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Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps

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Abstract Coarse spaces are instrumental in obtaining scalability for domain decomposition methods for partial differential equations (PDEs). However, it is known that most popular choices of coarse spaces perform rather weakly in the presence of heterogeneities in the PDE coefficients, especially for systems of PDEs. Here, we introduce in a variational setting a new coarse space that is robust even when there are such heterogeneities. We achieve this by solving local generalized eigenvalue problems in

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the overlaps of subdomains that isolate the terms responsible for slow convergence. We prove a general theoretical result that rigorously establishes the robustness of the new coarse space and give some numerical examples on two and three dimensional heterogeneous PDEs and systems of PDEs that confirm this property.

Keywords Coarse spaces · Overlapping Schwarz method · Two-level methods · Generalized eigenvectors · Problems with large coefficient variation

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1 Introduction

The effort to achieve scalability in domain decomposition methods has led to the design of so called two-level methods. Each of these methods is characterized by two ingredients: a coarse space and a formulation of how this coarse space is incorporated into the domain decomposition method. We will work in the already extensively studied framework of the overlapping additive Schwarz preconditioner [29,31], and focus on the definition of a suitable coarse space with the aim to achieve robustness with respect to heterogeneities in any of the coefficients in the PDEs. Heterogeneous problems arise in many applications, such as subsurface flow or linear elasticity. One way to avoid long stagnation in Schwarz methods is to build the subdomains in such a way that the variations in the coefficients are small or nonexistent inside each subdomain. In this context, classical coarse spaces based on these subdomain partitions are known to be robust, see e.g. [5,6,9,10,21]. In many cases, this is not feasible and so recently, for scalar elliptic problems, these results have been extended to operator dependent coarse spaces and to coefficients that are not resolved by the subdomain partition, see e.g. [7,8,12,14–16,24,27,28], as well as [32] and the references therein.

A very useful tool for building coarse spaces for which the corresponding two-level method is robust, regardless of the partition into subdomains and of the coefficient distribution, is the solution of local generalized eigenvalue problems. They allow to select suitable coarse vectors that satisfy certain local stability estimates and guarantee that the selected coarse vectors are orthogonal. This idea was first used in the pioneering work [2] within the multigrid community and the same ideas were incorporated into the spectral Algebraic Multigrid Method [3,13]. In the framework of two level additive Schwarz methods, [14,16,27] identify the bottleneck for proving a convergence bound which is independent of the jumps in the coefficients to be the so called stable decomposition property. For the scalar elliptic (Darcy) problem, [14] successfully proposes to solve local generalized eigenvalue problems on overlapping coarse patches that identify the modes that must be put into the coarse space in order for the stable splitting property to hold. The local coarse spaces are then 'glued together' via a partition of unity to obtain a global coarse space.

More recently [12,15] and [7,8] have built on these ideas and proposed different coarse spaces based on different generalized eigenvalue problems. The choice of generalized eigenvalue problem is a delicate compromise between ensuring stability and a moderate size of the coarse space. In this spirit, for the scalar elliptic equation, [12,15]

use multiscale partition of unity functions to eliminate some of the 'bad' eigenmodes a priori. While very effective in the scalar elliptic case, this may prove tricky in cases where there are several PDEs with several jumping coefficients. In the same context of the scalar elliptic equation, [7,8] propose a different generalized eigenvalue problem, associated with the Dirichlet-to-Neumann map, which is posed only on the boundary of each subdomain. In this aproach, the interior degrees of freedom in each subdomain are eliminated leaving only those that are shared with neighbouring subdomains, reducing the problem of finding the components that locally slow down convergence to eigenproblems on the interfaces. Introduced first in [8,22], this approach was rigorously analysed in [7]. The proof relies on uniform (in the coefficients) weighted Poincare inequalities [25]. While this allows full robustness for the small overlap case (cf. [7]), in a completely general setting it has two drawbacks: (i) for larger overlap some assumptions are needed on the coefficient distribution in the overlaps and (ii) the arguments cannot be generalized easily to the case of systems of PDEs. It is in fact not essential to construct the orthogonal local coarse bases via generalized eigenproblems. In [28], an abstract Bramble-Hilbert lemma is proved that opens up the possibility for a whole range of other local functionals to achieve stability.

In this article, we propose a coarse space construction based on *Generalized Eigenproblems in the Overlap* (which we will refer to as the *GenEO coarse space*). The method was first and briefly introduced in [30] by the same authors and we give here a detailed proof of the previously stated convergence result along with some numerical results. The coarse space construction applies to systems of PDEs discretized by finite elements with only a few extra assumptions. It only relies on having access to element stiffness matrices and the connectivity graph between elements. The subdomain partition is carried out using Metis. Overlap is added based on the connectivity graph and the coarse space is constructed automatically solving a generalized eigenproblem on each subdomain. In our analysis, we identify that in the small overlap case the abstract Schwarz framework allows to reduce the stability condition to an energy bound in the overlap, and thus the second matrix in the pencil of our generalized eigenvalue problem is a matrix that has zero blocks corresponding to the interior of the subdomain at hand.

The resulting generalized eigenvalue problems are closely related, but different to the ones proposed in [12]. Therefore the analysis and the final estimate are also different. In particular, the approach in [12] focuses on the generous overlap case. It requires a hexahedral coarse mesh and a procedure for computing multiscale partitions of unity subordinate to this coarse mesh. While it is clear that this is not a fundamental requirement in [12], and that their method could also be extended to the small overlap case and to more general coarse meshes, the same could be said about the GenEO coarse space. For implementational convenience we choose simple partition of unity operators that are sufficient in the small overlap case, but for larger overlap we could also use multiscale partitions of unity. Secondly (as in [12]), the requirement that the overlapping subdomains coincide with the supports of the coarse space partition of unity is not essential (cf. [27] in the case of our approach). Which of the two approaches is better in practice in terms of stability versus coarse space size is still the object of ongoing research. The recent work of [33] is a first step.

The major theoretical advance with respect to [12] is that our analysis applies to practical choices of the discrete approximation space. The theory in [12] is given in an

abstract variational setting that unfortunately does not allow for finite element spaces, since [12, Assumption A4(c)] cannot be satisfied in that case. In order for the proof to go through for classical finite element spaces, a stable interpolation operator with a constant independent of the coefficients would be needed. In many cases (elasticity for instance), such a stable interpolator does not yet exist to our knowledge. We overcame this problem by introducing partition of unity operators that work directly on the degrees of freedom instead of partition of unity functions. From a practical point of view, thanks to the partition of unity operators, the right hand side of the generalized eigenproblems can be constructed fully automatically from element stiffness matrices and diagonal weighting matrices. We only require access to some topological information to build a suitable partition of unity, as well as to the element stiffness matrices (as in AMGe methods, cf. [3]). This is reasonable in standard FE packages such as FreeFEM++ [17].

The rest of this article is organized as follows. In Sect. 2 we define the problem that we solve and introduce the two-level additive Schwarz framework along with some elements of generalized eigenvalue problem theory. In Sect. 3 we define the abstract procedure to construct our coarse space and give the main convergence result (Theorem 3.22).

Section 4 gives detailed guidelines on how to implement the two-level Schwarz preconditioner with the GenEO coarse space in a finite element code. Finally in Sect. 5 we test our method for Darcy and linear elasticity and make sure that it indeed converges robustly even for highly varying coefficients in two and three dimensions.

2 Preliminaries and notations

2.1 Problem description

Given a Hilbert space V, a symmetric and coercive bilinear form $a : V \times V \to \mathbb{R}$ and an element f in the dual space V', we consider the abstract variational problem: Find $v \in V$ such that

$$a(v,\ w) = \langle f, w\rangle, \quad \text{for all} \quad w \in V, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing. This variational problem is associated with an elliptic boundary value problem (BVP) on a given domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with suitable boundary conditions posed in a suitable space V of functions on Ω.

We consider a discretization of the variational problem (2.1) with finite elements based on a mesh $\mathscr{T}_h$ of Ω:

$$\overline{\Omega} = \bigcup_{\tau \in \mathscr{T}_h} \tau.$$

Let $V_h \subset V$ denote the chosen conforming space of finite element functions. In the case where $a(\cdot, \cdot)$ is a bilinear form derived from a system of PDEs, V_h is a space of vector functions. The discretization of (2.1) then reads: Find $v_h \in V_h$ such that

$$a(v_h, w_h) = \langle f, w_h\rangle, \quad \text{for all} \quad w_h \in V_h. \tag{2.2}$$

Let $\{\phi_k\}_{k=1}^n$ be a basis for V_h with $n := \dim(V_h)$, then from (2.2) we can derive a linear system

$$\mathbf{A}\mathbf{v} = \mathbf{f}, \tag{2.3}$$

where the coefficients of the stiffness matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and the load vector $\mathbf{f} \in \mathbb{R}^n$ are given by $A_{k,l} = a(\phi_l, \phi_k)$ and $f_k = \langle f, \phi_k\rangle$, where $k, l = 1, \ldots, n$, and $\mathbf{v}$ is the vector of coefficients corresponding to the unknown finite element function v_h in (2.2).

The basis $\{\phi_k\}_{k=1}^n$ can be quite arbitrary but it should fulfil a unisolvence property, such that the basis functions supported on each element $\tau \in \mathscr{T}_h$ are linearly independent when restricted to τ. This is the case for standard finite element bases.

The only significant assumption we make on the problem is that the stiffness matrix $\mathbf{A}$ is assembled from positive semi-definite element stiffness matrices.

Assumption 2.1 Let $V_h(\tau) = \{v|_\tau : v \in V_h\}$. We assume that there exist positive semi-definite bilinear forms $a_\tau : V_h(\tau) \times V_h(\tau) \to \mathbb{R}$, for all $\tau \in \mathscr{T}_h$, such that

$$a(v, w) = \sum_{\tau\in\mathscr{T}_h} a_\tau(v|_\tau, w|_\tau), \quad \text{for all} \quad v, w \in V_h.$$

Remark 2.2 If the variational problem is obtained from integrating local forms on the domain then this is not a problem at all. For instance in the case of the Darcy equation we can write for all $v, w \in H_0^1(\Omega)$:

$$a(v, w) = \int_\Omega \kappa\nabla v \cdot \nabla w = \sum_{\tau\in\mathscr{T}_h} \int_\tau \kappa\nabla v \cdot \nabla w = \sum_{\tau\in\mathscr{T}_h} a_\tau(v|_\tau, w|_\tau).$$

2.2 Additive Schwarz setting

In order to automatically construct a robust two-level Schwarz preconditioner for (2.3), we first partition our domain Ω into a set of non-overlapping subdomains $\{\Omega_j'\}_{j=1}^N$ resolved by $\mathscr{T}_h$ using for example a graph partitioner such as METIS [18] or SCOTCH [4]. Each subdomain Ω_j' is then extended to a domain Ω_j by adding one or several layers of mesh elements in the sense of Definition 2.3, thus creating an overlapping decomposition $\{\Omega_j\}_{j=1}^N$ of Ω.

Definition 2.3 Given a subdomain $D' \subset \Omega$ which is resolved by $\mathscr{T}_h$, the extension of D' by one layer of elements is

$$D = \mathrm{Int}\left(\bigcup_{\{k:\mathrm{supp}(\phi_k)\cap D'\neq\emptyset\}} \mathrm{supp}(\phi_k)\right)$$

and $\mathrm{Int}(\cdot)$ denotes the interior of a domain. Extensions by more than one layer can then be defined recursively.

The proof of the following lemma is a direct consequence of Definition 2.3.

Lemma 2.4 *For every degree of freedom k, with $1 \leq k \leq n$, there is a subdomain Ω_j, with $1 \leq j \leq N$, such that* $\mathrm{supp}(\phi_k) \subset \overline{\Omega}_j$.

Now, for each $j = 1, \ldots, N$, let

$$V_h(\Omega_j) := \{v|_{\Omega_j} : v \in V_h\}$$

denote the space of restrictions of functions in V_h to Ω_j. Furthermore, let

$$V_{h,0}(\Omega_j) := \{v|_{\Omega_j} : v \in V_h,\ \mathrm{supp}(v) \subset \Omega_j\}$$

denote the space of finite element functions supported in Ω_j. By definition, the extension by zero of a function $v \in V_{h,0}(\Omega_j)$ to Ω lies again in V_h. We denote the corresponding extension operator by

$$R_j^\top : V_{h,0}(\Omega_j) \to V_h\,. \tag{2.4}$$

Lemma 2.4 guarantees that $V_h = \sum_{j=1}^N R_j^\top V_{h,0}(\Omega_j)$. The adjoint of $R_j^\top$

$$R_j : V_h' \to V_{h,0}(\Omega_j)'\,,$$

called the restriction operator, is defined by $\langle R_j g, v\rangle = \langle g, R_j^\top v\rangle$, for $v \in V_{h,0}(\Omega_j)$, $g \in V_h'$. However, for the sake of simplicity, we will often leave out the action of $R_j^\top$ and view $V_{h,0}(\Omega_j)$ as a subspace of V_h.

The final ingredient is a coarse space $V_H \subset V_h$ which will be defined later. Let $R_H^\top : V_H \to V_h$ denote the natural embedding and R_H its adjoint. Then the two-level additive Schwarz preconditioner (in matrix form) reads

$$\mathbf{M}_{AS,2}^{-1} = \mathbf{R}_H^T \mathbf{A}_H^{-1} \mathbf{R}_H + \sum_{j=1}^N \mathbf{R}_j^T \mathbf{A}_j^{-1} \mathbf{R}_j, \quad \mathbf{A}_H := \mathbf{R}_H \mathbf{A} \mathbf{R}_H^T \quad \text{and} \quad \mathbf{A}_j := \mathbf{R}_j \mathbf{A} \mathbf{R}_j^T, \tag{2.5}$$

where $\mathbf{R}_j$, $\mathbf{R}_H$ are the matrix representations of R_j and R_H with respect to the basis $\{\phi_k\}_{k=1}^n$ and the chosen basis of the coarse space V_H. As usual for standard elliptic BVPs, $\mathbf{A}_j$ corresponds to the original (global) system matrix restricted to subdomain Ω_j with Dirichlet conditions on the artificial boundary $\partial\Omega_j \backslash \partial\Omega$.

To simplify the notation, if D is the union of elements of $\mathscr{T}_h$ and

$$V_h(D) := \{v|_D : v \in V_h\},$$

we write, for any $v, w \in V_h(D)$,

$$a_D(v, w) := \sum_{\tau \in D} a_\tau(v|_\tau, w|_\tau) \quad \text{and} \quad |v|_{a,D} = \sqrt{a_D(v, v)},$$

where the latter is the energy seminorm. The definition of $a_D(\cdot,\cdot)$ extends naturally to $v, w \in V_h(D')$, for any $D \subset D' \subset \Omega$ which simplifies notations. On each of the local spaces $V_{h,0}(\Omega_j)$ the bilinear form $a_{\Omega_j}(\cdot,\cdot)$ is positive definite since

$$a_{\Omega_j}(v,w) = a(R_j^\top v, R_j^\top w), \quad \text{for all } v, w \in V_{h,0}(\Omega_j),$$

and because $a(\cdot,\cdot)$ is coercive on V. For the same reason, the matrix $\mathbf{A}_j$ in (2.5) is invertible. Hence, $|\cdot|_{a,\Omega_j}$ becomes a norm on $V_{h,0}(\Omega_j)$ and so we write

$$\|v\|_{a,\Omega_j} = \sqrt{a_{\Omega_j}(v,v)}, \quad \text{for all } v \in V_{h,0}(\Omega_j).$$

If $D = \Omega$, we omit the domain from the subscript and write $\|\cdot\|_a$ instead of $\|\cdot\|_{a,\Omega}$.

We use here the abstract framework for additive Schwarz (see [31, Chapter 2]). In the following we summarize the most important ingredients.

Definition 2.5 We define $k_0 = \max_{\tau\in\mathcal{T}_h} \left(\#\{\Omega_j : 1 \le j \le N, \tau \subset \Omega_j\}\right)$.

This means that each point in Ω belongs to at most k_0 of the subdomains Ω_j.

Lemma 2.6 *With k_0 as in Definition 2.5, the largest eigenvalue of $\mathbf{M}_{AS,2}^{-1}\mathbf{A}$ satisfies*

$$\lambda_{max}\left(\mathbf{M}_{AS,2}^{-1}\mathbf{A}\right) \le k_0 + 1.$$

Proof See, e.g., [11, Section 4]. □

Definition 2.7 (*Stable decomposition*) Given a coarse space $V_H \subset V_h$, local subspaces $\{V_{h,0}(\Omega_j)\}_{1\le j\le N}$ and a constant C_0, a C_0-stable decomposition of $v \in V_h$ is a family of functions $\{z_j\}_{0\le j\le N}$ that satisfies

$$v = \sum_{j=0}^{N} z_j, \quad \text{with} \quad z_0 \in V_H, \quad z_j \in V_{h,0}(\Omega_j), \text{ for } j \ge 1, \tag{2.6}$$

and

$$\|z_0\|_a^2 + \sum_{j=1}^{N} \|z_j\|_{a,\Omega_j}^2 \le C_0^2 \|v\|_a^2 . \tag{2.7}$$

Theorem 2.8 *If every $v \in V_h$ admits a C_0-stable decomposition (with uniform C_0), then the smallest eigenvalue of $\mathbf{M}_{AS,2}^{-1}\mathbf{A}$ satisfies*

$$\lambda_{min}(\mathbf{M}_{AS,2}^{-1}\mathbf{A}) \ge C_0^{-2}.$$

Therefore, the condition number of the two-level Schwarz preconditioner (2.5) can be bounded by

$$\kappa(\mathbf{M}_{AS,2}^{-1}\mathbf{A}) \le C_0^2(k_0+1).$$

Proof The statement is a direct consequence of [31, Lemma 2.5] and Lemma 2.6. □

In the following, we will construct a C_0-stable decomposition in a specific framework, but prior to that we will provide in an abstract setting, a sufficient and simplified condition of stability.

Lemma 2.9 *Using the notations introduced in Definition 2.7, if there exists a constant C_1 such that*

$$\|z_j\|_{a,\Omega_j}^2 \leq C_1 |v|_{a,\Omega_j}^2 \quad \text{for all} \quad j = 1, \ldots, N, \tag{2.8}$$

then the decomposition (2.6) is C_0-stable with $C_0^2 = 2 + C_1 k_0 (2k_0 + 1)$ where k_0 is given in Definition 2.5.

Proof From (2.8) and Definition 2.5 we get successively

$$\sum_{j=1}^{N} \|z_j\|_{a,\Omega_j}^2 \leq C_1 \sum_{j=1}^{N} |v|_{a,\Omega_j}^2 \leq C_1 k_0 \|v\|_a^2 . \tag{2.9}$$

We also have:

$$\|z_0\|_a^2 = \left\| v - \sum_{j=1}^{N} z_j \right\|_a^2 \leq 2\|v\|_a^2 + 2 \left\| \sum_{j=1}^{N} z_j \right\|_a^2 , \tag{2.10}$$

and from Definition 2.5 and (2.9) we get

$$\left\| \sum_{j=1}^{N} z_j \right\|_a^2 \leq k_0 \sum_{j=1}^{N} \|z_j\|_{a,\Omega_j}^2 \leq C_1 k_0^2 \|v\|_a^2 . \tag{2.11}$$

Using (2.11) in (2.10) yields

$$\|z_0\|_a^2 \leq 2(1 + C_1 k_0^2) \|v\|_a^2 . \tag{2.12}$$

By adding (2.9) and (2.12) we get (2.7) with $C_0^2 = 2 + C_1 k_0 (2k_0 + 1)$. □

When $\|z_0\|_a^2$ can be bounded directly in terms of $\|v\|_a^2$ (independently of the coefficient variation), this lemma is superfluous and leads to a suboptimal quadratic dependence on k_0. In general, however, it is not possible to provide such a uniform bound on $\|z_0\|_a^2$, which is why Lemma 2.9 is in fact absolutely crucial for our analysis.

2.3 Abstract generalized eigenproblems

In order to construct the coarse space we will use generalized eigenvalue problems in each subdomain. Since several variations of generalized eigenvalue problems exist in the literature (particularly concerning the interpretation of the 'infinite eigenvalue'), we state the definition that we use.

Definition 2.10 (*Generalized eigenvalue problem*) Let $\widetilde{V}$ be a finite-dimensional Hilbert space, let $\widetilde{a} : \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ and $\widetilde{b} : \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ be two symmetric bilinear forms. Then the generalized eigenvalues associated with the so called 'pencil' $(\widetilde{a}, \widetilde{b})$ are the following values $\lambda \in \mathbb{R} \cup \{+\infty\}$: either $\lambda \in \mathbb{R}$ and there exists $p \in \widetilde{V} \backslash \{0\}$ such that

$$\widetilde{a}(p, v) = \lambda \widetilde{b}(p, v), \quad \text{for all } v \in \widetilde{V}, \tag{2.13}$$

or $\lambda = +\infty$ and there exists $p \in \widetilde{V} \backslash \{0\}$ such that

$$\widetilde{b}(p, v) = 0, \quad \text{for all } v \in \widetilde{V}, \quad \text{and} \quad \widetilde{a}(p, v) \neq 0, \quad \text{for a certain} \quad v \in \widetilde{V}.$$

In both cases p is called a generalized eigenvector associated with the eigenvalue λ.

The definition above allows for infinite eigenvalues. This results from the fact that if $(+\infty, p)$ is an eigenpair for the pencil $(\widetilde{a}, \widetilde{b})$ then $(0, p)$ is an eigenpair for the pencil $(\widetilde{b}, \widetilde{a})$ and there is no reason to discriminate between both formulations. In cases where the bilinear form $\widetilde{b}$ is positive definite, the problem can be simplified and crucial properties on the eigenvalues and eigenvectors arise. In particular, it leads quite naturally to optimal projectors onto subspaces of the functional space, as the next lemma shows in an abstract setting.

Lemma 2.11 *Let $\tilde{a}$ be positive semi-definite and $\tilde{b}$ positive definite, and let the eigenpairs $\{(p_k, \lambda_k)\}_{k=1}^{\dim(\widetilde{V})}$ of the generalized eigenvalue problem (2.13) be ordered such that*

$$0 \leq \lambda_1 \leq \ldots \leq \lambda_{\dim(\widetilde{V})} \quad \text{and} \quad \widetilde{b}(p_k, p_l) = \delta_{kl}\,, \quad \text{for any} \quad 1 \leq k, l \leq \dim(\widetilde{V}).$$

Then, for any integer $1 \leq m < \dim(\widetilde{V})$, the projection

$$\widetilde{\Pi}_m v := \sum_{k=1}^{m} \widetilde{b}(v, p_k) p_k$$

is $\widetilde{a}$-orthogonal, and thus

$$|\widetilde{\Pi}_m v|_{\widetilde{a}} \leq |v|_{\widetilde{a}} \quad \text{and} \quad |v - \widetilde{\Pi}_m v|_{\widetilde{a}} \leq |v|_{\widetilde{a}}, \quad \text{for all } v \in \widetilde{V}. \tag{2.14}$$

Additionally, if m is such that $\lambda_{m+1} > 0$, we have the stability estimate

$$\|v - \widetilde{\Pi}_m v\|_{\widetilde{b}}^2 \leq \frac{1}{\lambda_{m+1}} |v - \widetilde{\Pi}_m v|_{\widetilde{a}}^2, \quad \text{for all } v \in \widetilde{V}.$$

Proof Due to the additional assumptions on $\widetilde{a}$ and $\widetilde{b}$, the generalized eigenvalue problem can be simplified to a standard eigenvalue problem, for which the existence of eigenvectors $\{p_k\}_{k=1}^{\dim(\widetilde{V})}$ with associated non-negative real eigenvalues $\{\lambda_k\}_{k=1}^{\dim(\widetilde{V})}$ is guaranteed by standard spectral theory. Moreover, $\{p_k\}_{k=1}^{\dim(\widetilde{V})}$ can be chosen such that it is a basis of $\widetilde{V}$ fulfilling the orthogonality conditions:

$$\widetilde{a}(p_k, p_l) = \widetilde{b}(p_k, p_l) = 0 \quad \forall k \neq l, \quad |p_k|_{\widetilde{b}}^2 = 1 \quad \text{and} \quad |p_k|_{\widetilde{a}}^2 = \lambda_k.$$

Now let $v \in \widetilde{V}$ be fixed. From the $\widetilde{b}$-orthonormality of the basis we get

$$v = \sum_{k=1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k) p_k.$$

For any index set $I \subset \{1, ..., \dim(\widetilde{V})\}$, the $\widetilde{a}$-orthogonality implies

$$\left| \sum_{k \in I} \widetilde{b}(v, p_k) p_k \right|_{\widetilde{a}}^2 = \sum_{k \in I} \widetilde{b}(v, p_k)^2 |p_k|_{\widetilde{a}}^2.$$

Thus

$$|v|_{\widetilde{a}}^2 = |\widetilde{\Pi}_m v|_{\widetilde{a}}^2 + |v - \widetilde{\Pi}_m v|_{\widetilde{a}}^2.$$

and (2.14) follows directly. Finally,

$$\begin{aligned}
\|v - \widetilde{\Pi}_m v\|_{\widetilde{b}}^2 &= \left\| \sum_{k=m+1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k)\, p_k \right\|_{\widetilde{b}}^2 \\
&= \sum_{k=m+1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k)^2 \qquad (\text{by the } \widetilde{b} - \text{orthonormality of } p_k) \\
&= \sum_{k=m+1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k)^2 \, \frac{1}{\lambda_k} \, |p_k|_{\widetilde{a}}^2 \qquad (\text{since } \lambda_k = |p_k|_{\widetilde{a}}^2) \\
&\leq \frac{1}{\lambda_{m+1}} \sum_{k=m+1}^{\dim(\widetilde{V})} \widetilde{b}(v, p_k)^2 \, |p_k|_{\widetilde{a}}^2 \quad (\text{since } \lambda_1 \leq \ldots \leq \lambda_{\dim(\widetilde{V})}) \\
&= \frac{1}{\lambda_{m+1}} |v - \widetilde{\Pi}_m v|_{\widetilde{a}}^2 \qquad (\text{by the } \widetilde{a} - \text{orthogonality of } p_k).
\end{aligned}$$

□

This lemma will be one of the core arguments to prove the existence of a stable decomposition onto the new GenEO (*generalized eigenproblems in the overlap*) coarse

space and the local subspaces. It is in fact the central part in all the approaches that relie on solving eigenvalue problems, cf. Lemma 3.2 in the pioneering work [2] where b is the l_2 (euclidean) inner product or Eq. (2.8) in [12] where b is a particular bilinear form defined there. The particular form of b is one of the defining elements that characterizes each of these methods and for GenEO it will be introduced in the next section.

3 Algebraic construction of a robust coarse space and its analysis

In this section we introduce the coarse space and give a bound on the condition number of the two-level additive Schwarz method with this coarse space along with a rigorous proof of this result. The proof will consist in proving the existence of a stable splitting for any function in V_h in the sense of Definition 2.7.

3.1 The coarse space

The GenEO coarse space is constructed as follows. In each subdomain we pose a suitable generalized eigenproblem and select a number of low frequency eigenfunctions. These local functions are converted into global coarse basis functions using a partition of unity operator. As mentioned before, the eigenproblems are restricted to the overlapping zone, which is introduced in the next definition. Following this definition, we will then define the partition of unity operator, which will appear both in the eigenproblems themselves and in the construction of the coarse basis functions.

Definition 3.1 (*Overlapping zone*) For each subdomain Ω_j $(1 \leq j \leq N)$, the overlapping zone is given by

$$\Omega_j^\circ = \{x \in \Omega_j : \exists\, j' \neq j \quad \text{such that} \quad x \in \Omega_{j'}\}.$$

We will also require the set of degrees of freedom associated with $V_h(\Omega_j)$, as well as those associated with $V_{h,0}(\Omega_j)$, for $1 \leq j \leq N$.

Definition 3.2 Given a subdomain D that is a union of elements from $\mathscr{T}_h$, let

$$\overline{\text{dof}}(D) := \{k = 1, \ldots, n : \text{supp}(\phi_k) \cap D \neq \emptyset\}$$

denote the set of degrees of freedom that are "active" in D, including those associated with the boundary. Similarly, we denote by

$$\text{dof}(D) := \{k :\ 1 \leq k \leq n \text{ and } \text{supp}(\phi_k) \subset \overline{D}\}$$

the set of internal degrees of freedom in D.

Remark 3.3 Since the basis functions ϕ_k of V_h fulfil a unisolvence property on each element they also fulfil a unisolvence property on each subdomain Ω_j, in other words the functions $\{\phi_k|_{\Omega_j}\}_{k \in \overline{\text{dof}}(\Omega_j)}$ (resp. $\{\phi_k|_{\Omega_j}\}_{k \in \text{dof}(\Omega_j)}$) are linearly independent. A direct consequence is that these functions form a basis of $V_h(\Omega_j)$ (resp. $V_{h,0}(\Omega_j)$).

Now we can introduce the partition of unity operators. Recall that, for any $v \in V_h$, we write $v = \sum_{k=1}^{n} v_k \, \phi_k$.

Definition 3.4 (*Partition of unity*) For each degree of freedom $k \in \mathrm{dof}(\Omega) := \{1, \ldots, n\}$, let $\{\mu_{j,k} : k \in \mathrm{dof}(\Omega_j),\ 1 \le j \le N\}$ be a family of weights such that

$$\mu_{j,k} \ge 1 \quad \text{and} \quad \sum_{\{j;k\in\mathrm{dof}(\Omega_j)\}} \frac{1}{\mu_{j,k}} = 1.$$

Then, for $1 \le j \le N$, the local partition of unity operator $\Xi_j : V_h(\Omega_j) \to V_{h,0}(\Omega_j)$ is defined by

$$\Xi_j(v) \ := \sum_{k\in\mathrm{dof}(\Omega_j)} \frac{1}{\mu_{j,k}} \, v_k \, \phi_k|_{\Omega_j}, \quad \text{for all } v \in V_h(\Omega_j).$$

Remark 3.5 A possible choice for the weights in Definition 3.4 is to use the multiplicity of each degree of freedom: for any degree of freedom $k \in \mathrm{dof}(\Omega)$, let μ_k denote the number of subdomains for which k is an internal degree of freedom, i.e.

$$\mu_k := \#\{j : 1 \le j \le N \text{ and } k \in \mathrm{dof}(\Omega_j)\}$$

and then use the equal weights $\mu_{j,k} := \mu_k$, for any $j = 1, \ldots, N$ with $k \in \mathrm{dof}(\Omega_j)$.

Lemma 3.6 *The operators Ξ_j from Definition 3.4 form a partition of unity in the following sense:*

$$\sum_{j=1}^{N} R_j^{\top} \, \Xi_j(v|_{\Omega_j}) \ = \ v, \quad \text{for all } v \in V_h. \tag{3.1}$$

Moreover,

$$\Xi_j(v)|_{\Omega_j\setminus\Omega_j^\circ} = v|_{\Omega_j\setminus\Omega_j^\circ}, \quad \text{forall } v \in V_h(\Omega_j) \quad \textit{and} \quad 1 \le j \le N. \tag{3.2}$$

Proof Property (3.1) follows directly from the definition. To show (3.2), let $v \in V_h(\Omega_j)$ and recall that by definition

$$\Xi_j(v)|_{\Omega_j\setminus\Omega_j^\circ} \ = \sum_{k\in\mathrm{dof}(\Omega_j)} \frac{1}{\mu_{j,k}} \, v_k \, \phi_k|_{\Omega_j\setminus\Omega_j^\circ}.$$

Now note that if $\mu_{j,k} > 1$, then $\phi_k|_{\Omega_j\setminus\Omega_j^\circ} = 0$, because $k \in \mathrm{dof}(\Omega_j')$ for $j \ne j'$. Hence,

$$\Xi_j(v)|_{\Omega_j\setminus\Omega_j^\circ} \ = \sum_{k\in\mathrm{dof}(\Omega_j) \text{ s.t. } \mu_{j,k}=1} v_k \, \phi_k|_{\Omega_j\setminus\Omega_j^\circ} \ = \sum_{k\in\overline{\mathrm{dof}}(\Omega_j\setminus\Omega_j^\circ)} v_k \, \phi_k|_{\Omega_j\setminus\Omega_j^\circ},$$

but this is also the definition of $v|_{\Omega_j\setminus\Omega_j^\circ}$. □

Next we define the local generalized eigenproblems for the GenEO coarse space.

Definition 3.7 *(Generalized eigenproblems in the overlaps)* For each $j = 1, \ldots, N$, we define the following generalized eigenvalue problem

$$a_{\Omega_j}(p, v) = \lambda\, b_j(p,\ v), \quad \text{for all } \ v \in V_h(\Omega_j). \tag{3.3}$$

where $b_j(p,\ v) := a_{\Omega_j^\circ}(\Xi_j(p), \Xi_j(v))$, for all $p,\ v \in V_h(\Omega_j)$.

Remark 3.8 Although the form of the bilinear forms $b_j(\cdot, \cdot)$ seems somewhat artificial, we will see below that it actually arises naturally in the analysis. It is clear that the actual eigenvalues and eigenvectors will depend on the choice of the partition of unity in Definition 3.4.

The GenEO coarse space is now constructed (locally) as the span of a suitable subset of the eigenfunctions in (3.3). Finally, to obtain a global coarse space we apply the partition of unity operators.

Definition 3.9 (*GenEO coarse space*) For each $j = 1, \ldots, N$, let $(p_k^j)_{k=1}^{m_j}$ be the eigenfunctions of the eigenproblem (3.3) in Definition 3.7 corresponding to the m_j smallest eigenvalues. Then,

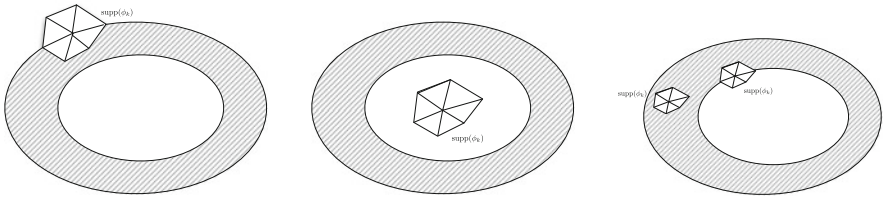
$$V_H := \text{span}\{R_j^\top \Xi_j(p_k^j) : k = 1, \ldots, m_j;\ j = 1, \ldots, N\},$$

where Ξ_j are the partition of unity operators from Definition 3.4 and $R_j^\top$ are the extension operators defined in (2.4).

Consequently, we can also make explicit the final component in Definition 2.5 of the matrix form $\mathbf{M}_{AS,2}^{-1}$ of the additive Schwarz preconditioner, namely the prolongation matrix $\mathbf{R}_H^T$. The columns of the rectangular matrix $\mathbf{R}_H^T \in \mathbb{R}^{n \times \dim(V_H)}$ are simply the vector representations of the functions $\{R_j^\top \Xi_j(p_k^j) : k = 1, \ldots, m_j;\ j = 1, \ldots, N\}$ with respect to the finite element basis $\{\phi_k\}_{k=1}^n$. Clearly $\dim(V_H) = \sum_{j=1}^N m_j$ and a strategy for selecting m_j will be given below. This completes the definition of $\mathbf{M}_{AS,2}^{-1}$.

3.2 Analysis of the preconditioner

To confirm the robustness of the above coarse space and to bound the condition number of $\mathbf{M}_{AS,2}^{-1}\mathbf{A}$ via Theorem 2.8 we will now show that there is a stable splitting for each $v \in V_h$ in the sense of Definition 2.7. First we will give some results on the local subspaces Ω_j, then we use them to show that the eigenproblems from Definition 3.7 are well defined and that the eigenpairs have some particular properties. In order to do this we define a subspace $\widetilde{V}_j$ of each $V_h(\Omega_j)$ on which the restriction of the local generalized eigenproblems satisfies the hypotheses of Lemma 2.11. This leads to local projectors onto subspaces of $V_h(\Omega_j)$ which satisfy stability estimates. These stability estimates will generalize to the whole of $V_h(\Omega_j)$ and enable us to split any $v \in V_h$ in a "C_0-stable" manner.



$k \in \beta_1^j$
$\text{supp}(\phi_k) \not\subset \overline{\Omega}_j$

$k \in \beta_2^j$
$\text{supp}(\phi_k) \subset \overline{\Omega_j \setminus \Omega_j^\circ}$

$k \in \beta_3^j$
$\text{supp}(\phi_k) \subset \overline{\Omega}_j$,
$\text{supp}(\phi_k) \not\subset \Omega_j \setminus \Omega_j^\circ$

Fig. 1 Three types of finite element basis functions on each subdomain Ω_j. The *hashed surface* is the overlap Ω_j°

Definition 3.10 We partition the set $\overline{\text{dof}}(\Omega_j)$ of degrees of freedom in $V_h(\Omega_j)$ into three sets (see also Fig. 1):

$$\begin{aligned}
\beta_1^j &:= \overline{\text{dof}}(\Omega_j) \setminus \text{dof}(\Omega_j) && \text{(the DOFs on the boundary of } \Omega_j\text{)},\\
\beta_2^j &:= \text{dof}(\Omega_j \setminus \Omega_j^\circ) && \text{(the interior DOFs in } \Omega_j \setminus \Omega_j^\circ\text{)},\\
\beta_3^j &:= \text{dof}(\Omega_j) \setminus \text{dof}(\Omega_j \setminus \Omega_j^\circ) && \text{(the DOFs in the overlap, incl. the inner boundary)}.
\end{aligned}$$

From these index sets we define subsets of functions of $V_h(\Omega_j)$

$$\mathscr{B}_1^j := \text{span}\left\{\phi_k|_{\Omega_j}\right\}_{k\in\beta_1^j}, \quad \mathscr{B}_2^j := \text{span}\left\{\phi_k|_{\Omega_j}\right\}_{k\in\beta_2^j} \quad \text{and} \quad \mathscr{B}_3^j := \text{span}\left\{\phi_k|_{\Omega_j}\right\}_{k\in\beta_3^j},$$

such that

$$V_h(\Omega_j) = \mathscr{B}_1^j \oplus \mathscr{B}_2^j \oplus \mathscr{B}_3^j.$$

The following simple properties will be used frequently in the following.

Lemma 3.11 *For any $1 \le j \le N$, the following properties are true*

1. $\text{supp}(v) \subset \Omega_j^\circ$, *for all* $v \in \mathscr{B}_1^j$,
2. $\mathscr{B}_1^j = \text{Ker}(\Xi_j)$,
3. $\mathscr{B}_2^j = \{v \in V_h(\Omega_j) : v|_{\Omega_j^\circ} = 0\}$,
4. a_{Ω_j} *is coercive on* $\mathscr{B}_2^j$.

Proof 1. For any basis function ϕ_k with $k \in \beta_1^j$, Lemma 2.4 implies that there is another subdomain $\Omega_{j'}$ with $\text{supp}(\phi_k) \subset \overline{\Omega}_{j'}$, and so $\text{supp}(\phi_k) \cap (\Omega_j \setminus \Omega_j^\circ) = \emptyset$.

2. Let $v \in V_h(\Omega_j)$. Then

$$v \in \text{Ker}(\Xi_j) \Leftrightarrow v_k = 0, \ \text{ for all } \ k \in \text{dof}(\Omega_j) \Leftrightarrow v = \sum_{k\in\beta_1^j} v_k \phi_k|_{\Omega_j} \in \mathscr{B}_1^j.$$

3. It is clear from the definition of $\mathscr{B}_2^j$ that $\mathscr{B}_2^j \subset \{v \in V_h(\Omega_j) : v|_{\Omega_j^\circ} = 0\}$. Conversely, if $v|_{\Omega_j^\circ} = 0$, then from the unisolvence property, $v_k = 0$, for all $k \in \overline{\text{dof}}(\Omega_j^\circ) = \beta_1^j \cup \beta_3^j$, and therefore $\{v \in V_h(\Omega_j) : v|_{\Omega_j^\circ} = 0\} \subset \mathscr{B}_2^j$ also.
4. The previous property implies that $\mathscr{B}_2^j \subset V_{h,0}(\Omega_j)$ and so

$$a_{\Omega_j}(v, w) = a(R_j^\top v, R_j^\top w) \quad \text{for all } v, w \in \mathscr{B}_2^j.$$

The coercivity of $a_{\Omega_j}(\cdot, \cdot)$ on $\mathscr{B}_2^j$ follows from the coercivity of $a(\cdot, \cdot)$. □

To carry out a robustness analysis we need to make the following two assumptions.

Assumption 3.12 For any $1 \leq j \leq N$, a_{Ω_j} is coercive on $\mathscr{B}_1^j$.

Assumption 3.13 For any $1 \leq j \leq N$, $a_{\Omega_j^\circ}$ is coercive on $\mathscr{B}_3^j$.

Note that by the first property in Lemma 3.11, Assumption 3.12 is equivalent to assuming that, for any $1 \leq j \leq N$, $a_{\Omega_j^\circ}$ is coercive on $\mathscr{B}_1^j$.

Remark 3.14 Assumptions 3.12 and 3.13 are not too restrictive. If all the element stiffness matrices are positive definite, then a_{Ω_j} and $a_{\Omega_j^\circ}$ are positive definite on the whole of $V_h(\Omega_j)$. For the Darcy equation or linear elasticity, the element stiffness matrices are not positive definite. However, any function $v \in \mathscr{B}_1^j$ satisfies $v_k = 0$, for $k \notin \beta_1^j$, and any function $v \in \mathscr{B}_3^j$ vanishes on the boundary of Ω_j (i.e. $v_k = 0$, for $k \in \beta_1^j$). Therefore, in the Darcy case and in the case of standard H^1-conforming finite elements, Assumptions 3.12 and 3.13 hold if each of the sets β_1^j and β_3^j contains at least one DOF. To make the assumptions hold for linear elasticity, the sets β_1^j and β_3^j need to contain enough DOFs to fix the rigid body modes in Ω_j°, i.e., at least $3(d-1)$ DOFs. Hence, for standard H^1-conforming finite elements, it is sufficient to have d non-collinear points (with associated DOFs for all components of the vector function) that lie on the outer boundary $\partial\Omega_j$, respectively in $\overline{\Omega_j^\circ} \backslash \partial\Omega_j$.

The final technical hurdle to construct a stable splitting is that we cannot apply the abstract Lemma 2.11 to the specific eigenproblems used in the construction of the GenEO coarse space V_H directly, because the bilinear forms $b_j(\cdot, \cdot) := a_{\Omega_j^\circ}(\Xi_j(\cdot), \Xi_j(\cdot))$ from Definition 3.3 are not necessarily positive definite on all of $V_h(\Omega_j) \times V_h(\Omega_j)$, for all $1 \leq j \leq N$. To complete the analysis we thus need to define a suitable subspace $\widetilde{V}_j \subset V_h(\Omega_j)$ such that b_j is positive definite on $\widetilde{V}_j \times \widetilde{V}_j$.

Definition 3.15 Let the spaces $\widetilde{V}_j$ and $\widetilde{W}_j$ be defined by

$$\widetilde{V}_j := \{v \in V_h(\Omega_j) : a_{\Omega_j}(v, w) = 0, \text{ for all } w \in \widetilde{W}_j\} \quad \text{where} \quad \widetilde{W}_j := \mathscr{B}_1^j \oplus \mathscr{B}_2^j.$$

Lemma 3.16 *Under Assumption 3.12,*

$$V_h(\Omega_j) = \widetilde{V}_j \oplus \widetilde{W}_j.$$

Proof Since a_{Ω_j} is coercive on $\mathscr{B}_1^j$ (cf. Assumption 3.12) and on $\mathscr{B}_2^j$ (cf. Lemma 3.11 (4)) and since functions in $\mathscr{B}_1^j$ and $\mathscr{B}_2^j$ have disjoint supports, we also have that a_{Ω_j} is coercive on $\widetilde{W}_j$. It follows from the definition of $\widetilde{V}_j$ (via some simple linear algebra) that $\widetilde{V}_j \cap \widetilde{W}_j = \{0\}$ and that $\dim(\widetilde{V}_j) = \dim(V_h(\Omega_j)) - \dim(\widetilde{W}_j)$. □

Remark 3.17 While this lemma shows that $\widetilde{V}_j$ and $\mathscr{B}_3^j$ contain the same degrees of freedom, it does not imply that $\widetilde{V}_j = \mathscr{B}_3^j$. Indeed having chosen values for the degrees of freedom in β_3^j, the corresponding function in $\widetilde{V}_j$ is the discrete PDE-harmonic extension to the whole of Ω_j while the corresponding function in $\mathscr{B}_3^j$ is the extension by zero. The discrete harmonic extension into $\Omega_j \backslash \Omega_j^\circ$ is always well defined because of the coercivity of a_{Ω_j} on $\mathscr{B}_2^j$ (cf. Lemma 3.11 (4)). The fact that the discrete harmonic extension onto $\mathscr{B}_1^j$ is well defined is a consequence of Assumption 3.12.

The role of Assumption 3.13 becomes clear in the next lemma.

Lemma 3.18 *Under Assumptions 3.12 and 3.13, for $j = 1, \ldots, N$, the bilinear form $b_j(\cdot,\cdot) := a_{\Omega_j^\circ}(\Xi_j(\cdot), \Xi_j(\cdot))$ is positive definite on $\widetilde{V}_j \times \widetilde{V}_j$.*

Proof Let $v \in \widetilde{V}_j$ such that $\widetilde{b}_j(v, v) = 0$. We need to show that necessarily $v = 0$.

There exists a unique decomposition $v = v_1 + v_2 + v_3$, such that $v_i \in \mathscr{B}_i^j$. The second property in Lemma 3.11 states that $\mathscr{B}_1^j = \mathrm{Ker}(\Xi_j)$, and so

$$\Xi_j(v_1) = 0.$$

From the definition of Ξ_j it is obvious that $\Xi_j|_{\mathscr{B}_2^j} : \mathscr{B}_2^j \to \mathscr{B}_2^j$ is the identity, and so $\Xi_j(v_2) \in \mathscr{B}_2^j$ and in particular from the third property in Lemma 3.11

$$\mathrm{supp}(\Xi_j(v_2)) \cap \Omega_j^\circ = \emptyset.$$

From these two remarks and the definition of b_j it follows that

$$b_j(v, v) = a_{\Omega_j^\circ}(\Xi_j(v_3), \Xi_j(v_3)). \tag{3.4}$$

Moreover, from the definition of Ξ_j it is also obvious that $\Xi_j|_{\mathscr{B}_3^j} : \mathscr{B}_3^j \to \mathscr{B}_3^j$ is a bijection, and so $\Xi_j(v_3) \in \mathscr{B}_3^j$. Now, (3.4) and Assumption 3.13 imply that $\Xi_j(v_3) = 0$. The fact that $\Xi_j|_{\mathscr{B}_3^j}$ is a bijection in turn implies that $v_3 = 0$, and so $v \in \widetilde{W}_j$. From Lemma 3.16, we know that $\widetilde{V}_j \cap \widetilde{W}_j = \{0\}$, and so $v = 0$ which ends the proof. □

We can now apply Lemma 2.11 to the restriction of the GenEO eigenproblems to $\widetilde{V}_j \times \widetilde{V}_j$ and characterize the entire spectrum (including the infinite eigenvalues).

Lemma 3.19 *For each $j = 1, ..., N$, consider the generalized eigenproblem (3.3) in Definition 3.7.*

(i) *There are $\dim(\widetilde{V}_j)$ finite eigenvalues $0 \leq \lambda_1^j \leq \lambda_2^j \leq \ldots \leq \lambda_{\dim(\widetilde{V}_j)}^j < \infty$ (counted according to multiplicity) with corresponding eigenvectors denoted by $\{p_k^j\}_{k=1}^{\dim(\widetilde{V}_j)}$ and normalized to form an orthonormal basis of $\widetilde{V}_j$ with respect to $b_j(\cdot,\cdot)$.*

(ii) *There are $\dim(\widetilde{W}_j)$ infinite eigenvalues $\lambda_{\dim(\widetilde{V}_j)+1}^j = \ldots = \lambda_{\dim(V_h(\Omega_j))}^j = \infty$ with associated eigenvectors denoted by $\{p_k^j\}_{\dim(\widetilde{V}_j)+1}^{\dim(V_h(\Omega_j))}$ forming a basis of $\widetilde{W}_j$.*

Proof Since $V_h(\Omega_j) = \widetilde{V}_j \oplus \widetilde{W}_j$ (cf. Lemma 3.16) and $a_{\Omega_j}(v, w) = b_j(v, w) = 0$, for all $v \in \widetilde{V}_j$ and $w \in \widetilde{W}_j$, the eigenproblem (3.3) can be decoupled into two eigenproblems: one on $\widetilde{V}_j$ and one on $\widetilde{W}_j$.

Since, according to Lemma 3.18, $b_j(\cdot,\cdot)$ is coercive on $\widetilde{V}_j \times \widetilde{V}_j$, we can apply Lemma 2.11 with $\widetilde{V} \mapsto \widetilde{V}_j$, $\widetilde{a} \mapsto a_{\Omega_j}$, and $\widetilde{b} \mapsto b_j$ to analyse the restriction of (3.3) to $\widetilde{V}_j$. This completes the proof of (i).

For the restriction of (3.3) to $\widetilde{W}_j$, we prove that all vectors in $\widetilde{W}_j$ are eigenvectors associated with the eigenvalue $+\infty$ in the sense of Definition 2.10. Let $v \in \widetilde{W}_j$. Then $\Xi_j(v)|_{\Omega_j^\circ} = 0$ and so in particular

$$a_{\Omega_j^\circ}(\Xi_j(v), \Xi_j(w)) = 0 \quad \text{for all } v,\ w \in \widetilde{W}_j\,. \tag{3.5}$$

Moreover, we have already seen in the proof of Lemma 3.16 that a_{Ω_j} is coercive on $\widetilde{W}_j$, and so

$$a_{\Omega_j}(v, v) \neq 0 \quad \text{for all } v \in \widetilde{W}_j \backslash \{0\}. \tag{3.6}$$

Due to (3.5) and (3.6), any $v \in \widetilde{W}_j$ is indeed an eigenvector to the eigenvalue $+\infty$ in the sense of Definition 2.10. We can use any set of linearly independent vectors in $\widetilde{W}_j$ to form a basis, e.g. $\{p_k^j\}_{k=\dim(\widetilde{V}_j)+1}^{\dim(V_h(\Omega_j))} = \{\phi_k|_{\Omega_j}\}_{k\in\beta_1^j\cup\beta_2^j}$. □

We are now ready to define the crucial projection operators onto the local components of the GenEO coarse space that satisfy suitable stability estimates.

Lemma 3.20 (Local stability estimate) *Let $j \in \{1, ..., N\}$ and let $\{(p_k^j, \lambda_k^j)\}_{k=1}^{\dim(V_h(\Omega_j))}$ be as defined in Lemma 3.19. Suppose that $m_j \in \{1, \ldots, \dim(V_h(\Omega_j)) - 1\}$ such that $0 < \lambda_{m_j+1}^j < \infty$. Then, the local projection operator*

$$\Pi_{m_j}^j v := \sum_{k=1}^{m_j} a_{\Omega_j^\circ}(\Xi_j(v), \Xi_j(p_k^j))\, p_k^j$$

satisfies

$$|\Pi^j_{m_j} v|_{a,\Omega_j} \leq |v|_{a,\Omega_j} \quad \textit{and} \quad |v - \Pi^j_{m_j} v|_{a,\Omega_j} \leq |v|_{a,\Omega_j}, \quad \textit{for all } v \in V_h(\Omega_j), \tag{3.7}$$

as well as the stability estimate

$$\left|\Xi_j(v - \Pi^j_{m_j} v)\right|^2_{a,\Omega_j^\circ} \leq \frac{1}{\lambda^j_{m_j+1}} \left|v - \Pi^j_{m_j} v\right|^2_{a,\Omega_j}, \quad \textit{for all } v \in V_h(\Omega_j). \tag{3.8}$$

Proof The condition $\lambda^j_{m_j+1} < \infty$, ensures that $m_j \leq \dim(\widetilde{V}_j)$, so $\Pi^j_{m_j}$ maps to $\widetilde{V}_j$. Therefore, for all $v \in \widetilde{V}_j$, the estimates in (3.7) and (3.8) can be deduced from Lemma 2.11 again, with $\widetilde{V} \mapsto \widetilde{V}_j$, $\widetilde{a} \mapsto a_{\Omega_j}$, $\widetilde{b} \mapsto b_j$, and $m \mapsto m_j$.

To prove the result for all $v \in V_h(\Omega_j)$, we use again the fact that $V_h(\Omega_j) = \widetilde{V}_j \oplus \widetilde{W}_j$ and that $a_{\Omega_j}(v, w) = 0$, for all $v \in \widetilde{V}_j$ and $w \in \widetilde{W}_j$. Let $v = v_V + v_W \in V_h(\Omega_j)$ with $v_V \in \widetilde{V}_j$ and $v_W \in \widetilde{W}_j$. Then $\Pi^j_{m_j} v = \Pi^j_{m_j} v_V$ and so (3.7) follows due to the a_{Ω_j}-orthogonality of $\widetilde{V}_j$ and $\widetilde{W}_j$. Estimate (3.8) follows similarly from $\Xi_j(v_W)|_{\Omega_j^\circ} = 0$. □

Lemma 3.21 (Stable decomposition) *Let $v \in V_h$ and suppose the definitions and notations of Lemma 3.20 hold. Then, the decomposition*

$$z_0 := \sum_{j=1}^N \Xi_j(\Pi^j_{m_j} v|_{\Omega_j}), \qquad z_j := \Xi_j(v|_{\Omega_j} - \Pi^j_{m_j} v|_{\Omega_j}), \quad \textit{for } j = 1, \ldots, N,$$

is C_0-stable with

$$C_0^2 = 2 + k_0(2k_0 + 1) \max_{1 \leq j \leq N} \left(1 + \frac{1}{\lambda^j_{m_j+1}}\right).$$

Proof By definition $\|z_j\|^2_{a,\Omega_j} = |\Xi_j(v - \Pi^j_{m_j} v|_{\Omega_j})|^2_{a,\Omega_j^\circ} + |\Xi_j(v - \Pi^j_{m_j} v|_{\Omega_j})|^2_{a,\Omega_j \setminus \Omega_j^\circ}$. However, due to property (3.2) in Lemma 3.6, Ξ_j is the identity for restrictions of functions to $\Omega_j \setminus \Omega_j^\circ$, and so

$$\|z_j\|^2_{a,\Omega_j} = \left|\Xi_j(v - \Pi^j_{m_j} v|_{\Omega_j})\right|^2_{a,\Omega_j^\circ} + \left|v - \Pi^j_{m_j} v|_{\Omega_j}\right|^2_{a,\Omega_j \setminus \Omega_j^\circ}.$$

Now we can apply Lemma 3.20 to get

$$\|z_j\|^2_{a,\Omega_j} \& \leq \left(1 + \frac{1}{\lambda^j_{m_j+1}}\right) \left|v - \Pi^j_{m_j} v|_{\Omega_j}\right|^2_{a,\Omega_j} \leq \left(1 + \frac{1}{\lambda^j_{m_j+1}}\right) |v|^2_{a,\Omega_j},$$

where in the last step we have used (3.7). □

With this stable decomposition we can now state our main result on the convergence of the two-level Schwarz preconditioner with the new GenEO coarse space. It follows immediately from Theorem 2.8 and Lemma 3.21.

Theorem 3.22 (Bound on the condition number) *Let Assumptions 2.1, 3.12, and 3.13 hold. Suppose that the coarse space V_H is given by Definition 3.9 and $\mathbf{M}_{AS,2}^{-1}$ is as defined in (2.5).*

Then we can bound the condition number for the two-level Schwarz method by

$$\kappa(\mathbf{M}_{AS,2}^{-1}\mathbf{A}) \;\leq\; (1+k_0)\left[2+k_0(2k_0+1)\max_{1\leq j\leq N}\left(1+\frac{1}{\lambda^j_{m_j+1}}\right)\right],$$

where k_0 is given in Definition 2.5.

The only parameters that need to be chosen in our coarse space are the numbers m_j of eigenmodes on each subdomain Ω_j, $1 \leq j \leq N$, to be included in the coarse space. We suggest the following choice which recovers the condition number estimate for problems with no strong coefficient variation.

Corollary 3.23 *For any j, $1 \leq j \leq N$, let*

$$m_j := \min\left\{m : \lambda^j_{m+1} > \frac{\delta_j}{H_j}\right\}, \tag{3.9}$$

where δ_j is a measure of the width of the overlap Ω_j° and $H_j = \operatorname{diam}(\Omega_j)$. Then

$$\kappa(\mathbf{M}_{AS,2}^{-1}\mathbf{A}) \;\leq\; (1+k_0)\left[2+k_0(2k_0+1)\max_{1\leq j\leq N}\left(1+\frac{H_j}{\delta_j}\right)\right].$$

Note that the number of subdomains and the coefficient variations do not appear in this bound on the condition number. This means that we have established rigorously that the algorithm is robust with respect to these two parameters. We will confirm this with some numerical tests in Sect. 5. The size of the coarse space induced by the criterion does however depend on the geometry of the coefficient variation in the overlaps and the choice of the partition of unity. In fact, for some problems it may happen that even for a very small criterion the number of eigenmodes which are selected is very large. This is the case for instance in the context of linear elasticity when one of the materials is almost incompressible (i.e. its Poisson ration approaches $1/2$), because then the bilinear form $a_{\Omega_j^\circ}(\Xi_j(\cdot), \Xi_j(\cdot))$ on the right hand side of eigenproblem (3.3) has very high energy.

4 Implementation

In this section we would like to address implementation issues of the proposed algorithm involving the GenEO coarse space. In the sections above, we have worked with function spaces as they are more convenient in the analysis. However, as we will

demonstrate below, our algorithm requires only abstract information of the problem in form of the element stiffness matrices and no further information on the mesh, the finite element spaces, or any coefficients. Indeed, for running the algorithm we need

(i) the list $\overline{\mathrm{dof}}(\tau)$ of degrees of freedom associated with each element $\tau \in \mathscr{T}_h$,
(ii) the element stiffness matrix $\mathbf{A}^\tau = (a_\tau(\phi_l, \phi_k))_{k,\, l \in \overline{\mathrm{dof}}(\tau)}$ associated with each element $\tau \in \mathscr{T}_h$.

Unless the overlapping subdomain partition is available a priori, we additionally need

(iii) the number ℓ of layers which determine the amount of overlap.

Before going into details, we note that as for the classical two-level overlapping Schwarz method (see, e.g. [31, Sect. 3]), our algorithm can be parallelized straightforwardly. In particular, the solution of the eigenproblems in the preprocessing step and the subdomain solves during each PCG iteration can be performed fully in parallel.

4.1 Preprocessing

We need the overlapping partition $\Omega = \bigcup_{j=1}^N \Omega_j$ in form of the list of elements associated with each subdomain Ω_j. To obtain this, we first create the connectivity graph of the elements (using the lists $\overline{\mathrm{dof}}(\tau)$ from (i)) and partition it into disjoint sets of elements which make up the non-overlapping subdomains Ω_j' using for instance METIS [18] or SCOTCH [4]. Then, for each (global) DOF k, we build the list

$$\mathrm{elem}(k) = \{\tau \in \mathscr{T}_h : k \in \overline{\mathrm{dof}}(\tau)\}$$

of elements where DOF k is active. This list realizes $\mathrm{supp}(\phi_k)$ without knowing the basis function ϕ_k itself. In a second step we add ℓ layers to each non-overlapping subdomain Ω_j' according to Definition 2.3, which finally results in a list of elements per (overlapping) subdomain Ω_j. From this, we construct

$$\overline{\mathrm{dof}}(\Omega_j) = \bigcup_{\tau \subset \bar{\Omega}_j} \overline{\mathrm{dof}}(\tau)$$

(cf. Definition 3.2). Then we can compute the set of *internal* degrees of freedom in Ω_j

$$\mathrm{dof}(\Omega_j) = \left\{ k \in \overline{\mathrm{dof}}(\Omega_j) : \bigcup_{\tau \in \mathrm{elem}(k)} \tau \subset \bar{\Omega}_j \right\}$$

(cf. Definition 3.4). Finally it is straightforward to get the list of elements that make up the overlapping zone Ω_j° for each $j = 1, \ldots, N$, namely $\{\tau \subset \bar{\Omega}_j : \tau \subset \bar{\Omega}_{j'},\ j' \neq j\}$.

4.2 The eigenproblems

For each subdomain Ω_j, $j = 1, \ldots, N$ we use a local renumbering of the degrees of freedom $\overline{\mathrm{dof}}(\Omega_j)$ of $V_h(\Omega_j)$. By assembling the element stiffness matrices for these DOFs over the elements $\tau \subset \bar{\Omega}_j$, we get the subdomain "Neumann" matrix $\widetilde{\mathbf{A}}_j$. This is the matrix formulation of $a_{\Omega_j}(\cdot,\cdot) : V_h(\Omega_j) \times V_h(\Omega_j) \to \mathbb{R}$. For the same renumbering of DOFs, we assemble only over the elements $\tau \subset \bar{\Omega}_j^\circ$ in the overlap and obtain matrix $\widetilde{\mathbf{A}}_j^\circ$ associated with the bilinear form $a_{\Omega_j^\circ}(\cdot,\cdot) : V_h(\Omega_j) \to V_h(\Omega_j)$. Note that $\widetilde{\mathbf{A}}_j$ and $\widetilde{\mathbf{A}}_j^\circ$ have the same format, but $\widetilde{\mathbf{A}}_j^\circ$ usually contains a block of zeros corresponding to the degrees of freedom that are in the part of Ω_j which is not overlapped by other subdomains.

From Definition 3.4, we see immediately that the action of the operator Ξ_j can be coded by a diagonal matrix $\mathbf{X}_j$, where the diagonal entry corresponding to DOF k is equal to $1/\mu_{j,k}$.

With these notations, the eigenproblem given in Definition 3.7 reads: Find the eigenvectors $\mathbf{p}_k^j \in \mathbb{R}^{\#\overline{\mathrm{dof}}(\Omega_j)}$ and eigenvalues $\lambda_k^j \in \mathbb{R} \cup \{+\infty\}$ that satisfy

$$\widetilde{\mathbf{A}}_j \mathbf{p}_k^j = \lambda_k^j \mathbf{X}_j \widetilde{\mathbf{A}}_j^\circ \mathbf{X}_j \mathbf{p}_k^j. \tag{4.1}$$

To get the coarse basis functions, we need to solve these eigenproblems (at least we need sufficiently many eigenpairs corresponding to low frequent modes) and to then select m_j of these eigenfunctions for our coarse space. With the criterion suggested in (3.9), we need measures δ_j and H_j for the width of the overlapping zone and the subdomain diameter, respectively. If the mesh can be assumed to be quasi-uniform, we may replace the ratio δ_j/H_j by the number of layers of extension we applied in subdomain Ω_j divided by the number of layers Ω_j contains in total (which is available via the connectivity graph).

4.3 The preconditioner

Having selected the eigenvectors $\mathbf{p}_k^j$, the coarse basis functions are given by the vectors $\widetilde{\mathbf{R}}_j^T \mathbf{X}_j \mathbf{p}_k^j$, where the matrix $\widetilde{\mathbf{R}}_j^T$ maps the renumbered DOFs to the global DOFs and fills the rest of the vector with zeros. The columns of the matrix $\mathbf{R}_H^T$ are exactly the vectors $\widetilde{\mathbf{R}}_j^T \mathbf{X}_j \mathbf{p}_k^j$, where $j = 1, \ldots, N$, $k = 1, \ldots, m_j$. The coarse matrix $\mathbf{A}_H = \mathbf{R}_H \mathbf{A} \mathbf{R}_H^T$ can be efficiently assembled subdomain-wise by using the fact that the coarse basis functions corresponding to two subdomains only interact when the subdomains overlap. Thus, in a parallel regime, we basically only need next-neighbor communication.

As for the 'one level' part of the preconditioner we have made the list $\mathrm{dof}(\Omega_j)$ of internal degrees of freedom for subdomain Ω_j available in the preprocessing step. Then R_j is simply a Boolean matrix which renumbers local vectors into global vectors and the matrix counterpart A_j of $a_{\Omega_j}(\cdot,\cdot) : V_{h,0}(\Omega_j) \times V_{h,0}(\Omega_j) \to \mathbb{R}$ is computed by assembling the element matrices for elements τ in the ready made list $\{\tau \subset \bar{\Omega}_j\}$.

Clearly, once the information above is stored and the matrices $\mathbf{A}_j$ are factorized, each application of $\mathbf{M}_{AS,2}^{-1}$ (within the PCG) can be carried out efficiently.

4.4 An alternative way of solving the eigenproblems

The size of the (algebraic) eigenproblem (4.1) to be solved in each subdomain can be reduced. By rearranging the local DOFs $\overline{\text{dof}}(\Omega_j)$ with respect to the sets β_1^j (the boundary), β_2^j (the overlap), and β_3^j (the interior) (cf. Definition 3.10), the matrices $\widetilde{\mathbf{A}}_j$ and $\mathbf{B}_j := \mathbf{X}_j \widetilde{\mathbf{A}}_j^{\circ} \mathbf{X}_j$ take the following block form

$$\widetilde{\mathbf{A}}_j = \begin{pmatrix} \widetilde{\mathbf{A}}_j^{11} & 0 & \widetilde{\mathbf{A}}_j^{13} \\ 0 & \widetilde{\mathbf{A}}_j^{22} & \widetilde{\mathbf{A}}_j^{23} \\ (\widetilde{\mathbf{A}}_j^{13})^T & (\widetilde{\mathbf{A}}_j^{23})^T & \widetilde{\mathbf{A}}_j^{33} \end{pmatrix}, \qquad \mathbf{B}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{B}_j^{33} \end{pmatrix},$$

where $\widetilde{\mathbf{A}}_j^{kl} = a_{\Omega_j}(\phi_m, \phi_n)_{n \in \beta_k^j, m \in \beta_l^j}$. The two zero blocks in $\widetilde{\mathbf{A}}_j$ are due the fact that the supports of functions in $\mathscr{B}_1^j$ and $\mathscr{B}_2^j$ are always disjoint. Since $\widetilde{\mathbf{A}}_j^{11}$ is the matrix version of the bilinear form $a_{\Omega_j^{\circ}}(\cdot,\cdot) : \mathscr{B}_1^j \times \mathscr{B}_1^j \to \mathbb{R}$, and since Assumption 3.12 states that $a_{\Omega_j^{\circ}}(\cdot,\cdot)$ is coercive on $\mathscr{B}_1$, it follows that the block $\widetilde{\mathbf{A}}_j^{11}$ is positive definite and thus invertible. Similarly, $\mathbf{A}_j^{22}$ is positive definite due to Lemma 3.11 (4). This means that the Schur complement $\mathbf{S}_j = \widetilde{\mathbf{A}}_j^{33} - \widetilde{\mathbf{A}}_j^{13}[\widetilde{\mathbf{A}}_j^{11}]^{-1}\widetilde{\mathbf{A}}_j^{13} - \widetilde{\mathbf{A}}_j^{23}[\widetilde{\mathbf{A}}_j^{22}]^{-1}\widetilde{\mathbf{A}}_j^{23}$ is well defined and we can reduce eigenproblem (4.1) to an eigenproblem for the Schur complement

$$\mathbf{S}_j \, \mathbf{p}_k^{j,3} = \lambda_k^j \mathbf{B}_j^{33} \mathbf{p}_k^{j,3}. \tag{4.2}$$

The two remaining blocks in $\mathbf{p}^j$ can then be computed from

$$\mathbf{p}_k^{j,1} = -\left[\widetilde{\mathbf{A}}_j^{11}\right]^{-1} \widetilde{\mathbf{A}}_j^{13} \mathbf{p}_k^{j,3},$$
$$\mathbf{p}_k^{j,2} = -\left[\widetilde{\mathbf{A}}_j^{22}\right]^{-1} \widetilde{\mathbf{A}}_j^{23} \mathbf{p}_k^{j,3}$$

(i.e. via discrete harmonic extension). The only difference is that with this version of the eigenproblem there are no infinite eigenvalues. Because we are only interested in the small eigenvalues we can solve eigenproblem (4.2) instead of (4.1). Due to the appearance of the Schur complement $\mathbf{S}_j$ and because we are interested only in the first few eigenpairs, an iterative eigensolver could be applied, e.g., we could use the inverse power method [23], ARPACK [20] or the LOBPCG method [19], maybe using a suitable regularization of $\widetilde{\mathbf{A}}_{jj}^{33}$ or $\mathbf{S}_j$ as a preconditioner. This, however, will be the subject of future research and we will use a direct eigensolver in the next section. Note finally, that the blocks $\mathbf{p}_k^{j,2}$ never need to be calculated in practice as they are annihilated by the matrix $\mathbf{X}_j$.

5 Numerical results

We have introduced an algorithm for a wide range of problems. In this section we test its efficiency on the three-dimensional Darcy equation and on the two- and three-dimensional linear elasticity equations with heterogeneous coefficients. We have used FreeFem++ [17] to define the test cases and build all the finite element data. Throughout we have used standard piecewise linear ($\mathbb{P}_1$) finite elements. The eigenvalue problems were solved using LAPACK [1]. For the remainder (including the subdomain solves and the coarse solve) we have used Matlab. Throughout this section we compare three methods.

1. The first one is the one-level additive Schwarz method (referred to as AS), defined by the preconditioner $\mathbf{M}_{AS,1}^{-1} = \sum_{j=1}^{N} \mathbf{R}_j^T \mathbf{A}_j^{-1} \mathbf{R}_j$.
2. The second one (referred to as ZEM for Zero Energy Modes) is the two-level method given by (2.5) with the coarse space $V_H := \text{span}\{\mathbf{R}_j^T \Xi_j(\mathbf{q}_k^j)\}_{j,k}$ where the $\mathbf{q}_k^j$ span the kernel of the subdomain operator. For the Darcy equation these are the constant functions and for elasticity the rigid body modes. In the floating subdomains that do not touch the Dirichlet boundary, this basically coincides with choosing $m_j = \dim(\ker(a_{\Omega_j}))$ in our GenEO method.
3. The third method (referred to as GenEO) is the two-level method introduced here, with the number m_j, for $j = 1, \ldots, N$, chosen according to (3.9) (except for one test where we will explicitly state this). The partition of unity operators are chosen to be the ones in Remark 3.5 where the weights are the multiplicities of each degree of freedom.

For each of these methods we use the Preconditioned Conjugate Gradient (PCG) solver. As a stopping criterion we apply $\|v - \bar{v}\|_\infty < 10^{-6} \|\bar{v}\|_\infty$ where $\bar{v}$ is the solution of (2.2) obtained *via* a direct solver on the global problem (unless otherwise stated). Of course this criterion is not practical but in this context we have chosen it to ensure a fair comparison.

In the tables below, we provide the number of PCG iterations needed to reach convergence. We have also computed condition number estimates for each of the preconditioned matrices using the Rayleigh-Ritz procedure [26] on the Krylov subspaces within PCG. We do not give any detail on the maximal and minimal eigenvalue. However, we can report that adding/enriching the coarse space leads to larger minimal eigenvalues, whereas the maximal eigenvalue depends only on the geometry. This is in agreement with Lemma 2.6 and Theorem 2.8. Finally, we also display the dimension of the coarse space V_H in each case.

For both three-dimensional scalability test (Sects. 5.1 and 5.2), we use the domain $\Omega = [0, L] \times [0, 1] \times [0, 1]$ and a regular tetrahedral mesh of $(10L + 1 \times 11 \times 11)$ nodes which we divide into L subdomains, horizontally side by side. We will either use a regular partition into L unit cubes (Fig. 2 left) or an automatic partition into L subdomains using Metis (Fig. 2 right). In the two dimensional test cases (Sect. 5.3, we will use more general (two-dimensional) partitions.

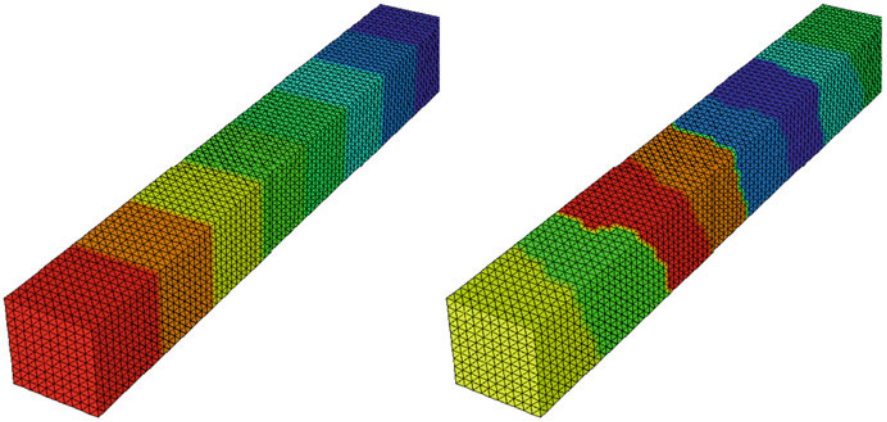


Fig. 2 Partition of Ω into $L = 8$ subdomains—regular (*left*) and Metis (*right*)

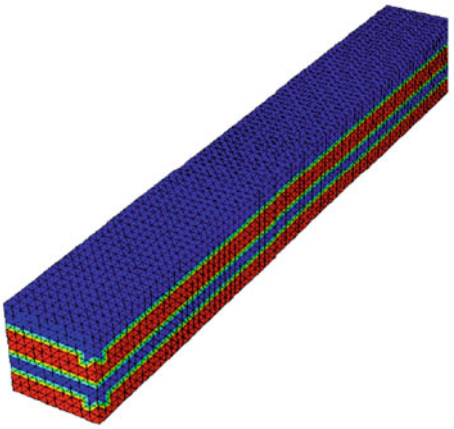


Fig. 3 Coefficient distribution (four alternating layers)

5.1 The Darcy equation

On the domain $\Omega \subset \mathbb{R}^3$ given above, we solve the following problem: Find $v \in H^1(\Omega)$ such that

$$-\nabla \cdot (\kappa \nabla v) = 0 \quad \text{in} \quad \Omega, \tag{5.1}$$

$v = 0$ on $\partial\Omega_D = \{(x, y, z) \in \partial\Omega : x = 0\}$ and $\kappa \nabla v \cdot \mathbf{n} = 0$ on the rest of $\partial\Omega$, where $\mathbf{n}$ is the outward unit normal. The coefficient distribution alternates between two different constant values κ_1 and κ_2 of κ on four horizontal layers (as shown in Fig. 3).

First, we study the robustness of our algorithm with respect to the coefficient variation. We partition Ω into $L = 8$ (non-overlapping) regular subdomains. Each sub-

Table 1 3D Darcy: number of PCG iterations (it), condition number (cond) and coarse space dimension (dim) vs. jump in κ for $\kappa_1 = 1$, $\ell = 1$ added layers, $L = 8$ regular subdomains

κ_2	AS		ZEM			GenEO		
	it	cond	it	cond	dim	it	cond	dim
1	16	229	11	6.3	8	11	8.4	7
10^2	27	230	19	22	8	13	8.4	14
10^4	29	230	23	210	8	15	8.4	14
10^6	26	230	22	230	8	11	8.4	14

domain is then extended by $\ell = 1$ layers in order to create the overlapping partition. Table 1 shows the iteration counts and condition numbers for fixed value $\kappa_1 = 1$ and various κ_2. As expected, for our algorithm the condition number and the number of PCG iterations are robust with respect to the jump κ_2/κ_1. Furthermore, for $\kappa_2 = \kappa_1$, the algorithm automatically selects seven eigenmodes (one per floating subdomain) to build the coarse space, this leads essentially to the same choice as for the ZEM method except for the subdomain in which the Dirichlet boundary condition is active, where GenEO does not select any coarse mode. In both cases 11 iterations are needed to reach convergence.

The second test that we conduct is the scalability with regard to the problem size and the number of subdomains. For simplicity, we make the problem parameter L vary. Recall that increasing L elongates the bar-shaped domain and at the same time increases the number of subdomains which equals L. Thus, the global number of degrees of freedom is also proportional to L. Table 2 gives the results for different problem sizes (we display the number of subdomains and the total number of degrees of freedom) and for regular and irregular partitions. For regular partitions we use (3.9); for irregular partitions, the choice of m_j becomes more tricky since there may be additional 'bad' eigenmodes close to the ratio δ_j/H_j that are due to the irregularity of the subdomains and not due to any coefficient variation. In particular, the ratio δ_j/H_j which is constant for regular partitions, as L gets increased, may differ significantly for two 'Metis' decompositions into L and L' subdomains with $L \neq L'$. In the regular case, (3.9) leads to $m_j = 2$ and $\lambda_3 = 0.5$. Thus, in order for the bound on the condition number given by Theorem 3.22 to be at least as strict in the irregular ('Metis') case we set

$$m_j := \min \left\{ m : \lambda^j_{m+1} > 0.5 \right\}, \tag{5.2}$$

in each subdomain in Table 2. We note that the condition numbers in both the regular and irregular subdomain cases are stable and consistently low.

Finally, Table 3 studies the dependence on the amount of overlap, or equivalently on the number ℓ of layers added to each non-overlapping subdomain. We can see that for this example, increasing the amount of overlap improves convergence without increasing the dimension of the coarse space.

Table 2 3D Darcy: number of PCG iterations (it), condition number (*cond*) and coarse space dimension (dim) vs. problem size for $\kappa_1 = 1$, $\kappa_2 = 10^6$, $\ell = 1$ added layers, L (sub) subdomains

sub	glob DOF	AS		ZEM			GenEO		
		it	cond	it	cond	dim	it	cond	dim
Regular									
4	4,840	14	51	15	51	4	10	8.4	6
8	9,680	26	230	22	230	8	11	8.4	14
16	19,360	51	980	36	970	16	13	8.4	30
32	38,720	103	4,000	61	3,900	32	13	8.4	62
Metis with criterion given by (5.2)									
4	4,840	21	67	18	63	4	9	3.0	19
8	9,680	36	290	29	280	8	9	3.0	40
16	19,360	65	1,200	45	1,200	16	11	3.1	81
32	38,720	123	4,900	79	4,700	32	11	3.1	171

Table 3 3D Darcy: number of PCG iterations (it), condition number (*cond*) and coarse space dimension (dim) vs. number ℓ of layers added to each domain, for $L = 8$ regular subdomains, $\kappa_1 = 1$ and $\kappa_2 = 10^6$

ℓ	AS		ZEM			GenEO		
	it	cond	it	*cond*	dim	it	cond	dim
1	26	230	22	230	8	11	8.4	14
2	22	150	18	150	8	9	5.4	14
3	16	110	15	110	8	9	4.0	14
4	15	92	13	92	8	7	3.3	14

5.2 The linear elasticity equations

For the second family of tests the equations are the following. Find $\mathbf{u} = (u_1, u_2, u_3)^{\mathrm{T}} \in H^1(\Omega)^3$ such that

$$-\mathrm{div}(\sigma(\mathbf{u})) = \mathbf{f}, \quad \text{in } \Omega,$$

$\mathbf{u} = (0, 0, 0)^{\mathrm{T}}$ on $\partial\Omega_D = \{(x, y, z) \in \partial\Omega : x = 0\}$ and $\sigma(\mathbf{u}) \cdot \mathbf{n} = 0$ on the rest of $\partial\Omega$, where the stress tensor $\sigma(\mathbf{u})$, the Lamé coefficients λ and μ and the right hand side are given by

$$\begin{cases} \sigma_{ij}(\mathbf{u}) = 2\mu\varepsilon_{ij}(\mathbf{u}) + \lambda\delta_{ij}\mathrm{div}(\mathbf{u}),\ \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \mathbf{f} = (0, 0, g)^T, \\ \mu = \frac{E}{2(1+\nu)},\ \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}. \end{cases}$$

Here E and ν denote respectively Young's modulus and Poisson's ratio, and we will let both parameters vary discontinuously over the domain. Again we use two sets of values of coefficients (E_1, ν_1) and (E_2, ν_2) alternating in four layers, as shown

Table 4 3D Elasticity: number of PCG iterations (it), condition number (*cond*), and coarse space dimension (dim) vs. number of regular subdomains, for $\ell = 1$ added layers, $g = 10$, $(E_1, \nu_1) = (2 \times 10^{11}, 0.3)$ and $(E_2, \nu_2) = (2 \times 10^7, 0.45)$

L	glob DOF	AS		ZEM			GenEO		
		it	cond	it	cond	dim	it	cond	dim
4	14,520	79	2.4×10^3	54	2.9×10^2	24	16	10	46
8	29,040	177	1.3×10^4	87	1.0×10^3	48	16	10	102
16	58,080	378	1.5×10^5	145	1.4×10^3	96	16	10	214

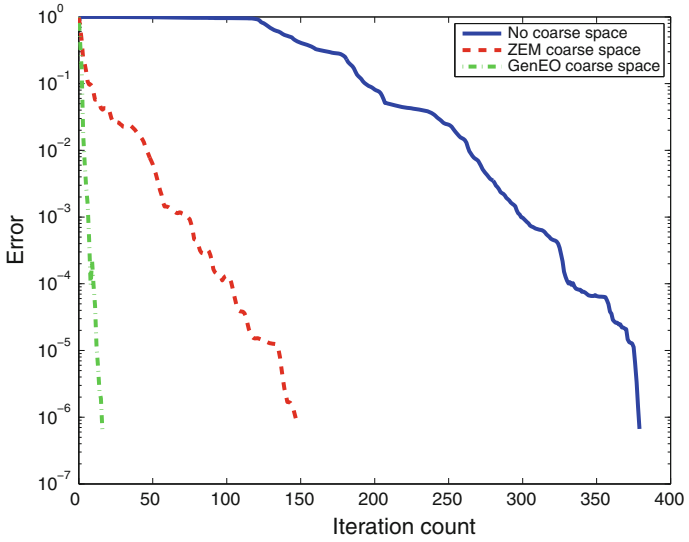


Fig. 4 3D Elasticity: Relative error vs. iteration count for $L = 16$ regular subdomains

in Fig. 3. Table 4 displays iteration counts, condition numbers, and coarse space dimensions for various partitions into regular subdomains (the parameter choices are given below the table). Note that for GenEO, we need only 16 PCG iterations in all cases. As an example, Fig. 4 shows the convergence profile for the case where Ω is split into 16 regular subdomains.

5.3 The two-dimensional linear elasticity equations

In this subsection, we look at the two-dimensional linear elasticity equations again with a Dirichlet boundary condition at $x = 0$ and Neumann conditions otherwise. In this case, the ZEM coarse space consists of three rigid body modes per subdomain. Here we choose $\Omega = (0, 1) \times (0, 1)$ and use a structured simplicial mesh with 81×81 nodes. The coefficient distribution is sketched on the left hand side of Fig. 5, where

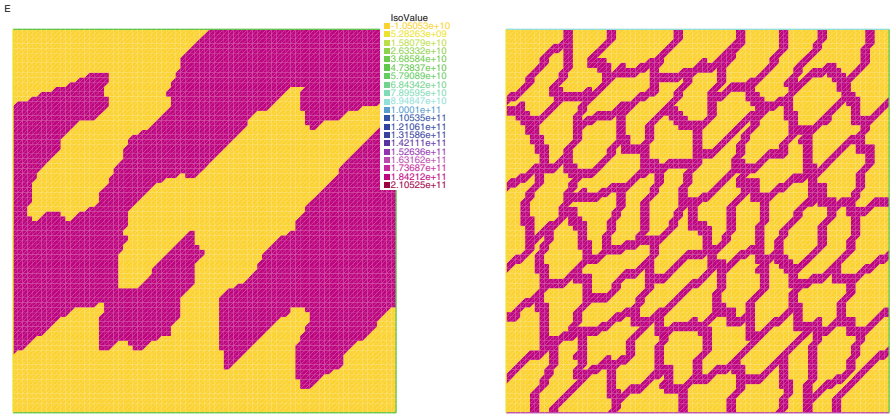


Fig. 5 2D Elasticity: coefficient distribution (*left*)—Metis decomposition into 64 subdomains (*right*)

Table 5 2D Elasticity: number of PCG iterations (it) and coarse space dimension (dim) vs. number of Metis subdomains for fixed problem size

sub	glob DOF	*AS*	ZEM		GenEO	
		it	it	dim	it	dim
4	13,122	90	94	12	36	36
16	13,122	169	179	48	39	112
25	13,122	222	157	75	40	166
64	13,122	317	196	192	39	343

on the two regions (indicated by the two different colors) we take the parameters $(E_1, \nu_1) = (2 \times 10^{11}, 0.3)$ and $(E_2, \nu_2) = (2 \times 10^7, 0.45)$.

This time, we keep the problem size fixed, but we make the number of subdomains vary. In all cases, we use a Metis partition and extend the non-overlapping subdomains by $\ell = 2$ layers. As shown in Fig. 5 (right) for a decomposition into 64 subdomains there are many floating subdomains. Table 5 shows the iteration counts and coarse space dimensions for different Metis partitions (the chosen parameters are given below the table). From the iteration counts we see that the GenEO method is scalable.

It is not surprising that the coarse space dimension grows with the number of subdomains because we construct local coarse basis functions per subdomain. Note however that for the case of 64 subdomains, the coarse space dimension of 343 is still comparable to the average dimension of 205 of a subdomain problem. To find the optimal partition in terms of CPU time, it would thus be necessary to take the cost of the subdomain solves into account, as well as the cost of the eigensolves in the setup of the method. In Table 5, no estimates for the condition number of the preconditioned matrices are given, as in some cases (but for all three types of preconditioners), the Rayleigh-Ritz procedure returned one or a few negative eigenvalues, which is probably due to rounding errors as a consequence of the high contrast. More extensive results for two dimensional elasticity can be found in [30].

6 Conclusion

In this article we have introduced a coarse space for symmetric positive definite variational problems. In order to remain as general as possible, we did so using an abstract formulation. We rigorously proved a bound for the condition number of the overlapping two-level additive Schwarz preconditioner for this coarse space. This bound does not depend on any of the coefficients in the equations or on the way the domain is split into subdomains. Numerical results on two-dimensional and three-dimensional problems are in agreement with the fact that the method is robust with regard to heterogeneities and rather irregular subdomains. We also gave details on how to implement the coarse space construction that relies only on having access to finite element stiffness matrices and the underlying connectivity graph. No additional data is required and no additional elementary matrices need to be computed. This means that the method is quite easily applicable to simulations of actual physical problems and it is our ambition to do so.

Along the way we have identified promising leads to further improve the efficiency of the method. In the near future there are three main ideas for further investigations. The first one is to take advantage of the fact that the partition of unity can be chosen differently since the proof holds as long as the partition of unity is defined by individual weights per interior degree of freedom in each subdomain. The second idea is to optimize the eigenvalue computations. Although this is a purely parallel task, this is the most costly part in the coarse space construction. Finally, the formulation of the GenEO coarse space makes it particularly well suited for a multilevel parallel implementation, which is of particular interest in cases where a two-level approach leads to excessively large coarse spaces.

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