $\langle B, \rightarrow_\beta \rangle$ is an arbitrary ARS and $\phi_i : B \rightarrow A_i$ are morphisms of ARS's, the map $\phi : B \rightarrow \prod_{i \in I} A_i$ defined as $\phi(b) = \{\phi_i(b)\}_{i \in I}$ is a morphism of ARS's and is the unique morphism of ARS's as above such that $\pi_i \circ \phi = \phi_i$.

In accordance to b) one can say that $\prod_{i \in I} \mathcal{A}_i$ is the categorical product of the \mathcal{A}_i .

Definition 1.1.8. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS with only one relation.

a) If $a \in A$ we define the following subsets of A :

$$\mathcal{G}_{\rightarrow}(a) = \{b \in A : a \rightarrow b\}$$

$$\mathcal{G}_{\twoheadrightarrow}(a) = \{b \in A : a \twoheadrightarrow b\}$$

We call them the simple and the transitive reduction graphs of a .

b) If $a \in A$ we define the following subsets of A :

$$\mathcal{E}_{\rightarrow}(a) = \{b \in A : b \rightarrow a\}$$

$$\mathcal{E}_{\twoheadrightarrow}(a) = \{b \in A : b \twoheadrightarrow a\}$$

We call them the simple and the transitive expansion graphs of a .

Note that $\mathcal{E}_{\rightarrow}(a) = \mathcal{G}_{\rightarrow^{-1}}(a)$ and that $\mathcal{E}_{\twoheadrightarrow}(a) = \mathcal{G}_{\twoheadrightarrow^{-1}}(a)$.

Property 1.1.9. Observe that the following properties are valid for $\mathcal{G}_{\rightarrow}(a)$ and $\mathcal{G}_{\twoheadrightarrow}(a)$.

a) $\mathcal{G}_{\rightarrow}(a) \subseteq \mathcal{G}_{\twoheadrightarrow}(a)$.

b) The element $a \in \mathcal{G}_{\twoheadrightarrow}(a)$. Observe that in general $a \notin \mathcal{G}_{\rightarrow}(a)$. In fact $a \in \mathcal{G}_{\rightarrow}(a)$ if and only if there is a loop of length one $a \rightarrow a$.

c) If the element $d \in A$ is a normal form (i.e. $\mathcal{G}_{\rightarrow}(d) = \emptyset$) then $\mathcal{G}_{\twoheadrightarrow}(d) = \{d\}$. Notice that the converse is not true. For example, the ARS : $a \rightarrow a$ has $\mathcal{G}_{\rightarrow}(a) = \mathcal{G}_{\twoheadrightarrow}(a) = \{a\}$.

Later on, with an eye to characterizing the concept of normalization in a topological way, we consider elements such that $\mathcal{G}_{\rightarrow}(a) = \{a\}$.

Property 1.1.10. Observe that the following properties are valid for $\mathcal{E}_{\rightarrow}(a)$ and $\mathcal{E}_{\twoheadrightarrow}(a)$.

a) $\mathcal{E}_{\rightarrow}(a) \subseteq \mathcal{E}_{\twoheadrightarrow}(a)$.

b) The element $a \in \mathcal{E}_{\twoheadrightarrow}(a)$. Observe that in general $a \notin \mathcal{E}_{\rightarrow}(a)$. In fact $a \in \mathcal{E}_{\rightarrow}(a)$ if and only if there is a loop of length one $a \rightarrow a$.

c) If the element $d \in A$ is a source (i.e. $\mathcal{E}_{\rightarrow}(d) = \emptyset$) then $\mathcal{E}_{\twoheadrightarrow}(d) = \{d\}$. Notice that the converse is not true. For example, the ARS : $a \rightarrow a$ has $\mathcal{E}_{\rightarrow}(a) = \mathcal{E}_{\twoheadrightarrow}(a) = \{a\}$.

1.2 From reduction to topology

We recall the definition and the basic properties of closure operators on a set (see [Kel55]).

Definition 1.2.1. A map $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is called a closure operator if it verifies:

- a) $C(\emptyset) = \emptyset$
- b) $X \subseteq C(X), \quad \forall X \subseteq A$
- c) $C(X) = C(C(X))$
- d) $C(X_1 \cup X_2) = C(X_1) \cup C(X_2)$

Recall that if $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ is a closure operator, $\mathcal{F} = \{X \subseteq A : X = C(X)\}$ and $\mathcal{T} = \{X \subseteq A : A - X \in \mathcal{F}\}$ form the family of closed and open sets of a topology in A .

Theorem 1.2.2. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an abstract reduction system. Consider the operator $C : \mathcal{P}(A) \rightarrow \mathcal{P}(A)$ defined as

$$C(X) = \{b \in A : \exists x \in X \quad x \rightarrow b\} \text{ for } X \subseteq A$$

Then C is a closure operator.

Proof: Properties a) , b), c) and d) are trivially verified for this case. □

In the case that $X = \{a\}$ we write $C(a)$ instead of $C(\{a\})$. Observe also that $C(X) = \bigcup_{a \in X} C(a)$.

Definition 1.2.3. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS. The operator C considered above is called the reduction closure operator and the associated topology is called the reduction topology on A . We call $\mathcal{F}_{\mathcal{A}}$ or $\mathcal{F}_{\rightarrow}$ and $\mathcal{T}_{\mathcal{A}}$ or $\mathcal{T}_{\rightarrow}$ the family of closed and open sets with respect to this topology.

This topology is sometimes called the Alexandroff topology associated to \rightarrow . It has been considered especially in the case in which \rightarrow is a partial ordering on A , see [ISH92].

Example 1.2.4. Let $\mathcal{A} = \langle \Lambda, \rightarrow_{\beta} \rangle$ where Λ is the set of λ -terms and \rightarrow_{β} is the β -reduction.

If $I = \lambda x.x$ then $C(I) = \{I\}$.

If $K = \lambda x.\lambda y.x$ and $\Omega = \lambda x.(x x) \lambda x.(x x)$ then $C(KI\Omega) = \{KI\Omega, I\}$.

If $W = \lambda x.h(x x)$ then $C(W W) = \{(W W), h(W W), h(h(W W)), \dots\}$.

We mention a few basic properties of this topology.

Property 1.2.5.

- a) A subset $X \subseteq A$ is closed if and only if it is invariant under \rightarrow reduction. In other words X is closed if and only if $\forall x, y \in A : x \in X, x \rightarrow y$ then $y \in X$.
- b) A subset $X \subseteq A$ is open if and only if it is invariant under \leftarrow reduction. In other words X is open if and only if $\forall x, y \in A : x \in X, y \rightarrow x$ then $y \in X$.
- c) Let \leftarrow be the converse relation of \rightarrow . Then a subset of A is \mathcal{T}_{\leftarrow} open if and only if it is $\mathcal{T}_{\rightarrow}$ closed, i.e. $\mathcal{T}_{\rightarrow} = \mathcal{F}_{\leftarrow}$ and $\mathcal{F}_{\rightarrow} = \mathcal{T}_{\leftarrow}$.

Example 1.2.6. Let be $\mathcal{A} = \langle \Lambda, \rightarrow_{\beta} \rangle$ and Λ^0 the set of closed λ -terms, i.e. λ -terms that do not contain free variables. If c does not contain free variables and $c \rightarrow_{\beta} d$ then d does not contain free variables. Then Λ^0 is a closed set in $\mathcal{T}_{\rightarrow_{\beta}}$. Observe that Λ^0 is not open.

Example 1.2.7. In the examples that follow we consider the concept of Pure Type System and the typing relation $\Gamma \vdash A : B$ as defined in [Bar92]. Let $\mathcal{B} \equiv \langle \mathbf{Terms}, \rightarrow_{\beta} \rangle$ where \mathbf{Terms} is the set of pseudoterms. The following sets are examples of closed sets in the topology $\mathcal{T}_{\mathcal{B}}$.

1. We consider the set of Γ -types of sort s , i.e. $\{D \in \mathbf{Terms} : (\Gamma \vdash D : s)\}$ for $\Gamma \in \mathbf{Ctx}$, $s \in \mathcal{S}$. It follows easily from the Subject Reduction Theorem that this set is closed. In fact, if $\Gamma \vdash D : s$ and $D \rightarrow_{\beta} D'$ then $\Gamma \vdash D' : s$.
2. Other examples of closed sets are: the set of Γ -types, i.e. $\{D \in \mathbf{Terms} : \exists s \in \mathcal{S}(\Gamma \vdash D : s)\}$ for $\Gamma \in \mathbf{Ctx}$; the set of Γ -elements of type D and sort s , i.e. $\{d \in \mathbf{Terms} : (\Gamma \vdash d : D : s)\}$ for $\Gamma \in \mathbf{Ctx}$, $D \in \mathbf{Terms}$ and $s \in \mathcal{S}$; the set of Γ -elements, i.e. $\{d \in \mathbf{Terms} : \exists D \in \mathbf{Terms}, s \in \mathcal{S}(\Gamma \vdash d : D : s)\}$ for $\Gamma \in \mathbf{Ctx}$; the set of Γ -terms, i.e. $\{A \in \mathbf{Terms} : \exists B(\Gamma \vdash A : B \vee \Gamma \vdash B : A)\}$ for $\Gamma \in \mathbf{Ctx}$; the set of legal terms, i.e. $\{A \in \mathbf{Terms} : \exists \Gamma \in \mathbf{Ctx}, B \in \mathbf{Terms}(\Gamma \vdash A : B \vee \Gamma \vdash B : A)\}$.

Property 1.2.8. If $\{\rightarrow_{\alpha} : \alpha \in I\}$ is a family of reductions in A then $\bigcup_{\alpha \in I} \rightarrow_{\alpha} \subseteq A \times A$ is a reduction in A . It is clear that

$$\text{a) } \mathcal{F}_{\bigcup_{\alpha \in I} \rightarrow_{\alpha}} = \bigcap_{\alpha \in I} \mathcal{F}_{\rightarrow_{\alpha}}$$

$$\text{b) } \mathcal{T}_{\bigcup_{\alpha \in I} \rightarrow_{\alpha}} = \bigcap_{\alpha \in I} \mathcal{T}_{\rightarrow_{\alpha}}$$

Last property explains the reason why in the definition of the reduction topology we considered an abstract reduction system with only one reduction relation.

Property 1.2.9.

- a) Note that if $a \in A$ then $\mathcal{C}(a) = \{x \in A : a \twoheadrightarrow x\} = \mathcal{G}_{\twoheadrightarrow}(a)$. In other words the closure of a point is the transitive reduction graph of the point. Note also that $a \twoheadrightarrow b \Leftrightarrow b \in \mathcal{C}(a) \Leftrightarrow \mathcal{C}(b) \subseteq \mathcal{C}(a)$. The family of all sets of the form $\mathcal{C}(a) : a \in A$ is called the family of principal closed sets and is denoted as $\mathcal{P}_{\twoheadrightarrow} \subseteq \mathcal{F}_{\twoheadrightarrow}$.
- b) Let $a \in A$. Then $\mathcal{E}_{\twoheadrightarrow}(a) = \{x \in A : x \twoheadrightarrow a\}$ is an open set. This is the smallest set that is open and contains the element a . Note also that $b \twoheadrightarrow a \Leftrightarrow b \in \mathcal{E}_{\twoheadrightarrow}(a) \Leftrightarrow \mathcal{E}_{\twoheadrightarrow}(b) \subseteq \mathcal{E}_{\twoheadrightarrow}(a)$. The family of all sets of the form $\mathcal{E}_{\twoheadrightarrow}(a) : a \in A$ is called the family of principal open sets and is denoted as $\mathcal{O}_{\twoheadrightarrow} \subseteq \mathcal{T}_{\twoheadrightarrow}$.
- c) The family of open sets $\mathcal{O}_{\twoheadrightarrow}$ is a basis of the topology $\mathcal{T}_{\twoheadrightarrow}$.

Example 1.2.10. Let be $\mathcal{A} = \langle \Lambda, \rightarrow_{\beta} \rangle$. In the topological space $\mathcal{T}_{\rightarrow_{\beta}}$ the sets $\mathcal{E}_{\rightarrow_{\beta}}(M)$ are always infinite because if $I = \lambda x.x$ then $IM \rightarrow_{\beta} M$ for all $M \in \Lambda$.

Now, we recall the following topological concept.

Definition 1.2.11. Let $\langle X_i, \mathcal{T}_i \rangle$ with $i \in I$ be a family of topological spaces. The subsets of $\prod_{i \in I} X_i$ of the form $\prod_{i \in I} A_i$ with A_i open in X_i form the basis of a topology of the set $\prod_{i \in I} X_i$. This topology of the product is called the **Box topology** and denoted as $\prod_{i \in I} \mathcal{T}_i$.

Observe that the Box topology has more open sets than the product topology.

Property 1.2.12.

- a) Let $\mathcal{A} = \langle A, \rightarrow_{\alpha} \rangle$ be an ARS and $\mathcal{B} = \langle B, \rightarrow_{\beta} \rangle$ a sub-ARS.
Then $\mathcal{F}_{\mathcal{B}} = \{X \cap B : X \in \mathcal{F}_{\mathcal{A}}\}$. Hence the reduction topology on B is the restriction of the reduction topology on A .
- b) Let $\mathcal{A}_i = \langle A_i, \rightarrow_{\alpha_i} \rangle$ with $i \in I$ be a family of ARS's. Let $\prod_{i \in I} \mathcal{A}_i$ be the product of the \mathcal{A}_i and $\rightarrow_{\prod_{i \in I} \alpha_i}$ its reduction as defined before. Then the topology associated to the product reduction is the box topology, i.e. $\mathcal{T}_{\rightarrow_{\prod_{i \in I} \alpha_i}} = \prod_{i \in I} \mathcal{T}_{\rightarrow_{\alpha_i}}$.

Proof: We prove only part b) that follows from the equality :

$$\prod_{i \in I} \mathcal{E}_{\rightarrow_{\alpha_i}}(a_i) = \mathcal{E}_{\rightarrow_{\prod_{i \in I} \alpha_i}}(\{a_i\}_{i \in I})$$

□

The correspondence established above between ARS's and topological spaces is functorial.

Property 1.2.13. *Let $\mathcal{A} = \langle A, \rightarrow_\alpha \rangle$ and $\mathcal{B} = \langle B, \rightarrow_\beta \rangle$ be two ARS's. If $f : A \rightarrow B$ preserves the reduction then f is a continuous function from $\langle A, \mathcal{T}_{\rightarrow_\alpha} \rangle$ into $\langle B, \mathcal{T}_{\rightarrow_\beta} \rangle$.*

The converse is only true if the reductions are reflexive and transitive.

Example 1.2.14. Let $\mathcal{A} = \langle \Lambda, \rightarrow_\beta \rangle$ where Λ is the set of λ -terms. The functions $Abs : \Lambda \rightarrow \Lambda$ and $Apl : \Lambda \times \Lambda \rightarrow \Lambda$ defined as $Abs(d) = \lambda x. d$ and $Apl(c, d) = (c d)$ are continuous because they preserve the reduction.

Definition 1.2.15. *The category **ARS** of ARS's is defined as follows:*

- $Obj(\mathbf{ARS}) = \{ \mathcal{A} \mid \mathcal{A} \text{ is an ARS} \}$
- $Mor_{\mathbf{ARS}}(\langle A, \rightarrow_\alpha \rangle, \langle B, \rightarrow_\beta \rangle) = \{ f : A \rightarrow B \mid \forall a, a' \in A, a \rightarrow_\alpha a' \Rightarrow f(a) \rightarrow_\beta f(a') \}$

Definition 1.2.16. *The functor \mathcal{H} from the category of ARS's to the category of topological spaces, i.e., $\mathcal{H} : \mathbf{ARS} \rightarrow \mathbf{Top}$ is defined as follows:*

- $\mathcal{H}(\langle A, \rightarrow \rangle) = \langle A, \mathcal{T}_{\rightarrow} \rangle$ for $\langle A, \rightarrow \rangle \in Obj(\mathbf{ARS})$.
- $\mathcal{H}(g) = g$ for $g : A \rightarrow B \in Mor(\mathbf{ARS})$.

1.3 From topology to reduction

Theorem 1.3.1.

Suppose that $\langle A, \mathcal{T} \rangle$ is a topological space. The following subset of $A \times A$, $\{(a, b) \in A \times A : b \in \mathcal{C}(a)\} \subseteq A \times A$ is denoted as $\rightarrow_{\mathcal{T}}$ and defines a reflexive transitive relation on A .

Definition 1.3.2. *Let $\langle A, \mathcal{T} \rangle$ be a topological space. The reduction $\rightarrow_{\mathcal{T}}$ is called the abstract reduction associated to \mathcal{T} and the associated ARS is called the abstract reduction system associated to \mathcal{T} .*

Property 1.3.3. *If $\{\mathcal{T}_i : i \in I\}$ is a family of topologies in A then $\bigcap_{i \in I} \mathcal{T}_i$ is a topology in A and $\rightarrow_{\bigcap_{i \in I} \mathcal{T}_i} = \bigcup_{i \in I} \rightarrow_{\mathcal{T}_i}$.*

Property 1.3.4.

- a) Let $\langle A, \mathcal{T} \rangle$ be a topological space and $\langle A', \mathcal{T}' \rangle$ a topological subspace of A . Then $\langle A', \rightarrow_{\mathcal{T}'} \rangle$ is a sub-ARS of $\langle A, \rightarrow_{\mathcal{T}} \rangle$.
- b) Let $\langle A_i, \mathcal{T}_i \rangle$ with $i \in I$ be a family of topological spaces and let $\prod_{i \in I} \mathcal{T}_i$ be the box product of the \mathcal{T}_i . Then the reduction associated to the box topology is the product reduction, i.e. $\rightarrow_{\prod_{i \in I} \mathcal{T}_i} = \prod_{i \in I} \rightarrow_{\mathcal{T}_i}$.

Property 1.3.5.

If f is a continuous function from $\langle A, \mathcal{T} \rangle$ into $\langle B, \mathcal{T}' \rangle$ then $f : A \rightarrow B$ preserves the reduction, i.e. $a \rightarrow_{\mathcal{T}} b \Rightarrow f(a) \rightarrow_{\mathcal{T}'} f(b)$

Proof: If f is continuous then $b \in \mathcal{C}_{\mathcal{T}}(a)$ implies that $f(b) \in \mathcal{C}_{\mathcal{T}'}(f(a))$. □

Definition 1.3.6. The functor \mathcal{G} from the category of topological spaces to the category of ARS's, i.e. $\mathcal{G} : \mathbf{Top} \rightarrow \mathbf{ARS}$ is defined as follows:

- $\mathcal{G}(\langle A, \mathcal{T} \rangle) = \langle A, \rightarrow_{\mathcal{T}} \rangle$ for $\langle A, \mathcal{T} \rangle \in \text{Obj}(\mathbf{Top})$.
- $\mathcal{G}(f) = f$ for $f : A \rightarrow B$ a continuous function of topological spaces.

1.4 Abstract reduction and topology

Definition 1.4.1. The functor "transitive closure" for ARS's is denoted as $\mathcal{TC} : \mathbf{ARS} \rightarrow \mathbf{ARS}$ and defined as follows:

- $\mathcal{TC}(\langle A, (\rightarrow_{\alpha})_{\alpha \in I} \rangle) = \langle A, (\rightarrow_{\alpha})_{\alpha \in I} \rangle$ for $\langle A, (\rightarrow_{\alpha})_{\alpha \in I} \rangle \in \mathbf{ARS}$.
- $\mathcal{TC}(f) = f$ for $f \in \mathbf{ARS}$.

Property 1.4.2. The functor \mathcal{TC} verifies that $\mathcal{TC}^2 = \mathcal{I}$ where \mathcal{I} denotes the identity functor.

We need the following construction from general topology.

Definition 1.4.3. Let $\langle A, \mathcal{T} \rangle$ be a topological space. The reductive topology associated to \mathcal{T} is denoted as \mathcal{T}_r and defined as follows. Let $\mathcal{T}_0 \subseteq \mathcal{T}$ be an arbitrary subfamily of \mathcal{T} and call $O_{\mathcal{T}_0}$ the set $O_{\mathcal{T}_0} = \bigcap_{O \in \mathcal{T}_0} O$. The set $\{O_{\mathcal{T}_0} : \mathcal{T}_0 \subseteq \mathcal{T}\}$ is a basis for the topology \mathcal{T}_r .

The construction above can be viewed from a functorial angle.

Definition 1.4.4. Define a functor \mathcal{R} from the category of topological spaces into itself, $\mathcal{R} : \mathbf{Top} \rightarrow \mathbf{Top}$, as follows:

- $\mathcal{R}(\langle A, \mathcal{T} \rangle) = \langle A, \mathcal{T}_r \rangle$ for $\langle A, \mathcal{T} \rangle \in \mathbf{Top}$
- $\mathcal{R}(f) = f$ for $f \in \mathbf{Top}$.

Property 1.4.5. The functor \mathcal{R} verifies that $\mathcal{R}^2 = \mathcal{I}$ where \mathcal{I} denotes the identity functor.

Definition 1.4.6. Let $\langle A, \mathcal{T} \rangle$ be a topology on A . We say that $\langle A, \mathcal{T} \rangle$ is a reduction topology if $\mathcal{T} = \mathcal{T}_r$. In other words if the family of open sets is closed by intersections.

Property 1.4.7.

a) Suppose that \mathcal{T} is a topology in A . Then \mathcal{T} is a reduction topology if and only if there exists another topology \mathcal{T}' in A , such that if \mathcal{F}' denotes the family of closed sets of \mathcal{T}' , then $\mathcal{F}' = \mathcal{T}$.

b) Suppose that \mathcal{T} is a topology in A that has the following property:

(M) For all $a \in A$, there exists a unique set $\mathcal{M}(a)$ that is the smallest open set that contains the point a , i.e. $\mathcal{M}(a) \subseteq X$ for all $X \in \mathcal{T}$ such that $a \in X$.

Then \mathcal{T} is a reduction topology if and only if it verifies the property (M).

c) Suppose that \mathcal{T} is a topology on the set A and \mathcal{T}_r the associated reduction topology. For every point $a \in A$ we have that $\mathcal{C}_{\mathcal{T}}(a) = \mathcal{C}_{\mathcal{T}_r}(a)$.

Proof: Assertions a) and c) are clear.

To prove b) we take a reduction topology $\langle A, \mathcal{T} \rangle$ and observe that it verifies (M) because $\bigcap_{\{U: a \in U \in \mathcal{T}\}} U$ is the smallest open set containing a .

Conversely, a topology $\langle A, \mathcal{T} \rangle$ that verifies (M) is a reduction topology. Suppose that $\mathcal{T}_0 \subseteq \mathcal{T}$ and $X = \bigcap_{U \in \mathcal{T}_0} U \neq \emptyset$. Take $a \in X$ then for any $U \in \mathcal{T}_0$, $\mathcal{M}(a) \subseteq U$ and then $\mathcal{M}(a) \subseteq X$. So that $X = \bigcup_{a \in X} \mathcal{M}(a)$ is an open set. \square

Property 1.4.8. Suppose that \mathcal{T} is a reduction topology on the set A .

a) For any $X \subseteq A$ there exists an open set $\mathcal{M}(X)$ that is the smallest open subset of A that contains X . Moreover $\mathcal{M}(X) = \bigcup_{a \in X} \mathcal{M}(a)$.

- b) Given $a, b \in A$, we have that $b \in \mathcal{C}(a) \Leftrightarrow a \in \mathcal{M}(b)$.
- c) For all $a, b \in A$, $b \in \mathcal{M}(a)$ if and only if $\mathcal{M}(b) \subseteq \mathcal{M}(a)$.
- d) The family of open sets $\mathcal{M}(a)$ with $a \in A$ form a basis for the topology \mathcal{T} .

Corollary 1.4.9.

- a) If $\langle A, \rightarrow \rangle$ is an ARS the topology \mathcal{T}_\rightarrow is a reduction topology and $\mathcal{M}(a) = \mathcal{E}_\rightarrow(a)$.
- b) If \mathcal{T} is a reduction topology on the set A and $\rightarrow_{\mathcal{T}}$ is the associated reduction relation, then $\mathcal{M}(a) = \{b \in A : b \rightarrow_{\mathcal{T}} a\}$

Proof: We only prove a). Clearly $\mathcal{E}_\rightarrow(a)$ is the minimal open set that contains a . Hence condition (M) is verified and thus, \mathcal{T}_\rightarrow is a reduction topology. \square

Lemma 1.4.10. Let A be a set and $\mathcal{T}, \mathcal{T}'$ two topologies on A . Suppose that for all $a \in A$ we have that $\mathcal{C}_{\mathcal{T}}(a) = \mathcal{C}_{\mathcal{T}'}(a)$. Then $\mathcal{T} = \mathcal{T}'$.

Proof: Using **Property 1.4.7** part c) we can assume that \mathcal{T} and \mathcal{T}' are reduction topologies. In this case the result follows immediately from **Property 1.4.8** parts b) and d). \square

Theorem 1.4.11. Let **ARS** and **Top** be the categories of Abstract Reduction Systems and topological spaces respectively. Let **ARS_t** be the subcategory of **ARS** consisting of the reflexive and transitive reduction systems and **Top_r** the subcategory of **Top** consisting of the topological spaces whose topology is a reduction topology. Let $\mathcal{H}, \mathcal{G}, \mathcal{TC}$ and \mathcal{R} be the functors defined before.

- a) The composition $\mathcal{G} \circ \mathcal{H} = \mathcal{TC}$.
- b) The composition $\mathcal{H} \circ \mathcal{G} = \mathcal{R}$
- c) The functors \mathcal{H} and \mathcal{G} are inverses of each other when respectively restricted to **ARS_t** and **Top_r**.
- d) In the category **Top_r** the Box topology is the categorical product topology.
- e) The functor $\mathcal{H} : \mathbf{ARS}_t \rightarrow \mathbf{Top}_r$ is an equivalence of categories and preserves infinite products. Its inverse is the functor $\mathcal{G} : \mathbf{Top}_r \rightarrow \mathbf{ARS}_t$.

Proof:

- a) Let $\langle A, \rightarrow \rangle$ be an ARS. Then $\mathcal{H}(\langle A, \rightarrow \rangle) = \langle A, \mathcal{T}_{\rightarrow} \rangle$ is a reductive topology with $\mathcal{M}(a) = \mathcal{E}_{\rightarrow}(a)$ by **Corollary 1.4.9** part a). Then $\mathcal{G} \circ \mathcal{H}(\langle A, \rightarrow \rangle) = \langle A, \rightarrow_{\mathcal{T}_{\rightarrow}} \rangle$. Using **Corollary 1.4.9** part b) we deduce that $\mathcal{M}(a) = \mathcal{E}_{\rightarrow_{\mathcal{T}_{\rightarrow}}}(a)$. Hence as for all $a \in A$, $\mathcal{E}_{\rightarrow_{\mathcal{T}_{\rightarrow}}}(a) = \mathcal{E}_{\rightarrow}(a)$ we deduce that $\rightarrow_{\mathcal{T}_{\rightarrow}} = \rightarrow$.
- b) Let $\langle A, \mathcal{T} \rangle$ be a topological space. Then $\mathcal{G}(\langle A, \mathcal{T} \rangle) = \langle A, \rightarrow_{\mathcal{T}} \rangle$ with $a \rightarrow_{\mathcal{T}} b$ if and only if $b \in \mathcal{C}(a)$. Then, $\mathcal{H} \circ \mathcal{G}(\langle A, \mathcal{T} \rangle) = \langle A, \mathcal{T}_{\rightarrow_{\mathcal{T}}} \rangle$ is a reductive topology such that $\mathcal{C}_{\mathcal{T}_{\rightarrow_{\mathcal{T}}}}(a) = \mathcal{C}_{\mathcal{T}}(a)$. Hence, using **Lemma 1.4.10** we conclude that $\mathcal{T}_{\rightarrow_{\mathcal{T}}} = \mathcal{T}$.
- c) It follows immediately from parts a), b) and the fact that \mathcal{R} and \mathcal{TC} are projection functors onto $\mathbf{Top}_{\mathbf{r}}$ and $\mathbf{ARS}_{\mathbf{t}}$ respectively.
- d) Suppose that B and $\{A_i\}_{i \in I}$ are reduction topological spaces. Suppose we have continuous maps $\phi_i : B \rightarrow A_i$, $i \in I$. The map $\Pi_i \phi_i : B \rightarrow \Pi_i A_i$ is continuous in the Box topology because if we take a basic open set of the form $\Pi_i X_i \subseteq \Pi_i A_i$, then $(\Pi_i \phi_i)^{-1}(\Pi_i X_i) = \bigcap_i (\phi_i)^{-1}(X_i)$.
- e) This follows immediately from the previous parts and from **Property 1.2.8**.

□

Definition 1.4.12. Let \mathcal{T} be a reduction topology on a set A . Let \mathcal{M} and \mathcal{C} be the operators on $\mathcal{P}(A)$ defined before. Given pair of points $a, b \in A$, a is said to be \mathcal{T} connectable with b iff $\exists n \in \mathbb{N}$ such that $b \in (\mathcal{CM})^n(a)$.

It is clear that in a reduction topology, $b \in \mathcal{CM}(a)$ if and only if $\mathcal{M}(b) \cap \mathcal{M}(a) \neq \emptyset$ and this happens if and only if $a \in \mathcal{CM}(b)$. By induction one can prove that the definition above is symmetric, i.e., a is \mathcal{T} connectable with b if and only if b is \mathcal{T} connectable with a . In fact $b \in (\mathcal{CM})^n(a) \Rightarrow \mathcal{M}(b) \cap \mathcal{M}((\mathcal{CM})^{n-1}(a)) \neq \emptyset \Rightarrow \exists b_1 \in (\mathcal{CM})^{n-1}(a) : \mathcal{M}(b) \cap \mathcal{M}(b_1) \neq \emptyset \Rightarrow a \in (\mathcal{CM})^{n-1}(b_1), b_1 \in \mathcal{CM}(b) \Rightarrow a \in (\mathcal{CM})^n(b)$.

Property 1.4.13. Let $\langle A, \rightarrow \rangle$ be an ARS and call \sim the equivalence relation generated by \rightarrow . Suppose that $\mathcal{T}_{\rightarrow}$ is the associated topology. Then, a and b are $\mathcal{T}_{\rightarrow}$ connectable, if and only if $a \sim b$.

For future reference we list some properties of the operators \mathcal{C} and \mathcal{M} .

Property 1.4.14.

- a) For all $a \in A$ we have that $\mathcal{CM}(a) \subseteq (\mathcal{CM})^2(a) \subseteq \dots \subseteq (\mathcal{CM})^n(a) \subseteq \dots$

b) The roles of \mathcal{C} and \mathcal{M} are interchangeable, i.e. $(\mathcal{C}\mathcal{M})^n(a) \subseteq (\mathcal{M}\mathcal{C})^{n+1}(a)$ and then $\bigcup_n (\mathcal{C}\mathcal{M})^n(a) = \bigcup_n (\mathcal{M}\mathcal{C})^n(a)$

c) The set $\bigcup_n (\mathcal{C}\mathcal{M})^n(a)$ is the smallest open and closed set in the topology \mathcal{T} that contains a .

Proof: We only prove b). As $a \in \mathcal{C}(a)$, then if $b \in (\mathcal{C}\mathcal{M})^n(a)$ we deduce that $b \in (\mathcal{C}\mathcal{M})^n(a) \subseteq (\mathcal{C}\mathcal{M})^n\mathcal{C}(a)$. Then $b \in \mathcal{M}(b) \subseteq \mathcal{M}(\mathcal{C}\mathcal{M})^n\mathcal{C}(a) \subseteq (\mathcal{M}\mathcal{C})^{n+1}(a)$. \square

Note that the set $\bigcup_n (\mathcal{C}\mathcal{M})^n(a)$ is the equivalence class of a respect to the relation $=$. We finish this Section with a Lemma that establishes the relationship between the operators \mathcal{C} and \mathcal{M} for the union of two reductions and the corresponding operators for the uniendum (see **Proposition 1.2.8**).

Lemma 1.4.15. *Let A be a set and α and β a pair of reductions on A . Call $\mathcal{C}_\alpha, \mathcal{C}_\beta, \mathcal{M}_\alpha$ and \mathcal{M}_β the corresponding operators on $\mathcal{P}(A)$. Then:*

i) $\mathcal{C}_{\alpha \cup \beta} = \bigcup_{n \geq 0} (\mathcal{C}_\alpha \mathcal{C}_\beta)^n = \bigcup_{n \geq 0} (\mathcal{C}_\beta \mathcal{C}_\alpha)^n$.

ii) $\mathcal{M}_{\alpha \cup \beta} = \bigcup_{n \geq 0} (\mathcal{M}_\alpha \mathcal{M}_\beta)^n = \bigcup_{n \geq 0} (\mathcal{M}_\beta \mathcal{M}_\alpha)^n$.

Chapter 2

Topological characterizations of confluence and normalization

In this Chapter we start with a reduction relation \rightarrow and consider the associated topology \mathcal{T} . We give, in terms of \mathcal{T} , characterizations of confluence, normalization and other concepts that are usually considered as relevant for abstract reduction systems. We also give topological proofs of some of the general theorems in the theory of ARS's.

2.1 Topological characterizations of confluence

Definition 2.1.1. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS with only one relation. We say that the reduction system \rightarrow is confluent or Church-Rosser – and write that \rightarrow is CR – if $\forall a, b, c \in A \exists d \in A : (c \xrightarrow{\alpha} a \xrightarrow{\alpha} b \Rightarrow c \xrightarrow{\alpha} d \xrightarrow{\alpha} b)$

Theorem 2.1.2. The abstract reduction system $\mathcal{A} = \langle A, \rightarrow \rangle$ is confluent (or verifies Church-Rosser) if and only if for all a in A and for every pair C and D of non empty \mathcal{T}_- closed subsets of $\mathcal{C}_{\mathcal{T}_-}(a)$, $C \cap D \neq \emptyset$.

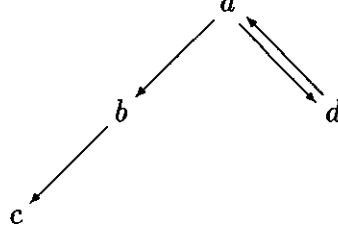
Proof: Suppose \mathcal{A} is confluent and C and D are as above. Take $c \in C$ and $d \in D$. As $c, d \in \mathcal{C}_{\mathcal{T}_-}(a)$ we have that $d \leftarrow a \rightarrow c$. As the ARS is confluent there exists an $x \in A$ such that $d \rightarrow x \leftarrow c$. As $d \in D$, $\mathcal{C}_{\mathcal{T}_-}(d) \subseteq D$ and as $d \rightarrow x$, $x \in \mathcal{C}_{\mathcal{T}_-}(d) \subseteq D$. Similarly, $x \in \mathcal{C}_{\mathcal{T}_-}(c) \subseteq C$. Hence $x \in C \cap D$.

Conversely, suppose that we have $a, d, c \in A$ such that $d \leftarrow a \rightarrow c$. Then $\mathcal{C}_{\mathcal{T}_-}(d) \subseteq \mathcal{C}_{\mathcal{T}_-}(a)$ and $\mathcal{C}_{\mathcal{T}_-}(c) \subseteq \mathcal{C}_{\mathcal{T}_-}(a)$. So that by hypothesis, there exists an element $x \in \mathcal{C}_{\mathcal{T}_-}(d) \cap \mathcal{C}_{\mathcal{T}_-}(c)$. That means that $d \rightarrow x \leftarrow c$ and hence that \mathcal{A} is confluent. \square

One can think of confluence as a property having to do with the "size" of open and closed subsets of principal closed subsets $\mathcal{C}_{\mathcal{T}_-}(a)$ of A . If \mathcal{A} is confluent then closed subsets of $\mathcal{C}_{\mathcal{T}_-}(a)$ are "large" because they always intersect, and open subsets of $\mathcal{C}_{\mathcal{T}_-}(a)$ are "small".

Example 2.1.3.

1. In the reduction system $b \leftarrow a \rightarrow c$, the closed subsets of $\mathcal{C}(a) = \{a, b, c\}$ are: $\{b\}$ and $\{c\}$. Note that they are disjoint.
2. The reduction system that follows is confluent.



The family of principal closed subsets is $\mathcal{P}_\rightarrow = \{\mathcal{C}(a) = \mathcal{C}(d) = \{a, b, c, d\}, \mathcal{C}(b) = \{b, c\}, \mathcal{C}(c) = \{c\}\}$. The closed subsets of $\mathcal{C}(a) = \mathcal{C}(d) = \{a, b, c, d\}$ are $\{c\}, \{b, c\}, \{a, b, c, d\}$. The closed subsets of $\mathcal{C}(b) = \{b, c\}$ are $\{c\}, \{b, c\}$.

Definition 2.1.4. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS. We say that it is converse Church-Rosser - and write that \rightarrow is CCR - if $\forall c, d, b \exists a : b \rightarrow d \leftarrow c \Rightarrow b \leftarrow a \rightarrow c$.

Theorem 2.1.5. The abstract reduction system $\mathcal{A} = \langle A, \rightarrow \rangle$ is converse Church-Rosser if and only if for all a in A and for every pair C and D of non empty \mathcal{T}_\rightarrow open subsets of $\mathcal{E}_\rightarrow(a)$, $C \cap D \neq \emptyset$.

Proof: The result follows from **Theorem 2.1.2** and the observation that $\langle A, \rightarrow \rangle$ is CCR iff $\langle A, \leftarrow \rangle$ is Church-Rosser. Also, we use that $\mathcal{T}_\leftarrow = \mathcal{F}_\leftarrow$, $\mathcal{F}_\rightarrow = \mathcal{T}_\leftarrow$ and $\mathcal{E}_\rightarrow(a) = \mathcal{C}_{\mathcal{T}_\leftarrow}(a)$. \square

Theorem 2.1.6. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an abstract reduction system and let \mathcal{M} and \mathcal{C} be the associated operators as defined in **Definition 1.2.1** and **Property 1.4.8**.

- a) The abstract reduction system $\mathcal{A} = \langle A, \rightarrow \rangle$ is Church-Rosser if and only if $\forall X \subseteq A$, $\mathcal{C}(\mathcal{M}(X)) \subseteq \mathcal{M}(\mathcal{C}(X))$.
- b) The abstract reduction system $\mathcal{A} = \langle A, \rightarrow \rangle$ is converse Church-Rosser if and only if $\forall X \subseteq A$, $\mathcal{M}(\mathcal{C}(X)) \subseteq \mathcal{C}(\mathcal{M}(X))$.

Proof: Observe that

$$\mathcal{C}(\mathcal{M}(X)) = \{z : \exists y, y \rightarrow z \& y \rightarrow x \& x \in X\}$$

$$\mathcal{M}(\mathcal{C}(X)) = \{u : \exists v, u \rightarrow v \& x \rightarrow v \& x \in X\}$$

Then propositions **a)** and **b)** are evident. \square

Theorem 2.1.7. *Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an abstract reduction system and let \mathcal{M} and \mathcal{C} be the associated operators as defined in **Definition 1.2.1** and **Property 1.4.8**.*

- a)** *The abstract reduction system $\mathcal{A} = \langle A, \rightarrow \rangle$ is Church-Rosser if and only if for all $X \subseteq A$ that is closed, then $\mathcal{M}(X)$ is also closed.*
- b)** *The abstract reduction system $\mathcal{A} = \langle A, \rightarrow \rangle$ is converse Church-Rosser if and only if for all $X \subseteq A$ that is open, then $\mathcal{C}(X)$ is also open.*

Proof: The proofs of **a)** and **b)** are identical. We prove only the first. Suppose that $X \subseteq A$ is a closed subset. By **Theorem 2.1.6** we have that $\mathcal{C}\mathcal{M}(X) \subseteq \mathcal{M}\mathcal{C}(X) = \mathcal{M}(X)$. Hence, we conclude that $\mathcal{C}(\mathcal{M}(X)) = \mathcal{M}(X)$. Conversely, suppose that \mathcal{M} takes closed sets into closed sets. Then $\mathcal{C}\mathcal{M}(X) \subseteq \mathcal{C}\mathcal{M}(\mathcal{C}(X)) = \mathcal{M}\mathcal{C}(X)$. \square

Example 2.1.8. For the abstract reduction system $\langle \Lambda, \rightarrow_\beta \rangle$ it is well-known that \rightarrow_β is confluent but not converse Church-Rosser.

The theorem that follows is a topological version of the proof of confluence for the abstract reduction system $\langle \Lambda, \rightarrow_\beta \rangle$ as appears in [Bar92], pg. 138.

Theorem 2.1.9. *Let $\mathcal{A} = \langle A, \rightarrow \rangle$ and $\mathcal{A}' = \langle A', \rightarrow' \rangle$ be a pair of abstract reduction systems. Call $\langle A, \mathcal{T} \rangle$ and $\langle A', \mathcal{T}' \rangle$ the corresponding topological spaces. Suppose there is a continuous function $f : A \rightarrow A'$ and a closed function $g : A \rightarrow A'$ such that $g(f^{-1}(X)) = \mathcal{M}(X)$, for all $X \subseteq A'$. Then \mathcal{A}' is Church-Rosser.*

Proof: Suppose $X \subseteq A'$. By hypothesis we know that $g(f^{-1}(X)) = \mathcal{M}(X)$. Then $\mathcal{C}\mathcal{M}(X) = \mathcal{C}g(f^{-1}(X))$. As g is closed : $\mathcal{C}g(f^{-1}(X)) \subseteq g(\mathcal{C}f^{-1}(X))$. The continuity of f implies that $g(\mathcal{C}f^{-1}(X)) \subseteq g(f^{-1}(\mathcal{C}X))$. Hence we have that $\mathcal{C}\mathcal{M}(X) = \mathcal{C}g(f^{-1}(X)) \subseteq g(\mathcal{C}f^{-1}(X)) \subseteq g(f^{-1}(\mathcal{C}X)) = \mathcal{M}\mathcal{C}(X)$. \square

In the notation of [Bar92], the system $\mathcal{A}' = \langle A', \rightarrow' \rangle$ is $\langle \Lambda, \rightarrow_\beta \rangle$, the system $\mathcal{A} = \langle A, \rightarrow \rangle$ is $\langle \underline{\Lambda}, \rightarrow_{\underline{\beta}} \rangle$, the function $f \equiv \phi$ and $g \equiv | \cdot |$. The facts that g is closed, f is continuous and that $g(f^{-1}(X)) = \mathcal{M}(X)$, $\forall X \subseteq A'$ are equivalent to Lemma 2.3.13, Lemma 2.3.14(3) and Lemma 2.3.15 in [Bar92].

The theorem that follows is a topological version of the proof of confluence for a Pure Type System with definitions as considered in [PS93].

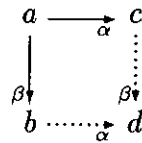
Theorem 2.1.10. Let $\mathcal{A} = \langle A, \rightarrow \rangle$ and $\mathcal{A}' = \langle A', \rightarrow' \rangle$ be an abstract reduction system and a subsystem. Call $\langle A, \mathcal{T} \rangle$ and $\langle A', \mathcal{T}' \rangle$ the corresponding topological spaces. Suppose there exists a continuous function $f : A \rightarrow A'$ such that $f(X) \subseteq \mathcal{C}_{\mathcal{T}}(X)$, $\forall X \subseteq A$. If \mathcal{A}' is CR then \mathcal{A} is CR.

Proof: To simplify notations we write $\mathcal{C}_{\mathcal{T}}$ and $\mathcal{C}_{\mathcal{T}'}$ as \mathcal{C} . Take C, D arbitrary closed sets in A . The hypothesis $f(X) \subseteq \mathcal{C}(X)$, $\forall X \subseteq A$ guarantees that $f(C) \subseteq \mathcal{C} \cap A'$ and $f(D) \subseteq D \cap A'$ and then: $\mathcal{C}(f(C)) \subseteq \mathcal{C} \cap A'$, $\mathcal{C}(f(D)) \subseteq D \cap A'$. Hence, if $\mathcal{C}(f(C)) \cap \mathcal{C}(f(D)) \neq \emptyset$ then $C \cap D \neq \emptyset$.

Suppose that $C, D \subseteq \mathcal{C}(a)$ are closed. We prove that $\mathcal{C}(f(C)) \cap \mathcal{C}(f(D)) \neq \emptyset$. As f is continuous $f(C), f(D) \subseteq f(\mathcal{C}(a)) \subseteq \mathcal{C}(f(a))$. Hence $\mathcal{C}(f(C)), \mathcal{C}(f(D)) \subseteq \mathcal{C}(f(a))$. Using the hypothesis that \mathcal{A}' is CR, we conclude that: $\mathcal{C}(f(C)) \cap \mathcal{C}(f(D)) \neq \emptyset$. □

We want to study conditions to guarantee the confluence of the union of two reductions (see [Bar85] pg. 64).

Definition 2.1.11. Let A be a set and $\rightarrow_{\alpha}, \rightarrow_{\beta}$ two reductions on A . We say that \rightarrow_{α} and \rightarrow_{β} commute (see picture bellow) if $\forall a, b, c \in A \exists d \in A : (c \xleftarrow{\alpha} a \rightarrow_{\beta} b \Rightarrow c \rightarrow_{\beta} d \xleftarrow{\alpha} b)$.



Lemma 2.1.12. If A is a set, $\rightarrow_{\alpha}, \rightarrow_{\beta}$ two reductions on A and $\mathcal{C}_{\alpha}, \mathcal{C}_{\beta}, \mathcal{M}_{\alpha}, \mathcal{M}_{\beta}$ the corresponding operators on $\mathcal{P}(A)$, then the following conditions are equivalent:

- a) The reductions \rightarrow_{α} and \rightarrow_{β} commute.
- b) For all $X \subseteq A$, $\mathcal{C}_{\alpha}\mathcal{M}_{\beta}(X) \subseteq \mathcal{M}_{\beta}\mathcal{C}_{\alpha}(X)$.
- c) For all $X \subseteq A$, $\mathcal{C}_{\beta}\mathcal{M}_{\alpha}(X) \subseteq \mathcal{M}_{\alpha}\mathcal{C}_{\beta}(X)$.

Proof: The proof is omitted because it is identical to the proof of **Theorem 2.1.6**. □

Lemma 2.1.13. If A is a set, $\rightarrow_{\alpha}, \rightarrow_{\beta}$ two reductions on A and $\mathcal{C}_{\alpha}, \mathcal{C}_{\beta}, \mathcal{M}_{\alpha}, \mathcal{M}_{\beta}$ the corresponding operators on $\mathcal{P}(A)$, then the following are equivalent:

- a) The reductions \rightarrow_{α} and \rightarrow_{β} commute.
- b) For all $X \subseteq A$ that is T_{α} closed, then $\mathcal{M}_{\beta}(X)$ is T_{α} closed.

Proof: See **Theorem 2.1.7**. □

In what follows we present a topological proof of the Lemma of Hindley-Rosen.

Lemma 2.1.14. *Let A be a set and α, β two reductions on A . Suppose that α and β are confluent and commute, then $\alpha \cup \beta$ is confluent.*

Proof: Suppose that $Z \subseteq A$ is closed in $\mathcal{T}_{\alpha \cup \beta}$. Then by **Proposition 1.2.8**, Z is \mathcal{T}_α and \mathcal{T}_β closed. Hence by **Lemma 2.1.13**, $\mathcal{M}_\beta(Z)$ is \mathcal{T}_α closed. As β is CR, by **Theorem 2.1.7** $\mathcal{M}_\beta(Z)$ is also \mathcal{T}_β closed. So that $\mathcal{M}_\beta(Z)$ is \mathcal{T}_α and \mathcal{T}_β closed. Repeating the argument taking α instead of β and $\mathcal{M}_\beta(Z)$ instead of Z , we conclude that $\mathcal{M}_\alpha \mathcal{M}_\beta(Z)$ is \mathcal{T}_α and \mathcal{T}_β closed. By induction we conclude that $\forall n$ $(\mathcal{M}_\alpha \mathcal{M}_\beta)^n(Z)$ is \mathcal{T}_α and \mathcal{T}_β closed. So that by **Proposition 1.2.8**, $\mathcal{M}_{\alpha \cup \beta}(Z)$ is $\mathcal{T}_{\alpha \cup \beta}$ closed (see **Proposition 1.2.8**). Hence using **Theorem 2.1.7** we conclude that $\alpha \cup \beta$ is a confluent reduction. □

2.2 Topological characterizations of normalization

We recall the following definitions from [Klo90].

Definition 2.2.1. *Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS with only one relation.*

- a) *We say that $d \in A$ is a normal form for \rightarrow , if there is no element $c \in A$ such that $d \rightarrow c$.
We say that $b \in A$ has a normal form if there exists $d \in A$ such that d is a normal form and $b \twoheadrightarrow d$.*
- b) *We say that \rightarrow weakly normalizes – and write \rightarrow is WN – if every element of A has a normal form.*

We will introduce the concept of zero loop.

Definition 2.2.2. *An element $d \in A$ such that $\mathcal{G}_\rightarrow(d) = \{d\}$ is called a zero loop.*

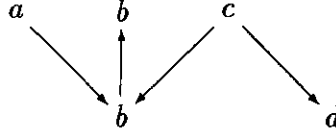
Example 2.2.3. Let be $\langle \Lambda, \rightarrow_\beta \rangle$ where Λ is the set of λ -terms. There is only one zero loop in this topology and is $\Omega = \lambda x.(x x) \lambda x.(x x)$.

Zero loops and normal forms are the only closed points in the reduction topology.

Lemma 2.2.4. *Let \mathcal{A} be an ARS. An element $a \in A$ is a zero loop or a normal form iff $\{a\}$ is a \mathcal{T}_\rightarrow closed set.*

Proof: If $d \in A$ is a zero loop or a normal form then $\mathcal{G}_{\rightarrow}(d) = \{d\} = \mathcal{C}(d)$. If $\{a\}$ is closed, $\mathcal{G}_{\rightarrow}(a) = \{a\}$ and so $\mathcal{G}_{\rightarrow}(a) \subseteq \{a\}$. Hence one of the following two alternatives holds: $\mathcal{G}_{\rightarrow}(a) = \{a\}$ and we have a zero loop or $\mathcal{G}_{\rightarrow}(a) = \emptyset$ and we have a normal form. \square

Example 2.2.5. Consider the following ARS



We have that: $\mathcal{C}(a) = \{a, b\}$, $\mathcal{C}(b) = \{b\}$, $\mathcal{C}(c) = \{b, c, d\}$ and $\mathcal{C}(d) = \{d\}$. Clearly d is a normal form and b is a zero loop.

The topological indistinguishability of zero loops and normal forms leads to the following definition.

Definition 2.2.6.

- a) Let \mathcal{A} be an ARS. An element $a \in A$ is an almost normal form if it is a normal form or a zero loop.
- b) We say that an element $b \in A$ has an almost normal form if there exists an $a \in A$ in almost normal form such that $b \rightarrow a$.
- c) We say that \rightarrow almost weakly normalizes – and write \rightarrow is AWN – if all the elements of A have an almost normal form.

Hence almost normal forms are the only closed points in the reduction topology. It is very simple to characterize topologically the property of AWN.

Theorem 2.2.7. An abstract reduction system \mathcal{A} almost weakly normalizes iff every non empty closed subset of A with respect to the reduction topology has a closed point.

Proof: Suppose \mathcal{A} verifies AWN. Take $C \neq \emptyset$ a closed subset of A . Take $c \in C$ and consider $a \in A$ such that a is in almost normal form and $c \rightarrow a$. Hence $a \in C$ because C is closed and a is a closed point because of **Lemma 2.2.4**.

Conversely, suppose that every non empty closed set has a closed point. Then for any $b \in A$ the closed set $\mathcal{C}(b)$ has a closed point c inside. Then $b \rightarrow c$ and c is in almost normal form. \square

The theorem that follows is a topological version of the proof of weak normalization for a PTS with definitions as considered in [PS93].

Theorem 2.2.8. *Let $\mathcal{A} = \langle A, \rightarrow \rangle$ and $\mathcal{A}' = \langle A', \rightarrow' \rangle$ be an abstract reduction system and a subsystem. Call $\langle A, \mathcal{T} \rangle$ and $\langle A', \mathcal{T}' \rangle$ the corresponding topological spaces. Suppose that A' is \mathcal{T} -closed and that there exists a function $f : A \rightarrow A'$ such that $f(X) \subseteq \mathcal{C}_{\mathcal{T}}(X)$, $\forall X \subseteq A$. If \mathcal{A}' is AWN then \mathcal{A} is AWN.*

Proof: To simplify notations we write $\mathcal{C}_{\mathcal{T}}$ and $\mathcal{C}_{\mathcal{T}'}$ as \mathcal{C} . Take $C \neq \emptyset$ a closed set in A . The hypothesis $f(X) \subseteq \mathcal{C}(X)$, $\forall X \subseteq A$ guarantees that $f(C) \subseteq C \cap A'$. As $C \neq \emptyset$, $\emptyset \neq f(C) \subseteq C \cap A'$. Since \mathcal{A}' is AWN there exists a \mathcal{T}' -closed point $\{a\}$ in $C \cap A'$. As A' is \mathcal{T} -closed in A then $\{a\}$ is a \mathcal{T} -closed point. \square

Definition 2.2.9. *Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS with only one relation. We say that \rightarrow almost strongly normalizes - and write \rightarrow is ASN - if every reduction sequence $a_1 \rightarrow_{\alpha} a_2 \rightarrow_{\alpha} \dots a_n \rightarrow_{\alpha} a_{n+1} \dots$ terminates in a zero loop or in a normal form. In equivalent terms we could say that the ARS is ASN if for every reduction sequence as above, there exists an $m \in \mathbb{N}$ such that $a_m \equiv a_{m+1} \equiv \dots$*

To characterize almost strong normalization we need the concept of Noetherian family of subsets.

Definition 2.2.10. *Let A be an arbitrary set and \mathcal{S} a family of subsets of A , i.e. $\mathcal{S} \subseteq \mathcal{P}(A)$. We say that \mathcal{S} is Noetherian¹ iff all decreasing subfamilies of \mathcal{S} stabilize, i.e. for an arbitrary family $\{S_i : i \in \mathbb{N}\} \subseteq \mathcal{S}$ such that $S_1 \supseteq S_2 \supseteq S_3 \supseteq \dots \supseteq S_n \supseteq \dots \Rightarrow \exists m \in \mathbb{N}$ such that $S_m = S_{m+1} = \dots$*

Theorem 2.2.11. *Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be an ARS. Then \mathcal{A} almost strongly normalizes if and only if the following two conditions are verified:*

1. *The family $\mathcal{P}_{\rightarrow} \subseteq \mathcal{F}_{\rightarrow}$ of principal closed sets of A is Noetherian.*
2. *$\mathcal{C}(a) = \mathcal{C}(b) \Leftrightarrow a \equiv b$.*

Proof: Suppose \rightarrow almost strongly normalizes. If $\mathcal{C}(a) = \mathcal{C}(b)$ with $a \not\equiv b$, we would have a reduction of infinite length $a \rightarrow b \rightarrow a \rightarrow b \dots$

Suppose now that we have a decreasing family of sets in $\mathcal{P}_{\rightarrow}$, i.e. a family of the form:

$$\mathcal{C}(a_1) \supseteq \mathcal{C}(a_2) \supseteq \dots \supseteq \mathcal{C}(a_n) \supseteq \mathcal{C}(a_{n+1}) \supseteq \dots$$

¹In accordance with the usual mathematical definitions one should use the word Artinian instead of Noetherian. We use Noetherian not to diverge too much from the existing literature on the subject. See [Klo90] Definition 1.3. (iii) page 5.

This family produces a family of reductions $a_1 \twoheadrightarrow a_2 \twoheadrightarrow \dots a_n \twoheadrightarrow a_{n+1} \dots$ that must terminate, i.e. there exists an $m \in \mathbb{N}$ such that $a_m \equiv a_{m+1} \equiv \dots$. Hence $\mathcal{C}(a_m) = \mathcal{C}(a_{m+1}) = \dots$

Conversely, any reduction $a_1 \twoheadrightarrow a_2 \twoheadrightarrow \dots a_n \twoheadrightarrow a_{n+1} \dots$ produces a family of principal closed sets : $\mathcal{C}(a_1) \supseteq \mathcal{C}(a_2) \supseteq \dots \supseteq \mathcal{C}(a_n) \supseteq \mathcal{C}(a_{n+1}) \supseteq \dots$. By the Noetherian hypothesis we conclude that there exists an $m \in \mathbb{N}$ such that $\mathcal{C}(a_m) = \mathcal{C}(a_{m+1}) = \dots$. Hence by hypothesis 2 we conclude that $a_m \equiv a_{m+1} \equiv \dots$ \square

In our topological formulation some of the known results for ARS become very easy to prove. We illustrate this with the proof that almost strong normalization implies almost weak normalization.

Theorem 2.2.12. *Let \mathcal{A} be ARS. If \mathcal{A} is almost strongly normalizable then \mathcal{A} is almost weakly normalizable. Equivalently (in topological terms) if in \mathcal{A} the family of principal closed sets is Noetherian and $\mathcal{C}(a) = \mathcal{C}(b) \Leftrightarrow a \equiv b$; then every closed subset of \mathcal{A} has a closed point.*

Proof: Let C be an arbitrary closed subset of \mathcal{A} and consider the subfamily of \mathcal{P}_\rightarrow consisting of the sets of the form $\{\mathcal{C}(x) : x \in C\}$. This subfamily has a minimal element $\mathcal{C}(m)$. Take $y \in \mathcal{C}(m)$. As $m \in C$ and C is closed $y \in C$. By minimality $\mathcal{C}(y) = \mathcal{C}(m)$ and so we conclude that $y \equiv m$. Hence $\mathcal{C}(m)$ has only one element and hence m is a closed point. \square

Note that we cannot deal topologically with the concepts of normalization. This is because topologically the zero loops and the normal forms are impossible to distinguish.

In a system in which there are no zero loops, the concept of almost normalization and of normalization coincide. Concrete examples of such systems are the set of typable terms of an arbitrary PTS in which the zero loop cannot be typed.

Definition 2.2.13.

- a) *Let \mathcal{A} be an ARS. We say that \mathcal{A} verifies the almost normal form property – and we write \rightarrow verifies ANF – if for all $a, b \in \mathcal{A}$ such a is in almost normal form and $a = b$; then $b \twoheadrightarrow a$.*
- b) *We say that \mathcal{A} verifies the unique almost normal form property – and we write \rightarrow verifies AUF – if for all $a, b \in \mathcal{A}$ such that a and b are in almost normal form and $a = b$; then $a \equiv b$.*

If we drop the word "almost" in a) and b) we say that \mathcal{A} verifies the normal form property and the unique normal form property respectively and we will use the abbreviations: NF and UF respectively.

Theorem 2.2.14.

- a) *The ARS \rightarrow verifies ANF if and only if for all $a, b \in A$ with $\{a\}$ closed and a, b \mathcal{T}_\rightarrow -connectable $\Rightarrow a \in \mathcal{C}(b)$.*
- b) *The ARS \rightarrow verifies AUF if and only if for all $a, b \in A$ with $\{a\}, \{b\}$ closed and \mathcal{T}_\rightarrow -connectable $\Rightarrow a \equiv b$.*

Obviously, $ANF \Rightarrow AUF$ because $a \in \mathcal{C}(b)$ and $\mathcal{C}(b) = \{b\}$.
Now we prove topologically that $CR \Rightarrow ANF$.

Theorem 2.2.15. *If \rightarrow is CR then \rightarrow verifies ANF. Equivalently in topological terms if for all $X \subseteq A$, $\mathcal{CM}(X) \subseteq \mathcal{MC}(X)$, then for all $a, b \in A$ with $\{a\}$ closed in \mathcal{T}_\rightarrow and a \mathcal{T}_\rightarrow -connectable with b we conclude that $a \in \mathcal{C}(b)$.*

Proof: Let $a, b \in A$ be a pair of \mathcal{T}_\rightarrow -connectable elements of A with $\{a\}$ closed. We start by observing that by CR condition, if X is a closed subset of A and \mathcal{M} and \mathcal{C} are as in **Definition 1.4.12**, then $\forall k \in \mathbb{N}$, $(\mathcal{CM})^k(X) \subseteq \mathcal{M}(X)$. This is because $\mathcal{CM}(X) \subseteq \mathcal{MC}(X) \subseteq \mathcal{M}(X)$ the first inclusion following from CR and the second because X is closed. Then $(\mathcal{CM})^2(X) \subseteq \mathcal{CMM}(X) = \mathcal{CM}(X) \subseteq \mathcal{M}(X)$, etc. As a and b are \mathcal{T}_\rightarrow -connectable, $b \in (\mathcal{CM})^n(a)$ for some $n \in \mathbb{N}$. As $\{a\}$ is closed $(\mathcal{CM})^n(a) \subseteq \mathcal{M}(a)$. Then $b \in \mathcal{M}(a)$ and then $a \in \mathcal{C}(b)$. □

In particular, we deduce that $CR \Rightarrow NF \Rightarrow UF$.

Chapter 3

Connectivity and irreducibility in the reduction topology

Recall the following definitions from general topology.

Definition 3.1.

Let (X, \mathcal{T}) be a topological space.

- a) X is said to be connected if for every pair C, D of disjoint closed subsets of X such that $C \cup D = X$ we have that C or D are equal to X .
- b) X is said to be irreducible if for every pair C, D of closed subsets of X such that $C \cup D = X$ we have that C or D are equal to X .

Observe that X is irreducible if and only if for every pair $S, T \subset X$ of non empty open subsets of X , we have that $S \cap T \neq \emptyset$. It is clear that, irreducible implies connected and it is known that the closure of an irreducible subset is irreducible. Hence, $\mathcal{C}(x)$ is irreducible –and connected– for all $x \in X$.

Observe that in case that \mathcal{T} is a reduction topology, the statement " $X \subseteq A$ is connected" is self dual, in other words X is \mathcal{T} connected if and only if X is \mathcal{T}' connected. Here \mathcal{T}' is as in **Property 1.4.7**, i.e. the topology whose open sets are the closed subsets in the topology \mathcal{T} .

For the rest of this Chapter we assume that $\mathcal{A} = \langle A, \rightarrow \rangle$ is an abstract reduction system and that \mathcal{T}_\rightarrow is the associated topology. Sometimes we abbreviate $\mathcal{T}_\rightarrow = \mathcal{T}$ and $\mathcal{T}_\leftarrow = \mathcal{T}'$. The associated operators will be denoted as \mathcal{C} , \mathcal{M} , \mathcal{C}' and \mathcal{M}' respectively. Observe that $\mathcal{C} = \mathcal{M}'$ and $\mathcal{M} = \mathcal{C}'$.

Theorem 3.2.

For all $a \in A$ the set $\mathcal{M}(a)$ is connected.

Proof: Clearly $\mathcal{M}(a) = \mathcal{C}'(a)$. We just observed that for an arbitrary topology the sets of the form $\mathcal{C}'(a)$ are \mathcal{T}' -connected. Hence they are \mathcal{T} -connected. \square

Now we identify the connected components of A .

In **Proposition 1.4.14** we established the main properties of the set $\bigcup_n (\mathcal{CM})^n(a)$. Here we prove that it coincides with the connected component of A that contains a .

Theorem 3.3. *For any $a \in A$ the set $\bigcup_n (\mathcal{CM})^n(a) = \bigcup_n (\mathcal{MC})^n(a)$, that will be denoted as $Cc(a)$, is the connected component of A that contains a .*

Proof: Suppose that $Cc(a) = C \cup D$ with C and D closed and disjoint in $Cc(a)$. As $Cc(a)$ is open and closed in A so are C and D . Suppose $a \in C$. Since $Cc(a)$ is the smallest open and closed set that contains a : $Cc(a) \subseteq C$. Hence $Cc(a) = C$ and $D = \emptyset$. Hence $Cc(a)$ is connected. Being open and closed it has to be a connected component. \square

Corollary 3.4. *Let $a, b \in A$, then the assertions that follow are equivalent: (a) The elements a and b are \mathcal{T} connectable, (b) If $=$ is the equivalence relation induced by \mathcal{A} , $a = b$, (c) $a \in Cc(b)$, (d) $b \in Cc(a)$, (e) $Cc(a) = Cc(b)$.*

In the case that \mathcal{A} is CR, one can describe $Cc(a)$ in a more precise way.

Theorem 3.5. *Assume that $\mathcal{A} = \langle A, \rightarrow \rangle$ is an ARS that verifies Church-Rosser and call \mathcal{C} and \mathcal{M} the associated operators as defined in **Definition 1.2.1** and **Property 1.4.8**. Then*

- i) For all integers $k \geq 1$, $(\mathcal{CM})^k(a) \subseteq \mathcal{MC}(a)$.
- ii) For all integers $k \geq 2$, $(\mathcal{CM})^k(a) = \mathcal{MC}(a)$.
- iii) $\mathcal{CM}(a) \subseteq \mathcal{MC}(a) = (\mathcal{CM})^2(a) = \dots = (\mathcal{CM})^n(a) = \bigcup_n (\mathcal{CM})^n(a)$.

Proof:

- i) We proceed by induction on k . If $k = 1$ the inclusion $\mathcal{CM}(a) \subseteq \mathcal{MC}(a)$ for a confluent abstract reduction system was proved in **Theorem 2.1.6**. If $k > 1$, $(\mathcal{CM})^k(a) = (\mathcal{CM})((\mathcal{CM})^{k-1}(a)) \subseteq \mathcal{CM}\mathcal{M}\mathcal{C}(a) = \mathcal{C}\mathcal{M}\mathcal{C}(a) \subseteq \mathcal{M}\mathcal{C}\mathcal{C}(a) = \mathcal{MC}(a)$.
- ii) As we proved in general in **Proposition 1.4.14**, $\mathcal{MC}(a) \subseteq (\mathcal{CM})^2(a)$. Using part i) we conclude that in the case that the ARS is confluent $(\mathcal{CM})^2(a) \subseteq \mathcal{MC}(a)$. Hence $(\mathcal{CM})^2(a) = \mathcal{MC}(a)$. For $k \geq 2$, $\mathcal{MC}(a) = (\mathcal{CM})^2(a) \subseteq (\mathcal{CM})^k(a) \subseteq \mathcal{MC}(a)$.
- iii) It follows immediately from i) and ii).

□

The theorem that follows is an immediate consequence of the one just proved.

Theorem 3.6. *Let \mathcal{A} be an abstract reduction system.*

- a) *If \mathcal{A} is CR then $Cc(a) = \mathcal{MC}(a)$ for all $a \in A$.*
- b) *If \mathcal{A} is CCR then $Cc(a) = \mathcal{CM}(a)$ for all $a \in A$.*
- c) *If \mathcal{A} is CR then $Cc(a) = \mathcal{M}(a)$ for all $a \in A$ that are closed points. In particular $\mathcal{M}(a)$ is closed.*

Example 3.7. In the case of λ -terms, i.e. $\langle \Lambda, \rightarrow_\beta \rangle$ we have that $Cc(a) = \{b : a =_\beta b\} = \mathcal{MC}(a)$.¹

If \mathcal{A} is an abstract reduction system, we call $\mathcal{N}(A) \subseteq A$ the set of closed points or in other words the set of terms in almost normal form.

Theorem 3.8.

Let $\mathcal{A} = \langle A, \rightarrow \rangle$ be a confluent and almost weakly normalizing ARS. Then the sets $\mathcal{M}(n), n \in \mathcal{N}(A)$ are the connected components of A .

Proof:

This follows from the results already proved by observing that if $a \in A$, and $n \in \mathcal{N}(A)$ is its normal form, then $Cc(a) = Cc(n) = \mathcal{M}(n)$. □

The sets $\mathcal{M}(a)$ are always connected, their irreducibility is related with the converse of Church-Rosser.

Let us see an example.

Example 3.9.

For reduction system $c \rightarrow a \leftarrow b$ the set $\mathcal{M}(a) = \{a, b, c\}$ is obviously connected. The closed sets are $\{a\}$, $\{a, c\}$ and $\{a, b\}$ and there is no way to write $\{a, b, c\}$ as the union of two disjoint closed sets.

However, it is not irreducible because $\{a, b, c\} = \{a, c\} \cup \{a, b\}$.

The point is that the system above is not converse Church-Rosser, i.e. even though $c \rightarrow a \leftarrow b$, we can not find d such that $c \leftarrow d \rightarrow b$.

¹In [Chu41] pg. 25, this result is stated as: "If A conv B , there is a conversion of A into B in which no expansion precedes any reduction".

Example 3.10. Consider the ARS $\langle \Lambda, \rightarrow_\beta \rangle$ where Λ is the set of λ -terms. If we take $M \equiv (b\ c)\ (b\ c)$, $M_1 \equiv (\lambda x. b\ x)\ (b\ c)$ and $M_2 \equiv \lambda x. (x\ x)\ (b\ c)$. The set $\mathcal{M}(M)$ is not irreducible because $\mathcal{M}(M_1)$ and $\mathcal{M}(M_2)$ are disjoint open subsets of $\mathcal{M}(M)$. See [Bar92], pg. 75, Ex. 3.5.11.

Theorem 3.11.

- a) *The abstract reduction system \mathcal{A} is Church-Rosser if and only if $\forall a \in A$ the sets $\mathcal{C}(a)$ are \mathcal{T}' irreducible.*
- b) *The abstract reduction system \mathcal{A} is converse Church-Rosser if and only if $\forall a \in A$ the sets $\mathcal{M}(a)$ are \mathcal{T} irreducible.*

Proof:

- a) In accordance with the comments that follow **Definition 3.1**, a subset X of A is \mathcal{T}' irreducible, if and only every pair of non empty \mathcal{T} closed subsets of X has non empty intersection. We conclude our result by applying this conclusion together with **Theorem 2.1.2** to the case that $X = \mathcal{C}_{\mathcal{T}^-}(a)$ for all a .
- b) It follows immediately from a).

□

Notice that the Theorem just proved provides a fourth topological characterization of confluence, the other ones appeared in **Theorem 2.1.2**, **Theorem 2.1.6** and **Theorem 2.1.7**.

Part b) of the Theorem that follows asserts that in the case of almost weak normalization, to guarantee CCR it is enough to look at the irreducibility of $\mathcal{M}(a)$ in the case that a is in almost normal form.

Theorem 3.12.

- a) *Let \mathcal{A} be an ARS and let $a, b \in A$ be such that $a \twoheadrightarrow b$. Then if $\mathcal{M}(b)$ is irreducible so is $\mathcal{M}(a)$.*
- b) *Suppose that \mathcal{A} is an almost weakly normalizing ARS. Then \mathcal{A} is CCR if and only if $\forall n \in \mathcal{N}(A)$ the sets $\mathcal{M}(n)$ are irreducible.*

Proof:

- a) As $a \twoheadrightarrow b$ we have that $\mathcal{M}(a) \subseteq \mathcal{M}(b)$. As $\mathcal{M}(a), \mathcal{M}(b)$ are open sets, if we take a pair of open non empty subsets C, D in $\mathcal{M}(a)$ they are also open in $\mathcal{M}(b)$. As $\mathcal{M}(b)$ is irreducible, the intersection $C \cap D \neq \emptyset$. So that $\mathcal{M}(a)$ is irreducible.

b) It follows immediately from what we just proved and **Theorem 3.11, b)**. This is because if $a \in A$ there exists an $n \in \mathcal{N}(A)$ such that $a \rightarrow n$.

□

Hence if the reduction relation is confluent and almost weakly normalizing, A can be written as the disjoint union of connected components that are open and closed in A . If moreover \rightarrow is converse Church-Rosser the connected components are irreducible.

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