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ABUNDANCE OF C^1-ROBUST HOMOCLINIC TANGENCIES

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To Carlos Gutierrez (1944–2008), in memoriam

ABSTRACT. A diffeomorphism f has a C^1-robust homoclinic tangency if there is a C^1-neighborhood $\mathcal{U}$ of f such that every diffeomorphism in $g \in \mathcal{U}$ has a hyperbolic set Λ_g, depending continuously on g, such that the stable and unstable manifolds of Λ_g have some non-transverse intersection. For every manifold of dimension greater than or equal to three we exhibit a local mechanism (blender-horseshoes) generating diffeomorphisms with C^1-robust homoclinic tangencies.

Using blender-horseshoes, we prove that homoclinic classes of C^1-generic diffeomorphisms containing saddles with different indices and that do not admit dominated splittings (of appropriate dimensions) display C^1-robust homoclinic tangencies.

1. INTRODUCTION

1.1. **Framework and general setting.** A homoclinic tangency is a dynamical mechanism which is at the heart of a great variety of non-hyperbolic phenomena: persistent coexistence of infinitely many sinks [22], Hénon-like strange attractors [5, 20], super-exponential growth of the number of periodic points [19], and non-existence of symbolic extensions [15], among others. Moreover, homoclinic bifurcations (homoclinic tangencies and heterodimensional cycles) are conjectured to be the main source of non-hyperbolic dynamics (Palis denseness conjecture; see [23]).

In this paper, we present a local mechanism generating C^1-robust homoclinic tangencies. Using this construction, we show that the occurrence of robust tangencies is a quite general phenomenon in the non-hyperbolic setting, especially when the dynamics do not admit a suitable dominated splitting.

Let us now give some basic definitions (in Section 2, we will state precisely the definitions involved in this paper). A transitive hyperbolic set Λ has a *homoclinic tangency* if there is a pair of points $x, y \in \Lambda$ such that the stable leaf $W^{\mathrm{s}}(x)$ of x and the unstable leaf $W^{\mathrm{u}}(y)$ of y have some non-transverse intersection.

Given a hyperbolic set Λ of a diffeomorphism f, for g close to f, we denote by Λ_g the hyperbolic set of g which is the *continuation* of Λ (i.e., Λ_g is close to Λ and the dynamics of f on Λ and g on Λ_g are conjugate).

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Definition 1.1 (Robust cycles)**.**

- *Robust homoclinic tangencies:* A transitive hyperbolic set Λ of a C^r-diffeomorphism f has a C^r*-robust homoclinic tangency* if there is a C^r-neighborhood $\mathcal{N}$ of f such that for every $g \in \mathcal{N}$ the continuation Λ_g of Λ for g has a homoclinic tangency.
- *Robust heterodimensional cycles:* A diffeomorphism f has a C^r*-robust heterodimensional cycle* if there are transitive hyperbolic sets Λ and Σ of f whose stable bundles have different dimensions and a C^r-neighborhood $\mathcal{V}$ of f such that $W^{\mathrm{s}}(\Lambda_g) \cap W^{\mathrm{u}}(\Sigma_g) \neq \emptyset$ and $W^{\mathrm{u}}(\Lambda_g) \cap W^{\mathrm{s}}(\Sigma_g) \neq \emptyset$, for every diffeomorphism $g \in \mathcal{V}$.

Note that, by the Kupka-Smale theorem, C^r-generically, invariant manifolds of periodic points are in general position. Hence, generically, the non-transverse intersections in a robust cycle (tangency or heterodimensional cycle) involve non-periodic points (i.e., at least a non-trivial hyperbolic set).

In [21], Newhouse constructed surface diffeomorphisms having hyperbolic sets (called *thick horseshoes*) exhibiting C^2-robust homoclinic tangencies. Later, he proved that, in dimension two, homoclinic tangencies of C^2-diffeomorphisms yield thick horseshoes with C^2-robust homoclinic tangencies, [22] (see also [24] for a broad discussion of homoclinic bifurcations on surfaces). With the same C^2-regularity assumption, theorems in [27, 25] extend the Newhouse result, proving that homoclinic tangencies in any dimension lead to C^2-robust homoclinic tangencies. In this paper, we study the occurrence of robust homoclinic tangencies in the C^1-setting.

Newhouse construction (thick horseshoes with robust tangencies) involves distortion estimates which are typically C^2. The results in [30] present some obstacles for carrying this construction to the C^1-topology: C^1-generic surface diffeomorphisms do not have thick horseshoes. Recent results by Moreira in [17] are a strong indication that there are no surface diffeomorphisms exhibiting C^1-robust homoclinic tangencies.[1]

Nevertheless, in higher dimensions, there are examples of diffeomorphisms having hyperbolic sets with C^1-robust tangencies. For instance, the product of a non-trivial hyperbolic attractor by a normal expansion gives a hyperbolic set Λ of saddle type, whose stable manifold has a topological dimension greater than the dimension of its stable bundle. Then the set Λ can play the role of thick horseshoes in Newhouse construction. Geometrical constructions using these kinds of "thick" hyperbolic sets provide examples of systems with C^1*-robust heterodimensional cycles*[2] (see [3]) or C^1-robust tangencies (see [28, 4]). But these constructions involve quite specific global dynamical configurations, thus they cannot translate to a general setting.

1.2. **Robust homoclinic tangencies.** The aim of this paper is to show that the existence of C^1-robust homoclinic tangencies is a common phenomenon in the non-hyperbolic setting. For instance, the next result is a consequence of the local mechanism for robust tangencies in Theorem 4.8.

[1] This question is closely related to the open problem of C^1-density of hyperbolic diffeomorphisms on compact surfaces (Smale's density conjecture). In fact, Moreira's result implies that there are no C^1-robust homoclinic tangencies associated to hyperbolic *basic* sets of surface diffeomorphisms. See [1] for a discussion of the current state of this conjecture.

[2] A *heterodimensional cycle* is a cycle associated to saddles having different indices.

Theorem 1.2. *Let M be a compact manifold with $\dim(M) \geq 3$. There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{R}$ and every periodic saddle P of f such that*

- **(index variability)** *the homoclinic class $H(P, f)$ of P has a periodic saddle Q with $\dim(E^{\mathrm{s}}(Q)) \neq \dim(E^{\mathrm{s}}(P))$,*
- **(non-domination)** *the stable/unstable splitting $E^{\mathrm{s}}(R) \oplus E^{\mathrm{u}}(R)$ over the set of saddles R homoclinically related with P is not dominated,*

the saddle P belongs to a transitive hyperbolic set having a C^1-robust homoclinic tangency.

For the precise definitions of homoclinic class and dominated splitting see Definitions 2.1 and 2.4. Let us reformulate Theorem 1.2 by focusing on the homoclinic class of a prescribed periodic orbit:

Corollary 1.3. *Let M be a compact manifold with $\dim(M) \geq 3$. Consider a diffeomorphism f with a saddle P_f whose continuation P_g is defined for all g in a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$. Assume that*

- **(generic index variability)** *there is a residual subset $\mathcal{G}$ of $\mathcal{U}$ such that, for every $g \in \mathcal{G}$, the homoclinic class of P_g of f contains a saddle Q of different index,*
- **(robust non-domination)** *for every $g \in \mathcal{U}$, the stable/unstable splitting $E^{\mathrm{s}}(R) \oplus E^{\mathrm{u}}(R)$ over the set of saddles R homoclinically related with P_g is not dominated.*

Then there is an open and dense subset $\mathcal{C}$ of $\mathcal{U}$ of diffeomorphisms g such that the saddle P_g belongs to a transitive hyperbolic set with a C^1-robust homoclinic tangency.

Remark 1.4. The diffeomorphisms f in the residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ in Theorem 1.2 satisfy the following properties (see [2, Section 2.1] and [13, Appendix B.1.1]):

- Every homoclinic class $H(P_f, f)$ of f depends continuously on $f \in \mathcal{R}$. Therefore, if $H(P_f, f)$ has a dominated splitting, then $H(P_g, g)$ also has a dominated splitting whose bundles have constant dimension for all $g \in \mathcal{R}$ close to f.
- Assume that a homoclinic class $H(P_f, f)$ of $f \in \mathcal{R}$ contains saddles of stable indices j and k, $j \neq k$. Then the homoclinic class $H(P_g, g)$ also contains saddles of stable indices j and k for every $g \in \mathcal{R}$ close to f.

In other words, the conditions in Theorem 1.2 are C^1-open in the residual set $\mathcal{R}$ of $\mathrm{Diff}^1(M)$.

The *index interval* of a homoclinic class H is the interval $[i, j]$, where i and j are the minimum and the maximum of the s-*indices* (dimension of the stable bundle) of the periodic points in H. The homoclinic class H has *index variation* if $i < j$. Given a transitive hyperbolic set Λ its s-*index* is the dimension of its stable bundle.

Corollary 1.5. *For every diffeomorphism f in the residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$, any homoclinic class H of f with index variation, and every $k \in [i, j]$, where $[i, j]$ is the index interval of H, one has:*

- *either there is a dominated splitting $E \oplus_< F$ (i.e., F dominates E) with* $\dim(E) = k$

- *or there is a hyperbolic transitive set $\Lambda \subset H$ with* s*-index k having a C^1-robust homoclinic tangency.*

When we are interested only in the existence of robust homoclinic tangencies, without paying attention to the index of the hyperbolic set involved in their generation, there is the following reformulation:

Corollary 1.6. *There is a residual subset $\mathcal{G}$ of* $\mathrm{Diff}^1(M)$ *such that for every diffeomorphism $f \in \mathcal{G}$ and every homoclinic class $H(P,f)$ of f with index interval $[i,j]$, $j > i$,*

- *either $H(P,f)$ has a dominated splitting*
$$T_{H(P,f)}M = E^{\mathrm{cs}} \oplus_< E_1 \oplus_< \cdots \oplus_< E_j \oplus_< E^{\mathrm{cu}},$$
where $\dim(E^{\mathrm{cs}}) = i$ *and $E_1, \ldots, E_j$ are one-dimensional*
- *or the homoclinic class $H(P,f)$ contains a transitive hyperbolic set with a robust homoclinic tangency.*

In the first case of Corollary 1.6, we say that $H(P,f)$ has an *indices adapted dominated splitting.*

The previous results have an interesting formulation for *tame diffeomorphisms,* i.e., the C^1-open set $\mathcal{T}(M)$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms having finitely many chain recurrence classes (see Definition 2.3) in a robust way. We define $\mathcal{W}(M) \stackrel{\mathrm{def}}{=} \mathrm{Diff}^1(M) \setminus \overline{\mathcal{T}(M)}$ as the set of *wild diffeomorphisms.* Let us observe that, for an open and dense subset of $\mathcal{T}(M)$, a chain recurrence class is either hyperbolic or has index variation; see [2].

Given a chain recurrence class C of f we first consider the *finest dominated splitting* over C (i.e., the bundles of this splitting cannot be decomposed in a dominated way). Then we let E^{s} (resp. E^{u}) be the sum of the uniformly contracting (resp. expanding) bundles of this splitting (these bundles may be trivial; see [14]). The bundles $E_1, \ldots, E_k$ are the remaining non-hyperbolic bundles of the finest dominated splitting of C. In this way, we get a dominated splitting over C,

$$T_C M = E^{\mathrm{s}} \oplus_< E_1 \oplus_< \cdots \oplus_< E_k \oplus_< E^{\mathrm{u}},$$

where E^{s} and E^{u} are uniformly contracting and expanding, and $E_1, \ldots, E_k$ are indecomposable and non-hyperbolic. We call this splitting the *finest central dominated splitting* of the chain recurrence C.

Remark 1.7. Let f be any tame diffeomorphisms and $H(P,f)$ any homoclinic class of f which is far from robust homoclinic tangencies. Then the finest central dominated splitting of $H(P,f)$ is indices adapted. For tame diffeomorphisms, the corollary below gives a more precise description of the relation between the finest central dominated splitting and the robust homoclinic tangencies associated to a homoclinic class.

Corollary 1.8. *There is a C^1-open and dense subset $\mathcal{O}$ of the set $\mathcal{T}(M)$ of tame diffeomorphisms such that, for every f in $\mathcal{O}$ and every chain recurrence class C of f whose finest central dominated splitting is*

$$T_C M = E^{\mathrm{s}} \oplus_< E_1 \oplus_< \cdots \oplus_< E_k \oplus_< E^{\mathrm{u}},$$

for every $i = 1, \dots, k$,

$$\dim(E_i) > 1 \Longleftrightarrow \begin{cases} \textit{for all } j \in \{1, \dots, \dim(E_i) - 1\} \\ \textit{there is a transitive hyperbolic set } K \textit{ of } \mathrm{s}\textit{-index} \\ \mathrm{ind}^{\mathrm{s}}(K) = \dim\left(E^{\mathrm{s}} \oplus E_1 \oplus \cdots \oplus E_{i-1}\right) + j \\ \textit{having a } C^1\textit{-robust homoclinic tangency.} \end{cases}$$

Remark 1.9.

(1) Under the hypotheses of Corollary 1.8, [9, Theorem 1.14] implies that (choosing appropriately the open and dense subset $\mathcal{O}$ of $\mathcal{T}(M)$) the hyperbolic set K with a C^1-robust homoclinic tangency is also involved in a C^1-robust heterodimensional cycle.
(2) Corollary 1.8 can also be stated for isolated chain recurrence classes of C^1-generic diffeomorphisms.[3]

This article continues a program for studying the generation of *robust cycles* (homoclinic tangencies and heterodimensional cycles) in the C^1-topology. In [9] we proved that homoclinic classes containing periodic points with different indices generate (by arbitrarily small C^1-perturbations) C^1-robust heterodimensional cycles. Here we show that these robust heterodimensional cycles generate *blender-horseshoes,* a sort of hyperbolic basic set with geometrical properties resembling the thick horseshoes; see Section 3.2 and Theorem 6.4. We next see that, in the context of critical dynamics (some suitable non-domination property), blender-horseshoes yield C^1-robust tangencies; see Theorem 4.8. In fact, the definition and construction of blender-horseshoes (a special class of cu-blenders defined in [8]) and Theorem 4.8 are the technical heart of our arguments and the main topic of this paper.

The results in this paper and the ones in [9] support the following conjecture:

Conjecture 1.10 (Bonatti, [6]). *Every C^1-diffeomorphism can be C^1-approximated either by a hyperbolic diffeomorphism (Axiom A and the no-cycle property) or by a diffeomorphism exhibiting a C^1-robust cycle (homoclinic tangency or heterodimensional cycle).*

This conjecture is a stronger version of the denseness conjecture by Palis in [23] (dichotomy hyperbolicity versus approximation by diffeomorphisms with homoclinic bifurcations). The novelty here is that the conjecture considers two disjoint open sets whose union is dense in the whole set of C^1-diffeomorphisms: the hyperbolic ones and those with robust cycles. In the setting of tame diffeomorphisms, a strong version of Conjecture 1.10 was proved in [9, Theorem 1.14]: every tame diffeomorphism can be C^1-approximated either by hyperbolic diffeomorphisms or by diffeomorphisms exhibiting robust heterodimensional cycles. Recall that the Palis conjecture for surface C^1-diffeomorphisms was proved in [26] (due to dimension deficiency, for surface diffeomorphisms the conjecture only involves homoclinic tangencies).

1.3. **Newhouse domains.** Following [19], we say that an open set $\mathcal{N}$ of $\mathrm{Diff}^r(M)$ is a *C^r-Newhouse domain* if there is a dense subset $\mathcal{D}$ of $\mathcal{N}$ such that every $g \in \mathcal{D}$ has a homoclinic tangency (associated to some saddle). A preliminary step toward Conjecture 1.10 is the following question.

[3]By C^1-generic diffeomorphisms we mean diffeomorphisms in a residual subset of $\mathrm{Diff}^1(M)$.

Question 1.11. Let M be a closed manifold and $\mathcal{N}$ be a C^1-Newhouse domain of $\mathrm{Diff}^1(M)$. Are the diffeomorphisms having C^1-robust homoclinic tangencies dense in $\mathcal{N}$?

If it is not possible to answer positively this question in its full generality, it would be interesting to provide sufficient conditions for a C^1-Newhouse domain to contain an open and dense subset of diffeomorphisms with C^1-robust homoclinic tangencies. If the dimension of the ambient manifold is at least three, one may also ask about the interplay between robust homoclinic tangencies and robust heterodimensional cycles.

We now briefly discuss Question 1.11. Before going to our setting, let us review the discussion in [1] about this question for C^1-surface diffeomorphisms. Let $\mathrm{Hyp}^1(M)$ denote the subset of $\mathrm{Diff}^1(M)$ consisting of Axiom A diffeomorphisms. By [26], for surface diffeomorphisms, the open set

$$\mathcal{N}^1(M^2) \stackrel{\text{def}}{=} \mathrm{Diff}^1(M^2) \setminus \overline{\mathrm{Hyp}^1(M^2)}$$

is a Newhouse domain. The set $\mathcal{N}^1(M^2)$ is the union of the closure of three pairwise disjoint open sets $\mathcal{O}_1(M^2), \mathcal{O}_2(M^2)$, and $\mathcal{O}_3(M^2)$ defined as follows.

- The set $\mathcal{O}_1(M^2)$ consists of diffeomorphisms having C^1-robust homoclinic tangencies.
- There is a residual subset $\mathcal{R}_2(M^2)$ of $\mathcal{O}_2(M^2)$ such that every $f \in \mathcal{R}_2(M^2)$ has a homoclinic class $H(P,f)$ that robustly does not admit any dominated splitting. However, for every hyperbolic set Λ contained in $H(P,f)$ the invariant manifolds of Λ meet transversely. In this case, we say that the diffeomorphism f has a *persistently fragile homoclinic tangency* associated to P.
- There is a residual subset $\mathcal{R}_3(M^2)$ of $\mathcal{O}_3(M^2)$ such that for every diffeomorphism $f \in \mathcal{R}_3(M^2)$ and every (hyperbolic) periodic point P of f the homoclinic class $H(P,f)$ is hyperbolic. But there is a sequence of periodic points $(P_n)_n$ of f such that the hyperbolic homoclinic classes $H(P_n,f)$ accumulate (Hausdorff limit) to an aperiodic class (i.e., a recurrence class without periodic points).

As mentioned above, Moreira's result in [17] provides strong evidence suggesting that $\mathcal{O}_1(M^2)$ is empty. On the other hand, we do not know whether or not the sets $\mathcal{O}_2(M^2)$ and $\mathcal{O}_3(M^2)$ are empty. In fact, Smale density conjecture (hyperbolic diffeomorphisms are dense in $\mathrm{Diff}^1(M^2)$) is equivalent to proving that these three sets are empty.

We now explain how the discussion above is translated to higher dimensions. As before, we first consider non-hyperbolic diffeomorphisms, that is, the set $\mathrm{Diff}^1(M) \setminus \overline{\mathrm{Hyp}^1(M)}$. If $\dim(M) \geq 3$ this set is not a Newhouse domain: it contains open sets of diffeomorphisms without homoclinic tangencies. Thus we consider the sets $\mathrm{Tang}^1(M)$ of diffeomorphisms having a homoclinic tangency associated to a saddle and $\mathcal{O}_0(M)$ of *non-hyperbolic diffeomorphisms far from homoclinic tangencies,*

$$\mathcal{O}_0(M) \stackrel{\text{def}}{=} \mathrm{Diff}^1(M) \setminus \overline{(\mathrm{Tang}^1(M) \cup \mathrm{Hyp}^1(M))}.$$

Note that this set is non-empty, and it is an open question whether it is contained in the set of tame diffeomorphisms (in fact, the first author conjectured that $\mathcal{O}_0(M)$ consists of tame diffeomorphisms, [6]). The diffeomorphisms in $\mathcal{O}_0(M)$ were studied in several papers; let us just refer to [31, 32, 33].

From now on, we will focus on the set

$$\mathcal{N}^1(M) \stackrel{\text{def}}{=} \operatorname{Diff}^1(M) \setminus \overline{\left(\mathcal{O}_0(M) \cup \operatorname{Hyp}^1(M)\right)}.$$

By definition, this set is a Newhouse domain. As in the case of surface diffeomorphisms, we split the set $\mathcal{N}^1(M)$ into three closed sets with pairwise disjoint interiors. We first define the set $\mathcal{O}_1(M)$ similarly as the set $\mathcal{O}_1(M^2)$,

$$\mathcal{O}_1(M) \stackrel{\text{def}}{=} \{f \in \operatorname{Diff}^1(M) \text{ with a transitive hyperbolic set with a robust homoclinic tangency}\}.$$

The results in this paper imply that $\mathcal{O}_1(M)$ is non-empty; see also [4, 28].

We define the set $\mathcal{O}_2(M)$ by

$$\mathcal{O}_2(M) \stackrel{\text{def}}{=} \{f \in (\operatorname{Diff}^1(M) \setminus \overline{\mathcal{O}_1(M)}) \text{ with a persistently fragile homoclinic tangency}\}.$$

Consider the residual set $\mathcal{G}$ of $\operatorname{Diff}^1(M)$ in Corollary 1.6. Then if f is a diffeomorphism in $\mathcal{G} \cap \mathcal{O}_2(M)$ with a persistently fragile homoclinic tangency associated to P, then the homoclinic class $H(P, f)$ has no index variation (otherwise one gets robust homoclinic tangencies).

Finally, define $\mathcal{O}_3(M)$ by

$$\mathcal{O}_3(M) \stackrel{\text{def}}{=} \left(\operatorname{Diff}^1(M) \setminus \overline{\operatorname{Hyp}^1(M) \cup \mathcal{O}_0(M) \cup \mathcal{O}_1(M) \cup \mathcal{O}_2(M)}\right).$$

Corollary 1.6 implies that if $f \in \mathcal{G} \cap \mathcal{O}_3(M)$, then every homoclinic class of f has an indices adapted dominated splitting. The description of the accumulation of homoclinic classes of diffeomorphisms in $\mathcal{O}_3(M)$ is a subtle issue. For instance, by shrinking $\mathcal{G}$, for diffeomorphisms $f \in \mathcal{G} \cap \mathcal{O}_3(M)$, there are k and a sequence of saddles P_n of index k such that every $H(P_n, f)$ has a dominated splitting $E \oplus_< F$ with $\dim(E) = k$ and the sequence of homoclinic classes $H(P_n, f)$ accumulates to a set Λ that does not admit a dominated splitting $E \oplus_< F$ with $\dim(E) = k$ (the set Λ is the Hausdorff limit of the sequence $(H(P_n, f))$).

We observe that, as a consequence of Corollary 1.8, there is an open and dense subset of $\mathcal{O}_2(M) \cup \mathcal{O}_3(M)$ consisting of wild diffeomorphisms. Note that we do not know whether or not the sets $\mathcal{O}_2(M)$ and $\mathcal{O}_3(M)$ are empty.

Summarizing, as in the case of surface diffeomorphisms, we have that the Newhouse domain $\mathcal{N}^1(M)$ is the closure of the union of the pairwise disjoint open sets $\mathcal{O}_1(M), \mathcal{O}_2(M)$, and $\mathcal{O}_3(M)$.

This paper is organized as follows. In Section 2, we recall some definitions and state some notation we will use throughout the paper. In Section 3, we review the notion of cu-blender in [8] and present the notion of blender-horseshoe, a key ingredient of our constructions. In Section 4, we introduce a class of submanifolds, called *folding manifolds*, relative to a blender-horseshoe Λ. The main result is that folding manifolds and the local stable manifold of the blender-horseshoe Λ have C^1-robust tangencies; see Theorem 4.8. Using this result, we state a sufficient condition for the generation of robust homoclinic tangencies by homoclinic tangencies associated to hyperbolic sets. In Section 5, we see that strong homoclinic intersections of non-hyperbolic periodic points (i.e., intersections between the strong stable and unstable manifolds) generate blender-horseshoes. We also see that such strong intersections naturally occur in the non-hyperbolic setting. Finally, in Section 6 we conclude the proof of Theorem 1.2. We also state a result about the occurrence of robust heterodimensional cycles inside non-hyperbolic chain recurrence classes; see Theorem 6.13, which is an extension of [9, Theorem 1.16].

2. Definitions and notation

In this section, we define precisely the notions involved in this paper and state some notation.

Given a closed manifold M, we denote by $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms endowed with the usual uniform topology.

A diffeomorphism f has a *homoclinic tangency* associated to a (hyperbolic) saddle R if the unstable manifold $W^{\mathrm{u}}(R,f)$ and the stable manifold $W^{\mathrm{s}}(R,f)$ of the orbit of R have some non-transverse intersection.

The s-index (resp. u-index) of a hyperbolic periodic point R, denoted by $\mathrm{ind}^{\mathrm{s}}(R)$ (resp. $\mathrm{ind}^{\mathrm{u}}(R)$), is the dimension of the stable bundle E^{s} (resp. dimension of E^{u}) of R. We similarly define the s-index and u-index of a transitive hyperbolic set Λ, denoted by $\mathrm{ind}^{\mathrm{s}}(\Lambda)$ and $\mathrm{ind}^{\mathrm{u}}(\Lambda)$, respectively.

A *heterodimensional cycle* of a diffeomorphism f consists of two hyperbolic saddles P and Q of f of different s-indices and two heteroclinic points $X \in W^{\mathrm{u}}(P,f) \cap W^{\mathrm{s}}(Q,f)$ and $Y \in W^{\mathrm{s}}(P,f) \cap W^{\mathrm{u}}(Q,f)$. In this case, we say that the cycle is associated to P and Q. Note that (due to insufficient dimensions) at least one of these intersections is not transverse. The heterodimensional cycle has *co-index* k if $|\mathrm{ind}^{\mathrm{s}}(Q) - \mathrm{ind}^{\mathrm{s}}(P)| = k$ (note that $k \geq 1$).

Definition 2.1 (Homoclinic class)**.** Consider a diffeomorphism f and a saddle P of f. The *homoclinic class of* P, denoted by $H(P,f)$, is the closure of the transverse intersections of the stable and unstable manifolds of the orbit of P.

Remark 2.2. The homoclinic class $H(P,f)$ can be alternatively defined as the closure of the saddles Q *homoclinically related with* P: the stable manifold of the orbit of Q transversely meets the unstable manifold of the orbit of P, and vice versa. Although all saddles homoclinically related with P have the same s-index as P, the homoclinic class $H(P,f)$ may contain periodic orbits of a different s-index from the one of P (i.e., there are homoclinic classes having index variation). Finally, a homoclinic class is a transitive set with dense periodic points.

Definition 2.3 (Chain recurrence class)**.** A point x is *chain recurrent* if for every $\varepsilon > 0$ there are ε-pseudo-orbits starting and ending at x. The *chain recurrence class* of x for f, denoted by $C(x,f)$, is the set of points y such that, for every $\varepsilon > 0$, there are ε-pseudo-orbits starting at x, passing ε-close to y and ending at x.

According to [7], for C^1-generic diffeomorphisms, the chain recurrence class of any periodic point is its homoclinic class.

Definition 2.4 (Dominated splitting)**.** Consider a diffeomorphism f and a compact f-invariant set Λ. A Df-invariant splitting $T_\Lambda M = E \oplus F$ over Λ is *dominated* if the fibers E_x and F_x of E and F have constant dimension and there exists $k \in \mathbb{N}$ such that

$$\frac{||D_x f^k(u)||}{||D_x f^k(w)||} < \frac{1}{2},$$

for every $x \in \Lambda$ and every pair of unitary vectors $u \in E_x$ and $w \in F_x$.

This definition means that vectors in the bundle F are uniformly more expanded than vectors in E by the derivative Df^k. If it occurs, we say that F *dominates* E and write $E \oplus_< F$.

Remark 2.5. In some cases, one needs to consider splittings with more than two bundles. A Df-invariant splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ over a set Λ is dominated if for all $j \in \{1, \dots, k-1\}$ the splitting $E_1^j \oplus E_{j+1}^k$ is dominated, where $E_i^r = E_i \oplus \cdots \oplus E_r$, $i < r$.

We use the notation $E_1 \oplus_< E_2 \oplus_< \cdots \oplus_< E_k$, meaning that E_{i+1} dominates E_i, or equivalently that $E_1^j \oplus_< E_{j+1}^k$.

As mentioned before, the main goal of this paper is to construct hyperbolic sets exhibiting homoclinic tangencies in a robust way. We need the following definition.

Definition 2.6 (Robust tangency)**.** Given a diffeomorphism $f\colon M \to M$, a hyperbolic set Γ of f with a hyperbolic splitting $E^{\mathrm{s}} \oplus E^{\mathrm{u}}$, and a submanifold $N \subset M$ with dimension $\dim(N) = \dim(E^{\mathrm{u}})$, we say that the stable manifold $W^{\mathrm{s}}(\Gamma)$ of Γ and the submanifold N have a C^1*-robust tangency* if for every diffeomorphism g C^1 close to f and every submanifold N_g C^1-close to N, the stable manifold $W^{\mathrm{s}}(\Gamma_g)$ of Γ_g has some non-transverse intersection with N_g.

We are especially interested in the case where N is the unstable manifold $W^{\mathrm{u}}(P)$ of a periodic point P of a non-trivial hyperbolic set Γ and $N_g = W^{\mathrm{u}}(P_g)$. In that case, one gets C^1-robust homoclinic tangencies (associated to Γ); recall Definition 1.1.

Standing notation. Throughout this paper we use the following notation:

- Given a diffeomorphism f and a hyperbolic set Λ_f of f there is a C^1-neighborhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ has a hyperbolic set Λ_g called the *continuation* of Λ_f. The set Λ_g is close to Λ_f, and the restrictions of f to Λ_f and of g to Λ_g are conjugate. If P_f is a hyperbolic periodic point, we denote by P_g the continuation of P_f for g close to f.
- Given a periodic point P of f we denote by $\pi(P)$ its period.
- The perturbations we consider are always arbitrarily small. Thus the sentence *there is a C^r-perturbation g of f* means *there is g arbitrarily C^r-close to f*.

3. Blender-horseshoes

In this section, we introduce precisely the definition of a *blender-horseshoe,* a particular case of the blenders in [8]. In fact, blender-horseshoes are the main ingredient of this paper and the key tool for getting robust homoclinic tangencies. We begin by reviewing the notion of a blender.

3.1. **Blenders.** The notion of a cu-blender was introduced in [8] as a class of examples, without a precise and formal definition. Blenders were used to get C^1-robust transitivity, [8], and robust heterodimensional cycles, [9]. The relevance of blenders comes from their internal geometry and not from their dynamics: a cu-blender is a (uniformly) hyperbolic transitive set whose stable set robustly has Hausdorff dimension greater than its stable bundle. In some sense, this property resembles and plays a similar role as the thick horseshoes introduced by Newhouse, [21]. Following [13, Definition 6.11], we now give a tentative formal definition of a cu-blender:

Definition 3.1 (cu-blender)**.** Let $f\colon M \to M$ be a diffeomorphism. A transitive hyperbolic set Γ of f with $\operatorname{ind}^{\mathrm{u}}(\Gamma) = k \geq 2$ is a cu-*blender* if there are a C^1-neighborhood $\mathcal{U}$ of f and a C^1-open set $\mathcal{D}$ of embeddings of $(k-1)$-dimensional disks D into M such that, for every diffeomorphism $g \in \mathcal{U}$, every disk $D \in \mathcal{D}$ intersects the local stable manifold $W^s_{\mathrm{loc}}(\Gamma_g)$ of the continuation Γ_g of Γ for g. The set $\mathcal{D}$ is called the *superposition* region of the blender.

By definition, the property of a diffeomorphism having a cu-blender is a C^1-robust property.

We do not know whether cu-blenders yield robust tangencies in the sense of Definition 2.6. This leads to the following questions:

Question 3.2. Let $f\colon M \to M$ be a diffeomorphism having a cu-blender Γ with $k = \mathrm{ind}^{\mathrm{u}}(\Gamma)$.

- Does there exist a submanifold $N \subset M$ with $\dim(N) = k$ such that $W^s(\Gamma)$ and N have a robust tangency?
- Suppose that a submanifold L of dimension k and $W^s(\Gamma)$ have a tangency. Does this tangency yield robust tangencies? More precisely, does there exist an open set $\mathcal{U}$ of $\mathrm{Diff}^1(M)$, f in the closure of $\mathcal{U}$, of diffeomorphisms g with robust tangencies associated to Γ_g and "continuations" of L?

We note that, even for the first cu-blenders constructed in [8, Section 1], these questions remain open. We will give a partial answer to this question in Theorem 4.8. For that we will introduce a special class of cu-blenders, conjugate to the usual Smale horseshoe, which we call *blender-horseshoes.*

3.2. **Blender-horseshoes.** In this section, we give the precise definition of a *blender-horseshoe*. This definition involves several concepts as invariant cone-fields, hyperbolicity, partial hyperbolicity, and Markov partitions, which we will present separately. Our presentation closely follows [8, Section 1], thus some details of our construction are just sketched.

3.2.1. *Cone-fields.* Consider $\mathbb{R}^n = \mathbb{R}^s \oplus \mathbb{R} \oplus \mathbb{R}^u$, where $s > 0, u > 0$, and $n = s+u+1$. For $\alpha \in (0,1)$, denote by $\mathcal{C}^{\mathrm{s}}_\alpha$, $\mathcal{C}^{\mathrm{u}}_\alpha$, and $\mathcal{C}^{\mathrm{uu}}_\alpha$ the following cone-fields:

$$\begin{array}{lll}
\mathcal{C}^{\mathrm{s}}_\alpha(x) & = \{v = (v^s, v^c, v^u) \in \mathbb{R}^s \oplus \mathbb{R} \oplus \mathbb{R}^u = T_xM & : \quad \|v^c + v^u\| \le \alpha\, \|v^s\|\},\\
\mathcal{C}^{\mathrm{u}}_\alpha(x) & = \{v = (v^s, v^c, v^u) \in \mathbb{R}^s \oplus \mathbb{R} \oplus \mathbb{R}^u = T_xM & : \quad \|v^s\| \le \alpha\, \|v^c + v^u\|\},\\
\mathcal{C}^{\mathrm{uu}}_\alpha(x) & = \{v = (v^s, v^c, v^u) \in \mathbb{R}^s \oplus \mathbb{R} \oplus \mathbb{R}^u = T_xM & : \quad \|v^s + v^c\| \le \alpha\, \|v^u\|\}.
\end{array}$$

As $\alpha \in (0,1)$, one gets that $\mathcal{C}^{\mathrm{s}}_\alpha$ is transverse to $\mathcal{C}^{\mathrm{u}}_\alpha$, that is, $\mathcal{C}^{\mathrm{s}}_\alpha(x) \cap \mathcal{C}^{\mathrm{u}}_\alpha(x) = 0_x \in T_xM$. Moreover, since $\{0^s\} \times \mathbb{R} \times \{0^u\}$ is orthogonal to $\mathbb{R}^s \times \{(0, 0^u)\}$ and to $\{(0^s, 0)\} \times \mathbb{R}^u$, one has that $\mathcal{C}^{\mathrm{uu}}_\alpha(x) \subset \mathcal{C}^{\mathrm{uu}}_\alpha(x)$ for all x.

Consider the cube

$$\mathbb{C} = [-1,1]^n = [-1,1]^s \times [-1,1] \times [-1,1]^u.$$

We split the boundary of $\mathbb{C}$ into three parts:

$$\begin{array}{ll}
\partial^{\mathrm{s}}\mathbb{C} & = \partial\big([-1,1]^s\big) \times [-1,1] \times [-1,1]^u,\\
\partial^{\mathrm{c}}\mathbb{C} & = [-1,1]^s \times \{-1,1\} \times [-1,1]^u, \quad \text{and}\\
\partial^{\mathrm{uu}}\mathbb{C} & = [-1,1]^s \times [-1,1] \times \partial\big([-1,1]^u\big).
\end{array}$$

We also consider

$$\partial^{\mathrm{u}}\mathbb{C} = [-1,1]^s \times \partial\big([-1,1] \times [-1,1]^u\big) = \partial^c\mathbb{C} \cup \partial^{\mathrm{uu}}\mathbb{C}.$$

We now consider a local diffeomorphism $f\colon \mathbb{C} \to \mathbb{R}^n$ and formulate conditions **BH1)**–**BH6)** for the maximal invariant set Λ of f in the cube $\mathbb{C}$,

$$\Lambda = \bigcap_{i\in\mathbb{Z}} f^i(\mathbb{C}),$$

to be a *blender-horseshoe,* see Definition 3.9.

BH1) The intersection $f(\mathbb{C}) \cap (\mathbb{R}^s \times \mathbb{R} \times [-1,1]^u)$ consists of two connected components, denoted $f(\mathcal{A})$ and $f(\mathcal{B})$. Furthermore,

$$f(\mathcal{A}) \cup f(\mathcal{B}) \subset (-1,1)^s \times \mathbb{R} \times [-1,1]^u \text{ and}$$
$$(\mathcal{A} \cup \mathcal{B}) \cap \partial^{\mathrm{uu}}(\mathbb{C}) = \emptyset.$$

We denote $f_{\mathcal{A}}\colon \mathcal{A} \to f(\mathcal{A})$ and $f_{\mathcal{B}}\colon \mathcal{B} \to f(\mathcal{B})$ as the restrictions of f to $\mathcal{A}$ and $\mathcal{B}$, respectively. See Figure 1.

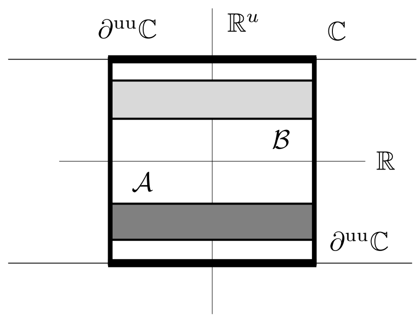

FIGURE 1. Projection in $\mathbb{R} \oplus \mathbb{R}^u$ of a blender-horseshoe. Condition **BH1)**.

BH2) Cone-fields: The cone-field $\mathcal{C}^{\mathrm{s}}_\alpha$ is strictly Df^{-1}-invariant and the cone-fields $\mathcal{C}^{\mathrm{u}}_\alpha$ and $\mathcal{C}^{\mathrm{uu}}_\alpha$ are strictly Df-invariant. More precisely, there is $0 < \alpha' < \alpha$ such that, for every $x \in f(\mathcal{A}) \cup f(\mathcal{B})$, one has

$$Df^{-1}(\mathcal{C}^{\mathrm{s}}_\alpha(x)) \subset \mathcal{C}^{\mathrm{s}}_{\alpha'}(f^{-1}x).$$

In the same way, for every $x \in \mathcal{A} \cup \mathcal{B}$, one has

$$Df(\mathcal{C}^{\mathrm{uu}}_\alpha(x)) \subset \mathcal{C}^{\mathrm{uu}}_{\alpha'}(f(x)) \quad \text{and} \quad Df(\mathcal{C}^{\mathrm{u}}_\alpha(x)) \subset \mathcal{C}^{\mathrm{u}}_{\alpha'}(f(x)).$$

Moreover, the cone-fields $\mathcal{C}^{\mathrm{u}}_\alpha$ and $\mathcal{C}^{\mathrm{s}}_\alpha$ are uniformly expanding and contracting, respectively.

Note that property **BH2)** is open: by increasing $\alpha' < \alpha$ slightly, it holds for every diffeomorphism g in a C^1 neighborhood of f.

Since $f(\partial^{\mathrm{uu}}\mathbb{C})$ is disjoint from $\mathbb{R}^{s+1} \times [-1,1]^u$ and $f(\mathcal{A}) \cup f(\mathcal{B}) \subset (-1,1)^s \times \mathbb{R} \times [-1,1]^u$, from condition **BH1)** one has that

$$f(\partial(\mathbb{C})) \cap \partial([-1,1]^s \times \mathbb{R} \times [-1,1]^u) \subset f(\partial^{\mathrm{s}}\mathbb{C} \cup \partial^{\mathrm{c}}\mathbb{C}).$$

Furthermore, this is a C^1-robust property.

By **BH2)**, the components of $f(\partial(\mathbb{C})) \cap \partial([-1,1]^s \times \mathbb{R} \times [-1,1]^u)$ are foliated by disks Δ tangent to $\mathcal{C}^{\mathrm{uu}}_\alpha$, i.e., $T_x\Delta \subset \mathcal{C}^{\mathrm{uu}}_\alpha(x)$. Hence these disks are transverse to $\partial([-1,1]^s \times \mathbb{R} \times [-1,1]^u)$. As a consequence, one gets the following:

Remark 3.3. Under the (C^1-robust) hypothesis **BH2)**, hypothesis **BH1)** is also a C^1-robust property.

Remark 3.4 (Hyperbolicity). Consider the maximal invariant set Λ of f in $\mathbb{C}$,

$$\Lambda = \bigcap_{i\in\mathbb{Z}} f^i(\mathbb{C}).$$

By **BH1)** and **BH2)** the set Λ is compact and satisfies

$$\Lambda \subset \operatorname{int}(\mathcal{A}\cup\mathcal{B}) \subset \operatorname{int}(\mathbb{C}).$$

Moreover, the set Λ has a dominated splitting $T_\Lambda M = E \oplus_< F \oplus_< F$, where $E\subset\mathcal{C}^{\mathrm{s}}$, $F\oplus G\subset\mathcal{C}^{\mathrm{u}}$, $G\subset\mathcal{C}^{\mathrm{uu}}$, and F is one-dimensional.

By **BH2)**, the set Λ has a hyperbolic splitting $E^{\mathrm{s}}\oplus E^{\mathrm{u}}$, where $E^{\mathrm{s}}=E$ and $E^{\mathrm{u}}=F\oplus G$ and $\dim(E^{\mathrm{s}})=s$ and $\dim(E^{\mathrm{u}})=u+1$. Furthermore, the set Λ also has a partially hyperbolic splitting

$$T_\Lambda M = E^{\mathrm{s}}\oplus_< E^{\mathrm{cu}}\oplus_< E^{\mathrm{uu}}$$

with three non-trivial directions, where $E^{\mathrm{cu}}=F$ and $E^{\mathrm{uu}}=G$ and $\dim(E^{\mathrm{uu}})=u$. Note that $E^{\mathrm{u}}=E^{\mathrm{cu}}\oplus E^{\mathrm{uu}}$. We say that E^{uu} is the *strong unstable bundle* of Λ.

3.2.2. *Markov partitions.* Write

$$\mathbb{A} = f^{-1}\left(f(\mathcal{A})\cap\mathbb{C}\right) \quad\text{and}\quad \mathbb{B} = f^{-1}\left(f(\mathcal{B})\cap\mathbb{C}\right).$$

BH3) Associated Markov partition:

- The sets $\mathbb{A}$ and $\mathbb{B}$ are both non-empty and connected. That is, the sets $\mathbb{A}$ and $\mathbb{B}$ are the connected components of $f^{-1}(\mathbb{C})\cap\mathbb{C}$.
- The sets $\mathbb{A}$ and $\mathbb{B}$ are *horizontal subcubes* of $\mathbb{C}$, and their images $f(\mathbb{A})$ and $f(\mathbb{B})$ are *vertical subcubes* of $\mathbb{C}$. More precisely,

$$f(\mathbb{A})\cup f(\mathbb{B}) \subset (-1,1)^s\times[-1,1]\times[-1,1]^u \text{ and}$$
$$\mathbb{A}\cup\mathbb{B}\subset[-1,1]^s\times(-1,1)\times(-1,1)^u.$$

In other words, $f(\mathbb{A})\cup f(\mathbb{B})$ is disjoint from $\partial^{\mathrm{s}}\mathbb{C}$ and $\mathbb{A}\cup\mathbb{B}$ is disjoint from $\partial^{\mathrm{u}}\mathbb{C}$.

As a consequence of **BH2)** and **BH3)**, one obtains that $\{\mathbb{A},\mathbb{B}\}$ is a Markov partition generating Λ. Therefore the dynamics of f in Λ are conjugate to the full shift of two symbols. In particular, the hyperbolic set Λ contains exactly two fixed points of f, $P\in\mathbb{A}$ and $Q\in\mathbb{B}$. See Figure 2.

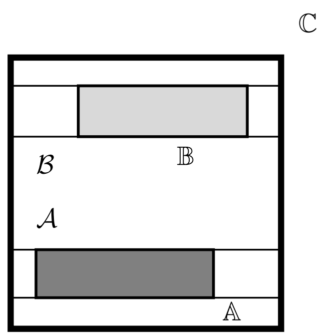

FIGURE 2. Projection in $\mathbb{R}\times\mathbb{R}^u$. **BH3)** Markov partition of a blender-horseshoe.

3.2.3. uu-*disks and their iterates.*

Definition 3.5 (s- and uu-disks)**.** A disk Δ of dimension s contained in $\mathbb{C}$ is an s-*disk* if

- it is tangent to $\mathcal{C}^{\mathrm{s}}_\alpha$, i.e., $T_x\Delta \subset \mathcal{C}^{\mathrm{s}}_\alpha(x)$ for all $x \in \Delta$, and
- its boundary $\partial\Delta$ is contained in $\partial^{\mathrm{s}}(\mathbb{C})$.

A disk $\Upsilon \subset \mathbb{R}^s \times \mathbb{R} \times [-1,1]^u$ of dimension u is a uu-*disk* if

- it is tangent to $\mathcal{C}^{\mathrm{uu}}_\alpha$, i.e. $T_x\Upsilon \subset \mathcal{C}^{\mathrm{uu}}_\alpha(x)$ for all $x \in \Upsilon$, and
- $\partial\Upsilon \subset \mathbb{R}^s \times \mathbb{R} \times \partial([-1,1]^u)$.

Given a point $x \in \Lambda$, there is a unique f-invariant manifold of dimension u tangent at x to the strong unstable bundle $E^{\mathrm{uu}}(x)$, the *strong unstable manifold* $W^{\mathrm{uu}}(x)$ *of* x. For points $x \in \Lambda$, the local invariant manifolds $W^{\mathrm{s}}_{\mathrm{loc}}(x)$, $W^{\mathrm{u}}_{\mathrm{loc}}(x)$, and $W^{\mathrm{uu}}_{\mathrm{loc}}(x)$ are the connected components of the intersections $W^{\mathrm{s}}(x) \cap \mathbb{C}$, $W^{\mathrm{u}}(x) \cap \mathbb{C}$, and $W^{\mathrm{uu}}(x) \cap \mathbb{C}$ containing x, respectively.

As a consequence of **BH1)**–**BH3)** one gets

Remark 3.6. For every $x \in \Lambda$, $W^{\mathrm{s}}_{\mathrm{loc}}(x)$ is an s-disk and $W^{\mathrm{uu}}_{\mathrm{loc}}(x)$ is a uu-disk.

BH4) uu-disks through the local stable manifolds of P and Q: Let D and D' be uu-disks such that $D \cap W^{\mathrm{s}}_{\mathrm{loc}}(P) \neq \emptyset$ and $D' \cap W^{\mathrm{s}}_{\mathrm{loc}}(Q) \neq \emptyset$. Then

$$D \cap \partial^{\mathrm{c}}(\mathbb{C}) = D' \cap \partial^{\mathrm{c}}(\mathbb{C}) = D \cap D' = \emptyset;$$

see Figure 3.

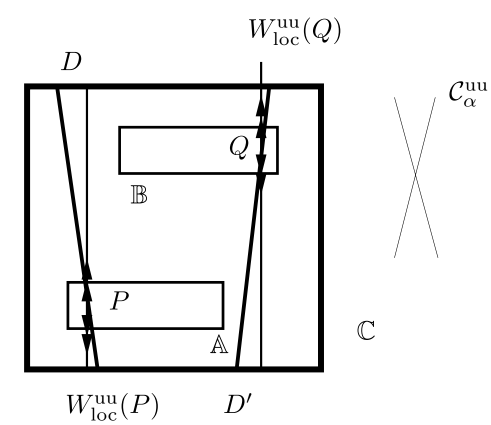

FIGURE 3. Projection in $\mathbb{R} \oplus \mathbb{R}^u$. uu-disks. Condition **BH4)**.

Given any s-disk Δ, there are two different homotopy classes of uu-disks contained in $[-1,1]^s \times \mathbb{R} \times [-1,1]^u$ and disjoint from Δ. We call these classes uu-*disks at the right* and *at the left* of Δ. We use the following criterion: the uu-disks disjoint from $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ in the homotopy class of $W^{\mathrm{uu}}_{\mathrm{loc}}(Q)$ are at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$. The uu-disks disjoint from $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ in the other homotopy class are at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$. We similarly define uu-disks at the left and at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$, where uu-disks at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ are those in the class of $W^{\mathrm{uu}}_{\mathrm{loc}}(P)$.

According to **BH4)**, uu-disks at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ are also at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. Analogously, uu-disks at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ are also at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$.

Summarizing, there are five possibilities for a uu-disk D in $[-1,1]^s \times \mathbb{R} \times [-1,1]^u$:

- either D is at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$,
- or $D \cap W^{\mathrm{s}}_{\mathrm{loc}}(P) \neq \emptyset$,
- or D is at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$,
- or $D \cap W^{\mathrm{s}}_{\mathrm{loc}}(Q) \neq \emptyset$,
- or else D is at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. In this case, we say that the uu-disk D is *in between* $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ *and* $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$.

As a consequence of **BH4)** one gets the following.

Remark 3.7 (uu-disks in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$).

(1) There is a non-empty open subset U of $\mathbb{C}$ such that any uu-disk through a point $x \in U$ is in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. In particular, there exist uu-disks in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$.
(2) Every uu-disk $\Delta \subset [-1,1]^s \times \mathbb{R} \times [-1,1]^u$ in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ is contained in $\mathbb{C}$ and is disjoint from $\partial^{\mathrm{c}}(\mathbb{C})$.

Consider a uu-disk $\Delta \subset \mathbb{C}$ and write

$$f_{\mathcal{A}}(\Delta) = f(\Delta \cap \mathcal{A}) \quad \text{and} \quad f_{\mathcal{B}}(\Delta) = f(\Delta \cap \mathcal{B}).$$

According to **BH1)** and **BH2)** one gets

Remark 3.8. For every uu-disk $\Delta \subset \mathbb{C}$, $f_{\mathcal{A}}(\Delta)$ and $f_{\mathcal{B}}(\Delta)$ are uu-disks in $[-1,1]^s \times \mathbb{R} \times [-1,1]^u$.

BH5) Positions of images of uu-**disks (I):** Given any uu-disk $\Delta \subset \mathbb{C}$, the following holds:

(1) if Δ is at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, then $f_{\mathcal{A}}(\Delta)$ is a uu-disk at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$,
(2) if Δ is at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, then $f_{\mathcal{A}}(\Delta)$ is a uu-disk at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$,
(3) if Δ is at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$, then $f_{\mathcal{B}}(\Delta)$ is a uu-disk at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$,
(4) if Δ is at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$, then $f_{\mathcal{B}}(\Delta)$ is a uu-disk at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$,
(5) if Δ is at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ or $\Delta \cap W^{\mathrm{s}}_{\mathrm{loc}}(P) \neq \emptyset$, then $f_{\mathcal{B}}(\Delta)$ is a uu-disk at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, and
(6) if Δ is at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ or $\Delta \cap W^{s}_{\mathrm{loc}}(Q) \neq \emptyset$, then $f_{\mathcal{A}}(\Delta)$ is a uu-disk at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$.

Finally, we state the last condition (which will play a key role) in the definition of a blender-horseshoe:

BH6) Positions of images of uu-**disks (II):** Let Δ be a uu-disk in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. Then either $f_{\mathcal{A}}(\Delta)$ or $f_{\mathcal{B}}(\Delta)$ is a uu-disk in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$.

Conditions **BH5)–BH6)** are depicted in Figure 4.

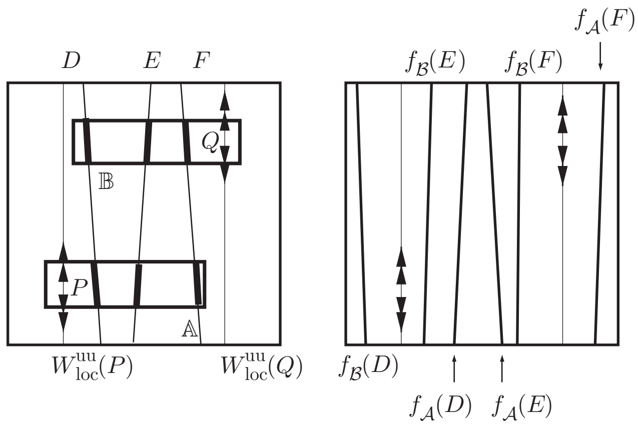

FIGURE 4. Projection in $\mathbb{R} \oplus \mathbb{R}^u$. Disks in between $W^s_{\mathrm{loc}}(P)$ and $W^s_{\mathrm{loc}}(Q)$ and their images.

3.2.4. *Definition of blender-horseshoe.* We now define blender-horseshoes:[4]

Definition 3.9 (Blender-horseshoe)**.** Consider a manifold M of dimension $n \geq 3$ and a diffeomorphism $f\colon M \to M$. A hyperbolic set Λ of f is a *blender-horseshoe* if (in some coordinate system) there are a cube $\mathbb{C}$ and families of cone-fields $\mathcal{C}^{\mathrm{s}}$, $\mathcal{C}^{\mathrm{u}}$, and $\mathcal{C}^{\mathrm{uu}}$ verifying conditions **BH1)–BH6)** above.

We say that $\mathbb{C}$ is the *reference cube* of the blender-horseshoe Λ and that the saddles P and Q are the *reference saddles* of Λ, where P *is the left saddle* and Q *is the right saddle.*

Recall that given a hyperbolic set Λ of a diffeomorphism f there is a C^1-neighborhood $\mathcal{U}_f$ of f such that every diffeomorphism $g \in \mathcal{U}_f$ has a hyperbolic set Λ_g which is close and conjugate to Λ, called the *continuation* of Λ for g. Following [8, Lemma 1.11], one can prove the following:

Lemma 3.10. *Let Λ be a blender-horseshoe of a diffeomorphism f with reference cube $\mathbb{C}$ and reference saddles P and Q. Then there is a neighborhood $\mathcal{U}_f$ of f in* $\mathrm{Diff}^1(M)$ *such that for all $g \in \mathcal{U}_g$ the continuation Λ_g of Λ for g is a blender-horseshoe with reference cube $\mathbb{C}$ and reference saddles P_g and Q_g.*

Arguing as in [8] (see also [12] for a simple toy model and the example in Section 5.1) one gets the following:

Remark 3.11. Every uu-disk in between $W^s_{\mathrm{loc}}(P)$ and $W^s_{\mathrm{loc}}(Q)$ intersects $W^s_{\mathrm{loc}}(\Lambda)$. Therefore the blender-horseshoe Λ is a cu-blender in the sense of Definition 3.1, where the uu-disks in between $W^s_{\mathrm{loc}}(P)$ and $W^s_{\mathrm{loc}}(Q)$ define its superposition region.

[4] In some cases, we will use the terminology cu-blender-horseshoe for emphasizing that the central one-dimensional direction is expanding. Using this terminology, a cs-blender-horseshoe is a blender-horseshoe for f^{-1}.

4. Robust tangencies

In this section, we introduce a class of submanifolds called folding manifolds relative to a blender-horseshoe Λ. The main technical step is Proposition 4.4, which claims that folding manifolds are tangent to the local stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$ of the blender-horseshoe Λ. Moreover, these tangencies are C^1-robust; see Theorem 4.8. Finally, using blender-horseshoes, in Theorem 4.9 we give sufficient conditions for the generation of robust tangencies by a homoclinic tangency.

4.1. **Folding manifolds and tangencies associated to blender-horseshoes.**

Let Λ be a blender-horseshoe with reference cube $\mathbb{C}$ as in Section 3.2. Recall that the dimension of the unstable bundle of Λ, $E^{\mathrm{u}} = E^{\mathrm{cu}} \oplus E^{\mathrm{uu}}$, is $(u+1)$. We say that *the* u*-index of the blender-horseshoe* Λ *is* $(u+1)$ (i.e., the u-index of Λ as a hyperbolic set). Define the *local stable manifold of* Λ by

$$W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda) = \bigcup_{x\in\Lambda} W^{\mathrm{s}}_{\mathrm{loc}}(x),$$

where $W^{\mathrm{s}}_{\mathrm{loc}}(x)$ is the connected component of $W^{\mathrm{s}}(x) \cap \mathbb{C}$ containing x.

Remark 4.1. The local stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$ of the blender-horseshoe Λ is the set of points $x \in \mathbb{C}$ whose forward orbit remains in the reference cube $\mathbb{C}$.

A new ingredient of this section is the notion of a folding manifold defined as follows:

Definition 4.2 (Folding manifold)**.** Consider a blender-horseshoe Λ of u-index $(u+1)$ with reference cube $\mathbb{C}$ and reference saddles P and Q. A submanifold $\mathcal{S} \subset \mathbb{C}$ of dimension $(u+1)$ is a *folding manifold of* Λ *(relative to the saddle* P*)* if there is a family $(\mathcal{S}_t)_{t\in[0,1]}$ of uu-disks depending continuously on t such that:

- $\mathcal{S} = \bigcup_{t\in[0,1]} \mathcal{S}_t$,
- $\mathcal{S}_0$ and $\mathcal{S}_1$ intersects $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, and
- for every $t \in (0,1)$, the uu-disk $\mathcal{S}_t$ is in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$.

We similarly define a *folding manifold (relative to* Q*)*. A *folding manifold* of the blender-horseshoe Λ is a folding manifold relative either to P or to Q.

Remark 4.3. Let $\mathcal{S}$ be a folding manifold of the blender-horseshoe Λ. Then there are a point $x \in \mathcal{S}$ and a non-zero vector $v \in T_x\mathcal{S}$ such that $v \in \mathcal{C}^{\mathrm{s}}(x)$.

A key property of folding manifolds of blender-horseshoes is the following:

Proposition 4.4. *Let* $\mathcal{S}$ *be a folding manifold of a blender-horseshoe* Λ*. Then* $\mathcal{S}$ *and* $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$ *are tangent at some point* z*.*

To prove this proposition we need the following lemma:

Lemma 4.5. *Consider a diffeomorphism* f *having a blender-horseshoe* Λ *as above. The image by* f *of a folding manifold* $\mathcal{S}$ *of* Λ *contains a folding manifold of* Λ*.*

Proof. Let us assume, for instance, that the folding manifold $\mathcal{S}$ is relative to P. We will prove that either $f_{\mathcal{A}}(\mathcal{S})$ is a folding manifold relative to P or $f_{\mathcal{B}}(\mathcal{S})$ contains a folding manifold relative to P.

If $f_{\mathcal{A}}(\mathcal{S})$ is a folding manifold relative to P we are done. So we can assume that $f_{\mathcal{A}}(\mathcal{S})$ is not a folding manifold. As the uu-disks $\mathcal{S}_0$ and $\mathcal{S}_1$ meet $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, by Remark 3.8, their images $f_{\mathcal{A}}(\mathcal{S}_0)$ and $f_{\mathcal{A}}(\mathcal{S}_1)$ are uu-disks intersecting $W^{\mathrm{s}}_{\mathrm{loc}}(P)$.

Furthermore, by item (1) in **BH5)**, the uu-disk $f_{\mathcal{A}}(\mathcal{S}_t)$ is at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, for every $t \in (0,1)$.

Since we are assuming that $f_{\mathcal{A}}(\mathcal{S})$ is not a folding manifold relative to P, by definition of a folding manifold relative to P, there is some $t_0 \in (0,1)$ such that $f_{\mathcal{A}}(\mathcal{S}_{t_0})$ is not at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ (i.e., it is either at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ or it meets $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$). Thus, by continuity of the disks $f_{\mathcal{A}}(\mathcal{S}_t)$, there is $t_1 \in (0,t_0)$ such that $f_{\mathcal{A}}(\mathcal{S}_{t_1}) \cap W^{\mathrm{s}}_{\mathrm{loc}}(Q) \neq \emptyset$. As $\mathcal{S}_{t_1}$ is in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$, by **BH6)**, the image $f_{\mathcal{B}}(\mathcal{S}_{t_1})$ is in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$.

By the definition of a folding manifold relative to P, every $\mathcal{S}_t$ is at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. Therefore, by item (4) in **BH5)**, $f_{\mathcal{B}}(\mathcal{S}_t)$ is a uu-disk at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$, for every $t \in (0,1)$. Moreover, by item (5) in **BH5)**, the images $f_{\mathcal{B}}(\mathcal{S}_0)$ and $f_{\mathcal{B}}(\mathcal{S}_1)$ are uu-disks at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$.

Now by continuity of the disks $f_{\mathcal{B}}(\mathcal{S}_t)$ and since $f_{\mathcal{B}}(\mathcal{S}_{t_1})$ is at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$), there are parameters t_2 and t_3, with $t_2 < t_1 < t_3$, such that $f_{\mathcal{B}}(\mathcal{S}_{t_2})$ and $f_{\mathcal{B}}(\mathcal{S}_{t_3})$ are uu-disks intersecting $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $f_{\mathcal{B}}(\mathcal{S}_t)$ is a uu-disk at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, for all $t \in (t_2,t_3)$. Since, by item (4) in **BH5)**, these disks are at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$, they are in between $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. This implies that

$$\bigcup_{t \in [t_2,t_3]} f_{\mathcal{B}}(\mathcal{S}_t) \subset f_{\mathcal{B}}(\mathcal{S})$$

is a folding manifold relative to P, ending the proof of the lemma. □

We are now ready to conclude the proof of Proposition 4.4.

Proof of Proposition 4.4. Write $\mathcal{S}_0 = \mathcal{S}$. By Lemma 4.5, there is a folding manifold $\mathcal{S}_1$ contained in $f(\mathcal{S}_0)$. Using Lemma 4.5 and arguing inductively, we define a sequence of folding manifolds $(\mathcal{S}_i)_i$ of the blender-horseshoe Λ such that, for every $i \geq 0$, $\mathcal{S}_{i+1}$ is contained in $f(S_i)$. Let

$$\tilde{\mathcal{S}}_i = f^{-i}(S_i).$$

In this way we get a nested sequence $(\tilde{\mathcal{S}}_i)_i$, $\tilde{\mathcal{S}}_{i+1} \subset \tilde{\mathcal{S}}_i \subset \mathcal{S}$, of connected and compact sets. Thus, by construction, the intersection set

$$\mathcal{S}_\infty = \bigcap_{i=0}^{\infty} \tilde{\mathcal{S}}_i \neq \emptyset$$

is connected and compact. Moreover, $\mathcal{S}_\infty \subset \mathcal{S}_0$.

By construction, the whole forward orbit of the set $\mathcal{S}_\infty$ is contained in the reference cube $\mathbb{C}$ of the blender. By Remark 4.1, the set $\mathcal{S}_\infty$ is contained in $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$. Note that, as Λ is totally disconnected (a Cantor set) and $\mathcal{S}_\infty$ is connected, there is some $z \in \Lambda$ such that

$$\mathcal{S}_\infty \subset W^{\mathrm{s}}_{\mathrm{loc}}(z).$$

Since $\mathcal{S}_i$ is a folding manifold for every i, Remark 4.3 implies that there is a point $x_i \in \mathcal{S}_i$ and a non-zero vector $v_i \in T_{x_i}\mathcal{S}_i$ such that $v_i \in \mathcal{C}^{\mathrm{s}}_\alpha(x_i)$.

Consider the point $\tilde{x}_i = f^{-i}(x_i) \in \tilde{\mathcal{S}}_i \subset \mathcal{S}$ and a unitary vector $\tilde{v}_i$ parallel to $D_{x_i}f^{-i}(v_i)$. Note that $\tilde{v}_i \in T_{\tilde{x}_i}\tilde{\mathcal{S}}_i$, thus $\tilde{v}_i \in T_{\tilde{x}_i}\mathcal{S}$. By the (Df^{-1})-invariance of the cone-field $\mathcal{C}^{\mathrm{s}}_\alpha$, condition **BH2)**, we have that $\tilde{v}_i \in \mathcal{C}^{\mathrm{s}}_\alpha(\tilde{x}_i)$. We can assume (taking a subsequence if necessary) that

$$\tilde{x}_i \to x_\infty \in \mathcal{S}_\infty \subset \mathcal{S} \cap W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda) \qquad \text{and} \qquad \tilde{v}_i \to v_\infty.$$

Hence $x_\infty \in \mathcal{S}_\infty$ and $x_\infty \in W^{\mathrm{s}}_{\mathrm{loc}}(z)$. Our construction also implies that $v_\infty \in T_{x_\infty}\mathcal{S}$. Finally, also by construction, the vector v_∞ belongs to the intersection

$$\bigcap_{i\geq 0} Df^{-i}(\mathcal{C}^{\mathrm{s}}_\alpha(f^i(x_\infty))) = T_{x_\infty} W^{\mathrm{s}}_{\mathrm{loc}}(z).$$

This completes the proof of the proposition. □

4.2. **Robust tangencies.** We now return to the problem of robust tangencies in Question 3.2. We need the following definition.

Definition 4.6 (Folded submanifolds with respect to a blender-horseshoe)**.** Let $f\colon M \to M$ be a diffeomorphism having a blender-horseshoe Λ with reference saddles P and Q and let $N \subset M$ be a submanifold of dimension $\mathrm{ind}^{\mathrm{u}}(\Lambda)$.

We say that N is *folded with respect to* Λ if the interior of N contains a folding manifold $\mathcal{S} = (\mathcal{S}_t)_{t\in[0,1]}$ relative to some reference saddle $A \in \{P, Q\}$ of the blender. Here $(\mathcal{S}_t)_{t\in[0,1]}$ is the family of uu-disks in Definition 4.2, such that:

- $\mathcal{S}_0 \cap W^{\mathrm{s}}_{\mathrm{loc}}(A)$ and $\mathcal{S}_1 \cap W^{\mathrm{s}}_{\mathrm{loc}}(A)$ are transverse intersection points of N with $W^{\mathrm{s}}_{\mathrm{loc}}(A)$.
- There is $0 < \alpha' < \alpha$ such that the uu-disks $\mathcal{S}_t$, $t \in [0, 1]$, are tangent to the cone-field $\mathcal{C}^{\mathrm{uu}}_{\alpha'}$.

To emphasize the reference saddle A of the blender we consider, we say that the submanifold N is *folded with respect to* (Λ, A).

Remark 4.7. A submanifold to be folded with respect to a blender-horseshoe is a C^1-open property.

As a direct consequence of Proposition 4.4 and Remark 4.7 one gets:

Theorem 4.8. *Let $N \subset M$ be a folded submanifold with respect to a blender-horseshoe Λ. Then N and $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$ have a C^r-robust tangency.*

4.3. **Robust homoclinic tangencies.** In this section we prove that homoclinic tangencies associated to blender-horseshoes yield C^r-robust homoclinic tangencies.

Theorem 4.9. *Consider a transitive hyperbolic set Σ of a C^r-diffeomorphism f containing a* cu-*blender-horseshoe and a saddle with a homoclinic tangency. Then there is a diffeomorphism g arbitrarily C^r-close to f such that the continuation Σ_g of Σ has a C^r-robust homoclinic tangency.*

We need the following lemma.

Lemma 4.10. *Consider a C^r-diffeomorphism f with a* cu-*blender-horseshoe Λ. Assume that there is a saddle R with $ind^{\mathrm{u}}(R) = ind^{\mathrm{u}}(\Lambda)$ and such that $W^{\mathrm{u}}(R)$ has a tangency with $W^{\mathrm{s}}(A)$, where A is a reference saddle of the blender Λ. Then there is a diffeomorphism g arbitrarily C^r-close to f such that $W^{\mathrm{u}}(R_g)$ is a folded manifold with respect to the continuation Λ_g of the blender-horseshoe Λ.*

By Remark 4.7 and Theorem 4.8 one gets:

Corollary 4.11. *In Lemma* 4.10*, the stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda_g)$ of the blender-horseshoe and $W^{\mathrm{u}}(R_g)$ have a C^r-robust tangency.*

Proof of Lemma 4.10. We suppose that $A = P$ is the left reference saddle of the blender. The proof involves a string of C^r-perturbations of the diffeomorphism f. For simplicity, we also denote these perturbations by f.

We begin by noting that *the center stable bundle* E^{cs} is well defined for every point x in the local stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$. Recall that $W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$ is the set of points whose forward orbit remains in the reference cube $\mathbb{C}$ of the blender-horseshoe, Remark 4.1. Given a point $x \in W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$, the subspace $E^{\mathrm{cs}}(x)$ is the set of vectors $v \in T_xM$ such that $Df^n(v) \notin (\mathcal{C}^{\mathrm{uu}}(f^n(x)) \setminus \{0\})$, for every $n \geq 0$. The space $E^{\mathrm{cs}}(x)$ has dimension $\operatorname{ind}^{\mathrm{s}}(\Lambda)+1$ and depends continuously on the point $x \in W^{\mathrm{s}}_{\mathrm{loc}}(\Lambda)$ and on the diffeomorphism f.

First, after considering forward iterations, we can assume that the tangency intersection point B between $W^{\mathrm{u}}(R)$ and $W^{\mathrm{s}}(P)$ is in $W^{\mathrm{s}}_{\mathrm{loc}}(P)$. Therefore the whole forward orbit of B is the reference cube $\mathbb{C}$. Recall that P is the left reference saddle of the blender-horseshoe, thus P is in the "rectangle" $\mathbb{A}$ of the Markov partition. Thus, for any $n \geq 0$, $f^n_{\mathcal{A}}(B)$ is defined and belongs to $\mathbb{C}$. Hence $E^{\mathrm{cs}}(f^i(B))$ is well defined for all $i \geq 0$ (recall the comment above). Note that

$$\begin{aligned}\dim(T_BW^{\mathrm{u}}(R)) + \dim(E^{\mathrm{cs}}(B)) &= \dim(T_BW^{\mathrm{u}}(R)) + \dim(T_BW^{\mathrm{s}}(P)) + 1\\ &= \dim(M) + 1.\end{aligned}$$

Thus after a perturbation, we can assume that $T_BW^{\mathrm{u}}(R)$ is transverse to $E^{\mathrm{cs}}(B)$. Hence there is a subspace $\mathbb{V} \subset T_BW^{\mathrm{u}}(R)$ of dimension u with $\mathbb{V} \oplus E^{\mathrm{cs}}(B) = T_BM$, where $(u+1) = \operatorname{ind}^{\mathrm{u}}(\Lambda)$.

Consider any $\alpha' \in (0,\alpha)$ (α is the constant in the definition of the cone-fields of the blender). Then, for every $n > 0$ large enough, one has that $Df^n(\mathbb{V})$ is contained in the cone $\mathcal{C}^{\mathrm{uu}}_{\alpha'}(f^n(B))$. For simplicity, let us assume that $n = 0$, that is, $\mathbb{V} \subset \mathcal{C}^{\mathrm{uu}}_{\alpha'}(B)$. This implies that (up to slightly increasing the constant $\alpha' < \alpha$) there is a small submanifold $\tilde{\mathcal{S}} \subset W^{\mathrm{u}}(R)$ such that the point B is in the interior of $\tilde{\mathcal{S}}$, and $\tilde{\mathcal{S}}$ is foliated by disks $(\tilde{\mathcal{S}}_t)_{t\in[-1,1]}$ of dimension u tangent to the cone-field $\mathcal{C}^{\mathrm{uu}}_{\alpha'}$.

Using the expansion by Df in the cone-field $\mathcal{C}^{\mathrm{uu}}_{\alpha'}$ and considering forward iterations of $\tilde{\mathcal{S}}$ by f, we get $k \geq 0$ and a submanifold $\mathcal{S} \subset f^k(\tilde{\mathcal{S}})$ such that:

- $\mathcal{S}$ contains $f^k(B)$ in its interior and is tangent to $W^{\mathrm{s}}(P)$ at $f^k(B)$, and
- $\mathcal{S}$ is foliated by uu-disks $\mathcal{S}_t \subset f^k(\tilde{\mathcal{S}}_t)$, $t \in [-1,1]$, where the disks $\mathcal{S}_t$ are tangent to $\mathcal{C}^{\mathrm{uu}}_{\alpha'}$.

Again for simplicity, we assume that $k = 0$ and that $B \in \mathcal{S}_0$.

After a new perturbation, we can assume that the contact between $\mathcal{S}$ and $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ at the point B is quadratic. In particular, there is small $\epsilon > 0$, such that either all the uu-disks $\mathcal{S}_t$, $t \neq 0$ and $t \in [-\epsilon,\epsilon]$, are at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ (case (a)), or all the uu-disks $\mathcal{S}_t$, $t \neq 0$ and $t \in [-\epsilon,\epsilon]$, are at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ (case (b)). So after discarding some disks and reparametrizing the family $\mathcal{S}_t$, we can assume that $\epsilon = 1$.

Case (a): for $t \neq 0$, every $\mathcal{S}_t$ is at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$ 1. After a new perturbation, we can assume that $\mathcal{S}$ is a folding manifold relative to P; see Figure 5. Since, by construction, $\mathcal{S}$ is contained in $W^{\mathrm{u}}(R)$, this concludes the proof in the first case.

Case (b): for $t \neq 0$ every $\mathcal{S}_t$ is at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$. By considering positive iterations of $\mathcal{S}$ by $f_{\mathcal{A}}$, one gets a large $i > 0$ such that $f^i_{\mathcal{A}}(\mathcal{S})$ meets $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$ transversely at some points in $f^i_{\mathcal{A}}(\mathcal{S}_{t_1})$ and $f^i_{\mathcal{A}}(\mathcal{S}_{t_2})$, where $t_1 < 0 < t_2$. Once again, let us assume that $i = 0$.

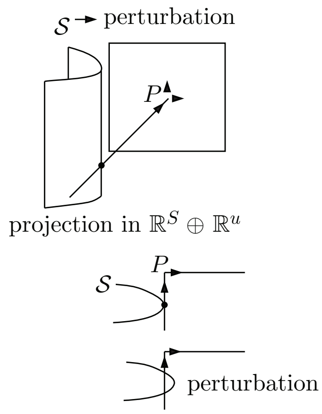

FIGURE 5. Folded manifold: case (a).

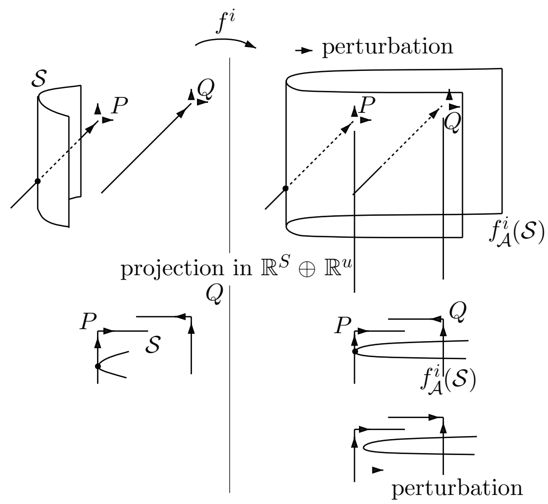

FIGURE 6. Folded manifold: case (b).

We can choose t_1 and t_2 such that the disks $\mathcal{S}_t$ are at the left of $W^s_{\mathrm{loc}}(Q)$ for every $t \in (t_1, t_2)$. Recall that, by hypothesis, for all $t \in [t_1, t_2] \setminus \{0\}$ the disks $\mathcal{S}_t$ are at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P)$. Therefore, we can perform a final perturbation so that $\hat{\mathcal{S}} = \bigcup_{t \in [t_1, t_2]} \mathcal{S}_t$ is a folding manifold relative to Q; see Figure 6. Since $\hat{\mathcal{S}}$ is contained in $W^{\mathrm{u}}(R)$ this ends the proof of the lemma. $\square$

We are now ready to prove Theorem 4.9.

Proof of Theorem 4.9. The theorem is a consequence of the following standard perturbation lemma:

Lemma 4.12. *Consider a C^r-diffeomorphism f, $r \geq 1$, a transitive hyperbolic compact set K (for f) that contains a pair of hyperbolic periodic points P and Q.*

Suppose that the invariant manifolds of Q have a homoclinic tangency. Then there exist diffeomorphisms g and h arbitrarily C^r-close to f such that

- $W^{\mathrm{s}}(Q_g, g)$ *and* $W^{\mathrm{u}}(P_g, g)$ *have a homoclinic tangency, and*
- $W^{\mathrm{u}}(Q_h, h)$ *and* $W^{\mathrm{s}}(P_h, h)$ *have a homoclinic tangency.*

Proof. First, the hyperbolicity of K implies that the saddles P and Q have the same s-index. Using transitivity and hyperbolicity of K, the shadowing property implies that P and Q are accumulated by orbits of saddles P_n (close to K) of f. The hyperbolicity of K implies that these saddles are homoclinically related to P and Q. In this way, we obtain a transitive hyperbolic set K_1 of f containing the saddles P and Q as well as transverse homoclinic intersection points of these saddles. Moreover, the set K_1 is locally maximal.

Let x be a tangency point between the invariant manifolds of Q. Since K_1 is hyperbolic, the point x does not belong to K_1. We fix a constant $L > 0$ such that the point x belongs to the invariant manifolds (with respect to the diffeomorphism f) of size L of Q. Denote these invariant manifolds by $W^{\mathrm{s}}_L(Q)$ and $W^{\mathrm{u}}_L(Q)$. Consider a small neighborhood U of x such that the forward (resp. backward) iterates of $W^{\mathrm{s}}_L(Q) \cap U$ (resp. $W^{\mathrm{u}}_L(Q) \cap U$) are pairwise disjoint.

We consider C^r-perturbations g of f of the form $g = f \circ \phi$, where ϕ is a C^r-diffeomorphism close to the identity whose support is contained in U. Note that if U is small enough, then

- the transitive hyperbolic set K_1 is not modified by the perturbation; thus $P_g = P$ and $Q_g = Q$;
- the intersection $W^{\mathrm{u}}_L(Q) \cap U$ is not modified by this perturbation; and
- the local stable manifold $W^{\mathrm{s}}_L(Q_g, g)$ is the image by ϕ of the initial manifold $W^{\mathrm{s}}(Q, f)$.

The first part of the lemma follows by noting that, since K_1 is locally maximal, transitive, and contains transverse homoclinic intersections of the invariant manifolds of P and Q, there is a sequence of points $y_n \in K_1 \cap W^{\mathrm{u}}(P_g, g)$ that accumulate on Q. Thus the unstable manifolds of size L of the points y_n, $W^{\mathrm{u}}_L(y_n, g)$, C^r-accumulate on $W^{\mathrm{u}}(Q_g, g)$. This allows us to choose a perturbation ϕ such that $W^{\mathrm{u}}_L(y_n, g)$ and $W^{\mathrm{s}}(Q_g, g)$ have a homoclinic tangency. Therefore $W^{\mathrm{u}}(P_g, g)$ and $W^{\mathrm{s}}(Q_g, g)$ have a homoclinic tangency.

A similar argument proves the second part of the lemma. □

Finally, to conclude the proof of Theorem 4.9 just consider the saddle R of the blender-horseshoe with a homoclinic tangency and the reference saddle A of the blender. There is a transitive hyperbolic set containing these two saddles. By Lemma 4.12, after a perturbation we can assume that the unstable manifold of R and the stable manifold of P have a homoclinic tangency. Now the theorem follows from Lemma 4.10 (and Corollary 4.11). □

5. Generation of blender-horseshoes

In this section we see how blender-horseshoes arise naturally in our non-hyperbolic setting. First, in Section 5.1, we review constructions in [12] providing simple examples of blender-horseshoes. In Section 5.2, using these constructions, we see that partially hyperbolic saddles (saddle-node and flip points) with strong homoclinic intersections (intersections between the strong stable and strong unstable manifolds of a non-hyperbolic saddle) yield blender-horseshoes. Finally, following

[9], in Sections 5.3 and 5.4 we prove that co-index one heterodimensional cycles generate blender-horseshoes. We will see in Section 6.2 that co-index one cycles occur naturally in the non-hyperbolic setting.

5.1. **Prototypical blender-horseshoes.** In this section, we consider a local diffeomorphism f having an affine horseshoe Λ with a dominated splitting with three non-trivial bundles, $E^{\rm s} \oplus_< E^{\rm cu} \oplus_< E^{\rm uu}$, where $E^{\rm cu}$ is one-dimensional and $E^{\rm u} = E^{\rm cu} \oplus E^{\rm uu}$ is the unstable bundle of Λ. We suppose that Λ is contained in a hyperplane Π tangent to $E^{\rm s} \oplus E^{\rm uu}$ and that the expansion along the direction $E^{\rm cu}$ is close to one. Under these assumptions we prove that there are perturbations g of f such that the continuations Λ_g of Λ for g are blender-horseshoes; see Proposition 5.1. We now proceed to the details of this construction; we borrow from [12].

Let $\mathbb{D} = [-1,1]^n$ and $n = s+u$, $s,u \geq 1$. Consider a diffeomorphism $F\colon \mathbb{R}^n \to \mathbb{R}^n$ having a horseshoe $\Sigma = \cap_{k\in\mathbb{Z}} F^k(\mathbb{D})$ such that:

- $F^{-1}(\mathbb{D}) \cap \mathbb{D}$ consists of two connected components $\mathbb{D}_1 = [-1,1]^s \times \mathbb{U}_1$ and $\mathbb{D}_2 = [-1,1]^s \times \mathbb{U}_2$, where $\mathbb{U}_1$ and $\mathbb{U}_2$ are disjoint topological compact disks of dimension u.
- The map F is affine on each rectangle $\mathbb{D}_i$: there are linear maps $S_i\colon \mathbb{R}^s \to \mathbb{R}^s$ and $U_i\colon \mathbb{R}^u \to \mathbb{R}^u$, $i = 1,2$, such that
$$DF|_{\mathbb{D}_i} = \begin{pmatrix} S_i & 0 \\ 0 & U_i \end{pmatrix}, \qquad ||S_i||, ||U_i^{-1}|| < 1/2, \quad i = 1,2,$$
where $||A||$ is the norm of the linear map A.

We suppose that, in the usual coordinates (x^s, x^u) in $\mathbb{R}^n = \mathbb{R}^s \times \mathbb{R}^u$, the fixed saddles of the horseshoe Σ are $p = (0^s, 0^u) \in \mathbb{D}_1$ and $q = (a^s, a^u) \in \mathbb{D}_2$.

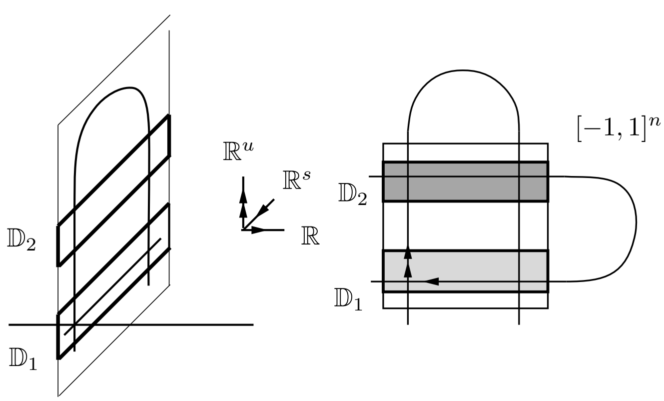

FIGURE 7. An affine horseshoe. The map $f_{\lambda,0}$.

Consider $\lambda \in (1,2)$ and the family of local diffeomorphisms $(f_{\lambda,\mu})_{\mu\in[-\epsilon,\epsilon]}$ of $\mathbb{R}^{n+1}$ given by:

$$f_{\lambda,\mu}(x^s, x^u, x) = \begin{cases} (F(x^s,x^u), \lambda\, x), & \text{if } x^u \in \mathbb{U}_1, \\ (F(x^s,x^u), \lambda\, x - \mu), & \text{if } x^u \in \mathbb{U}_2. \end{cases}$$

By definition, for small μ, the diffeomorphism $f_{\lambda,\mu}$ has two fixed saddles: $P = (0^s, 0^u, 0)$ (independent of λ and μ) and $Q_{\lambda,\mu} = (a^s, a^u, \mu/(\lambda-1))$.

Let $\Lambda_{\lambda,0}$ be the maximal invariant set of $f_{\lambda,0}$ in $(\mathbb{D}_1 \cup \mathbb{D}_2) \times [-1,1]$. Note that $\Lambda_{\lambda,0} = \Sigma \times \{0\}$ is a hyperbolic set of $f_{\lambda,0}$. We say that $\Lambda_{\lambda,0}$ is an *affine horseshoe* of $f_{\lambda,0}$ with central expansion λ. Observe that the hyperplane $\mathbb{R}^n \times \{0\}$ is not normally hyperbolic for $f_{\lambda,0}$.

We denote by $\Lambda_{\lambda,\mu}$ the maximal invariant set of $f_{\lambda,\mu}$ in $(\mathbb{D}_1 \cup \mathbb{D}_2) \times [-1,1]$. For small μ, the set $\Lambda_{\lambda,\mu}$ is the continuation of $\Lambda_{\lambda,0}$. More precisely, for fixed small $\delta > 0$ and the cube $\mathbb{C}_\delta = [-1,1]^s \times [-1,1]^u \times [-\delta,\delta]$, for $|\mu| < (\lambda-1)\,\delta$, the set $\Lambda_{\lambda,\mu}$ is the maximal invariant set of $f_{\lambda,\mu}$ in $\mathbb{C}_\delta$. Clearly, $P, Q_{\lambda,\mu} \in \Lambda_{\lambda,\mu}$.

Proposition 5.1. *For every $\lambda > 1$ close to 1 and $\mu > 0$, the set $\Lambda_{\lambda,\mu}$ is a blender-horseshoe with reference cube $\mathbb{C}_\delta$ and reference saddles P and $Q_{\lambda,\mu}$ (P is the left saddle and $Q_{\lambda,\mu}$ is the right one).*

In this section, for notational convenience, we write the central coordinates in the third position.

Proof. We fix $\lambda > 1$ and $\mu > 0$ and we simply write Λ, f, P, and Q, omitting the dependence on the parameters. The hyperbolicity of Λ follows from the hyperbolicity of F and from the normal expansion by $\lambda > 1$. Consider the constant bundles

$$E^{\mathrm{s}} = (\mathbb{R}^s \times \{(0^u,0)\})\,, \quad E^{\mathrm{c}} = (\{0^s,0^u\} \times \mathbb{R})\,, \quad E^{\mathrm{uu}} = (\{0^s\} \times \mathbb{R}^u \times \{0\})\,.$$

Since λ is less than 2, then

$$T_x\mathbb{R}^{n+1} = E^{\mathrm{s}} \oplus_< E^{\mathrm{c}} \oplus_< E^{\mathrm{uu}}, \quad x \in \Lambda,$$

is a dominated splitting over Λ. Furthermore, as the bundles above are constant, the cone-fields $\mathcal{C}^{\mathrm{cu}}_\alpha$ and $\mathcal{C}^{\mathrm{uu}}_\alpha$ are Df-invariant and $\mathcal{C}^{\mathrm{s}}_\alpha$ is (Df^{-1})-invariant, for every $\alpha \in (0,1)$. Finally, for small α, $\mathcal{C}^{\mathrm{s}}_\alpha$ is uniformly contracting and $\mathcal{C}^{\mathrm{u}}_\alpha$ is uniformly expanding. This gives condition **BH2)**.

To get conditions **BH1)** and **BH3)** let

$$\mathcal{A} = [-1,1]^s \times U_1^{-1}([-1,1]^u) \times [-\delta,\delta], \qquad \mathcal{B} = [-1,1]^s \times U_2^{-1}([-1,1]^u) \times [-\delta,\delta]$$

and

$$\mathbb{A} = [-1,1]^s \times U_1^{-1}([-1,1]^u) \times [-\lambda^{-1}\,\delta, \lambda^{-1}\,\delta],$$
$$\mathbb{B} = [-1,1]^s \times U_2^{-1}([-1,1]^u) \times \left[\frac{-\delta+\mu}{\lambda}, \frac{\delta+\mu}{\lambda}\right].$$

Observe that

$$f(\mathbb{A}) = S_1([-1,1]^s) \times [-1,1]^u \times [-\delta,\delta] \quad \text{and}$$
$$f(\mathbb{B}) = S_2([-1,1]^s) \times [-1,1]^u \times [-\delta,\delta].$$

Observe that the local invariant manifolds of P and Q are:

$$W^{\mathrm{s}}_{\mathrm{loc}}(P,f) = [-1,1]^s \times \{(0^u,0)\},$$
$$W^{\mathrm{s}}_{\mathrm{loc}}(Q,f) = [-1,1]^s \times \{(a^u, \frac{\mu}{\lambda-1})\},$$
$$W^{\mathrm{uu}}_{\mathrm{loc}}(P,f) = \{0^s\} \times [-1,1]^u \times \{0\},$$
$$W^{\mathrm{uu}}_{\mathrm{loc}}(Q,f) = \{a^s\} \times [-1,1]^u \times \{\frac{\mu}{\lambda-1}\}.$$

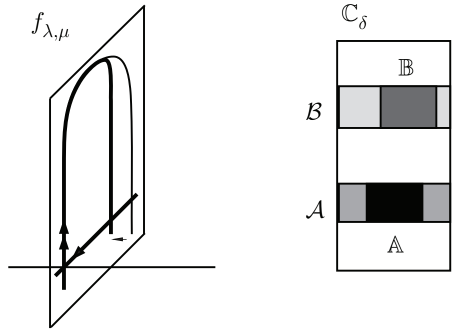

FIGURE 8. Prototypical blender-horseshoe.

It is immediate to check that vertical disks of the form $\{x^s\}\times[-1,1]^u\times\{x^c\}$ satisfy condition **BH4)**. To get **BH4)** for uu-disks it is enough to take $\alpha\in(0,1)$ small enough in the definition of the cone-fields.

Condition **BH5)** follows from the fact that $f_{\mathcal{A}}$ and $f_{\mathcal{B}}$ are affine maps preserving the dominated splitting and whose center eigenvalue λ is positive.

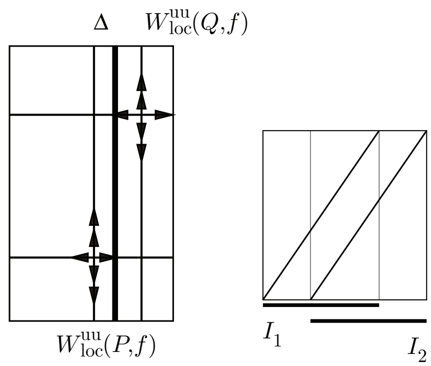

FIGURE 9. uu-disks and one-dimensional reduction.

It remains to check condition **BH6)**. We first consider vertical disks Δ parallel to E^{uu},

$$\Delta=\{x^s\}\times[-1,1]^u\times\{x^c\},$$

which are in between $W^{\mathrm{s}}_{\mathrm{loc}}(P,f)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q,f)$. This means that $x^c\in(0,\frac{\mu}{\lambda-1})$. We consider two cases:

Case 1: $x^c\in I_1=(0,\frac{\mu}{\lambda\,(\lambda-1)})$. In this case, one has that

$$f_{\mathcal{A}}(\Delta)=\{\bar{x}^s\}\times[-1,1]^u\times\{\lambda\,x^c\},\quad\text{where}\quad \lambda\,x^c\in\left(0,\frac{\mu}{\lambda-1}\right).$$

Thus $f_{\mathcal{A}}(\Delta)$ is in between $W^{\mathrm{s}}_{\mathrm{loc}}(P,f)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q,f)$.

Case 2: $x^c\in I_2=(\frac{\mu}{\lambda},\frac{\mu}{\lambda-1})$. Note that in this case one gets

$$f_{\mathcal{B}}(\Delta)=\{\bar{x}^s\}\times[-1,1]^u\times\{\lambda\,x^c-\mu\},\quad\text{where}\quad \lambda\,x^c-\mu\in\left(0,\frac{\mu}{\lambda-1}\right).$$

Hence $f_{\mathcal{B}}(\Delta)$ is in between $W^{\rm s}_{\rm loc}(P,f)$ and $W^{\rm s}_{\rm loc}(Q,f)$.

This completes the proof of **BH6)** for disks parallel to $E^{\rm uu}$.

We next consider general uu-disks. We begin with two claims.

Claim 5.2. Consider $\tau_1 < \frac{\mu}{\lambda\,(\lambda-1)}$ and any uu-disk Δ (i.e. tangent to the cone-field $\mathcal{C}^{\rm uu}_\alpha$) through a point (x^s,x^u,x^c) with $x^c \le \tau_1$. Then, for every $\alpha \in (0,1)$ small enough:

- $f_{\mathcal{A}}(\Delta)$ is at the left of $W^{\rm s}_{\rm loc}(Q)$,
- assume that the disk Δ is at the right of $W^{\rm s}_{\rm loc}(P)$; then $f_{\mathcal{A}}(\Delta)$ is in between $W^{\rm s}_{\rm loc}(P,f)$ and $W^{\rm s}_{\rm loc}(Q,f)$.

Proof. The first statement follows from the compactness of the set of vertical disks with $x^c \le \tau_1$ and the uniform convergence in α of the uu-disks to the vertical disks (parallel to $E^{\rm uu}$) as $\alpha \to 0$.

From the first part of the claim we know that $f_{\mathcal{A}}(\Delta)$ is at the left of $W^{\rm s}_{\rm loc}(Q,f)$. It remains to check that $f_{\mathcal{A}}(\Delta)$ is at the right of $W^{\rm s}_{\rm loc}(P,f)$. The disk Δ contains a point of the form $(x^s,0^u,x^c)$ with $x^c > 0$. Thus $f_{\mathcal{A}}(\Delta)$ contains the point $(S_1(x^s),0^u,\lambda\,x^c)$. This implies that $f_{\mathcal{A}}(\Delta)$ is at the right of $W^{\rm s}_{\rm loc}(P,f)$. □

Arguing as above and using Case 2, one also deduces the following:

Claim 5.3. Consider $\tau_2 > \frac{\mu}{\lambda}$ and any uu-disk Δ through a point (x^s,x^u,x^c) with $x^c \ge \tau_2$. Then, for every $\alpha \in (0,1)$ small enough:

- $f_{\mathcal{B}}(\Delta)$ is at the right of $W^{\rm s}_{\rm loc}(P)$,
- assume that Δ is at the left of $W^{\rm s}_{\rm loc}(Q)$; then $f_{\mathcal{B}}(\Delta)$ is in between $W^{\rm s}_{\rm loc}(P,f)$ and $W^{\rm s}_{\rm loc}(Q,f)$.

To get condition **BH6)**, note that, since $(\lambda-1)\in(0,1)$, one has $\frac{\mu}{\lambda} < \frac{\mu}{\lambda\,(\lambda-1)}$. Now it is enough to note that, for $\alpha\in(0,1)$ small enough, for every point (x^s,x^u,x^c) in a uu-disk Δ in between $W^{\rm s}_{\rm loc}(P,f)$ and $W^{\rm s}_{\rm loc}(Q,f)$ one has:

- either $x^c < \frac{\mu}{\lambda\,(\lambda-1)}$, and then, by Claim 5.2, $f_{\mathcal{A}}(\Delta)$ is in between $W^{\rm s}_{\rm loc}(P,f)$ and $W^{\rm s}_{\rm loc}(Q,f)$,
- or $x^c > \frac{\mu}{\lambda}$, and then, by Claim 5.3, $f_{\mathcal{B}}(\Delta)$ is in between $W^{\rm s}_{\rm loc}(P,f)$ and $W^{\rm s}_{\rm loc}(Q,f)$.

We have checked that the set Λ satisfies conditions **BH1)–BH6)**. Therefore the set Λ is a blender-horseshoe and the proof of Proposition 5.1 is complete. □

5.2. **Strong homoclinic intersections and the generation of blenders.** In this section, we state the generation of blender-horseshoes by saddle-node and flip periodic points with strong homoclinic intersections.

Let f be a diffeomorphism with a periodic point S such that the tangent bundle of M at S has a $Df^{\pi(S)}$-invariant dominated splitting $T_SM = E^{\rm ss}\oplus_< E^{\rm c}\oplus_< E^{\rm uu}$, where $E^{\rm ss}$ is uniformly contracting, $E^{\rm uu}$ is uniformly expanding, and $E^{\rm c}$ is one-dimensional. We say that S is a *saddle-node* (resp. *flip*) if the eigenvalue of $Df^{\pi(S)}$ corresponding to the (one-dimensional) central direction $E^{\rm c}$ is 1 (resp. -1).

Consider the strong stable and unstable manifolds of the orbit of S (denoted by $W^{\rm ss}(S)$ and $W^{\rm uu}(S)$). We say that S has a *strong homoclinic intersection* if there is a point $X\in W^{\rm ss}(S)\cap W^{\rm uu}(S)$ such that $X\neq f^i(S)$ for all i. The point X is a *strong homoclinic point of* f associated to S.

Let f be a diffeomorphism with a strong homoclinic intersection associated to a saddle-node S. In [9, Section 4.1] it is shown that there are $k \geq 1$ and a C^1-perturbation g of f such that g^k has an affine horseshoe Λ associated to S whose central expansion is arbitrarily close to 1 (recall Section 5.1). Considering perturbations similar to the ones in Proposition 5.1, [9, Sections 4.1.1-2] gives the following:

Proposition 5.4. *Consider a diffeomorphism f having a strong homoclinic intersection associated to a saddle-node S. There are a constant $k > 0$ and a diffeomorphism g arbitrarily C^1-close to f such that g^k has a* cu-*blender-horseshoe having S as a reference saddle.*

The same statement holds for cs-*blender-horseshoes, i.e.* cu-*blender-horseshoes for f^{-1}.*

We observe that in [9] the terminology blender-horseshoe is not used. However, the constructions in [9] provide prototypical blender-horseshoes exactly as the ones in Section 5.1. In fact, these constructions are the main motivation (and model) for our definition of a blender-horseshoe. Thus Proposition 5.4 just reformulates some results in [9] using this new terminology of blender-horseshoes.

For diffeomorphisms having flip points we need the following lemma (see [9, Remark 4.6]):

Lemma 5.5. *Consider a diffeomorphism f having a strong homoclinic intersection associated to a flip point S. There is a diffeomorphism g arbitrarily C^1-close to f having a saddle node S' with a strong homoclinic intersection and such that the orbit of S' remains in an arbitrarily small neighborhood of the orbit of the initial flip point S.*

Proof. Consider a 2-parameter family of deformations $f_{s,t}$ of $f = f_{0,0}$ such that:

- the parameter s corresponds to a (non-generic) unfolding of the flip, generating a saddle-node S' close to the flip S of period twice the period of S, and
- the parameter t corresponds to the unfolding of the strong homoclinic intersection of the flip S, the local strong stable manifold of S "passing from the left to the right" of the local unstable manifold of S.

Then for every $s \neq 0$ small enough, there is a small parameter $t = t(s)$, $t(s) \to 0$ as $s \to 0$, such that the saddle-node S' has a strong homoclinic intersection. □

5.3. **Co-index one cycles and blender-horseshoes.** In this section, we borrow some arguments and results from [2, 9] in order to prove that diffeomorphisms with co-index one heterodimensional cycles yield blender-horseshoes.

Proposition 5.6. *Let f be a diffeomorphism with a heterodimensional cycle associated to saddles P and Q with $\operatorname{ind}^{\mathrm{s}}(P) = \operatorname{ind}^{\mathrm{s}}(Q) + 1$. Then there is g arbitrarily C^1-close to f with a saddle R such that:*

1. *$\operatorname{ind}^{\mathrm{s}}(R) = \operatorname{ind}^{\mathrm{s}}(Q)$, the orbit of R has a dominated splitting $E^{\mathrm{ss}} \oplus_< E^{\mathrm{c}} \oplus_< E^{\mathrm{uu}}$ with three non-trivial bundles such that E^{ss} and E^{uu} are uniformly contracting and expanding, respectively, $\dim(E^{\mathrm{ss}}) = \operatorname{ind}^{\mathrm{s}}(Q)$, and $\dim(E^{\mathrm{c}})$ has dimension one and is expanding,*
2. *there is* cu-*blender-horseshoe of g having R as a reference saddle,*

(3) $W^{\mathrm{s}}(R)$ *transversely intersects* $W^{\mathrm{u}}(Q)$, *and*
(4) $W^{\mathrm{uu}}(R)$ *transversely meets* $W^{\mathrm{s}}(P)$.

The arguments for proving this proposition can be found scattered along several constructions in [9]. But, unfortunately, this result is not stated explicitly there, and its proof involves some adaptations of the constructions in [9]. As the proof of Proposition 5.6 is somewhat technical, we next explain the sequence of arguments we borrow from [9] and their adaptations in order to prove this proposition. The proof of Proposition 5.6 consists of several reductions to simpler cases that we proceed to explain. Let us begin with a definition.

Definition 5.7 (Strong-intermediate saddles)**.** Let f be a diffeomorphism having two periodic saddles P and Q of indices $\operatorname{ind}^{\mathrm{s}}(P) = \operatorname{ind}^{\mathrm{s}}(Q) + 1$. A periodic point R is *strong-intermediate* with respect to P and Q, denoted by $Q \prec_{\mathrm{u,ss}} R \prec_{\mathrm{uu,s}} P$, if:

- the orbit of R is partially hyperbolic and has a dominated splitting $E^{\mathrm{ss}} \oplus_< E^{\mathrm{c}} \oplus_< E^{\mathrm{uu}}$ with three non-trivial bundles such that E^{ss} and E^{uu} are uniformly contracting and expanding, $\dim(E^{\mathrm{ss}}) = \operatorname{ind}^{\mathrm{s}}(Q)$, and $\dim(E^{\mathrm{c}}) = 1$,
- the strong stable manifold of R meets transversely the unstable manifold of Q, and the strong unstable manifold of R meets transversely the stable manifold of P, in the formulas,

$$W^{\mathrm{ss}}(R) \pitchfork W^{\mathrm{u}}(Q) \neq \emptyset \quad \text{and} \quad W^{\mathrm{uu}}(R) \pitchfork W^{\mathrm{s}}(P) \neq \emptyset.$$

Note that if a (hyperbolic) saddle R with $\operatorname{ind}^{\mathrm{s}}(R) = \operatorname{ind}^{\mathrm{s}}(Q)$ is strong-intermediate with respect to P and Q, then it satisfies items (3) and (4) in Proposition (5.6).

We need the following lemma:

Lemma 5.8. *Consider two saddles P and Q in the same chain recurrence class C and a periodic point R which is strong-intermediate with respect to P and Q. Then $R \in C$.*

Proof. We construct a pseudo-orbit going from R to P (the other pseudo-orbits are obtained similarly). Take a point $X \in W^{\mathrm{ss}}(R) \cap W^{\mathrm{u}}(Q)$. Note that there are arbitrarily large n and m such that $\{f^{-n}(X), \dots, f^m(X)\}$ is a segment of the orbit starting (arbitrarily) close to R and ending close to Q. Since Q and P are in the same chain recurrent class, there is a finite pseudo-orbit going from Q to P. A pseudo-orbit going from R to P is obtained concatenating these two pseudo-orbits. This concludes the sketch of the proof of the lemma. $\square$

We now explain the generation of strong-intermediate saddles.

5.3.1. *Reduction to the case of cycles associated to saddles with real central eigenvalues.* Given a periodic point R of a diffeomorphism f, write $\lambda_1(R), \dots, \lambda_n(R)$ as the eigenvalues of $Df^{\pi(R)}(R)$ counted with multiplicity and ordered in increasing modulus ($|\lambda_i(R)| \le |\lambda_{i+1}(R)|$). We say that $\lambda_i(R)$ is the i-th multiplier of R.

Consider a diffeomorphism f having a co-index one cycle associated to period points A and B with $\operatorname{ind}^{\mathrm{s}}(P) = \operatorname{ind}^{\mathrm{s}}(Q) + 1 = s + 1$. The cycle *has real central eigenvalues* if $\lambda_{s+1}(A)$ and $\lambda_{s+1}(B)$ are both real and

$$|\lambda_s(A)| < |\lambda_{s+1}(A)| < 1 < |\lambda_{s+2}(A)| \quad \text{and} \quad |\lambda_s(B)| < 1 < |\lambda_{s+1}(B)| < |\lambda_{s+2}(B)|.$$

Before proving Proposition 5.6, we recall the following two facts:

Fact 1. [9, Theorem 2.1] claims that if f has a co-index one cycle associated to A and B, then there is g arbitrarily C^1-close to f with a co-index one cycle with real central eigenvalues. Moreover, this cycle can be chosen as associated to saddles A'_g and B'_g homoclinically related to the continuations A_g and B_g of A and B, respectively.

Fact 2. Assume that there is a diffeomorphism h arbitrarily C^1-close to g having a cu-blender-horseshoe Λ with a reference saddle R_h which is strong-intermediate to A'_h and B'_h. Since two saddles being homoclinically related is a C^1-robust relation, one has that R_h is strong-intermediate with respect to A_h and B_h. In this case, the proof of Proposition 5.6 is complete.

In view of these two facts, to prove Proposition 5.6 it is enough to consider the case where the saddles P and Q in the cycle have real central eigenvalues and to check that these cycles generate strong-intermediate saddles as in Fact 2; see Proposition 5.9. We now go to the details of this construction.

5.3.2. *Reduction to the generation of saddle-node or flip points.* In this section, we show that Proposition 5.6 is a consequence of the following result.

Proposition 5.9. *Let f be a diffeomorphism with a co-index one cycle associated to saddles P and Q with real central eigenvalues. Then there is g arbitrarily C^1-close to f having a saddle-node or flip periodic point R_g such that:*

- *R_g has a strong homoclinic intersection and*
- *R_g is strong-intermediate to P_g and Q_g.*

This proposition is a stronger version of [9, Theorem 2.3], adding the strong-intermediate property of R with respect to P and Q.

Proposition 5.9 *implies Proposition* 5.6. First, observe that the strong unstable and strong stable manifolds of R_g depend continuously on the diffeomorphism (while there is defined a continuation of R_g).

If $R = R_g$ is a saddle-node, then Proposition 5.4 gives a diffeomorphism h arbitrarily C^1-close to g (thus arbitrarily close to f) with a blender-horseshoe having R as a reference saddle. Therefore, for h close enough to g, one gets the announced intersections between the strong invariant manifolds (intermediate intersections).

If R is a flip, then Lemma 5.5 gives a perturbation h of g with a saddle-node with a strong homoclinic intersection and with the strong-intermediate property. Thus we are in the first case. This completes the proof of our claim. □

Therefore it is enough to prove Proposition 5.9. The proof of the proposition is similar to the one of [9, Theorem 2.3] and consists of several steps. We next explain and adapt these steps.

5.3.3. *Reduction to the creation of weak hyperbolic saddles.* We now see that Proposition 5.9 (hence of Proposition 5.6) follows from:

Proposition 5.10. *Let f be a diffeomorphism having a co-index one cycle associated to saddles P and Q, $\operatorname{ind}^s(P) = \operatorname{ind}^s(Q) + 1$, with real central eigenvalues.*

Then there are a constant $C > 1$ and a sequence of diffeomorphisms (f_n), $f_n \xrightarrow{C^1} f$, such that every f_n has a periodic point R_n such that:

- *R_n has a one-dimensional center-unstable direction whose corresponding multiplier $\lambda_n^{\rm c}$ satisfies $|\lambda_n^{\rm c}| \in [\frac{1}{C}, C]$,*
- *$W^{\rm uu}(R_n)$ and $W^{\rm ss}(R_n)$ have a quasi-transverse intersection, and therefore R_n has strong homoclinic intersections,*
- *the periods of R_n go to infinity as $n \to \infty$, and*
- *R_n is strong-intermediate with respect to P_n and Q_n (the continuations of P and Q for f_n).*

This proposition is a stronger version of [9, Proposition 3.3], adding the intersection property of the strong invariant manifolds.

Proposition 5.10 *implies Proposition* 5.9. We proceed exactly as in [9, page 484] (proof of Theorem 2.3 using Proposition 3.3). We just perform a local C^1-perturbation of f_n supported in a small neighborhood of R_n, turning to the central eigenvalue of R_n equal to ± 1 while keeping the strong homoclinic point of R_n and the transverse intersections $W^{\rm ss}(R_n) \pitchfork W^{\rm u}(Q_n) \neq \emptyset$ and $W^{\rm uu}(R_n) \pitchfork W^{\rm s}(P_n) \neq \emptyset$.

In this way, we get diffeomorphisms g (arbitrarily close to f) with saddle-node or flip points R_g (depending on whether $\lambda_n^{\rm c}$ is positive or negative) with strong homoclinic intersections and being strong-intermediate with respect to P_g and Q_g. This completes the proof of Proposition 5.9. □

5.4. **Proof of Proposition 5.10.** The following steps of the proof of [9, Proposition 3.3] are described in [9, page 484]:

Step 1. One first puts the cycle in a kind of normal form called a *simple cycle*. In fact, [9, Proposition 3.5] implies that, after a C^1-perturbation, one can assume that the cycle is simple.

Step 2. One shows that the dynamics in a simple cycle is given (up to a renormalization) by a model family, denoted by $F^{\pm,\pm}_{\lambda,\beta,t}$. Moreover, perturbations of this model family correspond to perturbations of the initial dynamics.

Therefore, to prove Proposition 5.10, it is enough to consider model families $F^{\pm,\pm}_{\lambda,\beta,t}$ and their perturbations. Hence it is enough to adapt the perturbations of these normal families in order to get the intersection properties between the strong invariant manifolds.

Proposition 3.8 in [9] claims that the unfolding of co-index one cycles generates sequences of saddles $A_{n,m}$ whose orbits are contained in a neighborhood of the cycle. We now see that these saddles can be taken with the strong-intermediate property (relative to saddles in the initial cycle).

Lemma 5.11. *In* [9, Proposition 3.8]*, for every integer n, m large enough (in fact larger than the integer N in the statement), all periodic points $A_{n,m}$ in the proposition are strong-intermediate with respect to P and Q.*

Note that we have the following string of implications:

$$\text{Lemma 5.11} \Rightarrow \text{Proposition 5.10} \Rightarrow \text{Proposition 5.9} \Rightarrow \text{Proposition 5.6}.$$

Therefore to prove Proposition 5.6 it remains to prove Lemma 5.11.

Proof. The model maps $F^{\pm,\pm}_{\lambda,\beta,t}$ are defined on some cubes, and their restrictions to each of these cubes are affine maps $\mathcal{A}_\lambda$, $\mathcal{B}_\beta$, $\mathcal{T}^{\pm}_{1,t}$, and $\mathcal{T}^{\pm}_2$ which preserve a constant dominated splitting: the strong stable bundle is the horizontal space $\mathbb{R}^s \times \{(0,0^u)\}$, the strong unstable bundle is the vertical one $\{(0^s,0)\}\times\mathbb{R}^u$, and the center bundle is one-dimensional $\{0^s\}\times\mathbb{R}\times\{0^u\}$. More precisely (see Figure 10):

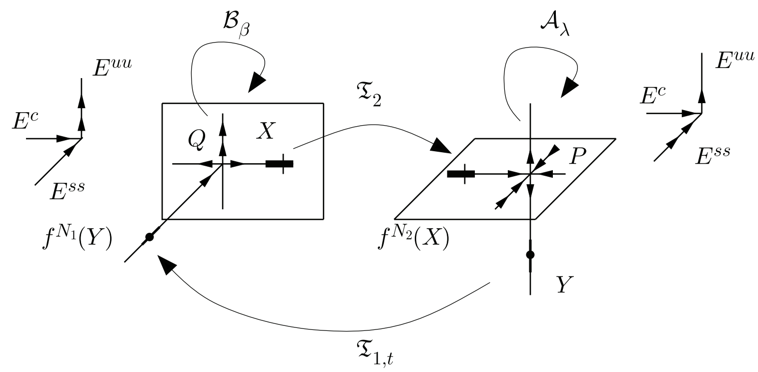

FIGURE 10. The model maps $F^{\pm,\pm}_{\lambda,\beta,t}$.

- The maps $\mathcal{A}_\lambda$ and $\mathcal{B}_\beta$ are $Df^{\pi(P)}(P)$ and $Df^{\pi(Q)}(Q)$ and correspond to iterates of f close to P and Q, respectively.
- The subscripts λ and β correspond to the central multipliers of $Df^{\pi(P)}(P)$ and $Df^{\pi(Q)}(Q)$.
- The maps $\mathcal{T}^{\pm}_{1,t}$ and $\mathcal{T}^{\pm}_2$ are the *transitions* of the cycle. The map $\mathcal{T}^{\pm}_2$ corresponds to a fixed number N_2 of iterates from a neighborhood of Q to a neighborhood of P following a segment of orbit of a transverse heteroclinic point X in $W^s(P)\pitchfork W^{\rm u}(Q)$.

  Similarly, the map $\mathcal{T}^{\pm}_{1,t}$ corresponds to a fixed number N_1 of iterates from a neighborhood of P to a neighborhood of Q following a segment of orbit of a fixed quasi-transverse heteroclinic point Y in $W^{\rm u}(P)\pitchfork W^{\rm s}(Q)$. The parameter t of $\mathcal{T}^{\pm}_{1,t}$ corresponds to the unfolding of the cycle.
- The super-script $\pm$ is positive if the transition map preserves the orientation in the central bundle and negative otherwise.

For details see [9, page 488].

By definition, the points $A_{m,n}=(a^s,a,a^u)$ are fixed points of the composition $\mathcal{B}^n_\beta\circ\mathcal{T}^{\pm}_{1,t}\circ\mathcal{A}^m_\lambda\circ\mathcal{T}^{\pm}_2$. In particular, the point $A_{m,n}$ belongs to the domain of the definition Σ_Q of $\mathcal{T}^{\pm}_2$; see Figure 11. This domain (defined in [9, page 488]) is the cube

$$\Sigma_Q=[-1,1]^s\times[b_Q-\delta,b_Q+\delta]\times[-1,1]^u,$$

where

$$[-1,1]^s\times[b_Q-\delta,b_Q+\delta]\times\{0^u\}\subset W^{\rm s}(P)\quad\text{and}$$
$$\{0^s\}\times[b_Q-\delta,b_Q+\delta]\times[-1,1]^u\subset W^{\rm u}(Q).$$

Moreover, the local strong stable and strong unstable manifolds of the point $A_{m,n}=(a^s,a,a^u)$ are (see [9, page 495, first paragraph]):

$$W^{\rm ss}_{\rm loc}(A_{m,n})=[-1,1]^s\times\{(a,a^u)\}\quad\text{and}\quad W^{\rm uu}_{\rm loc}(A_{m,n})=\{(a^s,a)\}\times[-1,1]^u.$$

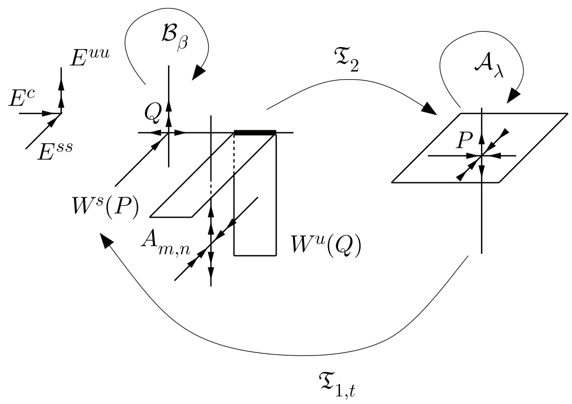

FIGURE 11. The intermediate saddle $A_{m,n}$.

This means that

$$W^{\mathrm{ss}}(A_{m,n}) \pitchfork W^{\mathrm{u}}(Q) \neq \emptyset \quad \text{and} \quad W^{\mathrm{uu}}_{\mathrm{loc}}(A_{m,n}) \pitchfork W^{\mathrm{s}}(P) \neq \emptyset.$$

Therefore the saddle $A_{m,n}$ is strong-intermediate with respect to P and Q, ending the proof of the lemma. □

6. Robust tangencies and heterodimensional cycles for C^1-generic diffeomorphisms

In this section, we prove Theorem 1.2. First, in Section 6.1, we state some properties about C^1-generic diffeomorphisms. In Section 6.2, we state the C^1-generic occurrence of blender-horseshoes in homoclinic classes containing saddles of different indices (Theorem 6.4). Finally, in Section 6.3, we state the existence of robust homoclinic tangencies inside homoclinic classes with index variation and lack of domination (Proposition 6.11), completing the proof of Theorem 1.2. We close this paper by presenting an extension of [9, Theorem 1.16] regarding the occurrence of robust heterodimensional cycles inside chain recurrence classes (see Theorem 6.13 in Section 6.4).

6.1. **C^1-generic properties of C^1-diffeomorphisms.** We now collect some properties of C^1-generic diffeomorphisms. According to [7, Remarque 1.10] and [2, Theorem 1], there is a residual subset set $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{G}$,

- every periodic point of f is hyperbolic,
- for every periodic point P of f, its homoclinic class $H(P,f)$ and its chain recurrence class $C(P,f)$ are equal,
- any homoclinic class $H(P,f)$ containing periodic points of u-indices α and β also contains saddles of u-index τ, for every $\tau \in [\alpha,\beta] \cap \mathbb{N}$.

Lemma 6.1 ([2, Lemma 2.1])**.** *There is a residual subset $\mathcal{G}_0 \subset \mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{G}_0$ and every pair of periodic points P_f and Q_f of f, there is a neighborhood $\mathcal{U}_f$ of f in $\mathcal{G}_0$ such that:*

- *either $H(P_g,g) = H(Q_g,g)$ for all $g \in \mathcal{U}_f \cap \mathcal{G}_0$*
- *or $H(P_g,g) \cap H(Q_g,g) = \emptyset$ for all $g \in \mathcal{U}_f \cap \mathcal{G}_0$.*

Remark 6.2 (Proof of Claim 2.2 in [2]). Using a filtration given by Conley theory, one has that the property of two hyperbolic saddles to be in different chain recurrence classes is C^1-robust.

The next lemma claims that, for C^1-generic diffeomorphisms, the property of the chain recurrence classes of two periodic points to be equal is also a C^1-robust property.

Lemma 6.3. *Let $\mathcal{G}_0$ be the residual set of* $\mathrm{Diff}^1(M)$ *in Lemma* 6.1. *For every $f \in \mathcal{G}_0$ and every pair of periodic points P_f and Q_f of f, the property of Q_f belonging to the chain recurrence class $C(P_f, f)$ of P_f is C^1-robust: if $Q_f \in C(P_f, f)$, then $Q_g \in C(P_g, g)$ for all g C^1-close to f.*

Proof. Let $f \in \mathcal{G}_0$ and suppose that $Q_f \in C(P_f, f)$. Since $f \in \mathcal{G}$ one has that $C(P_f, f) = H(P_f, f)$. As $f \in \mathcal{G}_0$, by Lemma 6.1, there is a C^1-open neighborhood $\mathcal{U}_f$ of f such that $H(P_g, g) = H(Q_g, g)$ for every $g \in \mathcal{U}_f \cap \mathcal{G}_0$. In particular, $Q_g \in H(P_g, g)$ for all $g \in \mathcal{U}_f \cap \mathcal{G}_0$.

Assume that there is $g \in \mathcal{U}_f$ such that $Q_g \notin C(P_g, g)$. By Remark 6.2, one has that $Q_h \notin C(P_h, h)$ for every h in a small neighborhood $\mathcal{V}_g$ of g contained in $\mathcal{U}_f$. Choosing $h \in \mathcal{G}_0 \cap \mathcal{V}_g$ one obtains that $Q_h \notin C(P_h, h) = H(P_h, h)$, a contradiction. $\square$

6.2. **Generation of blender-horseshoes.** In this section, we prove that blender-horseshoes occur C^1-generically for homoclinic classes with index variability (i.e., containing saddles with different indices).

Theorem 6.4. *There is a residual subset $\mathcal{R}$ of* $\mathrm{Diff}^1(M)$ *of diffeomorphisms f such that for every homoclinic class $H(P, f)$ containing a hyperbolic saddle Q with* $\mathrm{ind}^s(Q) > \mathrm{ind}^s(P)$ *there is a transitive hyperbolic set Σ containing P and a* cu-*blender-horseshoe* Λ.

Applying Theorem 6.4 to f^{-1} one obtains the following.

Remark 6.5. Under the assumptions of Theorem 6.4, every C^1-generic diffeomorphism f has a cs-blender-horseshoe containing Q and contained in a transitive hyperbolic set.

We first prove a version of Theorem 6.4 for a given fixed saddle P_f:

Proposition 6.6. *Let $\mathcal{U}$ be an open subset of* $\mathrm{Diff}^1(M)$ *and $f \mapsto P_f$ be a continuous map defined on $\mathcal{U}$ associating to each $f \in \mathcal{U}$ a hyperbolic periodic point P_f of f.*

There is a residual subset $\mathcal{R} = \mathcal{R}_P$ of $\mathrm{Diff}^1(M)$ *with the following property. For every diffeomorphism $f \in \mathcal{R} \cap \mathcal{U}$ such that $H(P_f, f)$ contains a saddle B_f with* $\mathrm{ind}^s(B_f) > \mathrm{ind}^s(P_f)$, *there is a transitive hyperbolic set Σ_f containing P_f and a* cu-*blender-horseshoe* Λ_f.

Theorem 6.4 will follow from this proposition using standard genericity arguments (the details can be found at the end of this subsection).

Proof of Proposition 6.4. Consider the residual subset $\mathcal{G}_0$ of $\mathrm{Diff}^1(M)$ in Lemma 6.1. Given any $f \in \mathcal{G}_0 \cap \mathcal{U}$, if $H(P, f)$ contains a saddle B_f with $\mathrm{ind}^s(B_f) > \mathrm{ind}^s(P_f)$, then there is a saddle $Q_f \in H(P_f, f)$ with $\mathrm{ind}^s(Q_f) = \mathrm{ind}^s(P_f) + 1$. Furthermore, by Lemma 6.3, if $f \in \mathcal{G}_0$ the point Q_f belongs robustly to the chain recurrence class $C(P_f, f)$. Moreover, $C(P_f, f) = H(P_f, f)$.

Let $\mathcal{W}$ be the set of diffeomorphisms

$$\mathcal{W} = \left\{ f \in \mathcal{U} \colon \exists \text{ a saddle } Q_f \in \mathrm{Per}(f) \text{ with } \begin{array}{l} \mathrm{ind}^{\mathrm{s}}(Q_f) = \mathrm{ind}^{\mathrm{s}}(P_f) + 1, \\ \text{and} \\ Q_f \text{ is } C^1\text{-robustly in } C(P_f, f) \end{array} \right\}.$$

By definition, the set $\mathcal{W}$ is an open subset of $\mathcal{U}$. Let

$$\mathcal{U}_1 = \mathcal{U} \setminus \overline{\mathcal{W}}.$$

By construction the set $\mathcal{U}_1$ is open and $\mathcal{U}_1 \cup \mathcal{W}$ is dense in $\mathcal{U}$.

Claim 6.7. Let $f \in \mathcal{G}_0 \cap \mathcal{U}_1$. Then $C(P_f, f)$ does not contain any hyperbolic periodic point $\tilde{Q}$ with $\mathrm{ind}^{\mathrm{s}}(\tilde{Q}) > \mathrm{ind}^{\mathrm{s}}(P_f)$.

Proof. The proof is by contradiction. Suppose that there is a saddle $\tilde{Q} \in C(P_f, f)$ with $\mathrm{ind}^{\mathrm{s}}(\tilde{Q}) > \mathrm{ind}^{\mathrm{s}}(P_f)$. As $f \in \mathcal{G}_0$, there is a saddle $Q_f \in C(P_f, f) = H(P_f, f)$ with $\mathrm{ind}^{\mathrm{s}}(Q_f) = \mathrm{ind}^{\mathrm{s}}(P_f) + 1$. As $f \in \mathcal{G}_0$, the saddle Q_f belongs C^1-robustly to $C(P_f, f)$. Therefore $f \in \mathcal{W}$, contradicting the definition of $\mathcal{U}_1$. □

Thus, by Lemma 6.3, it is enough to prove Proposition 6.6 for diffeomorphisms in $\mathcal{W}$. In other words, the next lemma implies Proposition 6.6.

Lemma 6.8. *The open set of diffeomorphisms f having a transitive hyperbolic set Σ_f containing P_f and a* cu-*blender-horseshoe Λ_f is dense in $\mathcal{W}$.*

Proof. Consider $f \in \mathcal{W}$ and a saddle Q_f with $\mathrm{ind}^{\mathrm{s}}(Q_f) = \mathrm{ind}^{\mathrm{s}}(P_f) + 1$ which belongs robustly to $C(P_f, f)$. The next lemma is an immediate consequence of the Hayashi connecting lemma in [18]:

Lemma 6.9 ([18])**.** *Let f be a diffeomorphism having a pair of (hyperbolic) saddles A_f and B_f whose orbits are different and such that $B_f \in H(A_f, f)$. Then there is g arbitrarily C^1-close to f such that $W^{\mathrm{u}}(A_g, g) \cap W^{\mathrm{s}}(B_g, g) \neq \emptyset$.*

Note that as $f \in \mathcal{W}$ we have $Q_f \in C(P_f, f)$ in a robust way. After a first perturbation, we can assume that $f \in \mathcal{G}_0$ so that $Q_f \in H(P_f, f)$. We apply Lemma 6.9 to the saddles P_f and Q_f to get a diffeomorphism g close to f with a transverse intersection between $W^{\mathrm{u}}(P_g, g)$ and $W^{\mathrm{s}}(Q_g, g)$. Note that this transverse intersection persists after perturbation, so that we can assume that $g \in \mathcal{W} \cap \mathcal{G}_0$. After a new application of Lemma 6.9 we can suppose that $W^{\mathrm{s}}(P_g, g) \cap W^{\mathrm{u}}(Q_g, g) \neq \emptyset$. Therefore there is g arbitrarily C^1-close to f, having a co-index one cycle associated to P_g and Q_g.

Applying Proposition 5.6 to the diffeomorphism g with a co-index one cycle (associated to P_g and Q_g), we get h close to g, thus close to f, with a periodic point R_h such that:

- R_h is strong-intermediate with respect to P_h and Q_h,
- R_h has the same index as P_h, and
- R_h is a reference saddle of a cu-blender-horseshoe Λ_h.

Recall that, by Lemma 3.10, the continuation of this blender-horseshoe is defined in a neighborhood of h. Thus we can assume that $h \in \mathcal{G}_0$.

As the saddle R_h is (robustly) intermediate with respect to P_h and Q_h and $Q_h \in C(P_h, h)$, Lemma 5.8 implies that $R_h \in C(P_h, h)$. Thus, from Lemma 6.3, we have that R_h belongs robustly to $C(P_h, h)$. As R_h and P_h have the same index, Lemma 6.9 gives a perturbation φ of h with transverse intersections between the invariant manifolds of these saddles. That is, the saddles R_φ and P_φ are homoclinically related, thus their homoclinic classes are equal. Therefore, there is a transitive hyperbolic set Σ_φ containing P_φ and the cu-blender-horseshoe Λ_φ (the continuation of Λ_h). This ends the proof of the lemma. □

The proof of Proposition 6.6 is now complete. □

Proof of Theorem 6.4. Given a diffeomorphism f, denote by $\mathrm{Per}_n(f)$ the set of periodic points P of f of period $\pi(P) \leq n$. To prove Theorem 6.4 it is enough to see that, for every $n \in \mathbb{N}$, there is a residual set $\mathcal{R}_{\leq n}$ of diffeomorphisms f such that the conclusion of the theorem holds for every periodic orbit $P \in \mathrm{Per}_n(f)$. Then it is enough to take $\mathcal{R} = \bigcap_n \mathcal{R}_{\leq n}$.

Note that there is a C^1-open and dense subset $\mathcal{O}_n \subset \mathrm{Diff}^1(M)$ of diffeomorphisms f such that every periodic point $P \in \mathrm{Per}_n(f)$ is hyperbolic. In particular, the cardinal of $\mathrm{Per}_n(f)$ is finite and locally constant in $\mathcal{O}_n$. Moreover, each periodic point $P \in \mathrm{Per}_n(f)$ has a hyperbolic continuation in each (open) connected component $\mathcal{U}$ of the open set $\mathcal{O}_n$. That is, there are a constant $k = k(\mathcal{U})$ and continuous maps $f \mapsto P_{i,f}$, $i = 1, \dots, k$, such that $\mathrm{Per}_n(f) = \{P_{1,f}, \dots, P_{k,f}\}$, for every $f \in \mathcal{U}$.

Note that the set $\mathrm{cc}(\mathcal{O}_n)$ of connected components of $\mathcal{O}_n$ is countable. Therefore to prove Theorem 6.4 for periods $\pi \leq n$ it is enough to see that this result holds in each connected component $\mathcal{U}$ of $\mathcal{O}_n$. More precisely, for each connected component $\mathcal{U}$ of $\mathcal{O}_n$, we first build a residual subset $\widetilde{\mathcal{R}_\mathcal{U}}$ of $\mathrm{Diff}^1(M)$ such that the conclusion holds in the set $\widetilde{\mathcal{R}_\mathcal{U}} \cap \mathcal{U}$. We now consider the set

$$\mathcal{R}_\mathcal{U} = \widetilde{\mathcal{R}_\mathcal{U}} \cup (\mathcal{O}_n \setminus \mathcal{U}).$$

Note that the set $\mathcal{O}_n \setminus \mathcal{U}$ is the union of the open connected components of $\mathcal{O}_n$ different from $\mathcal{U}$. Thus the set $\mathcal{O}_n \setminus \mathcal{U}$ is open (and closed) in $\mathcal{O}_n$, and therefore the set $\mathcal{R}_\mathcal{U}$ is residual in $\mathrm{Diff}^1(M)$. Finally, the announced residual subset $\mathcal{R}_{\leq n}$ of $\mathrm{Diff}^1(M)$ is the countable intersection of the residual subsets $\mathcal{R}_\mathcal{U}$ of $\mathrm{Diff}^1(M)$:

$$\mathcal{R}_{\leq n} = \bigcap_{\mathcal{U} \in \mathrm{cc}(\mathcal{O}_n)} \mathcal{R}_\mathcal{U}.$$

To complete the proof of the theorem it remains to define $\widetilde{\mathcal{R}_\mathcal{U}}$ for each component $\mathcal{U}$ of $\mathcal{O}_n$. Given $f \in \mathcal{U}$ consider $\mathrm{Per}_n(f) = \{P_{1,f}, \dots, P_{k,f}\}$. For each $i = 1, \dots, k$, Proposition 6.6 gives a residual subset $\mathcal{R}_{P_i}$ of $\mathrm{Diff}^1(M)$ where the conclusion holds. The residual set $\widetilde{\mathcal{R}_\mathcal{U}}$ is the finite intersection of the residual sets $\mathcal{R}_{P_i}$. The proof of Theorem 6.4 is now complete. □

6.3. **Robust homoclinic tangencies under lack of domination.** In this section, we conclude the proof of Theorem 1.2 regarding C^1-generic existence of robust homoclinic tangencies inside homoclinic classes with index variation and lack of domination. We first recall a key result stating the relation between lack of domination and homoclinic tangencies.

Theorem 6.10 (Theorem 1.1 in [16]). *Let P_f be a saddle of a diffeomorphism f such that the stable/unstable splitting defined over the set of periodic points homoclinically related with P_f is not dominated. Then there is a diffeomorphism h arbitrarily C^1-close to f with a homoclinic tangency associated to P_h.*

As in Section 6.2, we begin with a version of Theorem 1.2 for a given fixed saddle.

Proposition 6.11. *Consider a diffeomorphism g and a hyperbolic saddle P_g of g. Let $\mathcal{U}$ be an open subset of $\mathrm{Diff}^1(M)$ such that the map $f \mapsto P_f$ (P_f a hyperbolic saddle) is continuous and well defined. Then there is a residual subset $\mathcal{G}_{\mathcal{U}}$ of $\mathcal{U}$ with the following property. Let $f \in \mathcal{G}_{\mathcal{U}}$ be any diffeomorphism such that:*

- *the chain recurrence class $C(P_f, f)$ has a periodic point Q_f with $\mathrm{ind}^{\mathrm{s}}(Q_f) > \mathrm{ind}^{\mathrm{s}}(P_f)$ and*
- *$C(P_f, f)$ does not admit a dominated splitting $E \oplus_< F$ with $\dim(E) = \mathrm{ind}^{\mathrm{s}}(P_f)$.*

$C(P_f, f)$ has a transitive hyperbolic set containing P_f with a C^1-robust homoclinic tangency.

Proof. Let $\mathcal{W}_0 \subset \mathcal{U}$ be the set of diffeomorphisms f such that the chain recurrence class $C(P_f, f)$ of P_f robustly contains a hyperbolic periodic point Q_f with $\mathrm{ind}^{\mathrm{s}}(Q_f) > \mathrm{ind}^{\mathrm{s}}(P_f)$ (i.e., $Q_g \in C(P_g, g)$ for all g C^1-close to f). By definition and Remark 6.2, the set $\mathcal{W}_0$ is open and non-empty. Let

$$\mathcal{U}_0 = \mathcal{U} \setminus \overline{\mathcal{W}_0}.$$

Note that $\mathcal{U}_0 \cup \mathcal{W}_0$ is an open and dense subset of $\mathcal{U}$.

Let $\mathcal{U}_1 \subset \mathcal{U}$ be the set of diffeomorphisms f such that $C(P_f, f)$ has a dominated splitting $E \oplus_< F$ with $\dim(E) = \mathrm{ind}^{\mathrm{s}}(P_f)$. Since the map $f \mapsto C(P_f, f)$ is upper-semi-continuous and a dominated splitting persists in a neighborhood of $C(P_f, f)$ by perturbations (for instance, see [13, Chapter B.1]), one gets that the set $\mathcal{U}_1$ is open. Let

$$\mathcal{W}_1 = \mathcal{U} \setminus \overline{\mathcal{U}_1}.$$

Then the set $\mathcal{U}_1 \cup \mathcal{W}_1$ is open and dense in $\mathcal{U}$. As a consequence, the open sets $\mathcal{U}_0 \cup \mathcal{U}_1$ and $\mathcal{W}_0 \cap \mathcal{W}_1$ are disjoint and their union is dense in $\mathcal{U}$.

Note that we are interested in the subset of $\mathcal{U}$ of diffeomorphisms f whose chain recurrence class $C(P_f, f)$ contains points of different indices and does not have an appropriate dominated splitting, that is, the set set $\mathcal{W}_0 \cap \mathcal{W}_1$. Thus the next lemma implies the proposition.

Lemma 6.12. *The set $\mathcal{T}$ of diffeomorphisms g having a hyperbolic set Σ_g containing P_g and with a C^1-robust homoclinic tangency is open and dense in $\mathcal{W}_0 \cap \mathcal{W}_1$.*

Proof. It is enough to prove the density of the set $\mathcal{T}$. Let $g \in \mathcal{W}_0 \cap \mathcal{W}_1$. As $g \in \mathcal{W}_0$ there is a saddle Q_g with $\mathrm{ind}^{\mathrm{s}}(Q_g) > \mathrm{ind}^{\mathrm{s}}(P_g)$ such that Q_g belongs robustly to $C(P_g, g)$. After a C^1-perturbation, we can assume that the diffeomorphism g simultaneously belongs to the residual set $\mathcal{G}$ where $C(P_g, g) = H(P_g, g)$ and to the residual set $\mathcal{R}$ in Theorem 6.4. Note that:

- By Theorem 6.4, the set of diffeomorphisms g having a cu-blender-horseshoe Λ_g which is contained in a transitive hyperbolic set containing P_g is open and dense in $\mathcal{W}_0$.
- As $g \in \mathcal{W}_1$ and $H(P_g, g) = C(P_g, g)$, one has that the stable/unstable splitting defined over the set of periodic points homoclinically related with P_g is not dominated. Otherwise, this dominated splitting could be extended to the closure of these points (the whole $H(P_g, g)$) in a dominated way (see [13, Chapter B.1]), which is a contradiction.
- Since the stable/unstable splitting defined over the set of periodic points homoclinically related with P_g is not dominated, Theorem 6.10 implies

that there is a diffeomorphism h arbitrarily C^1-close to g with a homoclinic tangency associated to P_h.

Theorem 4.9 now implies that there is a diffeomorphism φ arbitrarily close to g with a transitive hyperbolic set containing P_φ and having a robust homoclinic tangency. This ends the proof of the lemma. $\square$

The proof of Proposition 6.11 is now complete. $\square$

6.3.1. *End of the proof of Theorem* 1.2. The proof of Theorem 1.2 using Proposition 6.11 is almost identical to the proof of Theorem 6.4 using Proposition 6.6. It is enough to see that, for every $n \in \mathbb{N}$, there is a residual set $\mathcal{G}_{\leq n}$ of diffeomorphisms f for which the conclusion of the theorem holds for the points in $\mathrm{Per}_n(f)$.

This proof is similar to the one of Theorem 6.4, so we will omit some details. As in Theorem 6.4, we consider the C^1-open and dense subset $\mathcal{O}_n \subset \mathrm{Diff}^1(M)$ of diffeomorphisms f such that every periodic point in $\mathrm{Per}_n(f)$ is hyperbolic. Recall that the number of elements of $\mathrm{Per}_n(f)$ is finite and locally constant, and that each periodic point in $\mathrm{Per}_n(f)$ has a hyperbolic continuation on each (open) connected component of $\mathcal{O}_n$.

To state Theorem 1.2 for periodic points in $\mathrm{Per}_n(f)$, it is enough to prove it in each connected component $\mathcal{U} \in \mathrm{cc}(\mathcal{O}_n)$ (recall that $\mathrm{cc}(\mathcal{O}_n)$ is countable): for each connected component $\mathcal{U}$, we construct a residual subset $\widetilde{\mathcal{G}_\mathcal{U}}$ such that the conclusion holds in the set $\widetilde{\mathcal{G}_\mathcal{U}} \cap \mathcal{U}$. Then we let

$$\mathcal{G}_\mathcal{U} = \widetilde{\mathcal{G}_\mathcal{U}} \cup (\mathcal{O}_n \setminus \mathcal{U})$$

and define

$$\mathcal{G}_{\leq n} = \bigcap_{\mathcal{U} \in \mathrm{cc}(\mathcal{O}_n)} \mathcal{G}_\mathcal{U}.$$

To define $\widetilde{\mathcal{G}_\mathcal{U}}$ for a component $\mathcal{U}$ of $\mathcal{O}_n$, given a diffeomorphism $f \in \mathcal{U}$ write $\mathrm{Per}_n(f) = \{P_{1,f}, \dots, P_{k,f}\}$ ($k = k(\mathcal{U})$) and consider the continuous maps $f \mapsto P_{i,f}$, $i \in \{1, \dots, k\}$, defined on $\mathcal{U}$. For each i, Proposition 6.11 provides a residual subset where the conclusion of the theorem holds for $P_{i,f}$. Now it is enough to define $\widetilde{\mathcal{G}_\mathcal{U}}$ as the intersection of these residual sets. The proof of Theorem 1.2 is now complete. $\square$

6.4. **Robust cycles in non-hyperbolic chain recurrence classes.** We close this paper by stating an extension of [9, Theorem 1.16]. The novelty of this version is that the hyperbolic sets involved in the robust cycle are contained in a prescribed chain recurrence class. We note that [9] does not give information about the relation between the hyperbolic set involved in the robust cycle and the saddles in the initial cycle.

Theorem 6.13. *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(M)$ with the following property. Consider any diffeomorphism $f \in \mathcal{R}$ having a chain recurrence class C with two saddles P and Q such that $\mathrm{ind}^s(P) = \mathrm{ind}^s(Q) + 1$. Then f has a C^1-robust heterodimensional cycle associated to hyperbolic sets Λ and Σ containing P and Q.*

Since this result follows from arguments similar to the ones in the previous sections and as robust heterodimensional cycles are not the main topic of this paper, we just give some hints for the proof.

As in the proofs above, it is enough to state a local version of the theorem for a given saddle P. Then the general version follows using standard genericity arguments identical to the ones in Sections 6.2 and 6.3.

To get the local version of the theorem, note that for generic diffeomorphisms f, the saddle Q is robustly in the chain recurrence class $C(P,f)$ and there is a cu-blender-horseshoe Σ associated to Q. In this step, the strongly intermediate points given by Proposition 5.6 play a key role. Finally, a perturbation gives a robust cycle with a transverse intersection between $W^{\mathrm{s}}(P)$ and $W^{\mathrm{u}}(Q)$ and a robust intersection of $W^{\mathrm{u}}(P)$ with $W^{\mathrm{s}}(\Sigma)$. This completes the brief sketch of the proof.

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