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Abstractions of Hamiltonian control systems

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Abstract

Given a control system and a desired property, an abstracted system is a reduced system that preserves the property of interest while ignoring modeling detail. In previous work, abstractions of linear and nonlinear control systems were considered while preserving reachability properties. In this paper, we consider the abstraction problem for Hamiltonian control systems, where, in addition to the property of interest we also preserve the Hamiltonian structure of the control system. We show how the Hamiltonian structure of control systems can be exploited to simplify the abstraction process. We then focus on local accessibility preserving abstractions, and provide conditions under which local accessibility properties of the abstracted Hamiltonian system are equivalent to the local accessibility properties of the original Hamiltonian control system.

Keywords

Comments

Abstractions of Hamiltonian Control Systems [*]

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Abstract

Given a control system and a desired property, an abstracted system is a reduced system that preserves the property of interest while ignoring modeling detail. In previous work, abstractions of linear and nonlinear control systems were considered while preserving reachability properties. In this paper, we consider the abstraction problem for Hamiltonian control systems where in addition to the property of interest we also preserve the Hamiltonian structure of the control system. We show how the Hamiltonian structure of control systems can be exploited to simplify the abstraction process. We then focus on local accessibility preserving abstractions, and provide conditions under which local accessibility properties of the abstracted Hamiltonian system are equivalent to the local accessibility properties of the original Hamiltonian control system.

Key words: Nonlinear control systems, Hamiltonian control systems, abstractions.

1 Introduction

Property preserving abstractions of control systems are important for reducing the complexity of their analysis or design. From an analysis perspective, given a large-scale control system and a property to be verified, one extracts a smaller *abstracted* system with equivalent properties. Checking the property on the abstraction is then equivalent to checking the property on the original system. From a design perspective, rather than designing a controller for the original large scale system, one designs a controller for the smaller abstracted system, and then refines the design to the original system while incorporating modeling detail.

This approach critically depends on whether we are able to *construct* hierarchies of abstractions as well as characterize conditions under which various properties of interest propagate from the original to the abstracted system and vice versa. In [19], hierarchical abstractions of linear control systems were extracted using computationally efficient constructions, and conditions under which controllability of the abstracted system implied controllability of the original system were obtained. This led to extremely efficient hierarchical controllability algorithms. In the same spirit, abstractions of nonlinear control affine systems were considered in [20], and the canonical construction for linear systems was generalized to nonlinear control affine systems. In [22] existence of abstractions (regarded as quotients in a category) was shown for fully nonlinear control systems. Furthermore, a characterization of the relation between the state/input space of the original system with the state/input space of its abstraction was also presented.

In this paper, we proceed in the spirit of [20], and consider abstractions of Hamiltonian control systems, which are control systems completely specified by controlled Hamiltonians. This additional structure allows to simplify the abstraction process by working with functions instead of vector fields or distributions as is the case for general nonlinear systems [20]. This is possible since the controlled Hamiltonian contains all the relevant information that must be captured by the abstracted system. On the other hand, to be able to relate the dynamics induced by the controlled Hamiltonians, we need to restrict the class of abstracting maps to those that preserve the Hamiltonian structure. More precisely, given a Hamiltonian control system on a Poisson manifold M, and a

(quotient) Poisson map $\phi : M \to N$, we present a canonical construction that extracts an abstracted Hamiltonian control system on N. We then characterize abstracting maps for which the original and abstracted system are equivalent from a local accessibility point of view.

Reduction of mechanical dynamical systems is a very rich and mature area and we point the reader to the standard references [14,15]. Similar ideas were applied to mechanical control systems [8,12,21,23] including also nonholonomic constraints [3,11]. The approach presented in this paper is quite different from these established notions of reduction for mechanical systems. When performing an abstraction one is interested in ignoring irrelevant modeling details. In this spirit, one quotients the original model by group actions that do *not* necessarily represent symmetries. Furthermore, the proposed abstraction methodology does not rely on the existence of a Lie group acting on the state space. This extra freedom when abstracting is balanced by the fact that information about the system is lost, whereas when reducing using symmetries no essential information is lost.

The structure of this paper is as follows : In Section 2 we review Poisson geometry and controlled Hamiltonian systems. Sections 3 and 4 contain the main contributions of the paper: in Section 3 a notion of abstraction is introduced for Hamiltonian control systems and an algorithm for its construction is given; while in Section 4 we present conditions for local accessibility equivalence between the original and the abstracted system. Section 5 illustrates the paper results on a spherical pendulum example and at Section 6 interesting topics for further research are discussed.

2 Hamiltonian Control Systems on Poisson Manifolds

In this section we review some basic facts from Poisson geometry as well as Hamiltonian control systems, in order to establish consistent notation. The reader may whish to consult numerous books on these subjects such as [7,16,17].

2.1 Poisson Geometry

Hamiltonian control systems can be defined on either symplectic or Poisson manifolds, however for the purposes of this paper, it will be more natural to work within the Poisson context. A Poisson structure on a smooth manifold M is a bilinear map $\{-, -\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ called *Poisson bracket*, satisfying the following identities:

$$\{f, g\} = -\{g, f\} \tag{2.1}$$
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{2.2}$$
$$\{fg, h\} = \{f, h\}g + f\{g, h\} \tag{2.3}$$

A Poisson manifold is a smooth manifold M equipped with a Poisson structure. Given a smooth function $h : M \to \mathbb{R}$, the Poisson bracket allows us to obtain a Hamiltonian vector field X_h with Hamiltonian h through the equality:

$$\mathcal{L}_{X_h} f = \{f, h\} \qquad \forall f \in C^\infty(M) \tag{2.4}$$

where $\mathcal{L}_{X_h} f$ is the Lie derivative of f along X_h. Note that the vector field X_h is well defined since the Poisson bracket verifies the Leibnitz rule and therefore defines a derivation on $C^\infty(M)$ ([16]). Furthermore $C^\infty(M)$ equipped with a Poisson bracket is a Lie algebra, called a Poisson algebra.

As an example of a Poisson bracket consider $M = \mathbb{R}^4$ with coordinates (q_1, q_2, p_1, p_2). We can define a Poisson bracket through the expression:

$$\{f, g\} = \frac{\partial f}{\partial q_1} \frac{\partial g}{\partial p_1} + \frac{\partial f}{\partial q_2} \frac{\partial g}{\partial p_2} - \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q_1} - \frac{\partial f}{\partial p_2} \frac{\partial g}{\partial q_2} \tag{2.5}$$

Given now a smooth Hamiltonian $h : M \to \mathbb{R}$, we use (2.5) to define the corresponding Hamiltonian vector field X_h through relation (2.4):

$$\dot{q}_1 = \mathcal{L}_{X_h} q_1 = \{q_1, h\} = \frac{\partial h}{\partial p_1}$$
$$\dot{q}_2 = \mathcal{L}_{X_h} q_2 = \{q_2, h\} = \frac{\partial h}{\partial p_2}$$
$$\dot{p}_1 = \mathcal{L}_{X_h} p_1 = \{p_1, h\} = -\frac{\partial h}{\partial q_1}$$
$$\dot{p}_2 = \mathcal{L}_{X_h} p_2 = \{p_2, h\} = -\frac{\partial h}{\partial q_2}$$

We thus see that a Poisson bracket provides a geometric, coordinate free way of describing the Hamiltonian dynamics.

We also introduce some additional notation used throughout the paper. Given a smooth map $f : M \to N$ we denote by $Tf : TM \to TN$ the tangent map of f, taking tangent vectors $X(x) \in T_x M$ at $x \in M$ to tangent vectors $T_x f \cdot X(x) \in T_{f(x)} N$ at $f(x) \in N$. Given a smooth map $g : N \to \mathbb{R}$ we also define de pullback of dg by f, denoted by $f^* dg$:

$$(f^* dg)(x)(X) = dg(f(x))(T_x f \cdot X) \qquad \forall X \in TM$$

and in the case of a covariant two-tensor $\omega : TN \times TN \to \mathbb{R}$, $f^* \omega$ is defined by:

$$(f^* \omega)(x)(X, Y) = \omega(f(x))(T_x f \cdot X, T_x f \cdot Y) \qquad \forall X, Y \in TM$$

When f is in fact a diffeomorphism, that is, a smooth map with a smooth inverse, we also define the pullback

2

of a vector field $Y \in TN$ to be:

$$(f^*Y)(x) = T_{f(x)}(f^{-1}) \cdot Y \circ f(x)$$

Associated with the Poisson bracket there is a contravariant anti-symmetric two-tensor:

$$B : T^*M \times T^*M \to \mathbb{R} \qquad (2.6)$$

such that:

$$B(x)(\mathrm{d}f, \mathrm{d}g) = \{f, g\}(x) \qquad (2.7)$$

where we have denoted by $\mathrm{d}f$ the exterior derivative of f. We say that the Poisson structure is nondegenerate if the map $B^\# : T^*M \to TM$ defined by:

$$\mathrm{d}g(B^\#(x)(\mathrm{d}f)) = B(x)(\mathrm{d}f, \mathrm{d}g)$$

is an isomorphism for every $x \in M$. When the Poisson bracket is nondegenerate it also induces a covariant anti-symmetric two-tensor [1] $\omega : TM \times TM \to \mathbb{R}$ defined by:

$$\omega(x)(X_f, X_g) = \{f, g\}(x) \qquad (2.8)$$

Given a map $\phi : M \to N$ between Poisson manifolds, we say that ϕ preserves the Poisson structure or that ϕ is a Poisson map if:

$$\{f \circ \phi, g \circ \phi\}_M = \{f, g\}_N \circ \phi \qquad (2.9)$$

for every $f, g \in C^\infty(N)$. Dually, we call a map $\phi : M \to N$ between Poisson manifolds, symplectic if:

$$\omega_M = \phi^* \omega_N \qquad (2.10)$$

2.2 Hamiltonian Control Systems

In this paper we will focus on affine Hamiltonian control systems that we now define:

Definition 2.1 *An affine Hamiltonian control system $\Sigma = (U, M, H)$ consists of a smooth manifold U (the input space), a Poisson manifold M with nondegenerate Poisson bracket (the state space), and a smooth function $H : M \times U \to \mathbb{R}$ (the controlled Hamiltonian). Furthermore, H is locally of the form $H = h_0 + \sum_{i=1}^{n} h_i u_i$, with h_i locally defined smooth real valued maps and $(u_1, u_2, \ldots, u_n)$ local coordinates for U.*

Using the controlled Hamiltonian and the Poisson structure on M we can recover the familiar system map $F : M \times U \to TM$ which is locally given by:

$$F = X_{h_0} + \sum_{i=1}^{n} X_{h_i} u_i$$

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[1] Such tensor is in fact a symplectic form.

and defines an affine distribution on M given by:

$$\mathcal{D}_M(x) = X_{h_0}(x) + \mathrm{span}\{X_{h_1}(x), X_{h_2}(x), \ldots, X_{h_n}(x)\}$$

This distribution captures all the possible directions of motion available at a certain point x, and therefore describes a control system, up to a parameterization by control inputs. This affine distribution will be the object of interest throughout the remaining paper, and we will assume that the rank of $\mathcal{D}_M$ does not change with x. Furthermore, we will denote an affine distribution $\mathcal{D}_M$ by $X + \Delta$, where X is a vector field and Δ a distribution. When this affine distribution is defined by a Hamiltonian control system we have $X = X_{h_0}$ and $\Delta = \mathrm{span}\{X_{h_1}, X_{h_2}, \ldots, X_{h_n}\}$. A similar reasoning is possible at the level of Hamiltonians. Locally, we can define the following affine space of smooth maps:

$$\mathcal{H}_M = h_0 + \mathrm{span}_{\mathbb{R}}\{h_1, h_1, \ldots, h_n\}$$

which defines $\mathcal{D}_M$ by the equality:

$$\mathcal{D}_M = B^\#(\mathrm{d}\mathcal{H}_M)$$

Here we used the notation $\mathrm{d}\mathcal{H}_M$ to denote the set $\cup_{h \in \mathcal{H}_M} \mathrm{d}h$. We will also use the notation $\mathcal{H}_M = h_0 + H_\Delta$ for an affine space of smooth maps where h_0 is a smooth map and H_Δ a linear space of smooth maps.

Having defined Hamiltonian control systems we turn to their trajectories or solutions:

Definition 2.2 *A smooth curve $c : I \to M$, $I \subseteq \mathbb{R}_0^+$ is called a* trajectory *of control system $\Sigma = (U, M, H)$, if there exists a curve $c^U : I \to U$ satisfying:*

$$\frac{d}{dt}c(t) = F(c(t), c^U(t)) \qquad (2.11)$$

for every $t \in I$.

3 Abstractions of Hamiltonian Control Systems

Given a Hamiltonian control system Σ_M on a manifold M and a map $\phi : M \to N$, our goal is to determine a new Hamiltonian control system Σ_N on the manifold N having as trajectories $\phi(c_M)$, where c_M are trajectories of Σ_M. This new control system Σ_N can be regarded as an abstraction of Σ_M if the manifold N is lower dimensional then M. In this case we are reducing the dimension of the control system, although capturing the relevant properties of the trajectories of Σ_M through the map ϕ. The choice of the map ϕ is problem dependent and reflects the knowledge one has about the system being analyzed. The following definition captures precisely this idea:

Definition 3.1 *Let Σ_M and Σ_N be two Hamilltonian control systems on manifolds M and N, respectively and $\phi : M \to N$ a Poisson map. Control system Σ_N is a ϕ-abstraction of control system Σ_M iff for every trajectory c_M of Σ_M, $\phi(c_M)$ is a trajectory of Σ_N.*

From the above definition it is clear that an abstraction captures all the trajectories of the original system. It is this relation between trajectories that will allow us to relate properties of control systems with properties of its abstractions. Clearly, it is difficult to determine whether a control system is an abstraction of another at the level of trajectories. One is then interested in a characterization of abstractions which is equivalent to Definition 3.1 but algebraic. Such a description is given in the next result adapted from [19]:

Theorem 3.2 *Let Σ_M and Σ_N be two Hamilltonian control systems defined on manifolds M and N, respectively and $\phi : M \to N$ a Poisson map. Control system Σ_N is a ϕ-abstraction of Σ_M if and only if Σ_N is ϕ-related to Σ_M, that is for every $x \in M$:*

$$T_x\phi\left(\mathcal{D}_M(x)\right) \subseteq \mathcal{D}_N \circ \phi(x) \qquad (3.1)$$

Making use of the above result we see that the notion of ϕ-related control system is a generalization of the notion of ϕ-related vector fields commonly found in differential geometry [1]. This justifies the name of ϕ-related control systems.

3.1 Computing ϕ-related control systems

Given a Poisson map, Definition 3.1 provides us with a geometric definition for Hamiltonian abstractions which is useful conceptually but not computationally. We now present a canonical construction that will allow us to construct an abstraction Σ_N from an Hamiltonian control system Σ_M and a Poisson map $\phi : M \to N$. From this point on, we will work locally and assume that the map ϕ is a surjective, Poisson submersion with connected fibers, that is, the submanifolds $\phi^{-1}(y) \subseteq M$ are connected for every $y \in N$.

The construction to be presented is similar, in spirit, to the canonical construction in [20], even though we work with Hamiltonians as opposed to distributions. This is natural for Hamiltonian systems since the differentials of the Hamiltonians capture all system information, that is, every X_h in a basis of $\mathcal{D}_M$ is uniquely determined from h, given the Poisson structure.

Given a distribution $\mathcal{D} \subseteq TM$ we say that $\mathcal{D}$ is invariant under a vector field $Z \in TM$ when $\gamma_t^*(\mathcal{D}) \subseteq \mathcal{D}$ for every $t \in \mathbb{R}$ such that the flow $\gamma_t : M \to M$ of Z is defined. This concept naturally extends to invariance under a distribution Δ by requiring $\mathcal{D}$ to be invariant under every

vector field belonging in Δ. Invariant affine distributions can alternatively be characterized as follows:

Proposition 3.3 ([20]) *An affine distribution $\mathcal{D} = X + \Delta$ on M is invariant under a vector field Z iff for every $Y \in \mathcal{D}$:*

$$[Z, Y] \subseteq \Delta \qquad (3.2)$$

When the affine distribution of interest $\mathcal{D}_M$ is invariant under the distribution $\mathrm{Ker}(T\phi)$, it is shown in [20] that $\mathcal{D}_M$ satisfies the equality:

$$T_x\phi(\mathcal{D}_M(x)) = T_{x'}\phi(\mathcal{D}_M(x')) \qquad (3.3)$$

for *every $x' \in \phi^{-1} \circ \phi(x)$*. This provides a simple method to compute a ϕ-related control system defined by an affine distribution $\mathcal{D}_N$ on N. To obtain $\mathcal{D}_N$ at a particular $y = \phi(x)$, it suffices to compute $T_x\phi(\mathcal{D}_M(x))$ for *some $x \in \phi^{-1}(y)$* in virtue of (3.3). This is, in fact, the rational behind the construction given in [20]. We now exploit the same idea, although we will work at the level of Hamiltonians. We start with an affine distribution defined by an affine space of Hamiltonians $\mathcal{H}_M = h_0 + H_\Delta$ and extend it to ensure that it defines a new affine distribution invariant under $\mathrm{Ker}(T\phi)$. This new affine space of Hamiltonians will allow one to determine the desired affine space of Hamiltonians on N by the use of a local section of ϕ.

Definition 3.4 (Canonical construction) *Let $\phi : M \to N$ be a Poisson map between manifolds with nondegenerate Poisson brackets, $\mathcal{H}_M = h_0 + H_\Delta$ a locally defined affine space of Hamiltonians, $\mathcal{G} = \mathrm{span}_\mathbb{R}\{g_1, g_2, \ldots, g_k\}$ a locally defined linear space of Hamiltonians such that $\mathrm{Ker}(T\phi) = B_M^\#(d\mathcal{G}) = \mathrm{span}\{X_{g_1}, X_{g_2}, \ldots, X_{g_k}\}$, and $\overline{\mathcal{H}_M}$ the following affine space of smooth maps:*

$$\overline{\mathcal{H}_M} = h_0 + \overline{H_\Delta} + \{\overline{H_\Delta}, \mathcal{G}\} + \{\{\overline{H_\Delta}, \mathcal{G}\}, \mathcal{G}\} + \ldots \quad (3.4)$$

for $\overline{H_\Delta} = H_\Delta + \{h_0, \mathcal{G}\}$. The affine space of Hamiltonians $\mathcal{H}_N$ defined by:

$$\mathcal{H}_N = \overline{\mathcal{H}_M} \circ i \qquad (3.5)$$

for any locally defined symplectic inclusion $i : N \to M$ satisfying $\phi \circ i = id_N$ is called canonically ϕ-related to $\mathcal{H}_M$.

Remark 3.5 *We note that the above construction can also be performed without using a basis of Hamiltonian vector fields for $\mathrm{Ker}(T\phi)$. Any other basis for $\mathrm{Ker}(T\phi)$ will equally work if we replace (3.4) with:*

$$\overline{\mathcal{H}_M} = h_0 + \overline{H_\Delta} + \mathcal{L}_X\overline{H_\Delta} + \mathcal{L}_X\mathcal{L}_X\overline{H_\Delta} + \ldots$$

for every X in the chosen basis of $\mathrm{Ker}(T\phi)$. This allows to compute $\overline{\mathcal{H}_M}$ even if $\mathcal{G}$ is not available. Nevertheless, it can be shown that, locally, such family $\mathcal{G}$ always exists.

The control system defined by $B_M^\#(\mathrm{d}\overline{\mathcal{H}_M})$ enjoys some properties which justify the use of the word canonical in Definition 3.4.

Proposition 3.6 *The affine distribution $\overline{\mathcal{D}_M} = B_M^\#(\mathrm{d}\overline{\mathcal{H}_M})$ is the smallest affine distribution, invariant under $\mathrm{Ker}(T\phi)$ which contains $\mathcal{D}_M$.*

Proof:

The distribution $\overline{\mathcal{D}_M}$ contains $\mathcal{D}_M$ since $\mathcal{H}_M \subseteq \overline{\mathcal{H}_M}$ and is affine by construction. Invariance under $\mathrm{Ker}(T\phi)$ follows from the equality $\{h, g\} = -[X_h, X_g]$ and Proposition 3.3. To show that $\overline{\mathcal{D}_M}$ is the smallest affine distribution with these properties we consider any other affine distribution $\mathcal{E}$ with the same properties. We now show that every vector field in $\overline{\mathcal{D}_M}$ also belongs to $\mathcal{E}$. Recall that, by construction, any vector field in $\overline{\mathcal{D}_M}$ can be written as a linear combination of vectors of the form:

$$Y = \sum_{j=0}^{k} ad_{X_{g(j)}}^j X$$

for some $X \in \mathcal{D}_M$, $X_{g(j)} \in \{X_{g_1}, X_{g_2}, \ldots, X_{g_k}\}$ and $ad_{X_{g(j)}}^j X$ defined by:

$$ad_{X_{g(0)}}^0 X = X$$
$$ad_{X_{g(j+1)}}^{j+1} X = [ad_{X_{g(j)}}^j X, X_{g(j+1)}]$$

In view of this, we now show that $Y \in \mathcal{E}$ by induction on the number k of Lie bracket depth:

If $k = 0$ then $X \in \mathcal{D}_M \subseteq \mathcal{E}$.

If $k = m$ and $k > 1$, then we can write Y as $Y = Z + [Z, X_g]$ for some $Z \in \overline{\mathcal{D}_M}$ and some $X_g \in \{X_{g_1}, X_{g_2}, \ldots, X_{g_k}\}$. Clearly Z has bracket depth $m - 1$ so that by the induction hypothesis $Z \in \mathcal{E}$. However $\mathcal{E}$ is $\mathrm{Ker}(T\phi)$ invariant so that Proposition 3.3 implies that $Y = Z + [Z, X_g] \in \mathcal{E}$. Since every vector field in $\overline{\mathcal{D}_M}$ is a linear combination of vector fields belonging to $\mathcal{E}$, the desired inclusion $\overline{\mathcal{D}_M} \subseteq \mathcal{E}$ follows.$\square$

The control system defined by $\mathcal{D}_N$ is the minimal abstraction of control system defined by $\mathcal{D}_M$:

Proposition 3.7 *The affine Hamiltonian control system Σ_N, obtained from Σ_M by the canonical construction is the smallest Hamiltonian control system on N ϕ-related to Σ_M.*

Proof: We start by showing that the Hamiltonian control system defined by $\mathcal{H}_N$ is ϕ-related to the Hamiltonian control system defined by $\mathcal{H}_M$. Let $x \in M$, $\phi(x) = y$ and $i(y) = x'$. Consider any $X \in \mathcal{D}_M(x)$, since $\overline{\mathcal{D}_M}$

is $\mathrm{Ker}(T\phi)$-invariant, it follows that $T_x\gamma \cdot X = X' \in \overline{\mathcal{D}_M}(x')$ where γ is the composition of local flows K_t^i of vector fields $K^i \in \mathrm{Ker}(T\phi)$ satisfying:

$$\gamma(x) = K_{t_1}^1 \circ K_{t_2}^2 \circ \ldots \circ K_{t_l}^l(x) = x' \qquad (3.6)$$

Such sequence of flows exists since the fibers of ϕ are connected. Note that $T_x\phi \cdot X = T_{x'}\phi \cdot X' = T_{x'}\phi(T_x\gamma \cdot X)$ since $\phi \circ \gamma = \phi$. This shows that the inclusion $T_{x'}\phi \cdot X' \in \mathcal{D}_N \circ \phi(x') = \mathcal{D}_N \circ \phi(x)$ is sufficient to ensure ϕ-relatedness of the control systems. Since $X' \in \overline{\mathcal{D}_M}(x')$ it follows that X' can be written as $X' = X_{\overline{h}}$ for some $\overline{h} \in \overline{\mathcal{H}_M}$. We now proceed by showing that $T_{x'}\phi \cdot X_{\overline{h}}(x') \in \mathcal{D}_N \circ \phi(x')$.

Recall that ϕ is a surjective submersion so that it locally factors uniquely through the Poisson projection $\pi : M \to M/\sim$, where $\sim$ is the regular equivalence relation on M defined by considering each leaf of the foliation induced by the integrable distribution $\mathrm{Ker}(T\phi)$ an equivalence class [13]. This means that there are coordinates $a = (a_1, \ldots, a_n)$ on N such that ϕ takes the form $\pi(a, b) = a$ for (a, b) coordinates on M with $b = (b_1, \ldots, b_m)$. Furthermore b can be chosen so as to satisfy $\{a_i, b_j\} = 0$, for any $i \in \{1, \ldots, n\}$ and any $j \in \{1, \ldots, m\}$ since the Poisson structure has constant rank [13], see also [26,25]. From $\{a_i, b_j\} = 0$ we conclude that the symplectic form associated with the Poisson bracket on M can be written as $\omega_M = \sum_{i \neq j} \alpha_{ij}(a)\mathrm{d}a_i \wedge \mathrm{d}a_j + \sum_{k \neq l} \beta_{kl}(b)\mathrm{d}b_k \wedge \mathrm{d}b_l$, while the symplectic form associated with the Poisson bracket on N equals $\omega_N = \sum_{i \neq j} \alpha_{ij}(a)\mathrm{d}a_i \wedge \mathrm{d}a_j$. Since in these local coordinates the map ϕ is the projection $\pi(a, b) = a$, the local section i of ϕ is of the form $i(a) = (a, \xi(a))$. Condition (2.10) now reduces to:

$$\sum_{i \neq j} \alpha_{ij}(a)\mathrm{d}a_i \wedge \mathrm{d}a_j = \sum_{i \neq j} \alpha_{ij}(a)\mathrm{d}a_i \wedge \mathrm{d}a_j + \sum_{k \neq l} \beta_{kl}(\xi(a))\mathrm{d}\xi_k \wedge \mathrm{d}\xi_l$$

from which we conclude that $\mathrm{d}\xi = 0$ or equivalently that ξ is constant. A vector field $X_{\overline{h}}$ is now given by:

$$\begin{aligned}
X_{\overline{h}}(a, b) &= B_M^\#\left(\mathrm{d}\overline{h}|_{(a,b)}\right) \\
&= B_M^\#\left(\frac{\partial \overline{h}}{\partial a}\mathrm{d}a|_{(a,b)} + \frac{\partial \overline{h}}{\partial b}\mathrm{d}b|_{(a,b)}\right) \\
&= B_M^\#\left(\frac{\partial \overline{h}}{\partial a}\mathrm{d}a|_{(a,b)}\right) + B_M^\#\left(\frac{\partial \overline{h}}{\partial b}\mathrm{d}b|_{(a,b)}\right)
\end{aligned}$$

Furthermore, from $\{a_i, b_j\} = 0$, we conclude that $B_M^\#(\frac{\partial \overline{h}}{\partial b}\mathrm{d}b|_{(a,b)})$ will be a linear combination of vectors $\{\frac{\partial}{\partial b_1}, \ldots, \frac{\partial}{\partial b_m}\} \subseteq \mathrm{Ker}(T\pi)$. This allows to express

$T_{(a,b)}\pi \cdot X_{\overline{h}}(a,b)$ as:

$$T_{(a,b)}\pi \cdot X_{\overline{h}}(a,b) = T_{(a,b)}\pi \cdot B_M^{\#}\Big(\frac{\partial \overline{h}}{\partial a}\mathrm{d}a|_{(a,b)}\Big)$$

$$+ T_{(a,b)}\pi \cdot B_M^{\#}\Big(\frac{\partial \overline{h}}{\partial b}\mathrm{d}b|_{(a,b)}\Big)$$

$$= T_{(a,b)}\pi \cdot B_M^{\#}\Big(\frac{\partial \overline{h}}{\partial a}\mathrm{d}(i \circ \pi)|_{(a,b)}\Big) + 0$$

$$= T_{(a,b)}\pi \cdot B_M^{\#}\Big(\pi^*\mathrm{d}(\overline{h} \circ i)|_{(a,b)}\Big)$$

$$= B_N^{\#}\Big(\mathrm{d}(\overline{h} \circ i)|_a\Big) = Y_{\overline{h} \circ i}(a) \qquad (3.7)$$

where the second equality follows from $\mathrm{d}\xi = 0$ and the last equality follows from the Poisson character of π. Since $\overline{h} \circ i \in \mathcal{H}_N$ we have shown, as desired, that $T_{x'}\phi \cdot X_{\overline{h}}(x') \in \mathcal{D}_N \circ \phi(x)$ and consequently that:

$$T_x\phi\big(\mathcal{D}_M\big) \subseteq T_x\phi\big(\overline{\mathcal{D}_M}\big) \subseteq \mathcal{D}_N \circ \phi(x) \qquad (3.8)$$

for every $x \in M$. Furthermore, since every $g \in \mathcal{H}_N$ is obtained from a $h \in \overline{\mathcal{H}_M}$ by restriction via $i : N \to M$, the following equality also holds:

$$T_x\phi\big(\overline{\mathcal{D}_M}\big) = \mathcal{D}_N \circ \phi(x) \qquad (3.9)$$

It remains to show that $\mathcal{D}_N$ is the smallest affine distribution satisfying (3.8). Consider any other affine distribution $\mathcal{E}$ on N satisfying $T_x\phi(\mathcal{D}_M) \subseteq \mathcal{E} \circ \phi$. From $\mathcal{E}$ we build a new distribution $\overline{\mathcal{E}}$ on M by $\overline{\mathcal{E}} = (T\phi)^{-1}(\mathcal{E})$, that is $\overline{\mathcal{E}} = \{X \in TM : T\phi \cdot X \in \mathcal{E}\}$. It is clear that $\mathcal{D}_M \subseteq \overline{\mathcal{E}}$ and it is also true that $\overline{\mathcal{E}}$ is $\mathrm{Ker}(T\phi)$ invariant since for any $x, x' \in M$ such that $\phi(x) = \phi(x')$, $T_x\phi(\overline{\mathcal{E}}(x)) = \mathcal{E} \circ \phi(x) = \mathcal{E} \circ \phi(x') = T_{x'}\phi(\overline{\mathcal{E}}(x'))$. However, by Proposition 3.6, $\overline{\mathcal{D}_M}$ is the smallest $\mathrm{Ker}(T\phi)$ invariant distribution containing $\mathcal{D}_M$. It then follows that:

$$\overline{\mathcal{D}_M} \subseteq \overline{\mathcal{E}} \;\Rightarrow\; T\phi\big(\overline{\mathcal{D}_M}\big) \subseteq T\phi(\overline{\mathcal{E}}) \;\Rightarrow\; \mathcal{D}_N \circ \phi \subseteq \mathcal{E} \circ \phi$$

which shows that $\mathcal{D}_N$ is, in fact, the smallest affine distribution satisfying (3.8).$\square$

The previous proposition is very important since it ensures that given *any* affine Hamiltonian control system Σ_M on a manifold M, and any Poisson surjective submersion $\phi : M \to N$, with connected fibers, it is always possible to compute the corresponding abstraction. Furthermore, it also ensures, that the resulting abstraction is the smallest over all affine control systems on N ϕ-related to Σ_M. This fact justifies the term canonical for the construction given in Definition 3.4.

4 Local Accessibility Equivalence

In addition to propagating trajectories and Hamiltonians from the original control system to the abstracted system, we will investigate how accessibility properties can be preserved in the abstraction process. We first review several (local) accessibility properties for control systems [9,10,17].

Definition 4.1 *Let Σ_M be a control system on a smooth manifold M. For each $T > 0$ and each $x \in M$, the set of points reachable from x at time T by trajectories which remain in a open set V containing x for $t \leq T$, denoted by $Reach^V(x, T)$, is equal to the set of terminal points $c_M(T)$ of Σ_M trajectories that originate at x and that satisfy $c_M(t) \in V$, for all $t \leq T$. The set of points reachable from x in T or fewer units of time while remaining in V, denoted by $Reach^V(x, \leq T)$ is given by $Reach^V(x, \leq T) = \cup_{t \leq T} Reach^V(x, T)$.*

Definition 4.2 *A control system Σ_M is said to be*

- *Locally accessible from x if $Reach^V(x, \leq T)$ contains a non-empty open set O of M for all open sets V and all $T > 0$.*
- *Locally accessible if it is locally accessible from all $x \in M$.*
- *Controllable if for any $x_1, x_2 \in M$, there exists a $T \in \mathbb{R}$ and a trajectory $c_M(t)$ of Σ_M such that $c_M(0) = x_1$ and $c_M(T) = x_2$.*

These local accessibility properties can be characterized by simple rank conditions on the Poisson algebra generated [2] by the affine space of Hamiltonians $\mathcal{H}_M$.

Proposition 4.3 (Accessibility Rank Conditions)
Let Σ_M be an Hamiltonian control system on a manifold M of dimension m and denote by $\mathcal{P}(\mathcal{H}_M)$ the Poisson algebra generated by the smooth maps in $\mathcal{H}_M$. Then:

- *If $\dim(\mathrm{d}\mathcal{P}(\mathcal{H}_M(x))) = m$, then control system Σ_M is locally accessible at $x \in M$.*
- *If $\dim(\mathrm{d}\mathcal{P}(\mathcal{H}_M(x))) = m$ for all $x \in M$, then control system Σ_M is locally accessible.*
- *If $\dim(\mathrm{d}\mathcal{P}(\mathcal{H}_M(x))) = m$ for all $x \in M$, $\mathcal{H}_M$ is symmetric, that is $h \in \mathcal{H}_M \Rightarrow -h \in \mathcal{H}_M$, and M is connected, then control system Σ_M is controllable.*

We now determine under what conditions on the abstracting maps, local accessibility of the original system Σ_M is equivalent to local accessibility of its canonical abstraction Σ_N. In particular, conditions to propagate accessibility from the abstracted system Σ_N to the original system Σ_M, are given in the next result.

Theorem 4.4 (Local Accessibility Equivalence)
Let Σ_M be a control system canonically ϕ-related to control system Σ_N, $\mathcal{G} = \mathrm{span}_{\mathbb{R}}\{g_1, g_2, \ldots, g_k\}$ a locally defined linear space of Hamiltonians such that

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[2] The Poisson algebra generated by a set of S smooth maps on M is the smallest Poisson subalgebra of $C^{\infty}(M)$ containing S.

$\mathrm{Ker}(T\phi) = B_M^\#(d\mathcal{G}) = \mathrm{span}\{X_{g_1}, X_{g_2}, \ldots, X_{g_k}\}$, and $\mathcal{P}(\mathcal{H}_M)$ the Poisson algebra generated by $\mathcal{H}_M$. If $\mathcal{G} \subseteq \mathcal{P}(\mathcal{H}_M)$ then Σ_M is locally accessible (at x) if and only if Σ_N is locally accessible (at $\phi(x)$).

Proof: If $\mathcal{D}_M$ is locally accessible (at x) then by Theorem 3.2, $\mathcal{D}_N$ is also locally accessible (at $\phi(x)$).

To show the converse we note that the equality $\{f, g\} = [X_g, X_f]$ implies $\dim(d\mathcal{P}(\mathcal{H}_M(x))) = \dim(\mathrm{Lie}(\mathcal{D}_M(x)))$ where $\mathrm{Lie}(\mathcal{D}_M)$ denotes the Lie algebra generated by the vector fields in $\mathcal{D}_M$. In virtue of this, we start by showing the following equality between the Lie algebras generated by $\mathcal{D}_M$ and its canonical ϕ-abstraction $\mathcal{D}_N$:

$$T_x\phi\big(\mathrm{Lie}(\mathcal{D}_M)(x)\big) = \mathrm{Lie}(\mathcal{D}_N) \circ \phi(x) \qquad (4.1)$$

Recall that $\overline{\mathcal{D}_M}$ (defined in Proposition 3.6) is $\mathrm{Ker}(T\phi)$-invariant, therefore there exists a local basis for $\overline{\mathcal{D}_M}$ of $\mathrm{Ker}(T\phi)$-invariant vector fields. Let us denote by $\overline{X_0} + \{\overline{X_1}, \overline{X_2}, \ldots, \overline{X_n}\}$ this basis. Since every vector field $\overline{X}$ in this basis is $\mathrm{Ker}(T\phi)$ invariant, it is ϕ-related to its projection $\overline{Y} \circ \phi = T\phi(\overline{X})$. It then follows that $\overline{X_1}$ and $\overline{X_2}$ being ϕ-related to $\overline{Y_1}$ and $\overline{Y_2}$, respectively, implies that $[\overline{X_1}, \overline{X_2}]$ is also ϕ-related to $[\overline{Y_1}, \overline{Y_2}]$ and an induction argument shows that:

$$T\phi(\mathrm{Lie}(\overline{\mathcal{D}_M})) = \mathrm{Lie}(\mathcal{D}_N) \circ \phi \qquad (4.2)$$

The assumption $\mathcal{G} \subseteq \mathcal{P}(\mathcal{H}_M)$ and the equality $\{f, g\} = [X_g, X_f]$ imply:

$$\mathrm{Ker}(T\phi) \subseteq \mathrm{Lie}(\mathcal{D}_M) \qquad (4.3)$$

and by construction of $\overline{\mathcal{D}_M}$ we have $\mathrm{Lie}(\overline{\mathcal{D}_M}) = \mathrm{Lie}(\mathcal{D}_M)$ which combined with (4.2) gives (4.1). Using now (4.1) we conclude:

$$\begin{aligned}
\dim(\mathrm{Lie}(\mathcal{D}_M)(x)) &= \dim(T_x\phi(\mathrm{Lie}(\mathcal{D}_M)(x))) \\
&\quad + \dim(\mathrm{Ker}(T\phi)(x)) \\
&= \dim(\mathrm{Lie}(\mathcal{D}_N) \circ \phi(x)) \\
&\quad + \dim(\mathrm{Ker}(T\phi)(x)) \qquad (4.4)
\end{aligned}$$

It is now easy to see that if Σ_N is locally accessible at $\phi(x)$, then $\dim(d\mathcal{P}(\mathcal{H}_N) \circ \phi(x)) = \dim(\mathrm{Lie}(\mathcal{D}_N) \circ \phi(x)) = \dim(N)$ and $\dim(d\mathcal{P}(\mathcal{H}_M)(x)) = \dim(\mathrm{Lie}(\mathcal{D}_M)(x)) = \dim(N) + \dim(\mathrm{Ker}(T\phi)(x)) = \dim(M)$ since ϕ is a submersion. Local accessibility (at x) now follows from $\dim(d\mathcal{P}(\mathcal{H}_M)(x)) = \dim(M)$ and Proposition 4.3.□

As an immediate corollary we have:

Corollary 4.5 *Let Σ_M be a control system canonically ϕ-related to control system Σ_N, $\mathcal{G} = \mathrm{span}_{\mathbb{R}}\{g_1, g_2, \ldots, g_k\}$*
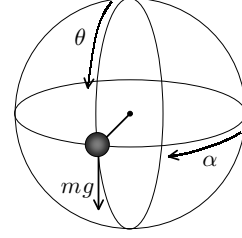

Fig. 1. Spherical pendulum with gravity force represented.

a linear space of Hamiltonians such that $\mathrm{Ker}(T\phi) = B_M^\#(d\mathcal{G}) = \mathrm{span}\{X_{g_1}, X_{g_2}, \ldots, X_{g_k}\}$, and $\mathcal{P}(\mathcal{H}_M)$ the Poisson algebra generated by $\mathcal{H}_M$. If $\mathcal{G} \subseteq \mathcal{P}(\mathcal{H}_M)$ and both $\mathcal{H}_M$ and $\mathcal{H}_N$ are symmetric and furthermore M is connected then Σ_M is controllable iff Σ_N is controllable.

Theorem 4.4 provides moderate conditions to propagate accessibility properties in a hierarchy of abstractions. This allows for a hierarchical approach to the determination of accessibility properties. As long as there are control inputs on the abstracted control system Σ_N, the Poisson algebra $\mathcal{P}(\mathcal{H}_N)$ contains non constant maps and consequently the system can be further abstracted to an equivalent system regarding local accessibility by the use of a map ϕ such that $Ker(T\phi) \subseteq B_N^\#(d\mathcal{P}(\mathcal{H}_N))$.

5 A spherical pendulum example

As an illustrative example, consider the spherical pendulum as a fully actuated mechanical control system. This system can be used to model, for example, the stabilization of the spinning axis of a satellite or a pan and tilt camera. Consider a massless rigid rod of length l fixed in one end by a spherical joint and having a bulb of mass m on the other end. The configuration space for this control system is S^2, however we will work locally on $C_2 \subset S^2$, $C_2 =]0, \pi[\times]0, 2\pi[$ parameterized by spherical coordinates (θ, α), see Figure 1.

The kinetic energy of the system is given by:

$$T = \frac{1}{2}ml^2(\dot{\theta}^2 + \sin^2\theta\ \dot{\alpha}^2) \qquad (5.1)$$

and the potential energy of the system is:

$$V = -mgl\cos\theta \qquad (5.2)$$

Through the Legendre transform of the Lagrangian $L = T - V$ one arrives at the Hamiltonian:

$$h_0 = \frac{1}{2ml^2}p_\theta^2 + \frac{1}{2ml^2\sin^2\theta}p_\alpha^2 - mgl\cos\theta \qquad (5.3)$$

where p_θ is given by $p_\theta = ml^2\dot{\theta}$ and $p_\alpha = ml^2\sin^2\theta\ \dot{\alpha}$. Since the system is fully actuated, the controlled Hamiltonian defined over $M = T^*C_2$ is given by:

$$H_M = h_0 + h_1 u_1 + h_2 u_2 \qquad (5.4)$$

with $h_1 = \theta$ and $h_2 = \alpha$ and where u_1 and u_2 are the control inputs. Such controlled Hamiltonian defines control system Σ_M through the use of the Poisson bracket:

$$\{f,g\} = \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial p_\theta} + \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\theta}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial \alpha} \qquad (5.5)$$

as described in Section 2. The drift vector field associated with h_0 is invariant under rotations around the vertical axis and could be reduced using this symmetry by using the methods in [15]. However to emphasize the advantages of the abstraction method we will abstract away precisely the directions where there are no symmetries. Consider the abstracting map:

$$\phi: T^*C_2 \to T^*C_1 \qquad (5.6)$$
$$(\theta, \alpha, p_\theta, p_\alpha) \mapsto (\alpha, p_\alpha) \qquad (5.7)$$

where $C_1 =]0, 2\pi[\subset S^1$ is equipped with the following Poisson bracket:

$$\{f,g\} = \frac{\partial f}{\partial \alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial \alpha} \qquad (5.8)$$

If one lets $g_1 = p_\theta$, $g_2 = \theta$ and follow the steps of the canonical construction one obtains:

$$\{h_0, g_1\} = \frac{\partial h_0}{\partial \theta} = \frac{\cos\theta}{ml^2\sin^3\theta}\, p_\alpha + mgl\sin\theta$$

$$\{\{h_0, g_1\}, g_1\} = \frac{\partial\{h_0, g_1\}}{\partial \theta} = -\frac{2 + \cos 2\theta}{ml^2\sin^4\theta} + mgl\cos\theta$$

$$\{h_0, g_2\} = \frac{\partial h_0}{\partial p_\theta} = \frac{1}{ml^2}p_\theta$$

$$\overline{H_\Delta} = H_\Delta + \mathrm{span}_{\mathbb{R}}\{\{h_0, g_1\}, \{h_0, g_2\}\}$$

$$= \mathrm{span}_{\mathbb{R}}\{\theta, \alpha, \frac{\cos\theta}{ml^2\sin^3\theta}\, p_\alpha + mgl\sin\theta, \frac{1}{ml^2}p_\theta\}$$

Note that $\overline{H_\Delta}$ is of rank 4 except when $\theta = \frac{\pi}{2}$, however by considering also $\{\{h_0, g_1\}, g_1\}$ we see that the affine distribution $\overline{H_\Delta} + \mathrm{span}_{\mathbb{R}}\{\{\{h_0, g_1\}, g_1\}\}$ has rank 4 on C_2. This implies that no further brackets need to be computed. Furthermore, we also see that $H_\Delta + \mathrm{span}_{\mathbb{R}}\{\{h_0, g_1\}, \{h_0, g_2\}\} = H_\Delta + \mathrm{span}_{\mathbb{R}}\{\{\{h_0, g_1\}, g_1\}, \{h_0, g_2\}\}$ so that we will work with $\overline{\mathcal{H}_M} = H_\Delta + \mathrm{span}_{\mathbb{R}}\{\{\{h_0, g_1\}, g_1\}, \{h_0, g_2\}\}$. The locally defined inclusion $i: N \to M$ mapping (α, p_α) to $i(\alpha, p_\alpha) = (\pi/4, \alpha, 0, p_\alpha)$ satisfies $\phi \circ i = id_N$. To show that it is also symplectic, we note that the covariant tensor associated with B_{C_2} is given by:

$$\omega_{C_2} = d\alpha \wedge dp_\alpha + d\theta \wedge dp_\theta \qquad (5.9)$$

while the covariant tensor associated with B_{C_1} is given by:

$$\omega_{C_1} = d\alpha \wedge dp_\alpha \qquad (5.10)$$

Computing now $i^*\omega_{C_2}$ we obtain:

$$i^*\omega_{C_2} = d\alpha \wedge dp_\alpha + d\pi/4 \wedge d0 = d\alpha \wedge dp_\alpha = \omega_{C_1} \quad (5.11)$$

which shows that i is symplectic. The abstracted system is now obtained through $\mathcal{H}_N$:

$$\mathcal{H}_N = \overline{\mathcal{H}_M} \circ i$$
$$= \frac{1}{ml^2\sin^2\theta}\, p_\alpha^2 - \frac{mgl}{\sqrt{2}} + \mathrm{span}_{\mathbb{R}}\left\{\frac{\pi}{3}, \alpha, -\frac{8}{ml^2}p_\alpha + \frac{mgl}{\sqrt{2}}\right\}$$

Discarding constant terms, which are associated with the zero vector field, we compute the abstraction to be:

$$\dot{\alpha} = \frac{2}{ml^2}(p_\alpha - 4v_1) \qquad (5.12)$$
$$\dot{p}_\alpha = v_2 \qquad (5.13)$$

which is a Hamiltonian control system on $N = T^*C_1$. Furthermore, since both g_1 and g_2 belong to $\mathcal{P}(\mathcal{H}_M)$ and Σ_N is locally accessible, by making use of Theorem 4.4 we infer that Σ_M is also locally accessible. This allows to use the abstracted model to do a preliminary control design to regulate the angle α on the smaller, abstracted model, which can be later extended to a complete design regulating both α and θ. This has the advantage of breaking the design in two phases, where only one variable is regulated in each phase. In concrete applications one would build an hierarchy of abstractions where Theorem 4.4 would be used to ensure that every control design at the upper levels can be implemented at the lower levels, therefore reducing the complexity of control design. Furthermore, since each level is an Hamiltonian control system, specific techniques for mechanical systems control design [18,4,2,5,6] can be used at each level of design.

6 Conclusions

In this paper, we have presented a hierarchical abstraction methodology for affine Hamiltonian control systems. We presented an algorithm for its determination which computes the smallest abstraction based on a given abstraction map. All the computations are performed at the level of Hamiltonians which simplifies the process. We have also characterized local accessibility preserving maps. This allows to build an hierarchy of control systems with different dimensions but equivalent local accessibility properties. These results are very encouraging for hierarchical control of mechanical systems.

Refining controller design from the abstracted to the original system is clearly important and will certainly build on the recent results [22] describing how

state/inputs of the original system are related to state/inputs of the abstracted system. Other research topics of great interest include the extension of present framework to port controlled Hamiltonian system (pcHs) and related notions of interconnection [24] as well as the propagation of nonholonomic constraints in the context of pcHs.

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