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ABSTRACT RELATIVE FOURIER TRANSFORMS OVER CANONICAL HOMOGENEOUS SPACES OF SEMI-DIRECT PRODUCT GROUPS WITH ABELIAN NORMAL FACTOR

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Abstract. This paper presents a systematic study for theoretical aspects of a unified approach to the abstract relative Fourier transforms over canonical homogeneous spaces of semi-direct product groups with Abelian normal factor. Let H be a locally compact group, K be a locally compact Abelian (LCA) group, and $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Let $G_\theta = H \ltimes_\theta K$ be the semi-direct product of H and K with respect to θ and G_θ/H be the canonical homogeneous space (left coset space) of G_θ. We introduce the notions of relative dual homogeneous space and also abstract relative Fourier transform over G_θ/H. Then we study theoretical properties of this approach.

1. Introduction

The mathematical theory of relative-convolution operators is a theoretical generalization for other classical operators in mathematical analysis and functional analysis such as two-sided convolutions and Toeplitz operators, see [15, 16, 17] and references therein. The abstract notion of relative-convolution operators over homogeneous spaces of locally compact groups introduced in [16] and studied comprehensively in [10, 18]. The class of locally compact semi-direct product groups as a large class of non-Abelian groups, has significant roles in theories connecting mathematical physics, mathematical theory of coherent states analysis [1, 8, 9, 12, 21] and covariant transforms, see [2, 3, 4, 5, 18, 19, 20] and standard references therein.

This research work consists of aspects of theoretical nature of abstract relative Fourier transforms over canonical homogeneous spaces of locally compact semi-direct product groups with Abelian normal factor. This article aims to present a unified approach to the abstract harmonic analysis of relative Fourier transforms over canonical homogeneous spaces of locally compact groups with Abelian normal factor. The main motivation to present the following approach

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to the abstract Fourier transform is to further develop theoretical aspects of coherent states analysis and covariant transforms over canonical homogeneous spaces of semi-direct product groups, see [10, 18, 20] and references therein.

The paper is organized as follows. Section 2 is devoted to fixing notations and a summary of classical harmonic analysis techniques on locally compact homogeneous spaces, locally compact semi-direct product groups, and standard Fourier analysis on locally compact Abelian (LCA) groups. In Section 3, we assume that H and K are locally compact groups and $\theta : H \to \mathrm{Aut}(K)$ is a continuous homomorphism. Further, it is assumed that $G_\theta = H \ltimes_\theta K$ is the semi-direct product of H and K with respect to θ. We briefly study abstract harmonic analysis properties of the locally compact canonical homogeneous space (left coset space) G_θ/H. Then we present the abstract notions of dual space for the canonical homogeneous space G_θ/H, abstract relative Fourier transform over G_θ/H, and we study theoretical aspects of the relative Fourier transform on function spaces of the canonical homogeneous space (left coset space) G_θ/H. Finally, we illustrate application of our results in the case of some well-known examples.

2. Preliminaries and notations

Let G be a locally compact group with the left Haar measure m_G and the modular function Δ_G. For $p \geq 1$ the notation $L^p(G)$ stands for the Banach function space $L^p(G, m_G)$. If $p = 1$, then the standard convolution for $f, g \in L^1(G)$ is defined via

$$f * g(x) = \int_G f(y)g(y^{-1}x)dm_G(y) \quad \text{for } x \in G. \tag{2.1}$$

The involution for $f \in L^1(G)$ is defined by $f^*(x) = \Delta_G(x^{-1})\overline{f(x)}$ for $x \in G$. Then the Banach function space $L^1(G)$ equipped with the above convolution and involution is a Banach $*$-algebra. The Banach $*$-algebra $L^1(G)$ is commutative if and only if G is Abelian, see [6, 13, 22].

Let H be a closed subgroup of a locally compact group G with the left Haar measures m_H and m_G, respectively. The left coset space $G/H = \{xH : x \in G\}$ is considered as a locally compact homogeneous space that G acts on it from the left. The locally compact left coset space G/H is called a locally compact pure homogeneous space if the closed subgroup H is not normal in G. The function space $\mathcal{C}_c(G/H)$ consists of all $P_H(f)$ functions, where $f \in \mathcal{C}_c(G)$ and

$$P_H(f)(xH) = \int_H f(xh)dm_H(h).$$

The mapping $P_H : \mathcal{C}_c(G) \to \mathcal{C}_c(G/H)$ defined by $f \mapsto P_H(f)$ is a surjective bounded linear operator. Let μ be a Radon measure on G/H and $x \in G$. The translation μ^x of μ is defined by $\mu^x(E) = \mu(xE)$ for each Borel subset E of G/H. The measure μ is called G-invariant if $\mu^x = \mu$ for all $x \in G$. The measure μ is called strongly quasi-invariant, if some continuous function

$\lambda : G \times G/H \to (0, \infty)$ satisfies $d\mu^x(yH) = \lambda(x, yH)d\mu(yH)$ for $x, y \in G$. If the function $\lambda(x, \cdot)$ reduces to constant, μ is called relatively invariant under G. A rho-function for the pair (G, H), is a continuous function $\rho : G \to (0, \infty)$ which satisfies

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(x) \quad \text{for } x \in G, \quad h \in H. \tag{2.2}$$

If H is a closed subgroup of G, the pair (G, H) admits a rho-function and for each rho-function ρ on G, there is a strongly quasi-invariant measure μ on G/H such that

$$\int_{G/H} P_H(f)(xH)d\mu(xH) = \int_G f(x)\rho(x)dm_G(x) \quad \text{for } f \in L^1(G),$$

and

$$d\mu^x(yH) = \frac{\rho(xy)}{\rho(y)}d\mu(yH) \quad \text{for } x \in G. \tag{2.3}$$

The homogeneous space G/H has a G-invariant measure if and only if $\Delta_G|_H = \Delta_H$. If μ is the strongly quasi invariant measure on G/H arising from the rho-function ρ, the mapping $T_H : L^1(G) \to L^1(G/H, \mu)$, given by

$$T_H(f)(xH) = \int_H \frac{f(xh)}{\rho(xh)}dm_H(h),$$

is a surjective bounded linear operator with $\|T_H\| \leq 1$, satisfying the Weil's formula [6, 13, 22]

$$\int_{G/H} T_H(f)(xH)d\mu(xH) = \int_G f(x)dm_G(x). \tag{2.4}$$

Let H and K be locally compact groups with identity elements e_H and e_K respectively and left Haar measures m_H and m_K respectively. Let $\theta : H \to \mathrm{Aut}(K)$ be a homomorphism such that the map $(h, k) \mapsto \theta_h(k)$ is continuous from $H \times K$ onto K.

The semi-direct product $G_\theta = H \ltimes_\theta K$ is the locally compact topological group with the underlying set $H \times K$ which is equipped by the product topology and the group operation is defined by

$$(h, k) \ltimes_\theta (h', k') = (hh', k\theta_h(k')) \quad \text{and} \quad (h, k)^{-1} = (h^{-1}, \theta_{h^{-1}}(k^{-1})). \tag{2.5}$$

The left Haar measure of the locally compact group G_θ is

$$dm_{G_\theta}(h, k) = \delta^\theta_{H,K}(h)dm_H(h)dm_K(k), \tag{2.6}$$

and the modular function Δ_{G_θ} is

$$\Delta_{G_\theta}(h, k) = \delta^\theta_{H,K}(h)\Delta_H(h)\Delta_K(k) \quad \text{for } (h, k) \in G_\theta, \tag{2.7}$$

where the positive continuous homomorphism $\delta^\theta_{H,K} : H \to (0, \infty)$ satisfies [13, 14]

$$dm_K(k) = \delta^\theta_{H,K}(h)dm_K(\theta_h(k)) \quad \text{for } h \in H. \tag{2.8}$$

The homomorphism $\theta : H \to \mathrm{Aut}(K)$ is called trivial if $\theta_h = I_K$ for all $h \in H$, where I_K is the identity automorphism. If $\theta : H \to \mathrm{Aut}(K)$ is trivial, then the semi-direct product of H and K with respect to θ is precisely the direct product of H and K. If $\widetilde{H} := \{(h, e_K) : h \in H\}$ and $\widetilde{K} := \{(e_H, k) : k \in K\}$, then $\widetilde{K}$ is a closed normal subgroup and $\widetilde{H}$ is a closed non-normal subgroup of G_θ. From now on we may use H, K instead of $\widetilde{H}, \widetilde{K}$, respectively.

If K is an LCA (locally compact Abelian) group, all irreducible representations of K are one-dimensional. Thus, if π is an irreducible unitary representation of K we have $\mathcal{H}_\pi = \mathbb{C}$. Hence, there exists a continuous homomorphism ω of K into the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, such that for each $k \in K$ and $z \in \mathbb{C}$ we have $\pi(k)(z) = \omega(k)z$. Such homomorphisms are called characters of K and the set of all such characters of K is denoted by $\widehat{K}$. If $\widehat{K}$ equipped with the topology of compact convergence on K which coincides with the w^*-topology that $\widehat{K}$ inherits as a subset of $L^\infty(K)$, then $\widehat{K}$ with respect to the product of characters is an LCA group which is called the dual group of K. The linear map $\widehat{\ }: L^1(K) \to \mathcal{C}(\widehat{K})$ defined by $v \mapsto \widehat{v}$ via

$$\widehat{v}(\omega) = \int_K v(k)\overline{\omega(k)}dm_K(k), \tag{2.9}$$

is the Fourier transform on K. It is a norm-decreasing $*$-homomorphism from $L^1(K)$ into $\mathcal{C}_0(\widehat{K})$ with a uniformly dense range in $\mathcal{C}_0(\widehat{K})$. The Fourier transform (2.9) on $L^1(K) \cap L^2(K)$ is an isometric transform and it extends uniquely to a unitary isomorphism from $L^2(K)$ onto $L^2(\widehat{K})$, and each $v \in L^1(K)$ with $\widehat{v} \in L^1(\widehat{K})$ satisfies the following Fourier inversion formula [6, 13, 22]

$$v(k) = \int_{\widehat{K}} \widehat{v}(\omega)\omega(k)dm_{\widehat{K}}(\omega) \quad \text{for } k \in K. \tag{2.10}$$

If $u \in L^1(\widehat{K})$, the function which is defined on K by

$$\check{u}(x) = \int_{\widehat{K}} u(\omega)\omega(x)dm_{\widehat{K}}(\omega), \tag{2.11}$$

belongs to $L^\infty(K)$ and for all $v \in L^1(K)$ we have the following orthogonality relation (Parseval formula)

$$\int_K v(k)\overline{\check{u}(k)}dm_K(k) = \int_{\widehat{K}} \widehat{v}(\omega)\overline{u(\omega)}dm_{\widehat{K}}(\omega). \tag{2.12}$$

3. Abstract harmonic analysis over canonical homogeneous spaces of semi-direct product groups

Throughout this section, we assume that H, K are locally compact groups with given left Haar measures m_H and m_K respectively, and $\theta : H \to \mathrm{Aut}(K)$ is a continuous homomorphism. Let $G_\theta = H \ltimes_\theta K$ be the semi-direct product of H and K with respect to θ. Then the left coset space

$$G_\theta/H = \{(h, k)H : (h, k) \in G_\theta\},$$

is a locally compact homogeneous space. The homogeneous space (left coset space) G_θ/H is called as *canonical homogeneous space* of the locally compact semi-direct product $G_\theta = H \ltimes_\theta K$. From now on, for $k \in K$ the notation kH stands for the left coset $(e_H, k)H$.

Next proposition states basic properties of the canonical homogeneous space G_θ/H.

Proposition 3.1. *Let H, K be locally compact groups and $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Let $G_\theta = H \ltimes_\theta K$ be the semi-direct product of H and K with respect to θ. Then*

(1) *H is normal in G_θ if and only if θ is the trivial homomorphism.*
(2) *For $k, k' \in K$, $kH = k'H$ if and only if $k = k'$.*
(3) *The canonical left coset space G_θ/H is precisely the set $\{kH : k \in K\}$.*

Remark 3.2. Proposition 3.1 shows that, if θ is not the trivial homomorphism, then the canonical left coset space G_θ/H is not a locally compact group. Thus, in this case classical tools and notions of abstract harmonic analysis such as convolution, involution, dual group, and Fourier transform are not well-defined for the pure homogeneous space G_θ/H.

Remark 3.3. Proposition 3.1 also asserts that the normal factor K parametrizes the canonical left coset space G_θ/H. It should be mentioned that the topological spaces K and G_θ/H are topologically isomorphic via the homeomorphism $k \mapsto kH$, although objects of these two spaces are different in general case, see Section 5.

As an immediate consequence of Proposition 3.1 we can conclude the following useful corollary.

Corollary 3.4. *Let ρ be a rho-function for the pair (G_θ, H). Then*

(1) *The linear map $P_H : \mathcal{C}_c(G_\theta) \to \mathcal{C}_c(G_\theta/H)$ is given by*

$$P_H(f)(kH) = \int_H f(h,k)dm_H(h) \quad \text{for } f \in \mathcal{C}_c(G_\theta). \tag{3.1}$$

(2) *The linear map $T_H : \mathcal{C}_c(G_\theta) \to \mathcal{C}_c(G_\theta/H)$ is given by*

$$T_H(f)(kH) = \int_H \frac{f(h,k)}{\rho(h,k)}dm_H(h) \quad \text{for } f \in \mathcal{C}_c(G_\theta). \tag{3.2}$$

Remark 3.5. The linear maps P_H and T_H have significant roles in abstract harmonic analysis over homogeneous spaces, see [6, 13, 22]. Corollary 3.4 asserts connections of partial integration over H with P_H and T_H.

The function $\rho_\theta : G_\theta \to (0, \infty)$ given by

$$\rho_\theta(h,k) = \Delta_H(h)\Delta_{G_\theta}(h)^{-1} = \delta^\theta_{H,K}(h)^{-1} \ \text{ for } (h,k) \in G_\theta = H \ltimes_\theta K. \tag{3.3}$$

is a rho-function for the pair (G_θ, H) which is called as *canonical rho-function*.

The canonical rho-function ρ_θ satisfies

$$\int_{G_\theta} f(h,k)\rho_\theta(h,k)dm_{G_\theta}(h,k) = \int_H \int_K f(h,k)dm_H(h)dm_K(k) \tag{3.4}$$

for all $f \in \mathcal{C}_c(G_\theta)$.

The following result shows that the induced strongly quasi-invariant measure μ_θ via the canonical rho-function ρ_θ defined in (3.3) is a relatively invariant measure.

Proposition 3.6. *The induced strongly quasi-invariant measure μ_θ via the canonical rho-function ρ_θ defined in* (3.3) *is a relatively invariant measure on the canonical homogeneous space G_θ/H.*

Next we present basic properties of the relatively invariant measure μ_θ.

Theorem 3.7. *Let μ_θ be the relatively invariant measure on the canonical left coset space G_θ/H which arises from the canonical rho-function ρ_θ defined in* (3.3). *Then*

$$\int_{G_\theta/H} \psi(kH)d\mu_\theta(kH) = \int_K \psi(kH)dm_K(k) \quad \text{for } \psi \in L^1(G_\theta/H, \mu_\theta), \tag{3.5}$$

$$\int_{G_\theta/H} v(kH)d\mu_\theta(kH) = \int_K v(k)dm_K(k) \quad \text{for } v \in L^1(K). \tag{3.6}$$

Proof. Let $\psi \in L^1(G_\theta/H, \mu_\theta)$ and $f \in L^1(G_\theta)$ with $T_H(f) = \psi$. Using the Weil's formula, we have

$$\int_{G_\theta/H} \psi(kH)d\mu_\theta(kH) = \int_{G_\theta/H} T_H(f)(kH)d\mu_\theta(kH) = \int_{G_\theta} f(h,k)dm_{G_\theta}(h,k).$$

By (3.2), we achieve

$$\begin{aligned}\int_{G_\theta} f(h,k)dm_{G_\theta}(h,k) &= \int_H \int_K f(h,k)\delta^\theta_{H,K}(h)dm_H(h)dm_K(k)\\ &= \int_H \int_K \frac{f(h,k)}{\rho(h,k)}dm_H(h)dm_K(k)\\ &= \int_K \left(\int_H \frac{f(h,k)}{\rho(h,k)}dm_H(h)\right)dm_K(k).\end{aligned}$$

Thus, we get

$$\begin{aligned}\int_K \left(\int_H \frac{f(h,k)}{\rho(h,k)}dm_H(h)\right)dm_K(k) &= \int_K T_H(f)(kH)dm_K(k)\\ &= \int_K \psi(kH)dm_K(k),\end{aligned}$$

which implies (3.5). The same argument implies (3.6). $\square$

Remark 3.8. Theorem 3.7 shows that, if we assume left Haar measures on H, K, then the relatively invariant measure μ_θ on G_θ/H, which arises from the canonical rho-function ρ_θ defined in (3.3) and satisfies the Weil's formula, is normalized automatically such that (3.5) and (3.6) hold.

The mapping $\Gamma = \Gamma_\theta : \mathcal{C}_c(K) \to \mathcal{C}_c(G_\theta/H)$ given by $v \mapsto \Gamma_\theta(v)$, where $\Gamma_\theta(v)$ is defined by

$$\Gamma_\theta(v)(sH) = v(s) \qquad \text{for } s \in K,$$

is well-defined, surjective and injective.

Next result shows that the linear map Γ_θ is a useful tool for analyzing functions on the canonical homogeneous space G_θ/H.

Corollary 3.9. *Let μ_θ be the relatively invariant measure on the canonical left coset space G_θ/H which arises from the canonical rho-function ρ_θ defined in* (3.3) *and $p \geq 1$. Then*

(1) *The linear map $\Gamma_\theta : \mathcal{C}_c(K) \to \mathcal{C}_c(G_\theta/H)$ satisfies*

$$\|\Gamma_\theta(v)\|_{L^p(G_\theta/H,\mu_\theta)} = \|v\|_{L^p(K)} \quad \text{for } v \in \mathcal{C}_c(K). \tag{3.7}$$

(2) *The linear map $\Gamma_\theta : \mathcal{C}_c(K) \to \mathcal{C}_c(G_\theta/H)$ has a unique extension to the linear map $\Gamma_\theta : L^p(K) \to L^p(G_\theta/H, \mu_\theta)$ which satisfies*

$$\|\Gamma_\theta(v)\|_{L^p(G_\theta/H,\mu_\theta)} = \|v\|_{L^p(K)} \quad \text{for } v \in L^p(K). \tag{3.8}$$

Proof. (1) Let $v \in \mathcal{C}_c(K)$. Then $\Gamma_\theta(v) \in \mathcal{C}_c(G_\theta/H)$, and hence we have $\psi := |\Gamma_\theta(v)|^p \in \mathcal{C}_c(G_\theta/H)$. Using Theorem 3.7, we get

$$\begin{aligned}\int_{G_\theta/H} |\Gamma_\theta(v)(kH)|^p d\mu_\theta(kH) &= \int_K |\Gamma_\theta(v)(kH)|^p dm_K(k)\\ &= \int_K |v(k)|^p dm_K(k),\end{aligned}$$

which implies (3.7).

(2) It is straightforward. □

For $\varphi, \varphi' \in \mathcal{C}_c(G_\theta/H)$, define the *$\theta$-convolution* of φ and φ' by

$$\varphi *_\theta \varphi'(sH) := \int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)d\mu_\theta(kH) \qquad \text{for } sH \in G_\theta/H. \tag{3.9}$$

Then the integral given in (3.9) converges and the mapping $(\varphi, \varphi') \mapsto \varphi *_\theta \varphi'$ is bilinear. Let $v, v' \in \mathcal{C}_c(K)$ with $\varphi = \Gamma_\theta(v)$ and $\varphi' = \Gamma_\theta(v')$. Then

$$\begin{aligned}\varphi *_\theta \varphi'(sH) &= \int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)d\mu_\theta(kH)\\ &= \int_{G_\theta/H} v(k)v'(k^{-1}s)d\mu_\theta(kH)\end{aligned}$$

for all $sH \in G_\theta/H$.

The following proposition states the relation of θ-convolution with the convolution on K.

Proposition 3.10. *Let $\varphi, \varphi' \in \mathcal{C}_c(G_\theta/H)$ and $v, v' \in \mathcal{C}_c(K)$ with $\varphi = \Gamma_\theta(v)$ and $\varphi' = \Gamma_\theta(v')$. Then*

$$\varphi *_\theta \varphi' = \Gamma_\theta(v * v'). \tag{3.10}$$

Proof. Let $\varphi, \varphi' \in \mathcal{C}_c(G_\theta/H)$, and let $v, v' \in \mathcal{C}_c(K)$ with $\varphi = \Gamma_\theta(v)$ and $\varphi' = \Gamma_\theta(v')$. Let $s \in K$. Then the function $\psi_s : G_\theta/H \to \mathbb{C}$ defined by

$$\psi_s(kH) := \varphi(kH)\varphi'(k^{-1}sH) = v(k)v'(k^{-1}s) \quad \text{for } kH \in G_\theta/H,$$

belongs to $\mathcal{C}_c(G_\theta/H)$. Invoking Theorem 3.7, we get

$$\begin{aligned}\int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)d\mu_\theta(kH) &= \int_{G_\theta/H} \psi_s(kH)d\mu_\theta(kH) \\ &= \int_K \psi_s(kH)dm_K(k).\end{aligned}$$

Then we can write

$$\begin{aligned}\varphi *_\theta \varphi'(sH) &= \int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)d\mu_\theta(kH) \\ &= \int_K \psi_s(kH)dm_K(k) \\ &= \int_K v(k)v'(k^{-1}s)dm_K(k) = v * v'(s),\end{aligned}$$

which implies (3.10). □

Similarly, one can define *θ-involution* of $\varphi \in \mathcal{C}_c(G_\theta/H)$ by

$$\varphi^{*_\theta}(sH) := \Delta_K(s^{-1})\overline{\varphi(s^{-1}H)} \quad \text{for } \; sH \in G_\theta/H. \tag{3.11}$$

Then we have

$$\varphi^{*_\theta} = \Gamma_\theta(v^*),$$

where $v \in \mathcal{C}_c(K)$ with $\varphi = \Gamma_\theta(v)$.

Next theorem guarantees that the θ-convolution and the θ-involution defined by (3.9) and (3.11) on $\mathcal{C}_c(G_\theta/H)$, have unique extensions to the Banach space $L^1(G_\theta/H, \mu_\theta)$, where μ_θ is the relatively invariant measure on G_θ/H which arises from the canonical rho-function given in (3.3).

Theorem 3.11. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by* (3.3). *The θ-convolution given in* (3.9) *and the θ-involution given in* (3.11) *have unique extensions to the Banach function space $L^1(G_\theta/H, \mu_\theta)$ in which the Banach function space $L^1(G_\theta/H, \mu_\theta)$ equipped with the extended θ-convolution and the extended θ-involution is a Banach $*$-algebra.*

Proof. Let $\varphi, \varphi' \in \mathcal{C}_c(G_\theta/H)$ and $v, v' \in \mathcal{C}_c(K)$ such that $\Gamma_\theta(v) = \varphi$ and $\Gamma_\theta(v') = \varphi'$. Then Proposition 3.10, implies $\varphi *_\theta \varphi' = \Gamma_\theta(v * v')$. Thus, we get

$$(\varphi *_\theta \varphi')^{*_\theta} = \varphi'^{*_\theta} *_\theta \varphi^{*_\theta}. \tag{3.12}$$

By (3.7), we can write

$$\begin{aligned}\|\varphi *_\theta \varphi'\|_{L^1(G_\theta/H,\mu_\theta)} &= \|\Gamma_\theta(v * v')\|_{L^1(G_\theta/H,\mu_\theta)}\\ &= \|v * v'\|_{L^1(K)}\\ &\leq \|v\|_{L^1(K)}\|v'\|_{L^1(K)}\\ &= \|\Gamma_\theta(v)\|_{L^1(G_\theta/H,\mu_\theta)}\|\Gamma_\theta(v')\|_{L^1(G_\theta/H,\mu_\theta)}\\ &= \|\varphi\|_{L^1(G_\theta/H,\mu_\theta)}\|\varphi'\|_{L^1(G_\theta/H,\mu_\theta)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|\varphi^{*_\theta}\|_{L^1(G_\theta/H,\mu_\theta)} &= \|\Gamma_\theta(v^*)\|_{L^1(G_\theta/H,\mu_\theta)}\\ &= \|v^*\|_{L^1(K)}\\ &= \|v\|_{L^1(K)} = \|\varphi\|_{L^1(G_\theta/H,\mu_\theta)}.\end{aligned}$$

Thus, for $\varphi, \varphi' \in \mathcal{C}_c(G_\theta/H)$, we achieve

$$\|\varphi *_\theta \varphi\|_{L^1(G_\theta/H,\mu_\theta)} \leq \|\varphi\|_{L^1(G_\theta/H,\mu_\theta)}\|\varphi'\|_{L^1(G_\theta/H,\mu_\theta)}, \tag{3.13}$$

$$\|\varphi^{*_\theta}\|_{L^1(G_\theta/H,\mu_\theta)} = \|\varphi\|_{L^1(G_\theta/H,\mu_\theta)}. \tag{3.14}$$

For $\varphi, \varphi' \in L^1(G_\theta/H, \mu_\theta)$, define the extended θ-convolution and θ-involution respectively, by

$$\varphi *_\theta \varphi' := \|.\|_{L^1(G_\theta/H,\mu_\theta)} - \lim_n \varphi_n *_\theta \varphi'_n, \tag{3.15}$$

$$\varphi^{*_\theta} := \|.\|_{L^1(G_\theta/H,\mu_\theta)} - \lim_n \varphi_n^{*_\theta}, \tag{3.16}$$

where $\{\varphi_n\}, \{\varphi'_n\} \subset \mathcal{C}_c(G_\theta/H)$ with $\varphi = \|\cdot\|_{L^1(G_\theta/H,\mu_\theta)} - \lim_n \varphi_n$ and $\varphi' = \|.\|_{L^1(G_\theta/H,\mu_\theta)} - \lim_n \varphi'_n$. The extended θ-convolution and θ-involution defined by (3.15) and (3.16) are well-defined and satisfy (3.12), (3.13) and (3.14). Thus, the extended θ-convolution and the extended θ-involution make the Banach space $L^1(G_\theta/H, \mu_\theta)$ into a Banach $*$-algebra. □

From now on, we may use the notations $*_\theta$ for extended θ-convolution and $^{*_\theta}$ for the extended θ-involution on $L^1(G_\theta/H, \mu_\theta)$.

Corollary 3.12. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by* (3.3). *Let*

$$\varphi, \varphi' \in L^1(G_\theta/H, \mu_\theta).$$

Then

$$\varphi *_\theta \varphi'(sH) = \int_{G_\theta/H} \varphi(kH)\varphi'(k^{-1}sH)d\mu_\theta(kH) \quad \textit{for} \quad sH \in G_\theta/H, \tag{3.17}$$

and

$$\varphi^{*_\theta}(sH) = \Delta_K(s^{-1})\overline{\varphi(s^{-1}H)} \quad \textit{for} \quad sH \in G_\theta/H. \tag{3.18}$$

Corollary 3.13. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by* (3.3). *Then*

(1) *The mapping* $\Gamma_\theta : L^1(K) \to L^1(G_\theta/H, \mu_\theta)$ *is an isometric* $*$*-isomorphism.*
(2) *The* θ*-convolution is commutative if and only if* K *is Abelian.*

Remark 3.14. Let $\mathbf{s}_\theta : G_\theta/H \to G_\theta$ be given by $\mathbf{s}_\theta(kH) := k$ for all $k \in k$. Then $\mathbf{s}_\theta$ is a continuous section, called as canonical section of the homogeneous space G_θ/H. Let μ_θ be the relatively invariant measure over the canonical homogeneous space G_θ/H which arises from the canonical rho function ρ_θ. The θ-convolution operator over the canonical homogeneous spaces G_θ/H is precisely the relative convolution operator associated to the canonical section $\mathbf{s}_\theta$ and the trivial unitary character $\imath : H \to \mathbb{C}$, see [6, 10, 13, 16, 18].

4. Abstract relative Fourier transform over canonical homogeneous spaces with Abelian normal factor

In this section, we present theoretical aspects of a unified approach to the notion of Fourier transform over the canonical left coset space G_θ/H. From now on in this article, we assume that H is a locally compact group with a left Haar measure m_H, K is an LCA group with the dual (character group) $\widehat{K}$, and $\theta : H \to \mathrm{Aut}(K)$ is a continuous homomorphism. For simplicity in notations we may use k^h instead of $\theta_h(k)$ for $h \in H$ and $k \in K$. Further, let m_K be a Haar measure on K and $m_{\widehat{K}}$ be the normalized Plancherel measure on $\widehat{K}$ associated to m_K.

For $\omega \in \widehat{K}$ and $h \in H$, define $\omega_h : K \to \mathbb{T}$ via

$$\omega_h(k) := \omega \circ \theta_{h^{-1}}(k) = \omega(\theta_{h^{-1}}(k)) \quad \text{for } k \in K. \tag{4.1}$$

If $\omega \in \widehat{K}$ and $h \in H$, then $\omega_h \in \widehat{K}$, because for $k, k' \in K$ we have

$$\begin{aligned}
\omega_h(kk') &= \omega \circ \theta_{h^{-1}}(kk') \\
&= \omega(\theta_{h^{-1}}(kk')) \\
&= \omega(\theta_{h^{-1}}(k)\theta_{h^{-1}}(k')) \\
&= \omega(\theta_{h^{-1}}(k))\omega(\theta_{h^{-1}}(k')) = \omega_h(k)\omega_h(k').
\end{aligned}$$

For $h \in H$, define $\widehat{\theta}_h : \widehat{K} \to \widehat{K}$ via

$$\widehat{\theta}_h(\omega) = \omega_h \quad \text{for } \omega \in \widehat{K}. \tag{4.2}$$

According to (4.1), for $h \in H$ we have $\widehat{\theta}_h \in \mathrm{Aut}(\widehat{K})$. Since for $k \in K$, $h \in H$, and $\omega, \omega' \in \widehat{K}$, we have

$$\begin{aligned}
(\omega.\omega')_h(k) &= (\omega.\omega') \circ \theta_{h^{-1}}(k) \\
&= \omega.\omega'(\theta_{h^{-1}}(k)) \\
&= \omega(\theta_{h^{-1}}(k))\omega'(\theta_{h^{-1}}(k)) = \omega_h(k)\omega'_h(k).
\end{aligned}$$

Thus, we get

$$\widehat{\theta}_h(\omega.\omega')(k) = \widehat{\theta}_h(\omega)(k)\widehat{\theta}_h(\omega')(k),$$

which implies

$$\widehat{\theta}_h(\omega.\omega') = \widehat{\theta}_h(\omega)\widehat{\theta}_h(\omega').$$

The following theorem gives the connection of the $\widehat{\theta}$-action and the normalized Plancherel measure over the dual group.

Theorem 4.1. *Let H be a locally compact group, K be an LCA group with Haar measure m_K, and $m_{\widehat{K}}$ be the normalized Plancherel measure on $\widehat{K}$. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism and $\delta^{\theta}_{H,K} : H \to (0, \infty)$ be the positive continuous homomorphism which satisfies* (2.8). *Then*

$$m_{\widehat{K}} \circ \widehat{\theta}_h = \delta^{\theta}_{H,K}(h) \cdot m_{\widehat{K}} \quad \textit{for } h \in H. \tag{4.3}$$

Proof. Let $h \in H$ be given. Then $m_{\widehat{K}} \circ \widehat{\theta}_h$ is a non-zero translation invariant measure (Haar measure) on the locally compact Abelian group $\widehat{K}$. To check this, let $E \subseteq \widehat{K}$ be a Borel subset and $\xi \in \widehat{K}$. By the translation invariance of the normalized Plancherel measure $m_{\widehat{K}}$, we can write

$$\begin{aligned}
m_{\widehat{K}} \circ \widehat{\theta}_h\,(\xi.E) &= m_{\widehat{K}} \circ \widehat{\theta}_h\,(\{\xi.\omega : \omega \in E\}) \\
&= m_{\widehat{K}} \left(\widehat{\theta}_h\{\xi.\omega : \omega \in E\}\right) \\
&= m_{\widehat{K}} \left(\left\{\widehat{\theta}_h(\xi.\omega) : \omega \in E\right\}\right) \\
&= m_{\widehat{K}}\,(\{\xi_h.\omega_h : \omega \in E\}) \\
&= m_{\widehat{K}}\,(\xi_h.\,\{\omega_h : \omega \in E\}) \\
&= m_{\widehat{K}}\,(\{\omega_h : \omega \in E\}) = m_{\widehat{K}} \circ \widehat{\theta}_h\,(E)\,.
\end{aligned}$$

Thus, by the uniqueness (up to scaling) of Haar measure on locally compact groups, we get $m_{\widehat{K}} \circ \widehat{\theta}_h = \beta_h \cdot m_{\widehat{K}}$, where β_h is a positive constant. Now we claim that $\beta_h = \delta^{\theta}_{H,K}(h)$. To prove this, let $f \in L^1(K)$. Then using (2.8), we have $f \circ \theta_h \in L^1(K)$ with $\|f \circ \theta_h\|_{L^1(K)} = \delta^{\theta}_{H,K}(h)\|f\|_{L^1(K)}$. Thus, for $\omega \in \widehat{K}$, we obtain

$$\begin{aligned}
\widehat{f \circ \theta_h}(\omega) &= \int_K f \circ \theta_h(k)\overline{\omega(k)}dm_K(k) \\
&= \int_K f(\theta_h(k))\overline{\omega(k)}dm_K(k) \\
&= \int_K f(k)\overline{\omega(\theta_{h^{-1}}(k))}dm_K(\theta_{h^{-1}}(k)) \\
&= \int_K f(k)\overline{\omega_h(k)}dm_K(\theta_{h^{-1}}(k)) \\
&= \delta^{\theta}_{H,K}(h) \int_K f(k)\overline{\omega_h(k)}dm_K(k) = \delta^{\theta}_{H,K}(h)\widehat{f}(\omega_h).
\end{aligned}$$

Let $f \in L^1(K) \cap L^2(K)$ be non-zero. Then, by Plancherel theorem, we can write

$$\begin{aligned}
\int_{\widehat{K}} |\widehat{f}(\omega)|^2 dm_{\widehat{K}}(\omega_h) &= \int_{\widehat{K}} |\widehat{f}(\omega_{h^{-1}})|^2 dm_{\widehat{K}}(\omega) \\
&= \delta^{\theta}_{H,K}(h)^2 \int_{\widehat{K}} |\widehat{f \circ \theta_{h^{-1}}}(\omega)|^2 dm_{\widehat{K}}(\omega) \\
&= \delta^{\theta}_{H,K}(h)^2 \int_{K} |f \circ \theta_{h^{-1}}(k)|^2 dm_K(k) \\
&= \delta^{\theta}_{H,K}(h)^2 \int_{K} |f(k)|^2 dm_K(\theta_h(k)) \\
&= \delta^{\theta}_{H,K}(h) \int_{K} |f(k)|^2 dm_K(k) \\
&= \delta^{\theta}_{H,K}(h) \int_{\widehat{K}} |\widehat{f}(\omega)|^2 dm_{\widehat{K}}(\omega),
\end{aligned}$$

which implies

$$\begin{aligned}
\beta_h \|\widehat{f}\|^2_{L^2(\widehat{K})} &= \beta_h \int_{\widehat{K}} |\widehat{f}(\omega)|^2 dm_{\widehat{K}}(\omega) \\
&= \int_{\widehat{K}} |\widehat{f}(\omega)|^2 dm_{\widehat{K}}(\omega_h) \\
&= \delta^{\theta}_{H,K}(h) \int_{\widehat{K}} |\widehat{f}(\omega)|^2 dm_{\widehat{K}}(\omega) = \delta^{\theta}_{H,K}(h) \|\widehat{f}\|^2_{L^2(\widehat{K})}.
\end{aligned}$$

Since f and hence $\|\widehat{f}\|_{L^2(K)}$ are non-zero, we can conclude that $\beta_h = \delta^{\theta}_{H,K}(h)$. □

Corollary 4.2. *The continuous homomorphism $\delta^{\widehat{\theta}}_{H,\widehat{K}} : H \to (0, \infty)$ is given by*

$$\delta^{\widehat{\theta}}_{H,\widehat{K}}(h) = \delta^{\theta}_{H,K}(h^{-1}) = \delta^{\theta}_{H,K}(h)^{-1} \quad \textit{for } h \in H.$$

Let $\widehat{\theta} : H \to \mathrm{Aut}(\widehat{K})$ be given via $h \mapsto \widehat{\theta}_h$. Then the map $h \mapsto \widehat{\theta}_h$ is a homomorphism from H into $\mathrm{Aut}(\widehat{K})$. To see this, let $h, h' \in H$, $\omega \in \widehat{K}$, and $k \in K$. Then we can write

$$\begin{aligned}
\omega_{hh'}(k) &= \omega(\theta_{(hh')^{-1}}(k)) \\
&= \omega(\theta_{h'^{-1}}\theta_{h^{-1}}(k)) = \omega_{h'}(\theta_{h^{-1}}(k)) = \widehat{\theta}_{h'}(\omega)(\theta_{h^{-1}}(k)).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
\widehat{\theta}_{hh'}(\omega)(k) &= \omega_{hh'}(k) \\
&= \widehat{\theta}_{h'}(\omega)(\theta_{h^{-1}}(k)) = \widehat{\theta}_h[\widehat{\theta}_{h'}(\omega)](k),
\end{aligned}$$

which guarantees that

$$\widehat{\theta}_{hh'}(\omega) = \widehat{\theta}_h[\widehat{\theta}_{h'}(\omega)].$$

Then we can present the following result.

Theorem 4.3. *Let $\delta^{\theta}_{H,K} : H \to (0,\infty)$ be the positive continuous homomorphism satisfying* (2.8)*. Then $\widehat{\theta} : H \to \mathrm{Aut}(\widehat{K})$ is a continuous homomorphism and the semi-direct product group $G_{\widehat{\theta}} := H \ltimes_{\widehat{\theta}} \widehat{K}$ is a locally compact group with a left Haar measure given by*

$$dm_{G_{\widehat{\theta}}}(h,\omega) = \delta^{\theta}_{H,K}(h)^{-1} dm_H(h) dm_{\widehat{K}}(\omega). \tag{4.4}$$

Proof. For $\sigma \in \mathrm{Aut}(K)$, let $\widehat{\sigma} \in \mathrm{Aut}(\widehat{K})$ be given for all $\omega \in \widehat{K}$ by $\widehat{\sigma}(\omega) := \omega\circ\sigma^{-1}$, where for all $k \in K$ we have $\omega\circ\sigma^{-1}(k) = \omega(\sigma^{-1}(k))$. Then by Theorem 26.9 and Theorem 26.5 of [13], the mapping $\widehat{\ }: \mathrm{Aut}(K) \to \mathrm{Aut}(\widehat{K})$ defined by $\sigma \mapsto \widehat{\sigma}$ is a topological group isomorphism and therefore it is continuous. Due to the following diagram

$$H \xrightarrow{\theta} \mathrm{Aut}(K) \xrightarrow{\widehat{\ }} \mathrm{Aut}(\widehat{K}), \tag{4.5}$$

the homomorphism $\widehat{\theta} : H \to \mathrm{Aut}(\widehat{K})$ defined in (4.1) is continuous, which consequently guarantees that the semi-direct product group $G_{\widehat{\theta}} = H \ltimes_{\widehat{\theta}} \widehat{K}$ is a locally compact group. Then Theorem 4.1, implies that

$$dm_{G_{\widehat{\theta}}}(h,\omega) = \delta_{H,K}(h)^{-1} dm_H(h) dm_{\widehat{K}}(\omega),$$

is a left Haar measure for $G_{\widehat{\theta}} = H \ltimes_{\widehat{\theta}} \widehat{K}$. □

Remark 4.4. The group law for $(h,\omega),(h',\omega') \in G_{\widehat{\theta}} = H \ltimes_{\widehat{\theta}} \widehat{K}$ is

$$(h,\omega) \ltimes_{\widehat{\theta}} (h',\omega') = (hh', \omega.\omega'_h).$$

Definition 4.5. Let H be a locally compact group, K be an LCA group with dual group $\widehat{K}$, and $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. The locally compact group $G_{\widehat{\theta}} = H \ltimes_{\widehat{\theta}} \widehat{K}$ is called as θ-*dual group* or the *semi-direct dual* of the locally compact semi-direct product group G_θ.

Due to the Pontrjagin duality theorem [6, 13], each $k \in K$ defines a character $\widehat{k}$ on $\widehat{K}$ via $\widehat{k}(\omega) = \omega(k)$ and the map $k \mapsto \widehat{k}$ is a topological group isomorphism from K onto $\widehat{\widehat{K}}$.

Proposition 4.6. *For $(h,k) \in G_\theta$ we have*

$$\widehat{\theta_h(k)} = \widehat{\widehat{\theta}}_h(\widehat{k}). \tag{4.6}$$

Proof. Let $(h,k) \in G_\theta$ and $\omega \in \widehat{K}$. Then we have

$$\widehat{\widehat{\theta}}_h(\widehat{k})(\omega) = \omega_{h^{-1}}(k). \tag{4.7}$$

Indeed, by (4.1), we can write

$$\begin{aligned}\widehat{\widehat{\theta}}_h(\widehat{k})(\omega) &= \widehat{k} \circ \widehat{\theta}_{h^{-1}}(\omega) \\ &= \widehat{k}\left(\widehat{\theta}_{h^{-1}}(\omega)\right)\end{aligned}$$

$$= \widehat{k}(\omega_{h^{-1}}) = \omega_{h^{-1}}(k).$$

Using duality notation and (4.7), we get

$$\begin{aligned}\widehat{\theta_h(k)}(\omega) &= \omega(\theta_h(k))\\ &= \omega \circ \theta_h(k)\\ &= \omega_{h^{-1}}(k) = \widehat{\widehat{\theta}}_h(\widehat{k})(\omega).\end{aligned}$$ □

Remark 4.7. The $\widehat{\theta}$-dual group operation, for $(h, \widehat{k}), (h', \widehat{k'}) \in G_{\widehat{\widehat{\theta}}} = H \ltimes_{\widehat{\widehat{\theta}}} \widehat{\widehat{K}}$, is

$$(h, \widehat{k}) \ltimes_{\widehat{\widehat{\theta}}} (h', \widehat{k'}) = (hh', \widehat{k}\widehat{\widehat{\theta}}_h(\widehat{k'})), \tag{4.8}$$

where $\widehat{\widehat{\theta}} : H \to \mathrm{Aut}(\widehat{\widehat{K}})$ is given by

$$\widehat{\widehat{\theta}}_h(\widehat{k})(\omega) = \omega_{h^{-1}}(k) \tag{4.9}$$

for all $\omega \in \widehat{K}$ and $(h, k) \in G_\theta$.

Next result is a type of Pontrjagin duality Theorem for the θ-dual structure of the locally compact semi-direct product group G_θ.

Theorem 4.8. *Let H be a locally compact group, K be an LCA group with dual group $\widehat{K}$, and $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Then*

$$(h, k) \mapsto \Theta(h, k) := (h, \widehat{k}), \tag{4.10}$$

is a topological groups isomorphism from G_θ onto $G_{\widehat{\widehat{\theta}}}$.

Proof. Let $(h, k), (h', k') \in G_\theta$. Using (4.6), and since the map $k \mapsto \widehat{k}$ is a topological group homomorphism, we have

$$\begin{aligned}\Theta\left((h, k) \ltimes_\theta (h', k')\right) &= \Theta\left(hh', k\theta_h(k')\right)\\ &= \left(hh', \widehat{k\theta_h(k')}\right)\\ &= \left(hh', \widehat{k}\widehat{\theta_h(k')}\right)\\ &= \left(hh', \widehat{k}\widehat{\widehat{\theta}}_h(\widehat{k'})\right)\\ &= (h, \widehat{k}) \ltimes_{\widehat{\widehat{\theta}}} (h', \widehat{k'}) = \Theta(h, k) \ltimes_{\widehat{\widehat{\theta}}} \Theta(h', k'),\end{aligned}$$

which guarantees that Θ is a homomorphism. The fact that, $k \mapsto \widehat{k}$ is a topological group isomorphism from K onto $\widehat{\widehat{K}}$, implies that the map Θ is a topological group isomorphism as well. □

Remark 4.9. Theorem 4.8 assures that we can identify elements of $G_{\widehat{\widehat{\theta}}}$ with G_θ via the topological group isomorphism Θ defined in (4.10). Form now on, we may identify an element $(h, \widehat{k}) \in G_{\widehat{\widehat{\theta}}}$ with (h, k), at times.

Definition 4.10. Let H be a locally compact group, and K be an LCA group with dual group $\widehat{K}$. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. The canonical left coset space $G_{\widehat{\theta}}/H = \{(h,\omega)H : (h,\omega) \in G_{\widehat{\theta}}\}$ is a locally compact homogeneous space, which is called as *dual homogeneous space* of the canonical locally compact homogeneous space G_θ/H.

Theorem 4.8 implies that dual of the canonical left coset space $G_{\widehat{\theta}}/H$ is the canonical left coset space G_θ/H.

Then we state basic properties of canonical dual homogeneous spaces.

Proposition 4.11. *Let H be a locally compact group, and K be an LCA group with dual group $\widehat{K}$. Let $\theta : H \to \mathrm{Aut}(K)$ be a continuous homomorphism. Then*

(1) *H is normal in $G_{\widehat{\theta}}$ if and only if θ is the trivial homomorphism.*
(2) *G_θ/H is a pure homogeneous space if and only if $G_{\widehat{\theta}}/H$ is a pure homogeneous space.*
(3) *For $\omega, \omega' \in \widehat{K}$, $\omega H = \omega' H$ if and only if $\omega = \omega'$.*
(4) *The canonical homogeneous space $G_{\widehat{\theta}}/H$ is precisely $\{\omega H : \omega \in \widehat{K}\}$.*

Corollary 4.12. *Let $\widehat{\rho}$ be a rho-function for the pair $(G_{\widehat{\theta}}, H)$. Then*

(1) *The linear map $P_H : \mathcal{C}_c(G_{\widehat{\theta}}) \to \mathcal{C}_c(G_{\widehat{\theta}}/H)$ is given by*

$$P_H(g)(\omega H) = \int_H g(h,\omega)dm_H(h) \quad \text{for } g \in \mathcal{C}_c(G_{\widehat{\theta}}). \tag{4.11}$$

(2) *The linear map $T_H : \mathcal{C}_c(G_{\widehat{\theta}}) \to \mathcal{C}_c(G_{\widehat{\theta}}/H)$ is given by*

$$T_H(g)(\omega H) = \int_H \frac{g(h,\omega)}{\widehat{\rho}(h,\omega)}dm_H(h) \quad \text{for } g \in \mathcal{C}_c(G_{\widehat{\theta}}). \tag{4.12}$$

The function $\rho_{\widehat{\theta}} : G_{\widehat{\theta}} \to (0,\infty)$ given by

$$\rho_{\widehat{\theta}}(h,\omega) = \Delta_H(h)\Delta_{G_{\widehat{\theta}}}(h)^{-1} = \delta^{\widehat{\theta}}_{H,\widehat{K}}(h)^{-1} = \delta^{\theta}_{H,K}(h), \tag{4.13}$$

for $(h,\omega) \in G_\theta = H \ltimes_\theta \widehat{K}$, is the canonical rho-function for the pair $(G_{\widehat{\theta}}, H)$.

Then we can present the following consequences, due to the structure of the θ-dual group $G_{\widehat{\theta}}$ and results of Section 3.

Theorem 4.13. *The induced strongly quasi-invariant measure $\mu_{\widehat{\theta}}$ on the canonical left coset space $G_{\widehat{\theta}}/H$ which arises from the rho-function defined in* (4.13) *is a relatively invariant measure on G_θ/H and satisfies*

$$\int_{G_{\widehat{\theta}}/H} \phi(\omega H)d\mu_{\widehat{\theta}}(\omega H) = \int_{\widehat{K}} \phi(\omega H)dm_{\widehat{K}}(\omega) \quad \text{for } \phi \in L^1(G_{\widehat{\theta}}/H, \mu_{\widehat{\theta}}). \tag{4.14}$$

$$\int_{G_{\widehat{\theta}}/H} u(\omega)d\mu_{\widehat{\theta}}(\omega H) = \int_{\widehat{K}} u(\omega)dm_{\widehat{K}}(\omega) \quad \text{for } u \in L^1(\widehat{K}). \tag{4.15}$$

Corollary 4.14. *Let $\mu_{\widehat{\theta}}$ be the relatively invariant measure on the canonical homogeneous space $G_{\widehat{\theta}}/H$ which arises from the canonical rho-function $\rho_{\widehat{\theta}}$ defined in* (4.13) *and $p \geq 1$. Then,*

(1) *The linear map $\Gamma_{\widehat{\theta}} : \mathcal{C}_c(\widehat{K}) \to \mathcal{C}_c(G_{\widehat{\theta}}/H)$ satisfies*

$$\|\Gamma_{\widehat{\theta}}(u)\|_{L^p(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})} = \|u\|_{L^p(\widehat{K})} \quad \text{for } u \in \mathcal{C}_c(\widehat{K}). \tag{4.16}$$

(2) *The linear map $\Gamma_{\widehat{\theta}} : \mathcal{C}_c(\widehat{K}) \to \mathcal{C}_c(G_{\widehat{\theta}}/H)$ has a unique extension to the linear map $\Gamma_{\widehat{\theta}}$ from $L^p(\widehat{K})$ onto $L^p(G_{\widehat{\theta}}/H, \mu_{\widehat{\theta}})$ which satisfies*

$$\|\Gamma_{\widehat{\theta}}(u)\|_{L^p(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})} = \|u\|_{L^p(\widehat{K})} \quad \text{for } u \in L^p(\widehat{K}). \tag{4.17}$$

Now we can introduce the abstract notion of relative Fourier transform over G_θ/H.

Definition 4.15. For $\varphi \in L^1(G_\theta/H, \mu_\theta)$, we define the θ-*Fourier transform* of φ at $\omega H \in G_{\widehat{\theta}}/H$ by

$$\mathcal{F}_\theta(\varphi)(\omega H) := \int_{G_\theta/H} \varphi(kH)\overline{\omega(k)}d\mu_\theta(kH). \tag{4.18}$$

Then the mapping $\varphi \mapsto \mathcal{F}_\theta(\varphi)$ is linear, and satisfies

$$\mathcal{F}_\theta(\varphi)(\omega H) = \int_{G_\theta/H} \varphi(kH)\overline{\omega(k)}d\mu_\theta(kH) = \int_{G_\theta/H} v(k)\overline{\omega(k)}d\mu_\theta(kH), \tag{4.19}$$

where $v \in L^1(K)$ with $\varphi = \Gamma_\theta(v)$.

The following result gives the relation of the θ-Fourier transform defined in (4.18) with the Fourier transform on K.

Proposition 4.16. *Let $v \in L^1(K)$ with $\varphi = \Gamma_\theta(v)$. Then*

$$\mathcal{F}_\theta(\varphi)(\omega H) = \int_K v(k)\overline{\omega(k)}dm_K(k) \quad \text{for } \omega H \in G_{\widehat{\theta}}/H. \tag{4.20}$$

Proof. Let $\varphi \in L^1(G_\theta/H, \mu_\theta)$ and $\omega \in \widehat{K}$. Let $v \in L^1(K)$ with $\varphi = \Gamma_\theta(v)$. Then the function $\psi_\omega : G_\theta/H \to \mathbb{C}$, defined by

$$\psi_\omega(kH) := \Gamma_\theta(v)(kH)\overline{\omega(k)} = v(k)\overline{\omega(k)} \quad \text{for} \quad kH \in G_\theta/H,$$

belongs to $L^1(G_\theta/H, \mu_\theta)$. Using Theorem 3.7, we have

$$\begin{aligned}\int_{G_\theta/H} \varphi(kH)\overline{\omega(k)}d\mu_\theta(kH) &= \int_{G_\theta/H} \psi_\omega(kH)d\mu_\theta(kH) \\ &= \int_K \psi_\omega(kH)dm_K(k) = \int_K \Gamma_\theta(v)\overline{\omega(k)}dm_K(k),\end{aligned}$$

which implies (4.20). □

Consequently, we can deduce the following proposition.

Proposition 4.17. *The θ-Fourier transform $\mathcal{F}_\theta : L^1(G_\theta/H, \mu_\theta) \to \mathcal{C}_0(G_{\widehat{\theta}}/H)$ is a norm-decreasing $*$-homomorphism.*

Next theorem can be considered as a Parseval formula for the θ-Fourier transform given in (4.18).

Theorem 4.18. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by* (3.3)*, and $\mu_{\widehat{\theta}}$ be the relatively invariant measure on $G_{\widehat{\theta}}/H$ which arises from the canonical rho-function $\rho_{\widehat{\theta}}$ given by* (4.13)*. Let $\phi \in L^1(G_{\widehat{\theta}}/H, \mu_{\widehat{\theta}})$. Then the function $\breve{\phi} : G_\theta/H \to \mathbb{C}$ defined by*

$$\breve{\phi}(kH) = \int_{G_{\widehat{\theta}}/H} \phi(\omega H)\omega(k)d\mu_{\widehat{\theta}}(\omega H) \tag{4.21}$$

belongs to $L^\infty(G_\theta/H, \mu_\theta)$, and for $\varphi \in L^1(G_\theta/H, \mu_\theta)$ we have

$$\int_{G_\theta/H} \varphi(kH)\overline{\breve{\phi}(kH)}d\mu_\theta(kH) = \int_{G_{\widehat{\theta}}/H} \mathcal{F}_\theta(\varphi)(\omega H)\overline{\phi(\omega H)}d\mu_{\widehat{\theta}}(\omega H). \tag{4.22}$$

Proof. By (4.20), for $\varphi = \Gamma_\theta(v) \in L^1(G_\theta/H, \mu_\theta)$ with $v \in L^1(K)$ we get

$$\mathcal{F}_\theta(\varphi)(\omega H) = \widehat{v}(\omega) = \Gamma_{\widehat{\theta}}(\widehat{v})(\omega H). \tag{4.23}$$

Then (4.21) implies $\breve{\phi} \in L^\infty(G_\theta/H, \mu_\theta)$. Let $\varphi \in L^1(G_\theta/H, \mu_\theta)$. Then $\varphi \cdot \overline{\breve{\phi}} \in L^1(G_\theta/H, \mu_\theta)$, and we can write

$$\begin{aligned}
&\int_{G_\theta/H} \varphi(kH)\overline{\breve{\phi}(kH)}d\mu_\theta(kH) \\
&= \int_{G_\theta/H} \varphi(kH)\left(\int_{G_{\widehat{\theta}}/H} \overline{\phi(\omega H)\omega(k)}d\mu_{\widehat{\theta}}(\omega H)\right)d\mu_\theta(kH) \\
&= \int_{G_{\widehat{\theta}}/H}\left(\int_{G_\theta/H} \varphi(kH)\overline{\omega(k)}d\mu_\theta(kH)\right)\overline{\phi(\omega H)}d\mu_{\widehat{\theta}}(\omega H) \\
&= \int_{G_{\widehat{\theta}}/H} \mathcal{F}_\theta(\varphi)(\omega H)\overline{\phi(\omega H)}d\mu_{\widehat{\theta}}(\omega H).
\end{aligned}$$

□

The following result is an L^1-inversion formula for the θ-Fourier transform.

Proposition 4.19. *The θ-Fourier transform $\mathcal{F}_\theta$ satisfies the following reconstruction formula*

$$\varphi(kH) = \int_{G_{\widehat{\theta}}/H} \mathcal{F}_\theta(\varphi)(\omega H)\omega(k)d\mu_{\widehat{\theta}}(\omega H) \tag{4.24}$$

if $\varphi \in L^1(G_\theta/H, \mu_\theta)$ with $\mathcal{F}_\theta(\varphi) \in L^1(G_{\widehat{\theta}}/H, \mu_{\widehat{\theta}})$.

For $\varphi = \Gamma_\theta(v) \in L^2(G_\theta/H, \mu_\theta)$ with $v \in L^2(K)$, we can simultaneously define the *θ-Fourier transform* of φ by

$$\mathcal{F}_\theta(\varphi)(\omega H) = \widehat{v}(\omega) = \Gamma_{\widehat{\theta}}(\widehat{v})(\omega H) \quad \text{for} \quad \omega H \in G_{\widehat{\theta}}/H. \tag{4.25}$$

Then the mapping $\varphi \mapsto \mathcal{F}_\theta(\varphi)$ is linear.

The following theorem can be considered as a Plancherel formula for the θ-Fourier transform given in (4.25).

Theorem 4.20. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by* (3.3)*, and let $\mu_{\widehat{\theta}}$ be the relatively invariant measure on $G_{\widehat{\theta}}/H$ which arises from the canonical rho-function $\rho_{\widehat{\theta}}$ given by* (4.13)*. The θ-Fourier transform $\mathcal{F}_\theta$ is a unitary transform from $L^2(G_\theta/H,\mu_\theta)$ onto $L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})$.*

Proof. Let $\varphi\in L^2(G_\theta/H,\mu_\theta)$. Let $v\in L^2(K)$ with $\Gamma_\theta(v)=\varphi$. Using (3.7) and (4.25), we have

$$\|\mathcal{F}_\theta(\varphi)\|_{L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})}=\|\Gamma_{\widehat{\theta}}(\widehat{v})\|_{L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})}=\|\widehat{v}\|_{L^2(\widehat{K})}. \tag{4.26}$$

Using Plancherel formula, we have $\|\widehat{v}\|_{L^2(\widehat{K})}=\|v\|_{L^2(K)}$. Then using (3.7), we get

$$\|v\|_{L^2(K)}=\|\Gamma_\theta(v)\|_{L^2(G_\theta/H,\mu_\theta)}=\|\varphi\|_{L^2(G_\theta/H,\mu_\theta)}. \tag{4.27}$$

Thus, we achieve

$$\|\mathcal{F}_\theta(\varphi)\|_{L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})}=\|\varphi\|_{L^2(G_\theta/H,\mu_\theta)}. \tag{4.28}$$

Invoking the fact that the standard Fourier transform $\widehat{\ }: L^2(K)\to L^2(\widehat{K})$ is unitary, and using (4.25), we obtain that the θ-Fourier transform maps $L^2(G_\theta/H,\mu_\theta)$ onto $L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})$. □

Corollary 4.21. *For $\varphi,\psi\in L^2(G_\theta/H,\mu_\theta)$ we have*

$$\int_{G_\theta/H}\varphi(kH)\overline{\psi(kH)}d\mu_\theta(kH)=\int_{G_{\widehat{\theta}}/H}\mathcal{F}_\theta(\varphi)(\omega H)\overline{\mathcal{F}_\theta(\psi)(\omega H)}d\mu_{\widehat{\theta}}(\omega H). \tag{4.29}$$

Remark 4.22. The construction of the measures μ_θ and $\mu_{\widehat{\theta}}$ is the main contribution of Theorem 4.20, where the relatively invariant measure μ_θ on G_θ/H (resp. $\mu_{\widehat{\theta}}$ on $G_{\widehat{\theta}}/H$) is normalized with respect to the Haar measure m_K (resp. $m_{\widehat{K}}$) such that (3.5) and (3.6) (resp. (4.14) and (4.15)) hold.

The following theorem presents an inversion formula for the relative Fourier transform in L^2-sense.

Theorem 4.23. *Let μ_θ be the relatively invariant measure on G_θ/H which arises from the canonical rho-function ρ_θ given by* (3.3)*, and let $\mu_{\widehat{\theta}}$ be the relatively invariant measure on $G_{\widehat{\theta}}/H$ which arises from the canonical rho-function $\rho_{\widehat{\theta}}$ given by* (4.13)*. Let $\Psi\in L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})$. Then the function $\psi: G_\theta/H\to\mathbb{C}$ defined by*

$$\psi(kH):=\int_{G_{\widehat{\theta}}/H}\Psi(\omega H)\omega(k)d\mu_{\widehat{\theta}}(\omega H) \tag{4.30}$$

belongs to $L^2(G_\theta/H,\mu_\theta)$, and we have $\mathcal{F}_\theta(\psi)=\Psi$.

Proof. It is straightforward to see that $\psi \in L^2(G_\theta/H, \mu_\theta)$. Let $\Phi \in L^2(G_{\widehat{\theta}}/H, \mu_{\widehat{\theta}})$ and also $\phi \in L^2(G_\theta/H, \mu_\theta)$ with $\Phi = \mathcal{F}_\theta(\phi)$. Using (4.29), we have

$$\begin{aligned}
\langle \Phi, \mathcal{F}_\theta(\psi)\rangle_{L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})} &= \langle \mathcal{F}_\theta(\phi), \mathcal{F}_\theta(\psi)\rangle_{L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})} \\
&= \langle \phi, \psi\rangle_{L^2(G_\theta/H,\mu_\theta)} \\
&= \int_{G_\theta/H} \phi(kH)\overline{\psi(kH)}d\mu_\theta(kH) \\
&= \int_{G_\theta/H} \phi(kH)\left(\int_{G_{\widehat{\theta}}/H} \overline{\Psi(\omega H)\omega(k)}d\mu_{\widehat{\theta}}(\omega H)\right)d\mu_\theta(kH) \\
&= \int_{G_{\widehat{\theta}}/H} \left(\int_{G_\theta/H} \phi(kH)\overline{\Psi(\omega H)\omega(k)}d\mu_\theta(kH)\right)d\mu_{\widehat{\theta}}(\omega H) \\
&= \int_{G_{\widehat{\theta}}/H} \mathcal{F}_\theta(\phi)(\omega H)\overline{\Psi(\omega H)}d\mu_{\widehat{\theta}}(\omega H) \\
&= \langle \mathcal{F}_\theta(\phi), \Psi\rangle_{L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})} = \langle \Phi, \Psi\rangle_{L^2(G_{\widehat{\theta}}/H,\mu_{\widehat{\theta}})},
\end{aligned}$$

which implies $\mathcal{F}_\theta(\psi) = \Psi$. □

Then we can prove the following result.

Proposition 4.24. *For $\varphi, \varphi' \in L^2(G_\theta/H, \mu_\theta)$ we have*

$$\mathcal{F}_\theta(\varphi.\varphi') = \mathcal{F}_\theta(\varphi) *_{\widehat{\theta}} \mathcal{F}_\theta(\varphi'). \tag{4.31}$$

Proof. Let $v, v' \in L^2(K)$ with $\varphi = \Gamma_\theta(v)$ and $\varphi' = \Gamma_\theta(v')$. Then we have $v.v' \in L^1(K)$ and $\varphi.\varphi' \in L^1(G_\theta/H, \mu_\theta)$. Thus, we can write

$$\begin{aligned}
\mathcal{F}_\theta(\varphi.\varphi') &= \mathcal{F}_\theta(\Gamma_\theta(v)\Gamma_\theta(v')) \\
&= \mathcal{F}_\theta(\Gamma_\theta(v.v')) = \widehat{v.v'} = \widehat{v} * \widehat{v'} \\
&= \Gamma_{\widehat{\theta}}(\widehat{v}) *_{\widehat{\theta}} \Gamma_{\widehat{\theta}}(\widehat{v}) = \mathcal{F}_\theta(\varphi) *_{\widehat{\theta}} \mathcal{F}_\theta(\varphi').
\end{aligned}$$

□

5. Examples

Throughout this section we study aspects of relative Fourier transforms over canonical homogeneous spaces of some semi-direct product groups with the Abelian normal factor.

5.1. Canonical homogeneous space of the affine group

Let $H := \mathbb{R}^+ = (0, +\infty)$ and $K := \mathbb{R}$. Let dx be the Haar measure of the additive group $\mathbb{R}$ and da/a be the Haar measure of the multiplicative group $\mathbb{R}^+$. The continuous affine group $a\mathbf{x}+b$ is the semi-direct product $H \ltimes_\theta K$ with respect to the homomorphism $\theta : H \to \mathrm{Aut}(K)$ given by $a \mapsto \theta_a$, where $\theta_a(x) = ax$ for $x \in \mathbb{R}$ and $a \in (0, \infty)$. The underlying manifold of the continuous affine group is $(0, \infty) \times \mathbb{R}$ and the group law is

$$(a, x) \ltimes_\theta (a', x') = (aa', x + ax'). \tag{5.1}$$

Then the canonical left coset space G_θ/H is $\{xH : x \in \mathbb{R}\}$, where $xH = \{(a,x) : a \in (0,+\infty)\}$ for all $x \in \mathbb{R}$. In geometric terms, each coset xH is precisely the half line with the end point $(0,x)$ which extended indefinitely in the direction of the positive part of the real axis, and hence the canonical left coset space G_θ/H is the locally compact space consists of all these half lines.

The continuous homomorphism $\delta^\theta_{H,K} : H \to (0,\infty)$ is given by $\delta^\theta_{H,K}(a) = a^{-1}$ for $a \in H$. The left Haar measure of G_θ is $d\mu_{G_\theta}(a,x) = a^{-2}dadx$. The linear map $P_H : \mathcal{C}_c(G_\theta) \to \mathcal{C}_c(G_\theta/H)$ is

$$P_H(f)(xH) = \int_0^{+\infty} \frac{f(a,x)}{a} da \quad \text{for } f \in \mathcal{C}_c(G_\theta) \text{ and } x \in \mathbb{R}. \tag{5.2}$$

The canonical rho-function $\rho : G_\theta \to (0,\infty)$ is $\rho(a,x) = \delta^\theta_{H,K}(a)^{-1} = a$ for $(a,x) \in G_\theta$. Thus, the linear map $T_H : \mathcal{C}_c(G_\theta) \to \mathcal{C}_c(G_\theta/H)$ is

$$T_H(f)(xH) = \int_0^{+\infty} \frac{f(a,x)}{a^2} da \quad \text{for } f \in \mathcal{C}_c(G_\theta) \text{ and } x \in \mathbb{R}. \tag{5.3}$$

Let μ_θ be the induced relatively invariant measure on the homogeneous space G_θ/H via the canonical rho-function ρ. Then

$$\int_{G_\theta/H} \phi(xH) d\mu_\theta(xH) = \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{f(a,x)}{a} dadx \quad \text{for } \phi \in \mathcal{C}_c(G_\theta/H), \tag{5.4}$$

$$\int_{G_\theta/H} \phi(xH) d\mu_\theta(xH) = \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{g(a,x)}{a^2} dadx \quad \text{for } \phi \in \mathcal{C}_c(G_\theta/H), \tag{5.5}$$

where $f, g \in \mathcal{C}_c(G_\theta)$ are satisfying $P_H(g) = \phi$ and $T_H(g) = \phi$.

Let $d\omega$ be the normalized Plancherel (Haar) measure on the character group $\widehat{\mathbb{R}}$. The character group $\widehat{\mathbb{R}}$ can be identified with $\mathbb{R}$ via the dual pairing $\omega(x) = \langle x, \omega\rangle = e^{2\pi i\omega x}$ for $x \in \mathbb{R}$ and $\omega \in \widehat{\mathbb{R}}$. The continuous homomorphism $\widehat{\theta} : H \to \mathrm{Aut}(\widehat{K})$ is given by $a \mapsto \widehat{\theta}_a$ where

$$\begin{aligned}\langle x, \omega_a\rangle &= \langle x, \widehat{\theta}_a(\omega)\rangle \\ &= \langle \theta_{a^{-1}}(x), \omega\rangle \\ &= \langle a^{-1}x, \omega\rangle = e^{2\pi i\omega a^{-1}x}.\end{aligned}$$

Thus, $G_{\widehat{\theta}}$ has the underlying manifold $(0,\infty) \times \widehat{\mathbb{R}}$, with the group law given by

$$(a,\omega) \ltimes_{\widehat{\theta}} (a',\omega') = (aa', \omega\omega'_a). \tag{5.6}$$

The continuous homomorphism $\delta^{\widehat{\theta}}_{H,\widehat{K}} : H \to (0,\infty)$ is given by $\delta^{\widehat{\theta}}_{H,\widehat{K}}(a) = a$ for $a \in H$. The left Haar measure $d\mu_{G_{\widehat{\theta}}}(a,\omega)$ of $G_{\widehat{\theta}}$ is precisely $dad\omega$. The linear map $P_H : \mathcal{C}_c(G_{\widehat{\theta}}) \to \mathcal{C}_c(G_{\widehat{\theta}}/H)$ is

$$P_H(\mathbf{f})(\omega H) = \int_0^{+\infty} \frac{\mathbf{f}(a,x)}{a} da \quad \text{for } \mathbf{f} \in \mathcal{C}_c(G_{\widehat{\theta}}) \text{ and } \omega \in \mathbb{R}. \tag{5.7}$$

The canonical rho-function $\widehat{\rho} : G_{\widehat{\theta}} \to (0,\infty)$ is $\widehat{\rho}(a,\omega) = \delta^{\widehat{\theta}}_{H,\widehat{K}}(a)^{-1} = a^{-1}$ for $(a,\omega) \in G_{\widehat{\theta}}$. Thus, the linear map $T_H : \mathcal{C}_c(G_{\widehat{\theta}}) \to \mathcal{C}_c(G_{\widehat{\theta}}/H)$ is

$$T_H(\mathbf{f})(\omega H) = \int_0^{+\infty} \mathbf{f}(a,\omega)da \quad \text{for } \mathbf{f} \in \mathcal{C}_c(G_{\widehat{\theta}}) \text{ and } \omega \in \mathbb{R}. \tag{5.8}$$

Let $\mu_{\widehat{\theta}}$ be the induced relatively invariant measure on the homogeneous space $G_{\widehat{\theta}}/H$ via the canonical rho-function $\widehat{\rho}$. Then

$$\int_{G_{\widehat{\theta}}/H} \Phi(\omega H)d\mu_{\widehat{\theta}}(\omega H) = \int_{-\infty}^{+\infty}\int_0^{+\infty} \frac{\mathbf{f}(a,\omega)}{a}dad\omega \quad \text{for } \Phi \in \mathcal{C}_c(G_{\widehat{\theta}}/H), \tag{5.9}$$

$$\int_{G_{\widehat{\theta}}/H} \Phi(\omega H)d\mu_{\widehat{\theta}}(\omega H) = \int_{-\infty}^{+\infty}\int_0^{+\infty} \mathbf{g}(a,\omega)dad\omega \quad \text{for } \Phi \in \mathcal{C}_c(G_{\widehat{\theta}}/H), \tag{5.10}$$

where $\mathbf{f},\mathbf{g} \in \mathcal{C}_c(G_{\widehat{\theta}})$ with $P_H(\mathbf{f}) = \Phi$ and $T_H(\mathbf{g}) = \Phi$.

Then the relative Fourier transform of $\varphi \in L^1(G_\theta/H,\mu_\theta)$ is given by

$$\mathcal{F}_\theta(\varphi)(\omega H) = \int_{G_\theta/H} \varphi(xH)e^{-2\pi i\omega x}d\mu_\theta(xH) \quad \text{for } \omega H \in G_{\widehat{\theta}}/H. \tag{5.11}$$

5.2. Canonical homogeneous spaces of Euclidean groups

Let $n \in \mathbb{N}$, $K := \mathbb{R}^n$, and $H := \mathrm{SO}(n)$. Let $E(n)$ be the group of rigid motions of K, the group generated by rotations and translations, that is the semi-direct product of H and K with respect to the continuous homomorphism $\theta : \mathrm{SO}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ which is given by $\sigma \mapsto \theta_\sigma$, where $\theta_\sigma(\mathbf{x}) = \sigma\mathbf{x}$ for all $\sigma \in \mathrm{SO}(n)$ and $\mathbf{x} \in \mathbb{R}^n$. The group operation for $E(n)$ is

$$(\sigma,\mathbf{x}) \ltimes_\theta (\sigma',\mathbf{x}') = (\sigma\sigma',\mathbf{x}+\theta_\sigma(\mathbf{x}')) = (\sigma\sigma',\mathbf{x}+\sigma\mathbf{x}'). \tag{5.12}$$

Since H is compact, we deduce that the continuous homomorphism $\delta^\theta_{H,K} : H \to (0,\infty)$ is the constant function 1. Therefore, $d\sigma d\mathbf{x}$ is a left Haar measure for $E(n) = G_\theta$ and the linear map $T_H = P_H : \mathcal{C}_c(E(n)) \to \mathcal{C}_c(X_n)$ is given by

$$T_H(f)(\mathbf{x}H) = P_H(f)(\mathbf{x}H) = \int_{\mathrm{SO}(n)} f(\sigma,\mathbf{x})d\sigma \quad \text{for } f \in \mathcal{C}_c(E(n)) \text{ and } \mathbf{x} \in \mathbb{R}^n. \tag{5.13}$$

Also, the canonical rho-function ρ_θ is the constant function 1 and hence the canonical invariant measure $\mu_n := \mu_\theta$ on the canonical homogeneous space $X_n := E(n)/H$ is $E(n)$-invariant. Thus, we can write

$$\int_{X_n} \phi(xH)d\mu_n(xH) = \int_{\mathbb{R}^n}\int_{\mathrm{SO}(n)} f(\sigma,\mathbf{x})d\sigma d\mathbf{x} \quad \text{for } \phi \in \mathcal{C}_c(X_n), \tag{5.14}$$

where $f \in \mathcal{C}_c(E(n))$ satisfies $T_H(f) = P_H(f) = \phi$.

Identifying $\widehat{\mathbb{R}^n}$ with $\mathbb{R}^n$, the continuous homomorphism $\widehat{\theta} : \mathrm{SO}(n) \to \mathrm{Aut}(\mathbb{R}^n)$ is $\sigma \mapsto \widehat{\theta}_\sigma$ via

$$\langle \mathbf{x}, \widehat{\theta}_\sigma(\mathbf{w})\rangle = \langle \mathbf{x}, \mathbf{w}_\sigma\rangle$$

$$= \langle \theta_{\sigma^{-1}}(\mathbf{x}), \mathbf{w}\rangle$$
$$= \langle \sigma^{-1}\mathbf{x}, \mathbf{w}\rangle = e^{-2\pi i(\sigma^{-1}\mathbf{x},\mathbf{w})},$$

where $(\cdot,\cdot)$ stands for the standard inner product of $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the dual pairing of $\mathbb{R}^n$ and $\widehat{\mathbb{R}^n}$. Thus, $G_{\widehat{\theta}}$ has the underlying manifold $\mathrm{SO}(n)\times\mathbb{R}^n = \mathrm{SO}(n)\times\widehat{\mathbb{R}^n}$ with the group operation

$$(\sigma,\mathbf{w}) \ltimes_{\widehat{\theta}} (\sigma',\mathbf{w}') = (\sigma\sigma', \mathbf{w}+\mathbf{w}'_\sigma). \tag{5.15}$$

Then Theorem 4.3 guarantees that $dm_{G_{\widehat{\theta}}}(\sigma,\mathbf{w}) = d\sigma d\mathbf{w}$ is a left Haar measure for $E(\widehat{n}) := G_{\widehat{\theta}}$, where $d\mathbf{w}$ is the normalized Plancherel (Haar) measure on the character group $\widehat{\mathbb{R}^n}$. From now on we denote the dual homogeneous space $G_{\widehat{\theta}}/H$ by $X_{\widehat{n}}$. Then the canonical rho-function $\rho_{\widehat{\theta}}$ is the constant function 1 and hence the canonical invariant measure $\mu_{\widehat{n}} := \mu_{\widehat{\theta}}$ on the canonical homogeneous space $X_{\widehat{n}} := E(\widehat{n})/H$ is $E(\widehat{n})$-invariant.

Then the relative Fourier transform of $\varphi \in L^1(X_n,\mu_n)$ is given by

$$\mathcal{F}_\theta(\varphi)(\mathbf{w}H) = \int_{X_n} \varphi(\mathbf{x}H)e^{-2\pi i(\mathbf{w},\mathbf{x})}d\mu_n(\mathbf{x}H) \quad \text{for } \mathbf{w}H \in X_{\widehat{n}}. \tag{5.16}$$

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