*Abstract Vibrating Systems**

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0. Introduction. With every Markov process there is an associated "heat equation"

$$\frac{du}{dt} = Au \tag{1}$$

where A is the infinitesimal generator of the process. For Brownian motion A is an extension of the Laplacian $\triangle$, and (1) becomes the heat equation of classical physics. If we take as the main feature of the heat equation the fact that a solution of it that has nonnegative initial data is nonnegative then the theory of Markov processes may be viewed as giving a general setting for the classical heat equation that preserves this main feature. An attempt will be made here to find a similar general setting for the wave equation

$$\frac{d^2u}{dt^2}(x) = \triangle u(x) \qquad (x \in R^n). \tag{2}$$

The main feature of (2) that we will be interested in preserving is the existence of an energy integral for the solutions of (2) that is constant in time. Explicitly, for dimension one, let

$$E(u, v) = \int_{-\infty}^{+\infty} \left|\frac{du}{dx}\right|^2 dx + \int_{-\infty}^{+\infty} |v|^2 \, dx \tag{3}$$

then if $u(x, t)$ satisfies (2) we have

$$E(u(\cdot, t_1), u_t(\cdot, t_1)) = E(u(\cdot, t_2), u_t(\cdot, t_2)) \tag{4}$$

for any t_1 , $t_2 \in R$.

In Section 1 we show that if A is a self adjoint operator in some Hilbert space H which is nonpositive, *i.e.*, (x, Ax) is nonpositive for all $x \in H$, then for the equation

$$\frac{d^2u}{dt^2} = Au \tag{5}$$

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an energy analogous to (3) can be introduced which is constant along solutions of (5). Together with this energy we obtain the solution of the initial value problem for (5) as a group of operators on the initial data. The special case (2) was treated by similar semi-group methods by Yosida [14]. In Section 3 it is shown that these self adjoint operators essentially exhaust the class of operators for which such an energy exists. More precisely, it is shown that if there is an energy associated with (5) in the manner spelled out in Section 1, then A is a nonpositive self adjoint operator on some Hilbert space.

A class of examples is given in Section 4 which associate with certain Markov processes "vibrating systems" whose evolution is governed by (5) for suitable A. The general vibrating string studied by W. Feller in [4] is included in these examples. The equation of this general vibrating string is (5) with

$$A = D_m D_x \tag{6}$$

which is the infinitesimal generator of a one dimensional diffusion and it was this equation that served as the starting point of these investigations.

If a membrane or string vibrates in a medium that retards its motion there is a dampening effect which depends upon the velocity so that its evolution is governed by an equation of the form

$$\frac{d^2u}{dt^2} = Au + B\frac{du}{dt}. \tag{7}$$

Making use of certain results of Trotter [13], we show in Section 2 that for a wide class of operators B the solutions of (7) satisfy

$$E(u(t_1), u_t(t_1)) \geqq E(u(t_2), u_t(t_2)) \qquad (0 \leqq t_1 \leqq t_2) \tag{8}$$

where E is the energy introduced in §1 for (5). As before, the solution of the initial value problem for (7) is obtained concomitantly. A curious example that comes under the scope of these results is the integro-differential equation (4.5).

In Section 5 we take up the generalization of (5) and (7) to higher order equations in time and show that one cannot associate a nonincreasing energy to these higher order equations.

1. The equation $u_{tt} = Au$. The operators A with which we shall deal can easily be described intrinsically as in the introduction, but we prefer to show first how they usually arise. Let $P(t)$ be a self adjoint contraction semi-group (abbreviated henceforth to c.s.g.) on a Hilbert space H; *i.e.*, for each $0 \leqq t < +\infty$, $P(t)$ is a self adjoint linear operator on H with norm less than or equal to one and satisfying

$$P(t)P(s) = P(t+s) \qquad (s, t \geqq 0). \tag{1.1}$$

Here and in the sequel all the semi-groups which we consider will be strongly continuous on $[0, +\infty)$ which means in view of (1.1) that for every $x \in H$

$$\lim_{t \downarrow 0} ||P(t)x - x|| = 0. \tag{1.2}$$

There is however no difficulty in extending the results to semi-groups in the weak topology as in [1]. The infinitesimal generator A of $P(t)$ is defined as

$$(1.3) \qquad \lim_{t \downarrow 0} \frac{P(t)x - x}{t} = Ax$$

whenever this limit exists in the norm of H. In general, A will not be bounded, but it is closed with dense domain denoted by dom (A). For these and other facts of the semi-group theory, see Hille and Phillips [5] which we shall use henceforth without explicit reference. Since $P(t)$ is self adjoint, A is clearly symmetric, but more is true; since

$$(1.4) \qquad (\lambda I - A)^{-1} = \int_0^\infty \exp(-\lambda t) P(t)\, dt \qquad (\lambda > 0)$$

we have that $(\lambda I - A)^{-1}$ is self adjoint and bounded and therefore $(\lambda I - A)$ is self adjoint which implies that A is self adjoint.

Now we will see that since $P(t)$ is a c.s.g., the generator A is necessarily nonpositive. Although for the moment we need the next lemma only for Hilbert spaces, it is given more generally for later use. It may be found in [9] and [10], but because the calculation is typical of some that we need later, a proof is included. The following notation is used throughout: if $\varphi \in B'$, the dual of a Banach space B, and $x \in B$ then $\varphi(x) = (\varphi, x)$.

Lemma 1. *Let $P(t)$ be a c.s.g. on a Banach space B with A its infinitesimal generator. If $x \in$ dom (A) and $\varphi_x \in B'$ satisfy*

$$(1.5) \qquad ||\varphi_x|| = 1 \quad \text{and} \quad (\varphi_x , x) = ||x||$$

then

$$(1.6) \qquad \text{Re}\,(\varphi_x , Ax) \leqq 0.$$

Proof. Let x be in dom (A) and φ_x satisfy (1.5), then

$$(1.7) \quad \text{Re}\,(\varphi_x , P(t)x) \leqq ||\varphi_x||\, ||P(t)||\, ||x|| \leqq ||x|| = (\varphi_x , x) = \text{Re}\,(\varphi_x , x).$$

Upon subtracting the extreme members of (1.7), dividing by t and letting t decrease to zero, we obtain (1.6).

Note that this implies that if $P(t)$ is a group of contractions, then

$$(1.8) \qquad \text{Re}\,(\varphi_x , Ax) = 0$$

which generalizes the fact that for a Hilbert space the generator A of a unitary transformation group is such that iA is self adjoint. Applying the lemma to the self adjoint c.s.g. above, we see that its generator is nonpositive and hence $-(x, Ax)$ defines a nonnegative definite bilinear form on dom (A). Since we shall want to use this form to define a second norm on dom (A) we must know when $(x, Ax) = 0$. Because A is also self adjoint, it has a spectral representation from which it follows that

$$(1.9) \qquad (x, Ax) = 0 \quad \text{iff} \quad Ax = 0 \quad \text{iff} \quad x \in N$$

where N is the null space of A. To avoid complicating the discussion, we will assume from now on that N reduces to $\{0\}$. In the contrary case, simply replace H by $H_0 = H \ominus N$, observe that H_0 is invariant under $P(t)$ and after everything is done, N can be added as a subspace of invariant elements for $P(t)$ and $U(t)$ (see below).

Set now a bilinear form on the domain of A by

$$(x, y)_A = -(x, Ay) \tag{1.10}$$

and observe that by the above discussion it defines a pre-Hilbert structure on dom (A).

Definition 1. The completion of dom (A) with respect to the inner product (1.10) is H_A and $H' = H_A \oplus H$.

The space H_A may be identified as a subspace of H in case A^{-1} is bounded, or in the terminology of potential theory, the semi-group $P(t)$ is integrable. The energy which will play the role of (3) is simply the norm on the space of pairs H'. Solutions of (5) will be obtained as the orbits of a group of unitary transformations $U(t)$ on H'. It is clear that this group should have as its generator an extension of A' which we now define.

Definition 2. On dom $(A) \times$ dom $(A) \subset H'$ let

$$A'\{x, y\} = \{y, Ax\}. \tag{1.11}$$

The group $U(t)$ will be obtained by appealing to the Hille-Yosida theorem; the argument is divided into several lemmas.

Lemma 2. *A' has a unique closed extension which is denoted again by A'.*

Proof. By the linearity of A' it suffices to show that

$$A'\{x_n, y_n\} \to \{x, y\}, \qquad \{x_n, y_n\} \to \{0, 0\} \tag{1.12}$$

implies $x = y = 0$, From the definition of A', (1.12) implies that $y_n \to x$ in the topology of H_A and $y_n \to 0$ in H. Thus for any $z \in$ dom (A)

$$(z, x)_A = \lim_n (z, y_n)_A = \lim_n -(z, Ay_n) = \lim_n -(Az, y_n) = 0, \tag{1.13}$$

and since dom (A) is dense in H_A we conclude that $x = 0$. Again $Ax_n \to y$ in H while $x_n \to 0$ in H_A, thus for any $z \in$ dom (A)

$$(z, y) = \lim_n (z, Ax_n) = \lim_n -(z, x_n)_A = 0, \tag{1.14}$$

and since dom (A) is dense in H, we conclude that $y = 0$.

Lemma 3. *For real $\lambda \neq 0$ the range of $(\lambda I + A)$ denoted by* ran $(\lambda I + A)$ *is dense in H'.*

Proof. Suppose that the range weren't dense, then there would exist $\{u, v\} \in H'$ such that for all $\{x, y\} \in$ dom (A')

$$(\lambda x + y, u)_A + (\lambda y + Ax, v) = 0. \tag{1.15}$$

Substitute $-\lambda x$ for y in (1.15) to obtain

$$((-\lambda^2 I + A)x, v) = 0. \tag{1.16}$$

But since A is the generator of a c.s.g. in H for all $\mu > 0$, ran $(\mu I - A)$ is H and thus $v = 0$. Returning to (1.15), it is immediate that $u = 0$.

Lemma 4. *$(f, A'f) = 0$ for all $f \varepsilon$ dom (A').*

Proof. It suffices to prove the lemma for $f = \{x, y\} \varepsilon$ dom $(A) \times$ dom (A). For such f we have

$$(f, A'f) = (x, y)_A + (y, Ax) = -(x, Ay) + (y, Ax) = 0 \tag{1.17}$$

by the self adjointness of A.

Theorem 1. *A' is the infinitesimal generator of a strongly continuous group of unitary transformations $U(t)$ on H'.*

Proof. A simple calculation based on Lemma 4 together with Lemma 3 shows that for all $\lambda \neq 0$ we have $(\lambda I + A')^{-1}$ everywhere defined and satisfying

$$\|(\lambda I + A')^{-1}\| = \frac{1}{|\lambda|}. \tag{1.18}$$

By the Hille-Yosida theorem, A' therefore generates a group of contractions on H'. If we now differentiate $(f, U(t)f)$ and make use of Lemma 4, we see that $U(t)$ is an isometry from which it follows easily that $U(t)$ is unitary.

This theorem shows that for solutions of (5) with a finite energy, that is for solutions such that $\{u, u_t\}$ lie in H', the energy which is simply the norm in H' is constant in time. It also yields the solution to the following problem: find u that satisfies (5) and which reduces to $u(0) = u_0$, $u_t(0) = v_0$ at time zero where u_0 and v_0 are prescribed initial data. We formulate the solution as:

Corollary 1. *If A is a self adjoint nonpositive operator on H, then the initial value problem for (5) has a unique solution for initial data $\{u_0, v_0\}$ in* dom (A') *given by*

$$U(t)\{u_0, v_0\} \tag{1.19}$$

where $U(t)$ is as in the theorem.

A simple integration by parts shows that (3) is identical with the square of the norm in $H_A \oplus H$ where $H = L^2(-\infty, +\infty)$ and $A = d^2/dx^2$, so that the energy introduced here is the classical one in case A is the Laplacian.

2. The dampening term. We turn now to the equation (7) and show that for a class of operators B, the norm of H', which is defined in terms of A as in Section 1, is nonincreasing along solutions of (7). The solutions of (7) will now be orbits of a c.s.g. on H' whose generator C' is an extension of C' which we proceed to define.

Definition 3. On dom $(A) \times (\text{dom}(A) \cap \text{dom}(B)) \subset H'$

$$C'\{x, y\} = \{y, Ax + By\}. \tag{2.1}$$

Just as in Lemma 2 one shows that if dom $(A) \cap$ dom (B) is dense in H then C' has a unique closed extension which we denote again by C'. In the proof of the next theorem, we need two results of Trotter [13] which we give as:

Theorem (Trotter). I. *If A and B generate c.s.g.'s on a Banach space F and* dom $(A) \cap$ dom (B) *is dense in F, then the closure of $A + B$ generates a c.s.g. on F if for some $\lambda > 0$ we have* ran $(\lambda I - A - B)$ *dense in F.*

II. *If A and B are the infinitesimal generators of c.s.g.'s on F and if* dom $(A) \subset$ dom (B) *then the set of real numbers c such that $A + cB$ generates a c.s.g. is an open subset of $[0, +\infty)$ that contains a neighborhood of zero.*

Theorem 2. *Assume that*

(i) *A and B generate c.s.g.'s on a Hilbert space H,*

(ii) *A is self adjoint,*

(iii) *either* dom $(A) \subset$ dom (B) *or* dom $(A) \supset$ dom (B),

then C' generates a c.s.g. on $H' = H_A \oplus H$.

Proof. From (i) and (iii) we can apply Trotter's theorem II to conclude that there exists some $c > 0$ such that $A + cB$ is the generator of a c.s.g. on H. In particular

$$\text{ran}\,(\lambda I - A - cB) \quad \text{is dense in } H \quad (\lambda > 0). \tag{2.2}$$

Let A' be as in Definition 2 and define B' on $H_A \times$ dom (B) by

$$B'\{u, v\} = \{0, Bv\}. \tag{2.3}$$

That A' generates a c.s.g. on H' follows from Theorem 1, while it is clear that (i) implies that B' also generates a c.s.g. on H'. Since

$$C' = A' + B' \tag{2.4}$$

and since clearly dom $(A') \cap$ dom (B') is dense in H' if for some $\lambda > 0$ we would have ran $(\lambda I - A' - B')$ dense in H', Trotter's theorem I would enable us to conclude the proof.

Finally we show that ran $(cI - A' - B')$ is dense in H'. For suppose it weren't. Then there would exist $\{u, v\} \in H'$ such that for all $\{x, y\} \in$ dom (C')

$$(u, cx - y)_A + (v, cy - Ax - By) = 0. \tag{2.5}$$

Setting $cx = y$ in (2.5) we see that

$$(v, (c^2 - A - cB)x) = 0. \tag{2.6}$$

But (2.2) for $\lambda = c^2$ implies now that $v = 0$. Substituting this back in (2.5) we obtain easily that $u = 0$. As we have remarked, this completes the proof.

In general, the semi-group of this theorem cannot be extended to a group, or what is the same the energy will actually decrease as can be expected on physical grounds. Frictional or drag forces invariably carry with them some energy losses. If however the operator B generates a group in H then the reasoning above will imply that so does C'. A simple example of this latter situation is

$$u_{tt} = \frac{d^2u}{dx^2} - \frac{du_t}{dx} \tag{2.7}$$

where the space H is $L^2(-\infty, +\infty)$. The condition (iii) of the theorem is readily verified for this example. We give two simple examples of operators B that satisfy the hypothesis.

1. $B = A$. This leads to examples like

$$u_{tt} = \Delta u + \Delta u_t , \tag{2.8}$$

which can be integrated by means of Theorem 2.

2. $B = -cI$ $(c > 0)$. This example includes the usual dampening term which is simply proportional to the velocity and includes equations such as

$$u_{tt}(x) = \Delta u(x) - c(x)u_t(x) \tag{2.9}$$

where $c(x)$ is any bounded nonnegative measurable function on R^n.

The translation of this theorem into an existence theorem for the initial value problem for (7) proceeds just as in Section 1 and yields:

Corollary 2. If A and B are as in Theorem 2, then the initial value problem for (7) *has a unique solution for initial data* $\{u_0 , v_0\}$ *in* dom (C') *given by*

$$Q(t)\{u_0 , v_0\} \tag{2.10}$$

where $Q(t)$ is the c.s.g. whose existence is asserted in Theorem 2.

3. The hypothesis of symmetry. In our treatment of (5) if we drop the assumption that A is symmetric, not only needn't an energy exist for the solutions of (5) but the initial value problem for it may not be well posed. This in spite of the fact that A is nonpositive or generates a c.s.g. Consider for example

$$A = \frac{d}{dx} \tag{3.1}$$

acting on functions defined on the real line. Although d/dx generates a c.s.g. on all L^p spaces, the initial value problem that we consider for

$$\frac{d^2u}{dt^2} = \frac{du}{dx} \tag{3.2}$$

violates the basic hypothesis of the Cauchy-Kowalesky theorem and is not well posed (see for example Picard [11] Lecon I).

From the point of view of the spectral theory, the operator (3.1) is as far away as possible from being symmetric since it has a pure imaginary spectrum. It might therefore be expected that for some operators between these extremes the corresponding wave equation (5) possesses an energy analogous to the one used in §1, but this expectation is not borne out. Instead we have:

Theorem 3. *Let A be a linear operator on a Hilbert space H, and let H_1 be a second Hilbert space in which* dom (A) *is imbedded. Then if there is a c.s.g. $P(t)$ on*

$$H' = H_1 \oplus H \tag{3.3}$$

which solves the initial value problem for

$$u_{tt} = Au \tag{3.4}$$

with initial position and velocity in H', then A is a nonpositive self adjoint operator.

Proof. The infinitesimal generator of $P(t)$ will be an extension of A' which acts on dom $(A) \times$ dom (A) by

$$A'\{u, v\} = \{v, Au\}. \tag{3.5}$$

Letting $(\ ,\ )_1$ denote the inner product in H_1 we obtain upon applying Lemma 1

$$\mathrm{Re}\ ((u, v)_1 + (v, Au)) \leqq 0. \tag{3.6}$$

Replacing v by $-v$ in (3.6) and combining inequalities yields

$$\mathrm{Re}\ ((u, v)_1 + (v, Au)) = 0. \tag{3.7}$$

Interchanging u and v in (3.7) and using the conjugate symmetry of the inner product, we get

$$\mathrm{Re}\ (v, Au) = \mathrm{Re}\ (Av, u), \tag{3.8}$$

from which it follows that A is a symmetric operator. That it is self adjoint follows easily from the fact that A' generates a c.s.g. on H'. Putting $u = v$ in (3.7) yields

$$(u, Au) = -(u, u)_1 \leqq 0 \tag{3.9}$$

so that A is indeed nonpositive.

The assumption made here that the energy is the norm on a direct sum of two Hilbert spaces can be relaxed without changing the conclusion that A is self adjoint. To wit, it suffices to assume that the energy is the norm in a Banach space of pairs $\{x, y\}$, where the norm of $\{x, y\}$ depends only on the norm of $\{x, 0\}$ and the norm of $\{0, y\}$. This means in effect that we have two Banach spaces B_1 and B_2 with

$$B = B_1 \times B_2 \tag{3.10}$$

and

$$\|\{x, y\}\| = k(\|x\|, \|y\|) \tag{3.11}$$

where k is a positive homogeneous convex function on R^2. For the proof of this strengthened version of Theorem 3, we need some preliminaries.

Definition 4. For a Banach space B and its dual B' the mapping $\mathfrak{J} : B \to B'$ is given by $x \to \tilde{x}$ where

$$\text{(i)} \qquad ||\tilde{x}|| = 1,$$

$$\text{(ii)} \qquad (\tilde{x}, x) = ||x||.$$

The existence of such a mapping is guaranteed by the Hahn-Banach theorem. The next lemma makes precise the statement that if the dual of a Banach space is the same as the space itself, the Banach space is a Hilbert space. We denote the complex numbers by C.

Lemma 5. *A Banach space B is a Hilbert space if the mapping $u \to ||u|| \, \tilde{u}$ is conjugate linear, i.e.,*

$$||ax + by|| \, (ax + by)^{\sim} = \bar{a} \, ||x|| \, \tilde{x} + \bar{b} \, ||y|| \, \tilde{y} \tag{3.12}$$

for all $x, y \in B$ and $a, b \in C$.

Proof. Define a mapping from $B \times B$ to C by

$$\langle x, y \rangle = (||y|| \, \tilde{y}, x). \tag{3.13}$$

The hypotheses imply that this map is linear in x and conjugate linear in y. Furthermore, we have that

$$\langle x, x \rangle = ||x||^2. \tag{3.14}$$

Applying (3.14) to $x + y$ and making use of the linearity we get that

$$\text{Im} \, (\langle x, y \rangle + \langle y, x \rangle) = 0. \tag{3.15}$$

Putting ix for x in (3.15) and combining, we conclude that $\langle \, , \, \rangle$ defines an inner product since only the conjugate symmetry remained unproved.

Any positive homogeneous convex function k as in (3.11) gives R^2 the structure of a Banach space by setting

$$||(x, y)|| = k(x, y). \tag{3.16}$$

This space is written R_k^2. Let f_1 and f_2 form a dual basis to the basis

$$\{(1, 0), (0, 1)\} = \{e_1, e_2\},$$

that is to say

$$(f_i, e_j) = \delta_{ij}. \tag{3.17}$$

Definition 5. The *dual* of k is the function defined by

$$k'(x, y) = ||xf_1 + yf_2|| \tag{3.18}$$

where the norm in (3.18) is the norm of the dual space to R_k^2.

Note that R_k^2 and $R_{k'}^2$ are dual spaces. If now the mapping $\mathfrak{J}$ is written out explicitly in terms of the dual basis we get

$$(x, y)^{\sim} = af_1 + bf_2 \tag{3.19}$$

where a and b are functions of x and y. Making use of the defining properties of $\mathfrak{J}$ we prove:

Lemma 6. *Any positive homogeneous convex function k on R^2 can be written in the form*

$$k(x, y) = a(x, y)x + b(x, y)y, \tag{3.20}$$

and if k' is the dual function of k

$$k'(a(x, y), b(x, y)) = 1. \tag{3.21}$$

For the usual L^p norm, the functions a and b can be written explicitly

$$a(x, y) = \frac{|x|^{p-1} \operatorname{sig}(x)}{(|x|^p + |y|^p)^{1/p'}}, \qquad b(x, y) = a(y, x) \tag{3.22}$$

where as usual $1/p + 1/p' = 1$.

Theorem 4. *Let A be a densely defined operator on a Banach space B_1, and B_2 a second Banach space in which* dom (A) *is imbedded; and let B be given by* (3.10) *and* (3.11). *Then if there is a c.s.g. on B which integrates the initial value problem for* (3.4), *the B_i are Hilbert spaces and A and is nonpositive and self adjoint.*

Proof. It is clear that B' may be identified with $B_1' \times B_2'$ together with the norm

$$||\{r, s\}|| = k'(||r||, ||s||) \tag{3.23}$$

where k' is dual to k. From Lemma 6 we have therefore

$$\{u, v\}^{\sim} = \{a(||u||, ||v||)\tilde{u}, b(||u||, ||v||)\tilde{v}\}. \tag{3.24}$$

Just as before, an application of Lemma 1 yields

$$\operatorname{Re}\, (a(||u||, ||v||)(\tilde{u}, v)_1 + b(||u||, ||v||)(\tilde{v}, Au)_2) \leqq 0 \tag{3.25}$$

where the subscripts indicate in what space the linear functionals $\tilde{u}$ and $\tilde{v}$ are. Replacing u by $-u$ in (3.25) yields

$$\operatorname{Re}\, (a(||u||, ||v||)(\tilde{u}, v)_1 + b(||u||, ||v||)(\tilde{v}, Au)_2) = 0. \tag{3.26}$$

Now replace v by $(||u||/||v||)v$ and observing that

$$a(x, x) = b(x, x) \tag{3.27}$$

we get finally from (3.26)

$$\operatorname{Re}\, ((||u||\, \tilde{u}, v)_1 + (||v||\, \tilde{v}, Au)_2) = 0. \tag{3.28}$$

Because A is a linear operator (3.28) implies that the hypotheses of Lemma 5 are satisfied and we conclude B_1 is a Hilbert space. One shows in a similar way that B_2 is a Hilbert space and then Theorem 3 completes the proof.

This last theorem indicates that there is a strong intrinsic connection between second order systems possessing an energy and symmetric operators in Hilbert space, and thus may provide further justification for the postulates that underlie quantum mechanics. Somewhat stronger results than our Theorem 4 were obtained by W. Littman [8] for the classical wave equations and the L_p spaces $p \neq 2$, by explicit constructions of solutions whose L_p norms grow in arbitrary way.

4. Vibrating systems associated with Markov processes. In [4] Feller derived the following equation

$$\frac{d^2u}{dt^2} = D_m D_x u \tag{4.1}$$

as the equation governing the motion of a vibrating string; here $D_m D_x$ is the general form of the infinitesimal generator of a one-dimensional diffusion. This suggests asking in general what generators, Ω, of Markov processes are such that

$$\frac{d^2u}{dt^2} = \Omega u \tag{4.2}$$

governs the motion of a vibrating system which enjoys some of the same properties as a vibrating string. If, as in the introduction, we take as the characteristic property the existence of an energy, we may, in view of Sections 1 and 3, phrase the question as which generators Ω satisfy the hypothesis of Theorem 1 and it is to this that we now turn.

A Markov process with state space X gives rise to a contraction semi-group P^t on the bounded measurable functions defined on X, which is denoted by $L^\infty(X)$. Initially, its infinitesimal generator Ω is defined on a certain dense subspace of $L^\infty(X)$ for which we write dom (Ω). In an adjoint fashion, the process gives rise to a semi-group on the probability measures defined on X. We denote this semi-group again by P^t. A measure $d\mu$ is said to be subinvariant with respect to P^t if $d\mu P^t \leqq d\mu$.

For a subinvariant measure $d\mu$, it is easily verified that P^t is a contraction semi-group on $L^1(X, d\mu)$, and from the Riesz-Thorin convexity theorem, it follows that P^t is a c.s.g. on $L^2(X, d\mu)$. Thus Ω may be defined densely on $L^2(X, d\mu)$. Clearly Ω will be self adjoint if and only if P^t, as an operator on $L^2(X, d\mu)$, is self adjoint. Probabilistically, this means that the reversed process with respect to $d\mu$ is the same as the original process. In this case a simple calculation shows that $d\mu$ must in fact be an invariant measure.

In case the state space is a finite set, a simple necessary and sufficient condition for the existence of such an invariant measure $d\mu$, in terms of Ω, was given by Kolmogorov [7]. This was extended to a countable, discrete space by Kendall [6] and may be formulated as follows. Write

$$\Omega = g(-I + P) \tag{4.3}$$

where g is a diagonal matrix with positive entries and P is a substochastic matrix. Then there exists a measure $d\mu$, with Ω symmetric on $L^2(d\mu)$, if and only if

$$p(i_0, i_1)p(i_1, i_2) \cdots p(i_{k-1}, i_k) = p(i_k, i_{k-1}) \cdots p(i_2, i_1)p(i_1, i_0) \tag{4.4}$$

for any set of points $i_0, i_1, \cdots i_k = i_0$, where naturally $p(i, j)$ is the (i, j) entry of P. In the terminology of [3] this may be expressed by saying that for any round trip on the road system the probability of going around in either direction is the same.

A simple example where (4.4) is satisfied is provided by the birth and death processes where the matrix P is such that $p(i, j)$ vanishes unless $j = i \pm 1$. Hence in this case there is always an invariant measure with respect to which the process is self adjoint. Since the one-dimensional diffusions are limits of birth and death processes (see for example [12]), we see why the one-dimensional diffusions always are self adjoint with respect to some measure. The discrete processes which approximate higher dimensional diffusions needn't satisfy (4.4) and thus one shouldn't expect to find the higher dimensional diffusions always giving rise to vibrating systems.

In the general state space there is no simple characterization such as that given by (4.4) for the self adjointness. We content ourselves with discussing one example. Let the state space X be a locally compact Abelian group, and suppose the Markov process commutes with translations of the group. Then the Haar measure dm is an invariant measure, the semi-group $P(t)$ may be represented by convolution with measures $dp(t)$. The process is now self adjoint with respect to the Haar measure if and only if the measures $dp(t)$ are symmetric. The comparability of the domains of the infinitesimal generators of two such convolution semi-groups becomes a simple condition on the behavior at infinity of the Fourier transforms of the respective generators. The operators A and B of Section 2 are certain constant coefficient differential operators or singular integral operators. An example of an equation for which Section 2 gives existence theorems for is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x \, \partial t} \cdot \frac{dy}{(y - x)} \tag{4.5}$$

which is obtained from the generators of Brownian motion and the Cauchy process.

5. Higher order equations. Classically the equations of mechanics involve only first and second order time derivatives, corresponding to the physically meaningful quantities of velocity and acceleration. A natural question to ask is whether or not one can set up a theory of mechanics involving time derivatives of order greater than two. While the question is in general not very well defined, in the context of the vibrating systems and their associated energies that we have been studying, a limited answer can be given.

Definition 6. By the *standard* initial value problem for

$$\frac{d^n u}{dt^n} = A_1 \frac{d^{n-1}u}{dt^{n-1}} + \cdots + A_n u \tag{5.1}$$

is meant finding a function $u(t)$ which satisfies (5.1) and for which we have at time $t = 0$

$$\left\{u, \frac{du}{dt}, \cdots \frac{d^{n-1}u}{dt^{n-1}}\right\} = \{u_0, u_1, \cdots u_{n-1}\} \tag{5.2}$$

prescribed initial data.

Our limited answer will be that the existence of a nonincreasing energy for the system governed by (5.1), or equivalently the existence of a c.s.g. that solves the standard initial value problem for (5.1), is only compatible with $n \leqq 2$. In order to see this clearly, a special case is considered first.

Theorem 5. *If the standard initial value problem for* (5.1) *is solved for initial data in a dense subspace of*

$$H = H_0 \oplus H_1 \oplus \cdots \oplus H_{n-1} \tag{5.3}$$

(where the H_i are Hilbert spaces) by a c.s.g. $P(t)$ on H then $n \leqq 2$.

Proof. The hypotheses imply that the infinitesimal generator of $P(t)$ is an extension of A' which is defined on

$$\text{dom}\,(A') = \text{dom}\,(A_n) \times \text{dom}\,(A_{n-1}) \times \cdots \times \text{dom}\,(A_1) \subset H \tag{5.4}$$

by the equation

$$A'\{u_0, u_1, \cdots u_{n-1}\} = \left\{u_1, u_2, \cdots u_{n-1}, \sum_{i=1}^{n} A_i u_{n-i}\right\}. \tag{5.5}$$

There are natural imbeddings of dom (A_i) in H_{n-i-1} which we use freely. Letting $(\ ,\ )_i$ denote the inner product in H_i and applying Lemma 1 we obtain

$$\text{Re}\left((u_0, u_1)_0 + \cdots + (u_{n-2}, u_{n-1})_{n-2} + \left(u_{n-1}, \sum_{i=1}^{n} A_i u_{n-i}\right)_{n-1}\right) \leqq 0. \tag{5.6}$$

Now let

$$u_{n-1} = 0 \tag{5.7}$$

and if $n \geqq 3$ we get from (5.6) the nontrivial

$$\text{Re}\,((u_0, u_1)_0 + \cdots + (u_{n-3}, u_{n-2})_{n-3}) = 0. \tag{5.8}$$

From (5.8) it follows that dom (A_n) must reduce to the zero element which means that A_n is trivial and thus the order of (5.1) is not honestly equal to n, a contradiction.

The restriction to Hilbert spaces is not essential and can be removed. Namely, if B_i are Banach spaces, form in analogy with (3.10) and (3.11)

$$B = B_0 \times B_1 \times \cdots \times B_{n-1} \tag{5.9}$$

and

$$||\{u_0, u_1, \cdots u_{n-1}\}|| = k(||u_0||, ||u_1||, \cdots ||u_{n-1}||) \tag{5.10}$$

where k is a positive homogeneous convex function on R^n. An analogue of Lemma 6 is easily proven for such functions on R^n and their duals and then the reasoning employed above can be applied *mutatis mutandis* to prove the following strengthening of Theorem 5:

Theorem 6. *If the standard initial value problem for* (5.1) *is integrated for initial data in a dense subspace of B by a c.s.g. on B then* $n \leqq 2$.

Both of these theorems can be immediately extended to the case where

$$||P(t)|| = \exp(ct) \tag{5.11}$$

rather than being c.s.g.'s on H or B. As Feller has shown [2], any bounded semi-group is a c.s.g. in an equivalent norm, and so in principle we need assume nothing about $P(t)$ except for its measurability. However, since our methods depend upon the norm in B being of the form (5.10), and since the equivalent norm may not be of this form, the restricted formulation of Theorem 6 is all that we can prove at the present. A somewhat related nonexistence theorem is given in [5]. The results there are obtained by considerations of the resolvent (see [5], p. 264, Theorem 23.9.3 and the remarks preceding).

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