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Abundance of strange attractors

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1. Introduction

In 1976 [He] Hénon performed a numerical study of the family of diffeomorphisms of the plane $h_{a,b}(x,y)=(1-ax^2+y, bx)$ and detected for parameter values $a=1.4$, $b=0.3$, what seemed to be a non-trivial attractor with a highly intricate geometric structure. This family has since then been the subject of intense research, both numerical and theoretical, but its dynamics is still far from being completely understood. In particular one could not exclude the possibility that the attractor observed by Hénon were just a periodic orbit with a very high period.

Recently, in a remarkable paper [BC2], Benedicks and Carleson were able to show that this is not the case, at least for a positive Lebesgue measure set of parameter values near $a=2$, $b=0$. More precisely, they showed that if $b>0$ is small enough then for a positive measure set of a-values near $a=2$ the corresponding diffeomorphism $h_{a,b}$ exhibits a strange attractor. Their argument is a very creative extension of the techniques they had previously developed in [BC1] for the study of the quadratic family on the real line and no doubt it will be important for the understanding of several other situations of complicated, nonhyperbolic dynamics.

When acquainted in 1985 with the work by Benedicks and Carleson, then in progress, Palis suggested that one should in this context think of the Hénon family as a particular, although important, model for the creation of a horseshoe and that the emphasis should be put on the occurrence of unfoldings of homoclinic tangencies. He proposed that the correct setting for Benedicks–Carleson's results is within this more general framework of homoclinic bifurcations and stated the following

Conjecture. Generic one-parameter families of surface diffeomorphisms unfolding a homoclinic tangency exhibit strange attractors or repellers in a persistent way in the measure-theoretic sense (i.e. for a positive measure set of values of the parameter).

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He also suggested the following program for the proof of this conjecture: to extend Benedicks–Carleson's methods and results to more general perturbations of the family of quadratic maps on the real line—Hénon-like families—and then make use of the fact that generic families of surface diffeomorphisms with homoclinic tangencies admit renormalizations which are Hénon-like, see [PT2, Chapter III]. The main goal of this paper is to carry on this program and so prove Palis' conjecture.

THEOREM A. *Let $(f_\mu)_\mu$ be a C^∞ one-parameter family of diffeomorphisms on a surface and suppose that f_0 has a homoclinic tangency associated to some periodic point p_0. Then, under generic (even open and dense) assumptions, there is a positive Lebesgue measure set E of parameter values near $\mu=0$, such that for $\mu\in E$ the diffeomorphism f_μ exhibits a strange attractor, or repeller, near the orbit of tangency.*

By an *attractor* we mean a compact invariant set Λ having a dense orbit and whose stable set $W^s(\Lambda)$ has non-empty interior. We call an attractor *strange* if it has a dense orbit with positive Lyapunov exponent ("sensitive dependence on initial conditions", Ruelle and Eckmann [RE]). Obviously, strange attractors are always non-trivial (i.e. non-periodic) and in our setting they are even non-hyperbolic as will be seen below. Similar definitions and comments hold for repellers.

The generic assumptions mentioned in the statement are those required by the renormalization construction described in Section 2 and are explicitly stated there. Our definition of Hénon-like families is through Theorem 2.1. Up to conjugacy this is the same notion as in [PT2], but we need some more information that we also provide in Section 2. Theorem A is a direct consequence of this and the following generalization of the main theorem in [BC2].

THEOREM B. *Let $0<c<\log 2$ and $\varphi=(\varphi_a)_a$ be a Hénon-like family (see Theorem 2.1). Then, there is $E=E(c,\varphi)\subset(1,2)$, with positive Lebesgue measure, such that for every $a\in E$ there is a compact, φ_a-invariant set $\Lambda=\Lambda_a$ satisfying:*

(a) *The stable set $W^s(\Lambda)$ of Λ has non-empty interior.*

(b) *There is $z_1\in\Lambda$ such that*

  (i) *$\{\varphi_a^n(z_1):n\geqslant 0\}$ is dense in Λ,*

  (ii) *$\|D\varphi_a^n(z_1)\cdot(1,0)\|\geqslant e^{cn}$ for all $n\geqslant 0$.*

Hénon-like maps are (strongly) area-contracting everywhere in their domain and so (Plykin [Py]) they can not have non-trivial hyperbolic attractors. Hence, as we have mentioned before, the strange attractors we find are always nonhyperbolic.

The proof of Theorem B occupies nearly all the paper and consists basically of variations of the arguments in [BC2], with several of them extended in order to fit into our more general setting. In the present paper we might have restricted ourselves to the description of the changes that we need to perform in [BC2], but this would make our presentation hardly readable. Instead, we insert these changes in a fairly detailed overview of Benedicks–Carleson's beautiful and yet difficult construction. In particular, by emphasizing their geometrical aspects, we clarify several of the hardest points in this construction: definition of critical points, induction structure, parameter exclusions, etc. A simplification of the arguments was also possible at some places, e.g. Section 10.

We also observe that the same arguments we use to obtain Theorems A and B apply (in a considerably easier form) to the one-dimensional setting and yield a proof of the abundance of strange attractors for families of endomorphisms of the circle or the interval (while unfolding a homoclinic bifurcation). Thus, we obtain as a corollary the next

THEOREM C. *Let $(f_\mu)_\mu$ be a smooth family of maps on $[0,1]$ or S^1 and p_0 be a hyperbolic periodic point for f_0. Suppose that the negative orbit of p_0 intersects the unstable set $W^u(p_0)$ in a non-degenerate critical point of f_0. Then, if this homoclinic tangency unfolds generically, there is a positive measure set of μ-values near $\mu=0$ for which f_μ exhibits strange attractors.*

Several comments are in order on the meaning and scope of our results. Palis has recently proposed a program for a theory of homoclinic bifurcations containing a fairly extensive description of the extraordinary richness of phenomena associated to the unfolding of a homoclinic tangency: cascades of bifurcations, persistent tangencies, infinitely many sinks, hyperbolicity, strange attractors, etc. As a scenario for this program, he stated the following (very difficult) conjecture: given a surface diffeomorphism, it can be approximated either by a stable (hyperbolic) diffeomorphism or by one exhibiting a homoclinic tangency. In view of Theorem A, this means that one expects strange attractors to be a very typical phenomenon under nonhyperbolicity of the diffeomorphism (i.e. of its limit set).

On the other hand, the strange attractors we find in the context of Theorem A (or B) are fairly structured and seem to share some of the properties of hyperbolic attractors. One such property concerns the eigenvalues of the periodic points contained in the attractor. Analogy to what happens in the one-dimensional case (Theorem C) suggests that all these eigenvalues are uniformly bounded away from 1 in norm; it seems of interest to decide whether this is indeed so.

Another important property regards the existence of invariant measures supported on the attractor. Benedicks and Young have recently announced the construction of SRB-measures for the strange attractors in [BC2]. In view of the arguments we present here it seems likely that their construction extends to general families of diffeomorphisms with homoclinic bifurcations.

The proof of Theorem B raises a few other questions which are of interest. A first one concerns the basin of the attractor Λ: for the orientation reversing case (including the Hénon family for $b>0$) we observe in Section 4 that $W^s(\Lambda)$ contains a neighborhood of Λ; the same seems to hold also in the orientation preserving case but so far we could not give a full argument to prove this.

For the one-dimensional quadratic family $Q_a(x)=1-ax^2$ the set of a-values for which Q_a has chaotic behaviour (positive Lyapunov exponent on the critical orbit) has density 1 at $a=2$, see Section 3. Now, $a=2$ corresponds to simultaneous homoclinic and heteroclinic tangencies for the family $Q=(Q_a)_a$ and so such tangencies exist also (at parameter values near $a=2$) for every family $\varphi=(\varphi_a)_a$ of surface diffeomorphisms sufficiently close to $\psi_a(x,y)=(1-ax^2,0)$. It is natural to ask for the bifurcation parameters denoted in Section 4 by $a_\pm(\varphi)$, whether they are points of positive density for the set of a-values corresponding to which φ_a has strange attractors. This is not known even for the Hénon family. Observe that one cannot expect this to hold for general homoclinic bifurcations, by Palis–Takens [PT1].

For the proof of Theorem A we assume, see Section 2, the homoclinic tangency to be non-degenerate (quadratic), but this seems unnecessary. Thus, most certainly, we can apply our results to the situation originally considered by Hénon, since very likely homoclinic bifurcations occur for the Hénon family at parameter values near $a=1.4$, $b=0.3$. In a small scale, this provides an explanation for the *chaotic behaviour* detected by Hénon. However, the question concerning the existence of a *global strange attractor*, as probably initially intended by Hénon, remains an open and very interesting question.

In a work in development, the second author is also proving Theorem A for higher-dimensional manifolds, under the assumptions of codimension one and *strong dissipativeness*. More precisely for a family $(f_\mu)_\mu$ of diffeomorphisms on an n-manifold, with a homoclinic tangency associated to a hyperbolic periodic point p_0 of f_0, we assume that $\dim E^u(p_0)=1$ and $|\sigma\lambda_{n-1}|<1$, where $|\lambda_1|\leqslant\ldots\leqslant|\lambda_{n-1}|<1<|\sigma|$ are the eigenvalues of $Df_0^k(p_0)$, $k=$ period of p_0.

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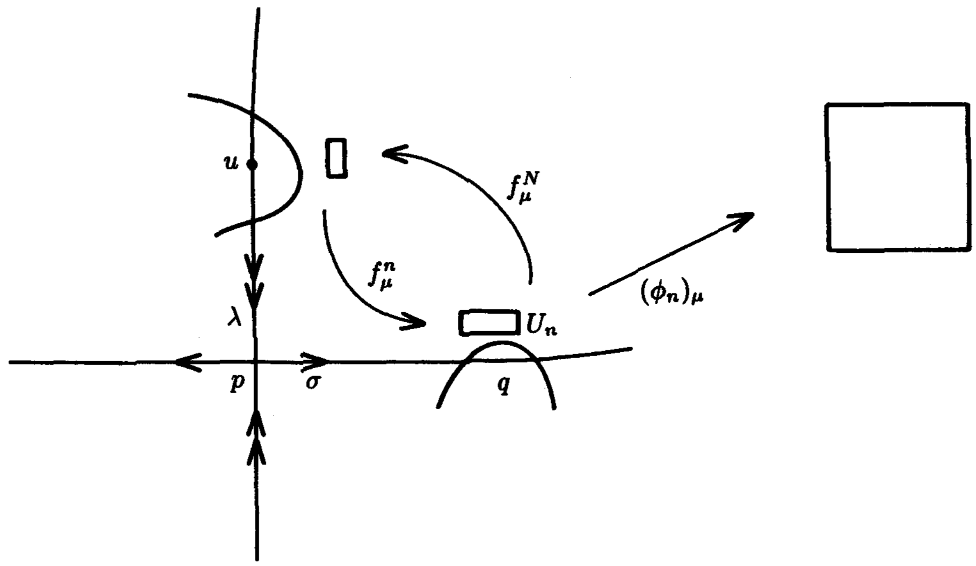

Fig. 1

2. Hénon-like families

Let $(f_\mu)_\mu$ be a C^∞ one-parameter family of diffeomorphisms on a surface, unfolding a homoclinic tangency associated to a hyperbolic fixed (or periodic) point p_0 of f_0. Let $1\leqslant r<\infty$ be fixed. We show that, under generic assumptions (including dissipativeness of f_0 at p_0), one can find renormalizations of $(f_\mu)_\mu$ which are arbitrarily C^r close to the family of endomorphisms $\psi_a(x,y)=(1-ax^2,0)$. By this we mean that there are (small) domains U_n on the surface converging to a point q of the orbit of tangency, (small) intervals I_n converging to $\mu=0$ in the parameter space and C^r n-dependent coordinates on $I_n\times U_n$, with the property that, as $n\to\infty$, the expression of $f^n|_{I_n\times U_n}$ in these coordinates converges to ψ in the C^r topology, see Figure 1. This construction is a variation of [PT2, §III.4] and so we only outline the main ideas, the details being easy to complete. On the other hand, we also derive additional properties of renormalized families for use in the forthcoming sections.

Assume first that there are μ-dependent C^r linearizing coordinates (ξ,η) for f_μ, defined in a neighbourhood U of p_μ, the analytic continuation of p_0. Up to a rescaling of these coordinates we may assume that U contains $\{(\xi,\eta):|\xi|\leqslant 2,|\eta|\leqslant 2\}$ and that $q=(1,0)$ and $u=(0,1)$ belong to the orbit of tangency, $u=f_0^N(q)$. Let f_μ be area contracting at p_μ for small μ: $|\det Df_\mu(p_\mu)|<1$.

The area expanding case is treated in the same way simply by replacing f by f^{-1} in what follows. We denote by $\lambda=\lambda_\mu$ and $\sigma=\sigma_\mu$ the eigenvalues of $Df_\mu(p_\mu)$ which we may assume to satisfy $0<\lambda<1<\sigma$.

Finally, the tangency is assumed to be non-degenerate (quadratic) and to unfold generically with μ. This means that

$$f_\mu^N(1+\xi,\eta)=(\alpha\xi^2+\beta\eta+v\mu+H_1(\mu,\xi,\eta),1+H_2(\mu,\xi,\eta))$$

with $\alpha\neq 0$ and $v\neq 0$ and $H_1=\partial_\xi H_1=\partial_{\xi\xi}H_1=\partial_\eta H_1=\partial_\mu H_1=H_2=0$ at $(0,0,0)$. By an affine change of coordinates we can even make $\partial_{\xi\mu}H_1(0,0,0)=0$ and, reparametrizing the family if necessary, $v=1$ and $\partial_{\mu\mu}H_1(0,0,0)=0$. Then we introduce new coordinates $(a,x,y)=\phi_n(\mu,\xi,\eta)$ given by

$$a=-\alpha(\sigma^{2n}\mu-\sigma^n+\beta\lambda^n\sigma^{2n});\qquad \mu=-\frac{a}{\alpha}\sigma^{-2n}+\sigma^{-n}-\beta\lambda^n;$$
$$x=-\frac{\alpha}{a}\sigma^n(\xi-1);\qquad \xi=-\frac{a}{\alpha}\sigma^{-n}x+1;$$
$$y=-\frac{\alpha}{a}\sigma^{2n}\left(\sqrt{\lambda\sigma}\right)^{-n}(\eta-\lambda^n);\qquad \eta=-\frac{a}{\alpha}\sigma^{-2n}\left(\sqrt{\lambda\sigma}\right)^n y+\lambda^n.$$

and define $\varphi_n=\phi_n\circ f^n\circ f^N\circ\phi_n^{-1}$, where we denote $f(\mu,x,y)=(\mu,f_\mu(x,y))$. Observe that if we let $R=\{(a,x,y):1\leqslant a\leqslant 3,|x|\leqslant 2,|y|\leqslant 2\}$ then $\phi_n^{-1}(R)$ converges to $(0,1,0)$ as $n\to\infty$. In particular the domain of definition of φ_n contains R, at least for n sufficiently large. It is fairly easy to check that

$$\|\varphi_n-\psi\|_{C^r(R)}\leqslant K\max\{\left(\sqrt{\lambda_0\sigma_0}\right)^n,\sigma_0^{-n}\}$$

and so $(\varphi_n)_n$ converges to ψ in the C^r topology, uniformly on R. Here and in what follows K always denotes a sufficiently large constant independent of n.

We use $D=D_{(x,y)}$ to denote derivative with respect to the (x,y) variables. Let us write

$$Df_\mu^N(\xi,\eta)=\begin{pmatrix}\alpha_\mu & \beta_\mu\\ \gamma_\mu & \delta_\mu\end{pmatrix}(\xi,\eta).$$

For $(\mu,\xi,\eta)\in\phi_n^{-1}(R)$ we have $|\alpha_\mu(\xi,\eta)|\leqslant(\|\partial_\mu\alpha\|\cdot|\mu|+\|\partial_\xi\alpha\|\cdot|\xi-1|+\|\partial_\eta\alpha\|\cdot|\eta|)\leqslant K\sigma^{-n}$ and also, just by continuity, $|\beta_\mu(\xi,\eta)|,|\gamma_\mu(\xi,\eta)|$, $|\delta_\mu(\xi,\eta)|\leqslant K$. Moreover $\det Df_0^N(q)=-\beta_0(q)\cdot\gamma_0(q)\neq 0$ and so $1/K\leqslant|\det Df_\mu^N(\xi,\eta)|\leqslant K$ and $|\beta_\mu(\xi,\eta)|,|\gamma_\mu(\xi,\eta)|\geqslant 1/K$, for $(\mu,\xi,\eta)\in\phi_n^{-1}(R)$, n big enough. As to the derivatives, clearly $\|D_{(a,x,y)}\alpha\|\leqslant\|D_{(\mu,\xi,\eta)}\alpha\|\cdot\|D\phi_n^{-1}\|\leqslant K\sigma^{-n}$ and, analogously, $\|D_{(a,x,y)}\beta\|$, $\|D_{(a,x,y)}\gamma\|$, $\|D_{(a,x,y)}\delta\|\leqslant K\sigma^{-n}$. In precisely the same way one obtains the following bounds for the second order derivatives $\|D^2_{(a,x,y)}\alpha\|$, $\|D^2_{(a,x,y)}\beta\|$, $\|D^2_{(a,x,y)}\gamma\|$, $\|D^2_{(a,x,y)}\delta\|\leqslant K\sigma^{-2n}$. Keeping in mind that

$$D\varphi_{n,a}(x,y)=\begin{pmatrix}\alpha_\mu\sigma^n & \beta_\mu\left(\sqrt{\lambda\sigma}\right)^n\\ \gamma_\mu\left(\sqrt{\lambda\sigma}\right)^n & \delta_\mu\lambda^n\end{pmatrix}(\xi,\eta)$$

we get

THEOREM 2.1. *Let $(f_\mu)_\mu$ be a C^∞ family of diffeomorphisms as above. Then there are $K>0$, $t>0$ and, given $b>0$ there is $n_0=n_0(b)\geqslant 1$ such that any $\varphi=\varphi_n$, $n\geqslant 0$, satisfies:*

(a) $\|\varphi-\psi\|_{C^r(R)}\leqslant Kb^t$. *In particular,* $\|\varphi\|_{C^r(R)}\leqslant 5$ $(\leqslant K)$;

(b) *Denote*

$$D\varphi_a(x,y)=\begin{pmatrix} A & B \\ C & D\end{pmatrix}(a,x,y).$$

Then, in R,

(i) $|A|\leqslant K$, $\sqrt{b}/K\leqslant|B|\leqslant K\sqrt{b}$, $\sqrt{b}/K\leqslant|C|\leqslant K\sqrt{b}$ *and* $|D|\leqslant Kb^{1+t}$. *Moreover* $b/K\leqslant|\det D\varphi_a|\leqslant Kb$, $\|D\varphi_a\|\leqslant K$ *and* $\|D\varphi_a^{-1}\|\leqslant K/b$.

(ii) $\|D_{(a,x,y)}A\|\leqslant K$, $\|D_{(a,x,y)}B\|\leqslant Kb^{1/2+t}$, $\|D_{(a,x,y)}C\|\leqslant Kb^{1/2+t}$, $\|D_{(a,x,y)}D\|\leqslant Kb^{1+2t}$. *Moreover* $\|D_{(a,x,y)}(\det D\varphi_a)\|\leqslant Kb^{1+t}$ *and* $\|D^2\varphi_a\|\leqslant K$.

(iii) $\|D^2_{(a,x,y)}A\|\leqslant Kb^t$, $\|D^2_{(a,x,y)}B\|\leqslant Kb^{1/2+2t}$, $\|D^2_{(a,x,y)}C\|\leqslant Kb^{1/2+2t}$ *and* $\|D^2_{(a,x,y)}D\|\leqslant Kb^{1+3t}$. *Finally* $\|D^2_{(a,x,y)}(\det D\varphi_a)\|\leqslant Kb^{1+2t}$ *and* $\|D^3\varphi_a\|\leqslant Kb^t$.

In Sections (4)–(12) we prove that the conclusion of Theorem B holds for all sufficiently smooth families $\varphi=(\varphi_a)_a$ satisfying (a)–(b) above for a sufficiently small b. These are what we call *Hénon-like families*. We fix the values of K and t from now on; for convenience we assume $K>10$ and $t<\frac{1}{2}$. We also let r be fixed, sufficiently large (Section 11). A few other parameters are also involved in the construction, namely $\frac{1}{2}<c<c_0<\log 2$ and small numbers $\varepsilon>0, \beta>0, \alpha>0$ and $\delta>0$. Formally speaking, they are chosen in the order we have listed them: the value of each parameter must be taken to satisfy a certain number of conditions which depend only on the ones listed previously to it. For the sake of clearness these conditions are stated throughout the proof in the order they are required. A small interval Ω in the a-space close to $a=2$ and a large integer N related to it are also fixed, depending on these constants. Finally b is assumed small with respect to everything else.

3. One-dimensional families

First we outline Benedicks–Carleson's proof ([BC1], [BC2, Section 2]) that, for a positive measure set of values of $a\in(1,2)$, the critical orbit of $Q_a(x)=1-ax^2$ has positive Lyapunov exponent. This is intended as a summary to be followed in Section 5 through 12 where their method is adapted to prove Theorem B. We also observe that the arguments (and the conclusion) here remain valid if $Q=(Q_a)_a$ is replaced by any smooth family (of one-dimensional transformations) sufficiently close to it (with 0, 1 replaced by the corresponding critical point and critical value, respectively). Theorem C follows by combining this with the one-dimensional version of the renormalization technique in the previous section (see also Sections 4 and 12).

Fix $\frac{1}{2}<c<c_0<\log 2$ and let $\varepsilon>0, \beta>0, \alpha>0$ and $\delta>0$ be sufficiently small constants, chosen in this order. We denote

$$D_n(a)=(Q_a^n)'(1)=\prod_1^n(-2aQ_a^j(0)) \tag{1}$$

and want to show that, there is a positive measure subset $E=E(c)$ of $(1,2)$, having 2 as a point of density, such that for $a\in E$

$$|D_n(a)|\geqslant e^{nc} \quad \text{for all } n\geqslant 1. \tag{2}$$

In particular, by Singer's theorem [Si] there are no attracting periodic orbits for Q_a if $a\in E$. The special role of $\log 2$ is due to the following lemma, which is a consequence of the fact that Q_2 is conjugated to the tent map $T(x)=1-2|x|$.

LEMMA 3.1. *Given* $0<c_0<\log 2$ *and* $\delta>0$ *there is* $a_0=a_0(c_0,\delta)<2$ *such that for* $a_0\leqslant a\leqslant 2$, *if*

(a) $|Q_a^j(x)|\geqslant\delta$ *for* $1\leqslant j\leqslant k$ *and*

(b) $|x|<\delta$ *or* $|Q_a^{k+1}(x)|<\delta$

then $|(Q_a^k)'(x)|\geqslant e^{kc_0}$.

In other words, (maximal) pieces of orbit outside $(-\delta,\delta)$ (*free periods*) have expanding behaviour. Now we must deal with the *returns*, i.e. the iterates ν for which $|Q_a^\nu(0)|<\delta$. Since ([BC1])

$$\inf_{k\geqslant 1}|Q_a^k(0)|\leqslant\delta \quad \text{for almost every } a\in[a_0,2] \tag{3}$$

we can not prevent the orbit of zero from returning close to itself. However this should not happen too fast and we make the basic assumption

$$|Q_a^\nu(0)|\geqslant e^{-\alpha\nu}. \tag{BA}$$

The values of $a\in(1,2)$ for which (BA) is not satisfied are excluded from the set E as we describe below.

Using (BA) one can show that the small factors introduced in (2) on returns are compensated by the growth of the derivative in the following iterates. The crucial idea here is that of binding period, which can be heuristically motivated as follows. If ν is a return then $Q_a^\nu(0)$ is close to zero and so their positive orbits remain close (bound) to each other for a period of time which depends essentially on how small $|Q_a^\nu(0)|$ is.

During this period the two orbits have similar behaviours. Hence, one may use, in an inductive way, information concerning the growth of the derivative on early iterates of zero (previous to time ν), in order to obtain the same kind of information for the iterates in the binding period following ν. For the induction to work the length of the binding period must be less than ν and this is a consequence of (BA). Let us come to a formal definition. The *binding period* associated to the return ν is the interval $[\nu+1, \nu+p]$ where $p \geqslant 1$ is defined by the binding condition

$$|Q_a^{\nu+j}(0) - Q_a^j(0)| \leqslant e^{-\beta j} \quad \text{for } 1 \leqslant j \leqslant p, \ p \text{ maximum.} \tag{BC}$$

Then it can be proved

LEMMA 3.2. *There are $\varrho=\varrho(c,\alpha,\beta)>0$ and $\sigma=\sigma(c,\alpha,\beta)>0$ such that if*

$$|Q_a^j(0)| \geqslant e^{-\alpha j} \ \textit{for } 1 \leqslant j \leqslant n \quad \textit{and} \quad |D_j(a)| \geqslant e^{cj} \ \textit{for } 1 \leqslant j \leqslant n-1$$

then for $\nu \leqslant n$ a return and p the length of the binding period associated to ν, we have:

(a) $1/\varrho \leqslant |(Q_a^j)'(\xi)|/|(Q_a^j)'(\eta)| \leqslant \varrho$ *for all* $\xi, \eta \in [Q_a^{\nu+1}(0), 1]$ *and* $1 \leqslant j \leqslant p$; *in particular* $|(Q_a^j)'(Q_a^{\nu+1}(0))| \geqslant e^{cj}/\varrho$ *for* $1 \leqslant j \leqslant p$.

(b) $p \in [r/(\beta+\log 4), 3r/(\beta+c)]$, *where* $r = -\log|Q_a^\nu(0)|$, *and so* $p \leqslant 3\alpha\nu/(\beta+c) \leqslant \nu/2$.

(c) $|(Q_a^{p+1})'(Q_a^\nu(0))| \geqslant \sigma e^{(p+1)c/3} \geqslant 1$.

The meaning of part (c) is that the growth of the derivative during the binding period (which follows from (a)) fully compensates the small factor $Q_a'(Q_a^\nu(0)) = -2aQ_a^\nu(0)$ and there is even some overall exponential gain in the interval of time $[\nu, \nu+p]$.

We say that a return ν is *free* if it does not belong to any binding period associated to some previous return. Let $N=\nu_1<\nu_2<\ldots<\nu_s \leqslant n$ be the free returns in $[1,n]$ and $p_1, p_2, \ldots, p_s$ be the lengths of the associated binding periods. We also denote by

$$q_0 = N-1, \quad q_1 = \nu_2 - (\nu_1 + p_1 + 1), \quad \ldots, \quad q_{s-1} = \nu_s - (\nu_{s-1} + p_{s-1} + 1)$$

(and $q_s = n - (\nu_s + p_s + 1)$, if $n > \nu_s + p_s$), the lengths of the complementary free periods and let $F_n = F_n(a) = q_0 + \ldots + q_{s-1}$ (respectively $F_n = q_0 + \ldots + q_{s-1} + q_s$) be the *total free time* in $[1,n]$. Then by Lemmas 3.1 and 3.2

$$|D_n(a)| \geqslant e^{c_0 F_n} \cdot e^{-\alpha\nu_s} \geqslant e^{c_0 F_n} \cdot e^{-\alpha n} \quad (\text{respectively } |D_n(a)| \geqslant e^{c_0 F_n}). \tag{4}$$

The next fundamental step is to make new exclusions of parameters, retaining only the values of $a \in (1,2)$ for which the critical orbit spends most of the time in a free iterate

$$F_n(a) \geqslant (1-\varepsilon)n. \tag{FA}$$

Then, from (4), $|D_n(a)| \geqslant e^{(c_0(1-\varepsilon)-\alpha)n} \geqslant e^{cn}$ if $\varepsilon > 0$ and $\alpha > 0$ are taken small enough.

Clearly, this completes an inductive procedure that proves (2) for the a-values satisfying (BA) and (FA) and so we are left to show that the set of such values has positive Lebesgue measure. This requires some notations and an important ingredient is the construction of a sequence $(\mathcal{P}_n)_n$ of families of disjoint intervals in the a-space with the property that the a-derivative $\partial_a Q_a^n(0)$ has bounded distortion on each $\omega \in \mathcal{P}_n$. We define a sequence $E_1 \supset E_2 \supset \ldots \supset E_{k-1} \supset E_k \supset \ldots$ of subsets of (1,2) such that parameters $a \in E_k$ satisfy the (BA) and the (FA) for all iterates $1 \leqslant j \leqslant k$ and so $|D_j(a)| \geqslant e^{jc}$ for $1 \leqslant j \leqslant k$. Each $\mathcal{P}_n$ is a partition of E_{n-1} and E_n is obtained as a union of intervals in $\mathcal{P}_n$. Finally we take $E = \bigcap_{n \geqslant 1} E_n$.

We begin by fixing $\omega_0 = (a_0, 2)$, a_0 close to 2. Denote $\gamma_k = \gamma_k(\omega_0) = \{Q_a^k(0) : a \in \omega_0\}$. Clearly $Q_2^k(0) = -1$ is in the boundary of γ_k, for $k \geqslant 2$, and from the fact that Q_2 is expanding at -1 (eigenvalue $=4$) one can show, in a fairly easy way, that γ_k eventually contains zero. Take N minimum such that $0 \in \gamma_N$. Observe that N can be made arbitrarily large by taking a_0 close enough to 2. For $2 \leqslant k \leqslant N$, $\gamma_k : \omega_0 \ni a \mapsto Q_a^n(0)$ is a diffeomorphism and $\gamma_k(\omega_0) = (-1, b_k)$ with $-1 < b_2 < \ldots < b_{N-1} < 0 < b_N$. Moreover, by slightly changing a_0, we may even suppose $\gamma_k(\omega_0) \cap (-\delta, \delta) = \varnothing$ for $2 \leqslant k \leqslant N-1$ and then, by Lemma 3.1, one obtains $|D_k(a)| \geqslant e^{kc_0}$ for all $1 \leqslant k \leqslant N-1$ and $a \in \omega_0$.

Now we start the construction of the E_n and $\mathcal{P}_n$, which is done by induction on n. For $n \leqslant N-1$ we take simply $E_n = \omega_0$ and $\mathcal{P}_n = \{\omega_0\}$, the trivial partition. Suppose now that E_k and $\mathcal{P}_k$ were already defined for $k \leqslant n-1$. We obtain $\mathcal{P}_n$ by refining $\mathcal{P}_{n-1}$ as follows. Let ω be any interval in $\mathcal{P}_{n-1}$. If $\gamma_n(\omega)$ does not intersect $(-\delta, \delta)$ we leave it unchanged: ω is also an element of $\mathcal{P}_n$. The same holds if n belongs to the binding period associated to some return $\nu < n$, i.e. if

$$|Q_a^{\nu+j}(0) - Q_a^j(0)| \leqslant e^{-\beta j} \quad \text{for all } j \leqslant n-\nu \text{ and } a \in \omega.$$

In this case still we leave ω unchanged even if $\gamma_n(\omega) \cap (-\delta, \delta) \neq \varnothing$ (we call that a *bound return situation*). Let now n be a *free return situation* for ω: $\gamma_n(\omega)$ intersects $(-\delta, \delta)$ and n does not belong to any binding period (the first such situation occurs for $n = N$ and $\omega = \omega_0$). First we write $\omega = \omega' \cup \omega''$, with $\omega' = \gamma_n^{-1}((-\delta, \delta))$ and $\omega'' = \gamma_n^{-1}((-\delta, \delta)^c)$. For $a \in \omega''$ the iterate n is not a return. By definition each connected component of ω'' is an element of $\mathcal{P}_n$. In order to describe the restriction of $\mathcal{P}_n$ to ω' we introduce the partition $\{I_r\}$ of $(-\delta, \delta)$ defined by $I_r = (e^{-r}, e^{-r+1}), I_{-r} = -I_r$ for $r > \Delta \equiv -\log \delta$. We also subdivide each I_r into r^2 intervals $I_{r,1}, \ldots, I_{r,r^2}$ of equal length. Two cases must be considered at this stage. If $\gamma_n(\omega')$ contains no interval $I_{r,i}$ then n is said to be an *inessential (free) return* and we take ω' to be an element of $\mathcal{P}_n$. Otherwise n is an *essential (free) return* and we decompose $\omega' = \bigcup \omega_{r,i}$ where each $\omega_{r,i}$ is an interval with

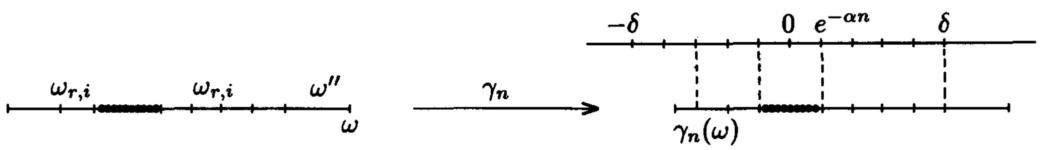

Fig. 2

$I_{r,i} \subset \gamma_n(\omega_{r,i}) \subset I_{r,i-1} \cup I_{r,i} \cup I_{r,i+1}$ (most of the times $\omega_{r,i} = \gamma_n^{-1}(I_{r,i})$ but if some $I_{s,j}$ is not fully contained in $\gamma_n(\omega')$ then $\gamma_n^{-1}(I_{s,j})$ is included in $\omega_{s,j\pm1}$, see Figure 2). By definition, the elements of $\mathcal{P}_n$ inside ω' are precisely these $\omega_{r,i}$.

The main goal of this construction is to obtain the following statement of uniformity of the derivative inside each interval of the partitions.

LEMMA 3.3. *There is $A_0 = A_0(c, \alpha, \beta, \delta) > 1$ such that if $\omega \in \mathcal{P}_{n-1}$ and $\omega \subset E_{n-1}$ then*

$$A_0^{-1} \leqslant \frac{|\gamma_n'(a)|}{|\gamma_n'(b)|} \leqslant A_0 \quad \textit{for all } a, b \in \omega.$$

In the proof of this, one first transfers the situation from the parameter space to the x-line through the following result, which is a particular case of the important general principle stating that under a sufficiently strong growth of the x-derivative then the x- and the a-derivative are comparable.

LEMMA 3.4. *Given $c > 0$ there is $N_0 = N_0(c) \geqslant 1$ such that if*

(a) $|\partial_x Q_a^j(1)| \geqslant 3^j$ *for* $1 \leqslant j \leqslant N_0 - 1$,

(b) $|\partial_x Q_a^j(1)| \geqslant e^{cj}$ *for* $1 \leqslant j \leqslant n - 1$,

then

$$\frac{1}{36} \leqslant \frac{|\partial_a Q_a^{n-1}(1)|}{|\partial_x Q_a^{n-1}(1)|} \leqslant 36.$$

Observe that $\gamma_n'(a) = \partial_a Q_a^n(0) = \partial_a Q_a^{n-1}(1)$, while $\partial_x Q_a^{n-1}(1) = D_{n-1}(a)$. We take a_0 close enough to 2 so that the first N_0 iterates of 1 are close to -1 and so (a) is satisfied ($Q_2'(-1) = 4 > 3$). Assumption (b) follows simply from the fact that $\omega \subset E_{n-1}$. This reduces the proof of Lemma 3.3 to proving

$$\frac{1}{A_0'} \leqslant \frac{|D_{n-1}(a)|}{|D_{n-1}(b)|} \leqslant A_0' \quad \text{for all } a, b \in \omega \tag{5}$$

and for this it is important that on returns the length of $\gamma_k(\omega)$ is not too big with respect to $\operatorname{dist}(\gamma_k(\omega), 0)$, as obtained above, see [BC1], [BC2] and Section 11.

Parameter values $a \in \omega_{r,i}$, with $|r| > \alpha n$ are excluded by the (BA) and we define $E_n' = E_{n-1} \setminus (\bigcup_{\omega'} \bigcup_{|r|>\alpha n} \omega_{r,i})$, where the first union is taken over all ω' corresponding to essentiall returns as above. A bound for the measure of these exclusions can now be provided. By the mean value theorem and Lemma 3.3

$$m(\omega \cap (E_{n-1} \setminus E'_n)) = m(\bigcup_{|r|>\alpha n} \omega_{r,i}) \approx \frac{e^{-\alpha n} \cdot m(\omega)}{m(\gamma_n(\omega))}. \tag{6}$$

In order to estimate the length of $\gamma_n(\omega)$ we need the following fact.

LEMMA 3.5. *There is $\tau = \tau(c, \alpha, \beta) > 0$ such that $m(\gamma_\mu(\omega)) \geqslant \tau e^{pc/3} m(\gamma_\nu(\omega))$ for any free returns $N \leqslant \nu < \mu \leqslant n$ of $\omega \in \mathcal{P}_{n-1}$, $\omega \subset E_{n-1}$, where p is the length of the binding period associated to ν.*

This is a consequence of the mean value theorem together with the expansiveness of the x-derivatives during binding periods (Lemma 3.2(c)) and free periods (Lemma 3.1) and the uniform equivalence of x- and a-derivatives at times ν and μ (Lemma 3.4). Let then k be the last essential return before n (when ω was created). The lemma says $m(\gamma_n(\omega)) \geqslant \text{const.}\, e^{pc/3} m(\gamma_k(\omega))$. Now, by definition $m(\gamma_k(\omega)) \geqslant m(I_{r,i}) \geqslant \text{const.}\, e^{-|r|/r^2}$ for some $\Delta \leqslant |r| \leqslant \alpha k \leqslant \alpha n$ and by Lemma 3.2(b) $p \geqslant (|r|)/(\beta + \log 4)$. It follows

$$m(\gamma_n(\omega)) \geqslant \text{const.} \exp\left(-\tfrac{19}{20}\alpha n\right). \tag{7}$$

Replacing in (6) we get $m(\omega \cap (E_{n-1} \setminus E'_n)) \leqslant \text{const.}\, e^{-\alpha n/20} m(\omega)$ and since this holds for all $\omega \in \mathcal{P}_{n-1}$, $\omega \subset E_{n-1}$ we have proved that

$$m(E_{n-1} \setminus E'_n) \leqslant \text{const.}\, e^{-\alpha n/20} m(E_{n-1}). \tag{8}$$

In order to bound the exclusions determined by the free period assumption (FA) Benedicks and Carleson introduce the notion of escape period. We return to the notations in the construction of $\mathcal{P}_n$. An essential free return situation n is said to be an *escape situation* for $\omega \in \mathcal{P}_{n-1}$ if

$$m(\gamma_n(\omega)) \geqslant \sqrt{\delta}. \tag{9}$$

Then the length of at least one of the connected components of $\gamma_n(\omega'')$ is greater than $\delta/3$ and we call that an *escaping component* of $\mathcal{P}_n$. By definition, an *escape period* for $a \in E_{n-1}$ is a maximal interval $[\nu, \nu+e)$ such that

ν is an escape situation and a belongs to an escaping component of $\mathcal{P}_\nu$; (10a)

$$|Q_a^j(0)| \geqslant \delta \quad \text{for all } j \in [\nu, \nu+e). \tag{10b}$$

We also consider $[1, N)$ to be an escape period for all $a \in \omega_0$, although the first escape situation is, clearly, $n = N$. Observe that once an escape period has begun it tends to persist: if $\tilde{\omega}\in\mathcal{P}_\nu$ is an escaping component then its next return μ is again an escape situation (by Lemma 3.5) and, due to the fact that $\sqrt{\delta}\gg\delta$, most of the values $a\in\tilde{\omega}$ belong to an escaping component of $\mathcal{P}_\mu$. On the other hand, after an escape period ends there is a positive probability that a new one will start at some subsequent return (i.e. there is a definite positive fraction of values of a for which this happens). This is a consequence of the exponential growth of lengths (Lemmas 3.1, 3.2(c) and 3.4). By combining these two facts one obtains the fundamental lemma below, stating that in the average the critical orbit spends most of the time in escape periods. Let

$$T_n(a)=\#\{j\in\{1,\dots,n\}: j \text{ does not belong to any escape period of } a\}.$$

Clearly, $T_n(a)$ is constant on each $\omega\in\mathcal{P}_n$ and we denote by $T_n(\omega)$ this constant value.

LEMMA 3.6. *Fix γ small enough* ($\gamma=1/1000$, *say*). *Then*

$$\frac{1}{m(\omega_0)}\sum_{\substack{\omega\in\mathcal{P}_n\\ \omega\subset E_{n-1}}} e^{\gamma T_n(\omega)}\cdot m(\omega)\leqslant e^{\gamma\varepsilon n/2} \quad \textit{and so} \quad m\Big(\bigcup_{T_n(\omega)\geqslant\varepsilon n}\omega\Big)\leqslant e^{-\gamma\varepsilon n/2}\cdot m(\omega_0).$$

We take $E_n=E'_n\setminus(\bigcup_{T_n(\omega)\geqslant\varepsilon n}\omega)$ so that the parameters $a\in E_n$ satisfy the (FA) (escape periods are, obviously, free). On the other hand Lemma 3.6 implies

$$m(E'_n\setminus E_n)\leqslant e^{-\gamma\varepsilon n/2}\cdot m(\omega_0) \tag{11}$$

and putting this together with (8) we get

$$m(E_{n-1}\setminus E_n)\leqslant B_0 e^{-\alpha_0 n}\cdot m(\omega_0) \tag{12}$$

where B_0 and α_0 depend on $c,\alpha,\beta,\varepsilon$ and δ but not on N. It follows, that

$$m(E)\geqslant m(\omega_0)-\sum_{n=N}^{\infty} m(E_{n-1}\setminus E_n)\geqslant m(\omega_0)\cdot\Big(1-B_0\sum_{n=N}^{\infty} e^{-\alpha_0 n}\Big)$$

is positive if N is large enough, i.e. if a_0 is close enough to 2. Moreover $m(E)/m(\omega_0)$ converges to 1 as $a_0\to 2$.

4. The attractor: basic properties

In this section we exhibit, for a Hénon-like map φ_a, a compact invariant set $\Lambda=\Lambda_a$ and we show that it is an attractor in the sense that its stable set has non-empty interior. The fact that (for a positive measure set of parameter values) Λ is transitive and has positive Lyapunov exponent is much harder to prove and this occupies the remainder of the paper.

The set Λ is obtained as the closure of the unstable manifold of one of the fixed points of φ_a, so we start by studying these points. An immediate computation shows that $\psi_2(x,y)=(1-2x^2,0)$ has exactly two fixed points $P=(\frac{1}{2},0)$ and $Q=(-1,0)$, and that the eigenvalues of $D\psi_2(P)$, respectively $D\psi_2(Q)$, are -2 and 0, respectively 4 and 0. Then P and Q have analytic continuations $P(\varphi_a)$ and $Q(\varphi_a)$ defined for φ_a in a neighbourhood of ψ_2 and (if this neighbourhood is small enough) these are the unique fixed points of φ_a. We also want to describe the unstable sets of these points and the way they unfold with the parameter. Let us begin by considering the family $(\psi_a)_a$. Recall that one defines $W^u(P(\psi_a))=\bigcap_{n\geqslant 0}\psi_a^n(I)$, with $I\subset\mathbf{R}\times\{0\}$ a small interval containing $P(\psi_a)$, and analogously for $Q(\psi_a)$. Again it is easy to check that $W^u(Q(\psi_a))$ always has an unbounded separatrix $(-\infty,Q(\psi_a)]$ and moreover

- for $1<a<2$: $\frac{1}{2}<P(\psi_a)<1$, $Q(\psi_a)<-1<1-a$,

  $$W^u(P(\psi_a))=[1-a,1],\quad \text{bounded separatrix of } W^u(Q(\psi_a))=[Q(\psi_a),1];$$

- for $a=2$: $P(\psi_a)=\frac{1}{2}$, $Q(\psi_a)=-1$,

  $$W^u(P(\psi_a))=\text{ bounded separatrix of } W^u(Q(\psi_a))=[-1,1];$$

- for $2<a<3$: $0<P(\psi_a)<\frac{1}{2}$, $1-a<-1<Q(\psi_a)$,
- the right-hand side separatrix of $W^u(Q(\psi_a))=[Q(\psi_a),1]$ and $W^u(P(\psi_a))=(-\infty,1]$.

The special role of $1-a$ comes from the fact that $1-a=\psi_a(1)=\psi_a^2(0)$. Figure 3 describes how the $\psi_a^k\colon I\to\mathbf{R}$ parametrize (compact parts of) these W^u and how the situation changes with the parameter. (In order to make the figures easier to read, multiple points of the unstable sets are represented in slightly different levels. The real picture is, of course, one dimensional and the folds at the positive orbit of zero correspond in fact to velocity zero turn-back points of the parametrization.)

Observe that the vertical straight line passing through Q being contained in the stable set of Q, one may think of the bifurcation at $a=2$ as the creation of a homoclinic *tangency* associated to Q and of a simultaneous heteroclinic *tangency* involving $W^u(P)$ and $W^s(Q)$. Due to the continuous dependence of the local stable and unstable sets on the map (see Proposition 7.1), Hénon-like families of diffeomorphisms have such homoclinic and heteroclinic tangencies for parameter values near $a=2$, as we now describe. As one goes from the endomorphism to the nearby diffeomorphism the turn-back points in the unstable sets become real folds and we need to know whether they turn *up* or *down*. This depends on the signs of the entries of

$$D\varphi_a=\begin{pmatrix} A & B \\ C & D \end{pmatrix}:$$

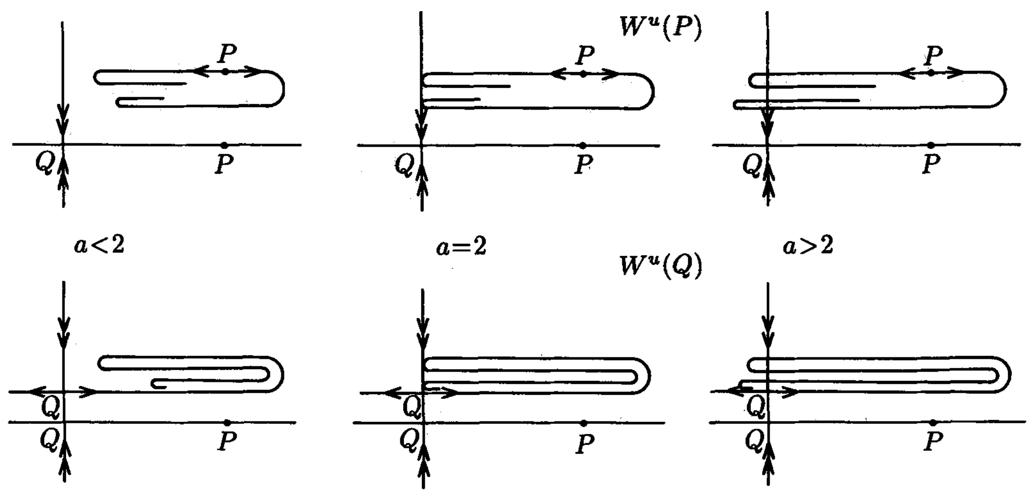

Fig. 3

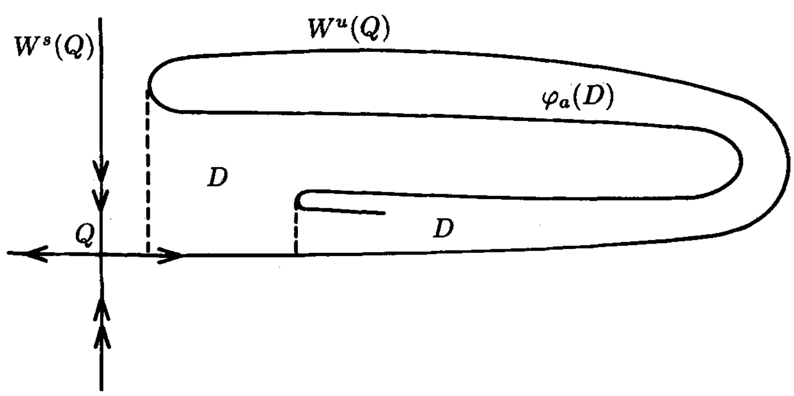

Fig. 4

it is easy to check that the first fold of $W^u(Q)$ (near $(1,0)$) turns up if $C>0$ and down otherwise; analogously the sign of B determines the orientation of the second fold (near $(-1,0)$).

Consider first the case $\det(D\varphi_a)>0$ ($\Leftrightarrow BC<0$). As a increases, this second fold moves to the left and we have a first homoclinic tangency associated to Q for $a=a_+(\varphi)$. Then for $a<a_+(\varphi)$ we may construct, as in Figure 4, a compact domain $D=D_a$ which is invariant for φ_a in the sense that $\varphi_a(D)\subset D$. Observe that by Brower's fixed point theorem we must have $P\in D$ and so even $P\in \text{int}(D)$, which implies $W^u(P)\subset D$.

The construction of an invariant domain for φ_a is slightly more complicated in the case $\det(D\varphi_a)<0$ ($\Leftrightarrow BC>0$). Now the tangency between $W^s(Q)$ and the second fold

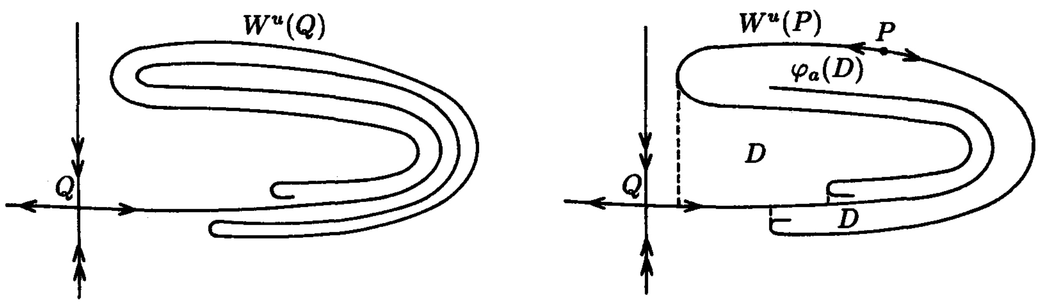

Fig. 5

of $W^u(Q)$ is not the first homoclinic bifurcation associated to Q, as Figure 5 shows.

Observe that the contracting eigenvalue of $D\varphi_a(Q)$ must be negative since the expanding one is near $\psi_2'(-1)=4$. We consider instead the unstable manifold of P. Another elementary reasoning shows that its first three folds look like as in the figure. We let $a_-(\varphi)$ be the parameter value corresponding to the first tangency between $W^u(P)$ and $W^s(Q)$ and assume $a<a_-(\varphi)$. Then we may take D as in the figure since, clearly, this is invariant and contains $W^u(P)$.

We summary this discussion in the following proposition.

PROPOSITION 4.1. *Let $\varphi=(\varphi_a)_a$ be a Hénon-like family preserving (respectively reversing) orientation. For $a<a_+(\varphi)$ (respectively $a<a_-(\varphi)$) there is a compact domain $D\subset[-2,2]^2$ which is positively invariant under φ_a and contains $\Lambda=\overline{W^u(P)}$.*

It is well known, and fairly easy to show, that the basin of Λ contains non-trivial open sets.

PROPOSITION 4.2 (see [BC2], [PT2]). *Let an open domain $\Omega\subset D$ be such that $\partial\Omega\subset W^s(P)\cup W^u(P)$. Then $\lim_{n\to+\infty}\operatorname{dist}(\varphi_a^n(z),\Lambda)=0$ for all $z\in\Omega$.*

Observe that domains Ω as above exist since P has transverse homoclinic points.

Remark. At least in the orientation reversing case one can show that $W^s(\Lambda)$ contains a neighbourhood of Λ. We just sketch this argument. Note first that for every a there is a sequence $P=P_0>P_1>P_2>\ldots>P_n>\ldots$ converging to Q and such that $P_n=\psi_a(P_{n+1})$. For a decreasing from 2 the point $1-a=\psi_a^2(0)$ crosses these P_n and this corresponds to a cascade of homoclinic tangencies associated to P. In fact, the stable set of P is formed by the vertical lines passing through each of its backward images. By continuity, for $\varphi=(\varphi_a)_a$ a Hénon-like family, the stable manifold of $P(\varphi_a)$ contains segments close to (compact parts of) these lines. When φ_a is orientation reversing one can see that the geometry of $W^s(P(\varphi_a))$ is as depicted: segments on the left connect to segments on

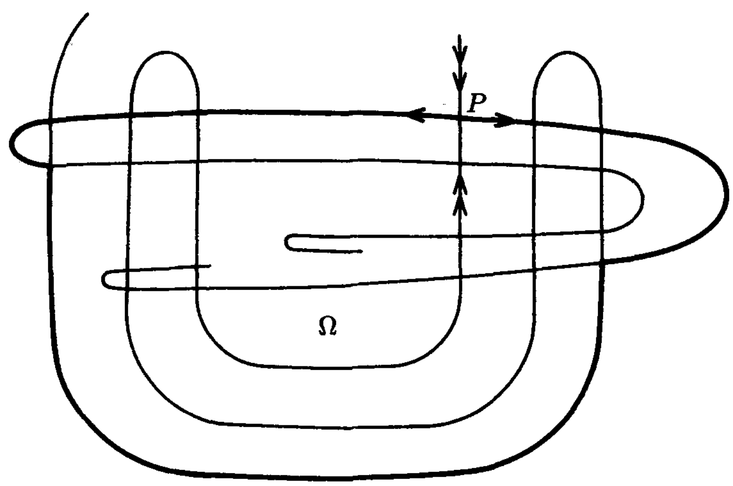

Fig. 6

the right *below* $W^u(P(\varphi_a))$ (Figure 6 corresponds to $B, C>0$, so that $P(\varphi_a)$ is *above*). This permits to construct, for $a<a_-(\varphi)$, domains Ω as in the proposition such that $\Lambda\subset(\Omega\cup W^u(P))$ and so $\Lambda\subset(\bigcup_{n\geqslant 0}\varphi_a^{-n}(\Omega))\cup W^s_{\text{loc}}(P)$. By Proposition 4.2 the open set on the right is contained in $W^s(\Lambda)$.

Observe also that even in the orientation preserving case $W^s(\Lambda)$ contains at least a neighbourhood of $W^u(P)$: this follows simply from Proposition 4.2 together with the fact that P has transverse homoclinic points in all its separatrices.

5. The induction: critical points

Now we want to extend the one-dimensional argument of Section 3 to general Hénon-like families in order to prove Theorem B. A crucial ingredient in this extension is the construction of a set of *critical* points for the Hénon-like maps. These are points in the unstable manifold $W^u=W^u(P)$ which play a role in the proof similar to that of the critical point $x=0$ in the one-dimensional case. This construction turns out to be quite complicated and the purpose of this section is to give a heuristic motivation and outline of it, as well as to advance some information on the global structure of the argument. Rigorous definitions and statements will be given later.

For $z_1\in W^u$ and $n\geqslant 0$ denote

$$z_{n+1}=(x_{n+1},y_{n+1})=\varphi_a^n(z_1) \quad \text{and} \quad w_n=w_n(z_1)=D\varphi_a^n(z_1)\cdot(1,0).$$

Due to the form of $D\varphi_a$ the vectors w_n stay nearly horizontal as long as $|x_n|\geqslant\delta$. During this period the action of φ_a on these vectors is essentially that of $Q_a(x)=1-ax^2$ and

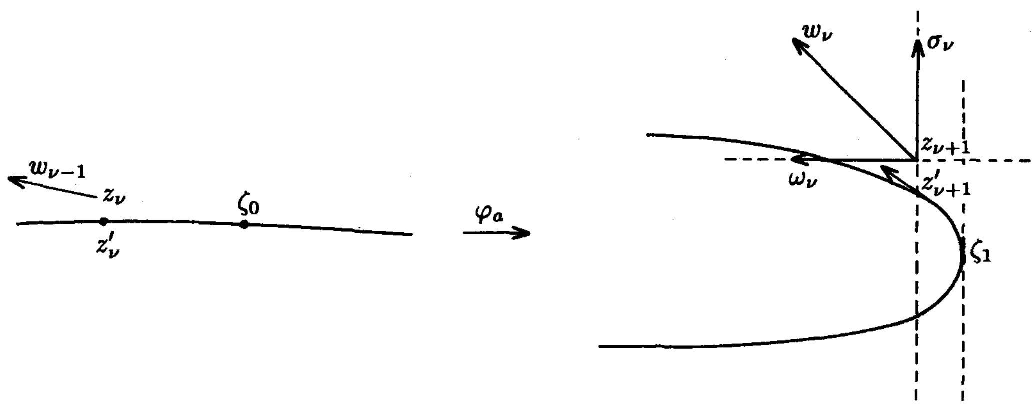

Fig. 7

in particular $\|w_n\| \approx 2a|x_n| \cdot \|w_{n-1}\|$. Let now $\nu \geqslant 1$ be a return iterate, i.e. let $|x_\nu| < \delta$. Clearly, w_ν can no longer be expected to be nearly horizontal: it may have large slope if x_ν is near zero. We suppose first that φ_a admits (say, in $\{x > \frac{1}{2}\}$) a contractive foliation, meaning a foliation by nearly vertical curves which are (exponentially) contracted by all positive iterates of φ_a. Although this is not the case in general, the rigorous proof of Theorem B will involve geometric ideas similar to these, as we explain below. Then we may split $w_\nu = \omega_\nu + \sigma_\nu$, as illustrated in Figure 7, where ω_ν is horizontal and σ_ν is in the contractive direction at $z_{\nu+1}$. Further iterates of σ_ν play no role in what concerns growth of $\|w_n\|$, since they are exponentially small. On the other hand, ω_ν being horizontal the action of $D\varphi_a$ on it will again be essentially the multiplication by $-2ax$, up to the next return where the whole process is repeated.

Clearly, one needs to estimate the amount $\|\omega_\nu\| / \|w_{\nu-1}\|$ of loss on the norms of vectors on the return time ν. This is related to the angle between w_ν and the contractive direction at $z_{\nu+1}$, which motivates that we introduce the following notion. A point $\zeta_0 = (\xi_0, \eta_0) \in W^u$, $|\xi_0| < \delta$, is said to be a *critical point* of φ_a if $D\varphi_a(\zeta_0)$ maps the tangent direction to W^u at ζ_0 into the contractive direction at $\varphi_a(\zeta_0)$ or, equivalently, if the contractive direction is tangent to W^u at $\varphi_a(\zeta_0)$. We also assume in the definition that W^u is almost horizontal and almost flat near ζ_0. We will see later (Section 9) that on each return ν a critical point ζ_0 can always be found such that z_ν is in tangential position with respect to ζ_0, in the sense that $\operatorname{dist}(z_\nu, \gamma) \ll |z_\nu - \zeta_0|$, where γ denotes a nearly flat piece of W^u containing ζ_0. This permits us to argue as if z_ν were in γ: replacing it by a suitable $z'_\nu \in \gamma$ introduces only an error which is neglectable with respect to $|z_\nu - \zeta_0|$. We may also think of $w_{\nu-1}$ as being tangent to W^u at z_ν (actually z'_ν): the angle it makes with the tangent direction is also shown to be neglectable relative to $|z_\nu - \zeta_0|$. From all this one obtains in a fairly easy way that $\|\omega_\nu\| \approx \|w_{\nu-1}\| \cdot |z_\nu - \zeta_0|$, as a consequence of the quadratic nature of the fold of W^u near $\zeta_1=\varphi_a(\zeta_0)$. This is completely analogous to the one-dimensional case, where we had on each return a loss on the derivative proportional to the distance to the critical point.

Proceeding with our heuristic overview of the proof, one shows then that, with positive probability in the parameter space, all critical values have positive Lyapunov exponent: there is $E\subset(1,3)$, with $m(E)>0$, such that for $a\in E$, $\|w_n(z_1)\|\geqslant e^{cn}$ for all critical values z_1 and $n\geqslant 1$. In order to get this we imitate the one-dimensional argument of Section 3 and parameter exclusions are made so that the basic assumption (BA) and the free period assumption (FA) hold for the orbit of each critical point. The fact that after all exclusions there remains a positive measure set of parameters is more delicate than in dimension one, since now we have infinitely many critical points. This follows from the observation that only a number $\leqslant e^{\varepsilon n}$ ($\varepsilon>0$ small) of critical points needs to be considered at each stage n (the other critical points remaining close—bound—to these ones during at least n iterations), together with the fact that the exclusions required by each critical point decrease exponentially with n ((3.12)). Finally one proves that for almost every $a\in E$ there is a critical point z_0 whose orbit is dense in W^u (see Section 12). Clearly, for such parameter values Λ can not contain periodic attractors.

Now we want to discuss some of the points in which the rigorous proof of the theorem differs from this heuristic outline. As we said before, contractive foliations as above do not exist in general. On the other hand, if a point z_1 is *λ-expanding up to time n*, i.e. $\|w_j(z_1)\|\geqslant\lambda^j$ for $1\leqslant j\leqslant n$, with $\lambda\gg b$ then (Section 6) a direction $e^{(n)}(z_1)$ can be constructed with the property of being exponentially contracted by the first n iterates $D\varphi_a^j(z_1)$, $1\leqslant j\leqslant n$. We think of $e^{(n)}(z_1)$ as an approximation to the contractive direction at z_1 (which would be contracted by all positive iterates of $D\varphi_a$) and in fact this contractive direction may be obtained as $\lim_{n\to\infty} e^{(n)}(z_1)$, if z_1 is expanding for all times. Using these approximations of contractive directions one may define approximations of critical points $z_0^{(n)}$ by the condition that $e^{(n)}(\varphi_a(z_0^{(n)}))$ be tangent to W^u at $\varphi_a(z_0^{(n)})$. Of course, this definition makes sense only if we have expansiveness which is precisely our goal (with $\lambda=e^c$) and this shows that an induction procedure involving simultaneously the construction of the critical set and the exclusion arguments is required for the proof. We formalize this procedure as follows. Fix (in a more or less arbitrary way, see Section 7) a compact neighbourhood G_0 of the fixed point P in $W^u(P)$. For $g\geqslant 1$ let $G_g=\varphi_a^g(G_0)\setminus\varphi_a^{g-1}(G_0)$. A critical (approximation) point is said to be of generation g if it belongs to G_g. First we construct critical approximations $z_0^{(i)}$ in G_0 and $w_0^{(i)}$ in G_1, $1\leqslant i\leqslant N-2$, corresponding to the unique critical points of generation 0 and 1, respectively. For $n\leqslant N-1$, the nth critical set is

$$\mathcal{C}_n=\{z_0^{(n-1)},w_0^{(n-1)}\}.$$

These approximations remain outside $(-\delta,\delta)$ for all iterates $1\leqslant n\leqslant N-1$ and so their images $z_1^{(i)}=\varphi_a(z_0^{(i)})$ and $w_1^{(i)}=\varphi_a(w_0^{(i)})$ are e^{c_0}-expanding, $0<c<c_0<\log 2$, up to $N-1$. Actually, the same is still true for every point ξ_0 which remains bound to some of the approximations up to time $N-1$. For $n\geqslant N$ the construction proceeds by induction. We assume that for all $k\leqslant n-1$ a kth critical set $\mathcal{C}_k$ has been constructed, containing $(k-1)$st order approximations $z_0^{(k-1)}$ of critical points of generation $\leqslant\theta k$ ($\theta=\theta(b)\ll 1$ to be fixed later). For any point ξ_0 which is bound up to k to some element of $\mathcal{C}_k$, it is assumed that $\xi_1=\varphi_a(\xi_0)$ is e^c-expanding up to time k. Then we construct the nth critical set $\mathcal{C}_n$, composed of $(n-1)$st order approximations of critical points of generation $\leqslant\theta n$. This corresponds to providing better approximations for the critical points already encountered (of generation $g\leqslant\theta(n-1)$) and, possibly, introducing approximations of new critical points of generation $\theta(n-1)<g\leqslant\theta n$. By construction all the points of $\mathcal{C}_n$ are very near $\mathcal{C}_{n-1}$. This has the consequence that if a point ξ_0 is bound to some $z_0^{(n-1)}\in\mathcal{C}_n$ up to time n then it is also bound up to $(n-1)$ to some element of $\mathcal{C}_{n-1}$ and so, by induction, ξ_1 is e^c-expanding up to $(n-1)$. In order to obtain the e^c-expansiveness at time n, parameter exclusions are made, determined by the 2-dimensional analogs of (BA) and (FA). This is described in more detail in Section 8, after appropriate notations and techniques have been introduced. This completes the induction. At this stage *true* critical points are, finally, defined (as the limit $\lim_{n\to\infty} z_0^{(n)}$ of increasingly accurate critical approximations) and the corresponding critical values are e^c-expanding for all times.

6. Contractive directions

We begin by constructing (approximate) contractive directions for a Hénon-like map φ_a. The crucial property of $M=D\varphi_a$, as far as this section is concerned, is the strong area contractiveness

$$|\det M|\leqslant Kb\ll 1. \tag{1}$$

We also make important use of the homogeneity of M

$$\frac{\|(\det M)'\|}{|\det M|}\leqslant K^2b^t\ll K. \tag{2}$$

Our construction is essentially different from that in [BC2, Section 5], which is based on a continuous fraction development. Instead, we define contractive approximations simply to be the maximally contracting directions of the iterates of M.

Recall that a point z_1 is said λ-expanding up to time n if $\|w_\nu\|\geqslant\lambda^\nu$ for $1\leqslant\nu\leqslant n$, where $w_\nu=w_\nu(z_1)=M^\nu(z_1)\cdot(1,0)$. We permit λ to be less than 1 but we always have $K>\lambda\geqslant(\delta/10K)^{10}\gg b$. For such a point z_1 and $1\leqslant\nu\leqslant n$ let the norm 1 vectors $e^{(\nu)}$ and

$f^{(\nu)}$ be, respectively, maximally contracting and maximally expanding for $M^\nu(z_1)$. These correspond to the solutions of

$$\frac{d}{d\theta}\|M^\nu(z_1)\cdot(\cos\theta,\sin\theta)\|=0$$

which are given by

$$\operatorname{tg}(2\theta)=\frac{2(A_\nu B_\nu+C_\nu D_\nu)}{(A_\nu^2+C_\nu^2)-(B_\nu^2+D_\nu^2)} \quad \text{where } M^\nu(z_1)=\begin{pmatrix} A_\nu & B_\nu \\ C_\nu & D_\nu \end{pmatrix}. \tag{3}$$

In particular $e^{(\nu)}$ and $f^{(\nu)}$ are orthogonal and the same holds for their images $M^\nu(z_1)\cdot e^{(\nu)}$ and $M^\nu(z_1)\cdot f^{(\nu)}$ (which are, respectively, maximally expanding and maximally contracting for $(M^\nu(z_1))^{-1}$). Therefore

$$\|M^\nu(z_1)\cdot e^{(\nu)}\|\cdot\|M^\nu(z_1)\cdot f^{(\nu)}\|=|\det M^\nu(z_1)|\leqslant(Kb)^\nu$$

and, since the expansion assumption implies

$$\|M^\nu(z_1)\cdot f^{(\nu)}\|\geqslant\lambda^\nu, \tag{4}$$

we conclude that

$$\|M^\nu(z_1)\cdot e^{(\nu)}\|\leqslant\left(\frac{Kb}{\lambda}\right)^\nu. \tag{5}$$

In what follows we denote by angle(u,v) the angle between the directions of two vectors u and v. This is a number in $\left(-\frac{1}{2}\pi,\frac{1}{2}\pi\right]$.

LEMMA 6.1. (a) $|\text{angle}(e^{(\mu)},e^{(\nu)})|\leqslant(3K/\lambda)(Kb/\lambda^2)^\mu$ *for all* $1\leqslant\mu\leqslant\nu\leqslant n$;
(b) $\|M^\mu(z_1)\cdot e^{(\nu)}\|\leqslant(4K/\lambda)(K^2b/\lambda^2)^\mu$ *for all* $1\leqslant\mu\leqslant\nu\leqslant n$.

Proof. Denote $e_\mu^{(\nu)}=M^\mu(z_1)\cdot e^{(\nu)}$ and $f_\mu^{(\nu)}=M^\mu(z_1)\cdot f^{(\nu)}$ for $1\leqslant\mu,\nu\leqslant n$. Let $\nu\geqslant2$ and write $e^{(\nu-1)}=\xi\cdot e^{(\nu)}+\eta\cdot f^{(\nu)}$, see Figure 8. Then

$$\xi^2\|e_\nu^{(\nu)}\|^2+\eta^2\|f_\nu^{(\nu)}\|^2=\|e_\nu^{(\nu-1)}\|^2=\|e_\nu^{(\nu-1)}\|^2\cdot(\xi^2+\eta^2),$$

giving

$$(\operatorname{tg}\phi^{(\nu)})^2=\left(\frac{\eta}{\xi}\right)^2=\frac{\|e_\nu^{(\nu-1)}\|^2-\|e_\nu^{(\nu)}\|^2}{\|f_\nu^{(\nu)}\|^2-\|e_\nu^{(\nu-1)}\|^2}, \tag{6}$$

where $\phi^{(\nu)}=\text{angle}(e^{(\nu-1)},e^{(\nu)})$. Now from (4), (5) and the fact that $b\ll\lambda$ it easily follows $|\phi^{(\nu)}|\leqslant|\operatorname{tg}\phi^{(\nu)}|\leqslant(2K/\lambda)\cdot(Kb/\lambda^2)^{\nu-1}$. Then, for $1\leqslant\mu<\nu\leqslant n$,

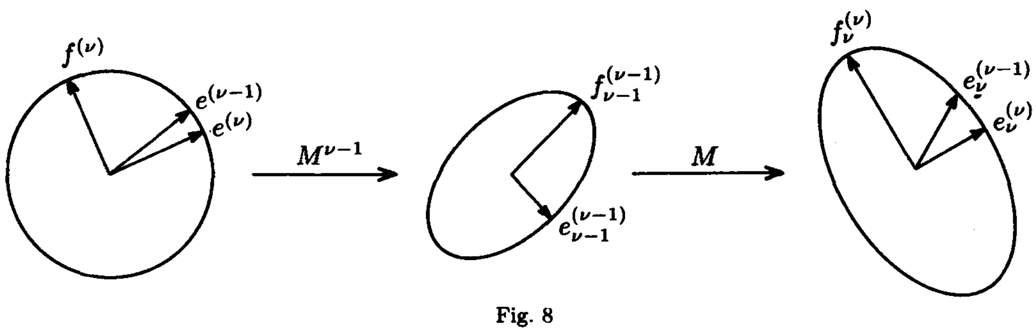

Fig. 8

$$|\text{angle}(e^{(\mu)}, e^{(\nu)})| \leqslant \sum_{\mu+1}^{\nu} |\phi^{(i)}| \leqslant \sum_{\mu}^{\nu-1} \frac{2K}{\lambda} \cdot \left(\frac{Kb}{\lambda^2}\right)^i \leqslant \frac{3K}{\lambda}\left(\frac{Kb}{\lambda}\right)^\mu$$

and this proves (a). Finally, (b) is now an easy consequence:

$$\begin{aligned}\|M^\mu(z_1)\cdot e^{(\nu)}\| &\leqslant \|M^\mu(z_1)\cdot(e^{(\nu)}-e^{(\mu)})\|+\|M^\mu(z_1)\cdot e^{(\mu)}\| \\ &\leqslant K^\mu \cdot \frac{3K}{\lambda}\left(\frac{Kb}{\lambda^2}\right)^\mu + \left(\frac{Kb}{\lambda}\right)^\mu \leqslant \frac{4K}{\lambda}\left(\frac{K^2 b}{\lambda^2}\right)^\mu. \qquad \square\end{aligned}$$

In particular, if z_1 is expanding up to n for all $n \geqslant 1$ then the $e^{(n)}$ converge extremely fast and the limit direction has the property of being exponentially contracted by all positive iterates of M. We call $e^{(n)}(z_1)$ the nth *contractive approximation* at z_1.

One may use (3) to give simple estimates for the contractive approximations $e^{(\nu)}$, $1 \leqslant \nu \leqslant n$, at least for points far from $x=0$. Let $z_1=(x_1, y_1)$ be such that $|x_1| \geqslant \delta$. For $b \ll \delta$, $|A_1+2ax_1| \leqslant Kb^t$ implies $|A_1| \geqslant |x_1| \geqslant \delta$. On the other hand (3) and Theorem 2.1 give

$$|\operatorname{tg} 2\theta^{(1)}| \leqslant \frac{4K^2\sqrt{b}}{\delta^2-2K^2 b} \leqslant \frac{8K^2\sqrt{b}}{\delta^2}. \tag{7}$$

From this, one concludes easily that $e^{(1)}$ is nearly vertical (the almost horizontal case in (7) is more expanding and so must correspond to $f^{(1)}$):

$$\left|\theta^{(1)} - \frac{\pi}{2}\right| \leqslant \frac{4K^2\sqrt{b}}{\delta^2}. \tag{8}$$

By Lemma 6.1(a) the same holds for all contractive approximations $e^{(\nu)}$

$$\left|\theta^{(\nu)} - \frac{\pi}{2}\right| \leqslant 4K^2\sqrt{b}\left(\frac{1}{\delta^2}+\frac{1}{\lambda^3}\right) \leqslant \sqrt[4]{b}. \tag{9}$$

Now, this implies that $w_0=(1,0)=\alpha_\nu e^{(\nu)}+\beta_\nu f^{(\nu)}$, with $|\alpha_\nu| \leqslant 2\sqrt[4]{b}$ and $|\beta_\nu| \geqslant 1-2\sqrt[4]{b}$, and thus $\|w_\nu\|=\|\alpha_\nu e_\nu^{(\nu)}+\beta_\nu f_\nu^{(\nu)}\| \geqslant \frac{1}{2}\cdot\|f_\nu^{(\nu)}\|$. Clearly this same argument proves

LEMMA 6.2. *Let $z_1=(x_1,y_1)$, $|x_1|\geqslant\delta$, be such that for some norm 1 vector u_0 we have $\|M^\nu(z_1)\cdot u_0\|\geqslant\lambda^\nu$ for all $1\leqslant\nu\leqslant n$. Then*

$$\|M^\nu(z_1)\cdot v_0\|\geqslant\tfrac{1}{2}\|M^\nu(z_1)\|$$

for all $1\leqslant\nu\leqslant n$ and all norm 1 vector v_0 with $|\operatorname{slope}(v_0)|\leqslant\frac{1}{10}$.

LEMMA 6.3. *Let z_0,ζ_0 and norm 1 vectors u,v satisfy $|z_0-\zeta_0|\leqslant\sigma^n$ and $\|u-v\|\leqslant\sigma^n$, with $\sigma\leqslant(\lambda/10K^2)^2$. Then for any $1\leqslant\nu\leqslant n$ such that $\|M^\nu(z_1)\cdot u\|\geqslant\lambda^\nu$, we have*

(a) $\frac{1}{2}\leqslant\|M^v(z_1)\cdot u\|/\|M^v(\zeta_1)\cdot v\|\leqslant 2$ *and*

(b) $|\operatorname{angle}(M^\nu(z_1)\cdot u, M^\nu(\zeta_1)\cdot v)|\leqslant(\sqrt{\sigma})^{2n-\nu}\leqslant(\sqrt{\sigma})^n$.

Proof. Clearly

$$\begin{aligned}\|M^\nu(z_1)\cdot u-M^\nu(\zeta_1)\cdot v\| &\leqslant \|M^\nu(z_1)-M^\nu(\zeta_1)\|+\|M^\nu(\zeta_1)\|\cdot\|u-v\|\\ &\leqslant\sum_1^\nu K^\nu|z_j-\zeta_j|+K^\nu\sigma^n\leqslant 2K^{2\nu}\sigma^n.\end{aligned}$$

Hence $\|M^\nu(z_1)\cdot u-M^\nu(\zeta_1)\cdot v\|\leqslant\frac{1}{2}(\sqrt{\sigma})^{2n-\nu}\|M^\nu(z_1)\cdot u\|$ and now (a) and (b) follow easily. □

LEMMA 6.4. *Let z_1,ζ_1 be such that $z_1=(x_1,y_1), |x_1|>\delta$, is λ-expanding up to time n and $|z_\nu-\zeta_\nu|\leqslant\sigma^\nu$ for every $1\leqslant\nu\leqslant n$, with $\sqrt{b}\leqslant\sigma\leqslant(\lambda/10K^2)^4$. Then*

(a) $\frac{1}{2}\leqslant\|M^\nu(z_1)\cdot u\|/\|M^\nu(\zeta_1)\cdot v\|\leqslant 2$,

(b) $|\operatorname{angle}(M^\nu(z_1)\cdot u, M^\nu(\zeta_1)\cdot v)|\leqslant(K^2\sqrt{\sigma}/\lambda)^{\nu+1}$,

for any $1\leqslant\nu\leqslant n$ and any norm 1 vectors u,v with $|\operatorname{slope}(u)|\leqslant\frac{1}{10}$ and $|\operatorname{slope}(v)|\leqslant\frac{1}{10}$.

Proof. Denote $u_\nu=M^\nu(z_1)\cdot u$ and $v_\nu=M^\nu(\zeta_1)\cdot v$; note that by the previous lemma $\|u_\nu\|\geqslant\lambda^\nu/2$ for $1\leqslant\nu\leqslant n$. We claim that we may write $v_v=\alpha_\nu u_\nu+\varepsilon_\nu$ with

$$|\alpha_\nu-1|\leqslant\frac{1}{10}+5K\sum_1^\nu\left(\frac{K^2\sqrt{\sigma}}{\lambda^2}\right)^i \quad\text{and}\quad \|\varepsilon_\nu\|\leqslant\left(K\sqrt{\sigma}\right)^{\nu+1}. \tag{10}$$

Let us prove first this claim and then show that it implies the lemma. We write $v=\alpha_0u+\varepsilon_0$ with ε_0 a vertical vector. The assumption on the slopes of u and v implies $|\alpha_0-1|\leqslant\frac{1}{50}$ and $\|\varepsilon_0\|\leqslant\frac{1}{5}$. We decompose $\varepsilon_0=\bar{\alpha}_0u+\bar{\beta}_0e^{(1)}(z_1)$ and then $v_1=\alpha_1u_1+\varepsilon_1$, where

$$\alpha_1=\alpha_0+\bar{\alpha}_0 \quad\text{and}\quad \varepsilon_1=(M(\zeta_1)-M(z_1))\cdot v+\bar{\beta}_0e_1^{(1)}(z_1).$$

From (8) (and assuming b small) we get $|\bar{\beta}_0|\leqslant\frac{1}{4}$ and $|\bar{\alpha}_0|\leqslant(2K^2\sqrt{b}/\delta^2)$. It follows that

$$\begin{aligned}|\alpha_1-1|&\leqslant\frac{1}{50}+\frac{2K^2\sqrt{b}}{\delta^2}\leqslant\frac{1}{10}\\ \|\varepsilon_1\|&\leqslant K\sigma+\frac{K^3b}{\lambda^3}\leqslant\left(K\sqrt{\sigma}\right)^2.\end{aligned}$$

This proves (10) for $\nu=1$. Now we proceed by induction. Let $\nu>1$ and take $\mu=[\nu/2]$; since $1\leqslant\mu<\nu$ we may assume that (10) holds for μ. Let $e^{(\nu-\mu)}=e^{(\nu-\mu)}(z_{\mu+1})$ and $f^{(\nu-\mu)}=f^{(\nu-\mu)}(z_{\mu+1})$ be norm 1 vectors, respectively maximally contracting and maximally expanding for $M^{\nu-\mu}(z_{\mu+1})$. Note that

$$\|f_{\nu-\mu}^{(\nu-\mu)}\| \geqslant \left\| M^{\nu-\mu}(z_{\mu+1})\cdot \frac{u_\mu}{\|u_\mu\|}\right\| \geqslant \frac{\lambda^\nu}{2K^\mu}$$

and so

$$\|e_{\nu-\mu}^{(\nu-\mu)}\| \leqslant \frac{2(Kb)^{\nu-\mu}K^\mu}{\lambda^\nu}. \tag{11}$$

Moreover, (11) is easily seen to imply

$$|\det(u_\mu, e^{(\nu-\mu)})| = |u_\mu\cdot f^{(\nu-\mu)}| \geqslant \frac{\lambda^\nu}{5K^{\nu-\mu}}.$$

Again we decompose $\varepsilon_\mu=\bar\alpha_\mu u_\mu+\bar\beta_\mu e^{(\nu-\mu)}$ and get $v_\nu=\alpha_\nu u_\nu+\varepsilon_\nu$ with

$$\alpha_\nu=\alpha_\mu+\bar\alpha_\mu \quad\text{and}\quad \varepsilon_\nu=(M^{\nu-\mu}(\zeta_{\mu+1})-M^{\nu-\mu}(z_{\mu+1}))\cdot v+\bar\beta_\mu e_{\nu-\mu}^{(\nu-\mu)}.$$

Now,

$$|\bar\alpha_\mu| = \frac{|\det(\varepsilon_\mu, e^{(\nu-\mu)})|}{|\det(u_\mu, e^{(\nu-\mu)})|} \leqslant \frac{5K^{\nu-\mu}\|\varepsilon_\mu\|}{\lambda^\nu} \leqslant 5K\left(\frac{K^2\sqrt{\sigma}}{\lambda^2}\right)^{\mu+1} \tag{12}$$

showing that α_ν satisfies (10). On the other hand

$$|\bar\beta_\mu| = \frac{|\det(u_\mu, \varepsilon_\mu)|}{|\det(u_\mu, e^{\nu-\mu)})|} \leqslant \frac{5K^{\nu-\mu}\|u_\mu\|\cdot\|\varepsilon_\mu\|}{\lambda^\nu} \leqslant \left(\frac{K^3\sqrt{\sigma}}{\lambda^2}\right)^{\mu+1}. \tag{13}$$

Note moreover that

$$\|M^{\nu-\mu}(\zeta_{\mu+1})-M^{\nu-\mu}(z_{\mu+1})\| \leqslant K^{\nu-\mu}\sum_{\mu+1}^{\nu}|z_j-\zeta_j| \leqslant 2K^{\nu-\mu}\sigma^{\mu+1}.$$

It follows that

$$\|\varepsilon_\nu\| \leqslant 2K^{\nu-\mu}\sigma^{\mu+1}\cdot K^\mu + \left(\frac{K^3\sqrt{\sigma}}{\lambda^2}\right)^{\mu+1}\cdot\frac{K^\mu(Kb)^{\nu-\mu}}{\lambda^\nu} \leqslant \left(K\sqrt{\sigma}\right)^{\nu+1}$$

(recall that $\nu-\mu\geqslant\nu/2\geqslant\mu$) and this completes the proof of the claim. Finally, this together with the assumption on σ gives

$$|\alpha_\nu-1|\leqslant\frac{1}{5} \quad\text{and}\quad \|\varepsilon_\nu\|\leqslant\frac{1}{5}\left(\frac{K^2\sqrt{\sigma}}{\lambda}\right)^{\nu+1}\|u_\nu\| \leqslant \frac{1}{5}\|u_\nu\|$$

and the lemma follows easily. □

We also need to show that the contractive approximations are nearly constant (uniformly const.$\sqrt{b}$-Lipschitz) functions of the point z_1. This requires some estimates that we collect in the following lemma.

LEMMA 6.5. *Write*

$$e^{\nu}=(\cos\theta^{(\nu)},\sin\theta^{(\nu)}),\quad f^{(\nu)}=(-\sin\theta^{(\nu)},\cos\theta^{(\nu)})$$

and

$$e_\nu^{(\nu)}=E_\nu(\cos\theta_\nu^{(\nu)},\sin\theta_\nu^{(\nu)}),\quad f_\nu^{(\nu)}=F_\nu(-\sin\theta_\nu^{(\nu)},\cos\theta_\nu^{(\nu)}).$$

Then

(a) $\|D\theta^{(\nu)}\|$, $\|D\theta_\nu^{(\nu)}\|\leqslant 100(K/\lambda)^{4\nu}\sqrt{b}$,

(b) $\|E_\nu'\|\leqslant\|De_\nu^{(\nu)}\|\leqslant K^{2\nu}(K/\lambda)^{4\nu}$ *and the same holds for* F_ν *and* $f_\nu^{(\nu)}$.

Proof. In (3) we obtained

$$\operatorname{tg}2\theta^{(\nu)}=\frac{2(A_\nu B_\nu+C_\nu D_\nu)}{(A_\nu^2+C_\nu^2)-(B_\nu^2+D_\nu^2)}\equiv\frac{2I_\nu}{H_\nu}. \tag{14}$$

Taking derivatives on both sides and using $(d/d\theta)\operatorname{tg}=1+\operatorname{tg}^2$ one gets

$$D\theta^{(\nu)}=\frac{H_\nu I_\nu'-H_\nu' I_\nu}{H_\nu^2+4I_\nu^2} \tag{15}$$

From the properties in Theorem 2.1 we have

$$|H_\nu I_\nu'-H_\nu' I_\nu|\leqslant 32K^{4\nu}\sqrt{b}. \tag{16}$$

On the other hand,

$$\begin{aligned}E_\nu^2-F_\nu^2&=((A_\nu^2+C_\nu^2)-(B_\nu^2+D_\nu^2),2(A_\nu B_\nu+C_\nu D_\nu))\cdot(\cos 2\theta^{(\nu)},\sin 2\theta^{(\nu)})\\&=(H_\nu,2I_\nu)\cdot(\cos 2\theta^{(\nu)},\sin 2\theta^{(\nu)}).\end{aligned}$$

The two vectors on the right being colinear, by (14), this proves

$$H_\nu^2+4I_\nu^2=(E_\nu^2-F_\nu^2)^2\geqslant\left(\lambda^{2\nu}-\left(\frac{Kb}{\lambda}\right)^{2\nu}\right)^2\geqslant\tfrac12\lambda^{4\nu}. \tag{17}$$

Replacing (16) and (17) in (15) we obtain (a) for $\theta^{(\nu)}$. Observe now that $(\cos\theta_\nu^{(\nu)},\sin\theta_\nu^{(\nu)})$ and $(-\sin\theta_\nu^{(\nu)},\cos\theta_\nu^{(\nu)})$ are, respectively, maximally expanding and maximally contracting for

$$M^{-\nu}(\varphi_a^\nu(z_1))\cdot\det M^\nu(z_1)=\begin{pmatrix}D_\nu & -B_\nu\\ -C_\nu & A_\nu\end{pmatrix}.$$

Therefore, analogously to (14),

$$\operatorname{tg}2\theta_\nu^{(\nu)}=\frac{-2(D_\nu B_\nu+C_\nu A_\nu)}{(D_\nu^2+C_\nu^2)-(B_\nu^2+A_\nu^2)} \tag{18}$$

and now the bound for $\|D\theta_\nu^{(\nu)}\|$ is obtained in precisely the same way as above for $\|D\theta^{(\nu)}\|$. As for the proof of (b), we take derivatives on $e_\nu^{(\nu)}(z_1)=M(z_\nu)\cdot e_{\nu-1}^{(\nu)}(z_1)$, $z_\nu=\varphi_a^{\nu-1}(z_1)$, and get by induction

$$\|De_\nu^{(\nu)}\|\leqslant K^{2\nu-1}+K^{2\nu-2}+\ldots+K^{\nu+1}+K^\nu\|De^{(\nu)}\|\leqslant K^{2\nu}\left(\frac{K}{\lambda}\right)^{4\nu}.$$

Clearly, the same argument also works for $f_\nu^{(\nu)}$. □

LEMMA 6.6. *There is* $K_0 = K_0(K, \lambda) > 0$ *such that*

$$\|De^{(\nu)}(z_1)\| \leqslant K_0\sqrt{b}, \qquad \text{for all } 1 \leqslant \nu \leqslant n.$$

Proof. In view of Lemma 6.5(a), we may restrict here to $\nu \geqslant 5$ (say). Let

$$\phi^{(\nu)} = \text{angle}(e^{(\nu)}, e^{(\nu-1)}) = \theta^{(\nu)} - \theta^{(\nu-1)},$$
$$\phi_\nu^{(\nu)} = \text{angle}(e_\nu^{(\nu)}, e_\nu^{(\nu-1)}).$$

Observe that

$$\operatorname{tg} \phi^{(\nu)} = \frac{E_\nu}{F_\nu} \cdot \operatorname{tg} \phi_\nu^{(\nu)} \tag{19}$$

and so

$$(1 + \operatorname{tg}^2 \phi^{(\nu)}) D\phi^{(\nu)} = \left(\frac{E_\nu}{F_\nu}\right)' \operatorname{tg} \phi_\nu^{(\nu)} + \left(\frac{E_\nu}{F_\nu}\right)(1 + \operatorname{tg}^2 \phi_\nu^{(\nu)}) D\phi_\nu^{(\nu)}. \tag{20}$$

Recalling that $E_\nu \cdot F_\nu = \det M^\nu(z_1)$ we find

$$\left(\frac{E_\nu}{F_\nu}\right)' = \frac{1}{F_\nu^2} \cdot \left(\det M^\nu(z_1) \cdot \sum_{j=1}^{\nu} \frac{(\det M)'}{\det M}(z_j) \cdot M^{j-1}(z_1) - 2E_\nu F_\nu'\right).$$

From Lemma 6.5 and the properties in Theorem 2.1 it follows that

$$\left\| \left(\frac{E_\nu}{F_\nu}\right)' \right\| \leqslant 4 \left(\frac{K}{\lambda}\right)^{6\nu} \left(\frac{Kb}{\lambda}\right)^{\nu}. \tag{21}$$

On the other hand, by (6) and (19),

$$\operatorname{tg}^2 \phi_\nu^{(\nu)} = \frac{(\|e_\nu^{(\nu-1)}\| / \|e_\nu^{(\nu)}\|)^2 - 1}{1 - (\|e_\nu^{(\nu-1)}\| / \|f_\nu^{(\nu)}\|)^2} \leqslant 2\left(\frac{K^2}{b}\right)^2, \tag{22}$$

where we also use $\|e_\nu^{(\nu-1)}\| \leqslant K \|e_{\nu-1}^{(\nu-1)}\| \leqslant K \|e_{\nu-1}^{(\nu)}\| \leqslant (K^2/b) \|e_\nu^{(\nu)}\|$. Now, we bound $\|D\phi_\nu^{(\nu)}\|$. Write $e_\nu^{(\nu-1)} = \bar{E}_\nu(\cos\theta_\nu^{(\nu-1)}, \sin\theta_\nu^{(\nu-1)})$, so that $\phi_\nu^{(\nu)} = \theta_\nu^{(\nu)} - \theta_\nu^{(\nu-1)}$. By Lemma 6.5

$$\|D\theta_\nu^{(\nu)}\| \leqslant 100\sqrt{b} \cdot \left(\frac{K}{\lambda}\right)^{4\nu}. \tag{23}$$

On the other hand,

$$\operatorname{tg} \theta_\nu^{(\nu-1)} = \frac{C(z_\nu) \cdot \cos\theta_{\nu-1}^{(\nu-1)} + D(z_\nu) \cdot \sin\theta_{\nu-1}^{(\nu-1)}}{A(z_\nu) \cdot \cos\theta_{\nu-1}^{(\nu-1)} + B(z_\nu) \cdot \sin\theta_{\nu-1}^{(\nu-1)}}.$$

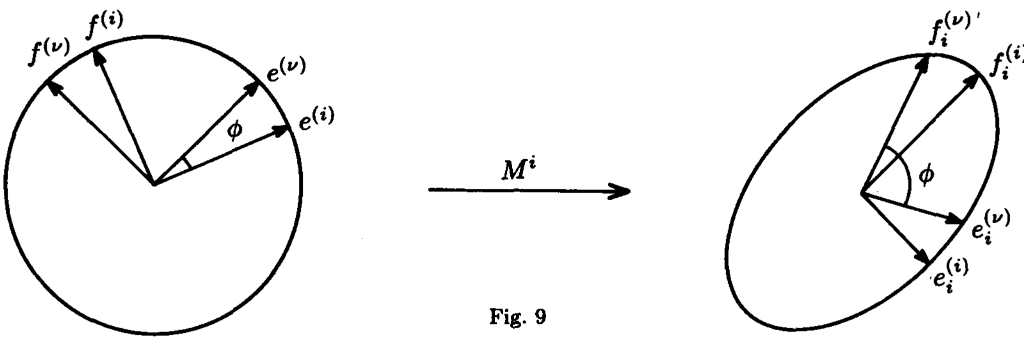

Fig. 9

Again we take derivatives and, using Lemma 6.5, obtain

$$\|D\theta_\nu^{(\nu-1)}\| \leqslant \frac{1000K^4}{b\sqrt{b}}\cdot K^\nu\left(\frac{K}{\lambda}\right)^{4\nu}, \tag{24}$$

which, together with (23) implies

$$\|D\phi_\nu^{(\nu)}\| \leqslant \frac{K^\nu}{b^2}\left(\frac{K}{\lambda}\right)^{4\nu}. \tag{25}$$

Replacing (21), (22) and (25) in (20) we get

$$\|D\phi^{(\nu)}\| \leqslant \frac{10K^4}{b^4}\cdot K^\nu\left(\frac{K}{\lambda}\right)^{6\nu}\left(\frac{Kb}{\lambda}\right)^{\nu}. \tag{26}$$

Finally the proof is completed by observing that for $\nu\geqslant 5$

$$\|D\theta^{(\nu)}\| \leqslant \|D\theta^{(4)}\| + \sum_5^\nu \|D\phi^{(\nu)}\| \leqslant 200\left(\frac{K}{\lambda}\right)^{16}\sqrt{b}. \qquad \square$$

Observe that the statements and proofs of Lemmas 6.5 and 6.6 remain true as they are, if one thinks of the derivatives as being taken with respect to all three variables (a,x,y). Moreover, the argument extends to the second order derivative in a laborious but totally straightforward way and we get the following statement to be used in Section 11.

LEMMA 6.7. *There is* $K_0=K_0(K,\lambda)>0$ *such that*

$$\|e^{(\nu)}\|_{C^2(a,x,y)} \leqslant K_0\sqrt{b} \quad \textit{for all } 1\leqslant\nu\leqslant n.$$

Sections 10 and 11 require a good control on the variation of $e_i^{(\nu)}$ for $i\leqslant\nu$. The following estimate is sufficient for that purpose. Observe that for $i=\nu$ this is much better than Lemma 6.5.

LEMMA 6.8. *There is $K_1=K_1(K,\lambda)$ such that for all $4\leqslant i\leqslant \nu$*

$$\|D_{(a,x,y)}e_i^{(\nu)}\|\leqslant (K_1 b)^{i-3}.$$

Proof. We denote $D=D_{(a,x,y)}$ and

$$\phi=\text{angle}(e^{(i)},e^{(\nu)})=\text{angle}(f^{(i)},f^{(\nu)}),$$
$$\psi=\text{angle}(e_i^{(\nu)},f_i^{(\nu)})=\tfrac{1}{2}\pi-\phi_e+\phi_f$$

with $\phi_e=\text{angle}(e_i^{(i)},e_i^{(\nu)})$ and $\phi_f=\text{angle}(f_i^{(i)},f_i^{(\nu)})$, see Figure 9. We get

$$(1+\text{tg}^2\,\phi_e)D\phi_e=\left(\frac{F_i}{E_i}\right)'\text{tg}\,\phi+\left(\frac{F_i}{E_i}\right)(1+\text{tg}^2\,\phi)D\phi.$$

Trivially $|F_i/E_i|\leqslant (K^2/b)^i$. Moreover, by (21),

$$\left\|\left(\frac{F_i}{E_i}\right)'\right\|=\left\|\left(\frac{E_i}{F_i}\right)'\left(\frac{F_i}{E_i}\right)^2\right\|\leqslant 4\left(\frac{K^{11}}{\lambda^7 b}\right)^i.$$

On the other hand, from (26),

$$\|D\phi\|\leqslant\sum_{i+1}^{\nu}10\left(\frac{K}{b}\right)^4K^s\left(\frac{K}{\lambda}\right)^{6s}\left(\frac{Kb}{\lambda}\right)^s\leqslant 20\left(\frac{K}{b}\right)^4\left(\frac{K^8b}{\lambda^7}\right)^{i+1}.$$

Thus, for $i\geqslant 4$,

$$\|D\phi_e\|\leqslant\frac{K^7}{2b^3\lambda^7}\left(\frac{K^{12}}{\lambda^9}\right)^i.$$

The same argument gives an even better estimate for $D\phi_f$ and we conclude that

$$\|D\psi\|\leqslant\frac{K^7}{b^3\lambda^7}\left(\frac{K^{12}}{\lambda^9}\right)^i. \tag{27}$$

Observe now that $\|e_i^{(\nu)}\|\cdot\|f_i^{(\nu)}\|\sin\psi=\det M^i(z_i)$ and so

$$\frac{\|e_i^{(\nu)}\|'}{\|e_i^{(\nu)}\|}=\frac{(\det M^i)'}{\det M^i}(z_1)-\frac{\|f_i^{(\nu)}\|'}{\|f_i^{(\nu)}\|}-\cos\psi\cdot D\psi.$$

As in the proof of Lemma 6.5,

$$\left\|\frac{\|f_i^{(\nu)}\|'}{\|f_i^{(\nu)}\|}\right\|\leqslant\frac{K^{2i}(K/\lambda)^{4i}}{\lambda^i}=\left(\frac{K^6}{\lambda^5}\right)^i.$$

It follows that

$$\| \, \|e_i^{(\nu)}\|' \, \| \leqslant \frac{2K^7}{b^3\lambda^7}\left(\frac{K^{12}}{\lambda^9}\right)^i \|e_i^{(\nu)}\|. \tag{28}$$

Write also $f_i^{(\nu)} = \|f_i^{(\nu)}\|(\cos\tau, \sin\tau)$ and note that

$$\|D\tau\| \leqslant \frac{\|Df_i^{(\nu)}\|}{\|f_i^{(\nu)}\|} \leqslant \left(\frac{K^6}{\lambda^5}\right)^i. \tag{29}$$

On the other hand, $e_i^{(\nu)} = \|e_i^{(\nu)}\|(\cos(\tau+\psi), \sin(\tau+\psi))$ and so

$$\|De_i^{(\nu)}\| \leqslant \| \, \|e_i^{(\nu)}\|' \, \| + \|e_i^{(\nu)}\| \cdot \|D(\tau+\psi)\|.$$

Now the lemma follows from (27), (28), (29) and Lemma 6.1. □

7. Algorithms for the construction of critical points

7A. Generation zero

We restrict from now on to an interval Ω_0 of a-values close to $a=2$ but with $\sup \Omega_0 < 2$ (compare Section 3). We assume b small enough so that $a_+(\varphi) > \sup \Omega_0$ (respectively $a_-(\varphi) > \sup \Omega_0$) and so Section 4 applies to $\varphi_a, a \in \Omega_0$, for all Hénon-like families under consideration.

Let us explain how approximations $z_0^{(n)}$ to the critical points are obtained, using the contractive approximations $e^{(n)}$. We say that a segment γ is a $C^2(b)$ curve if it is the graph of a function $y=y(x)$ with $|\dot{y}|, |\ddot{y}| \leqslant b^{t/2}$, $0<t<\frac{1}{2}$ as in Theorem 2.1. The critical approximations $z_0^{(n)}$ are always constructed in $C^2(b)$ pieces of W^u. First we let $z_0^{(0)}$ be the point of $W^u \cap \{x=0\}$ closest to P in W^u and denote $z_i^{(0)} = \varphi_a^i(z_0^{(0)})$. Define $G_0 = [z_1^{(0)}, z_2^{(0)}] \subset W^u$ and, for $g \geqslant 1$, $G_g = \varphi_a^g(G_0) \setminus \varphi_a^{g-1}(G_0)$. By assuming Ω_0 close enough to $a=2$ (and b sufficiently small) we have that, for

$$\delta_0 = 10(2 - \sup \Omega_0), \tag{1}$$

the pieces of G_0 and G_1 inside $\{\|x\| \leqslant 1-\delta_0\}$ are $C^2(b)$ curves. This follows simply from the Lipschitz (even smooth) dependence on the map of (compact parts of) stable and unstable manifolds.

PROPOSITION 7.1. *Let U be a neighbourhood of $0 \in \mathbf{R}^n$ and $g=(g_t)_t$ be a C^k m-parameter family of (not necessarily invertible) maps $g_t: U \to \mathbf{R}^n$. Suppose that $0 \in U$ is a hyperbolic fixed point for g_0 and let $\mathbf{R}^n = E^u \oplus E^s$ be the corresponding splitting. Then,*

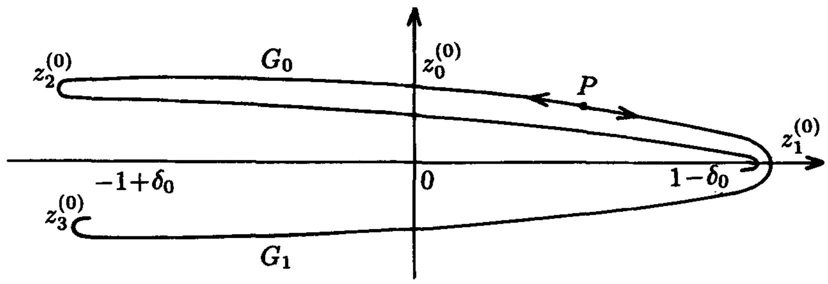

Fig. 10

for h close to g and t close to $0\in\mathbf{R}^m$, the map h_t has a unique hyperbolic fixed point near $0\in U$ and its local unstable manifold may be written as $\mathrm{graph}(\phi_h(t,\cdot))$, *with*

$$\phi_h(t,\cdot): x\in B_\varepsilon(0)\subset E^u\mapsto\phi_h(t,x)\in E^s$$

a C^k map. Moreover this can be done in such a way that

$$\phi:(h,t,x)\mapsto\phi_h(t,x)$$

is C^k (with bounded C^k norm) in all variables on a neighbourhood of $(g,0\in\mathbf{R}^m,0\in E^u)$. In particular

$$h\mapsto\phi_h(\cdot,\cdot)\in C^{k-1}(V,E^s)$$

is a Lipschitz (even C^1) function, if we consider it as taking values in the space of C^{k-1} maps $V\to E^s$, V a neighbourhood of $0\in\mathbf{R}^m\times E^u$, endowed with the C^{k-1} topology, Finally, analogous facts hold for the local stable manifold.

The first part of the proposition is quite standard. The main idea to prove the smoothness of ϕ in all variables, is to consider $F:(h,t,p)\mapsto(h,t,h_t(p))$ and to obtain graph(ϕ) as a C^k local center-unstable (respectively center-stable) manifold associated to the fixed point $(g,0\in\mathbf{R}^m,0\in U)$ of F, see [PT2, Appendices I, IV], [Ru, Chapters 6, 7], [Sh, Chapter 5] for details.

In our setting this gives $G_0\cap\{|x|\leqslant 1-\delta_0\}$ as the graph of $y(x)=y_\varphi(a,x)$ with

$$\|y_\varphi\|_{C^2(a,x)}\leqslant\text{const.}\, b^t, \tag{2}$$

and analogously for G_1, see Figure 10.

First we want to construct approximations $z_0^{(n)}$ for a critical point in G_0. We begin with $z_0^{(0)}$ above. Observe that $z_1^{(0)}=\varphi_a(z_0^{(0)})$ is e^{c_0}-expanding up to time $N-1$ as a consequence of part (b) of the following two-dimensional version of Lemma 2.1.

LEMMA 7.2. *Given $0<c_0<\log 2$ and $\delta>0$, there are $\varepsilon_0=\varepsilon_0(c_0,\delta)>0$ and $b_0=b_0(c_0,\delta)>0$ such that for $a\in[2-\varepsilon_0,2+\varepsilon_0]$ and if $b<b_0(c_0,\delta)$ the following holds. Let*

v be a unit vector with $|\operatorname{slope}(v)| \leqslant \frac{1}{10}$ *and* $z_i = (x_i, y_i)$, $0 \leqslant i \leqslant k+1$, *be a piece of orbit with* $|x_i| \geqslant \delta$ *for* $1 \leqslant i \leqslant k$. *Then we have*:

(a) *For all* $1 \leqslant i \leqslant k$, $|\operatorname{slope}(D\varphi_a^i(z_1)\cdot v)| \leqslant \sqrt[4]{b}$ *and* $\|D\varphi_a^i(z_1)\cdot v\| \geqslant a|x_i|\cdot\|D\varphi_a^{i-1}(z_1)\cdot v\|$.

(b) *If* $|x_0| \leqslant \delta$ *or* $|x_{k+1}| \leqslant \delta$ *then* $\|D\varphi_a^k(z_1)\cdot v\| \geqslant e^{kc_0}$.

Proof. (a) is an immediate consequence of our definition of Hénon-like map and (b) is analogous to Lemma 4.6 of [BC2]. □

Let $z(x) = (x, y(x))$ parametrize the $C^2(b)$ piece of G_0 in $|x| \leqslant 1-\delta_0$. We take $q^{(n)}(x)$ such that $(q^{(n)}(x), 1)$ is colinear to $e^{(n)}(\varphi_a(z(x)))$, $1 \leqslant n \leqslant N-1$. This is defined in some interval $|x| \leqslant \sigma$. By (6.9)

$$|q^{(n)}(x)| \leqslant \sqrt[4]{b} \tag{3}$$

and Lemma 6.6 gives

$$|\dot{q}^{(n)}(x)| \leqslant 2KK_0\sqrt{b} \tag{4}$$

(dot representing derivative with respect to x). On the other hand, the tangent space to W^u at $\varphi_a(z(x))$ is generated by $(t(x), 1)$, where

$$t(x) = \frac{A(z(x)) + B(z(x))\dot{y}(x)}{C(z(x)) + D(z(x))\dot{y}(x)}.$$

Now

$$\dot{t}(x) = \frac{A'(z(x))\cdot(1, \dot{y}(x))C(z(x)) + \text{ other terms}}{(C(z(x)) + D(z(x))\dot{y}(x))^2}$$

where (Theorem 2.1)

$$|A'(z(x))\cdot(1, \dot{y}(x))C(z(x))| \geqslant \frac{2\sqrt{b}}{K}, \quad |C(z(x))| \in \sqrt{b}\,[1/K, K]$$

and all the other terms are of order $\leqslant b^{1/2+t}$. Therefore, for b small

$$|\dot{t}(x)| \geqslant \frac{1}{2}\left|\frac{A'(z(x))\cdot(1, \dot{y}(x))}{C(z(x))}\right| \geqslant \frac{1}{K\sqrt{b}}. \tag{5}$$

Finally, note that Theorem 2.1 (a) implies $|A(z_0^{(0)})| \leqslant Kb^t$ and this leads to

$$|t(0)| \leqslant 2K^2 b^{t-1/2}. \tag{6}$$

Now we are in position to exhibit the critical approximations $z_0^{(n)}$. First, by (3)–(6),

$$|t(0) - q^{(1)}(0)| \leqslant 3K^2 b^{t-1/2} \quad \text{and} \quad |\dot{t}(x) - \dot{q}^{(1)}(x)| \geqslant \frac{1}{2K\sqrt{b}} \quad \text{for } |x| \leqslant \sigma.$$

Hence, there is a unique $x^{(1)}\in[-\sigma,\sigma]$ such that

$$t(x^{(1)})=q^{(1)}(x^{(1)}) \tag{7}$$

and, moreover,

$$|x^{(1)}|\leqslant 6K^3\sqrt{b}. \tag{8}$$

We take $z_0^{(1)}=(x^{(1)},y(x^{(1)}))$ and (7) means that $e^{(1)}$ is tangent to W^u at $\varphi_a(z_0^{(1)})$. Now, by Lemma 6.1(a), $|q^{(2)}(x^{(1)})-q^{(1)}(x^{(1)})|\leqslant 3K(Kb)$ and so, using also (4)–(5),

$$|t(x^{(1)})-q^{(2)}(x^{(1)})|\leqslant 3K^2b \quad\text{and}\quad |\dot t(x)-\dot q^{(2)}(x)|\geqslant\frac{1}{2K\sqrt{b}}\quad\text{on } |x|\leqslant\sigma.$$

As above we conclude the existence of a unique $x^{(2)}$ such that

$$t(x^{(2)})=q^{(2)}(x^{(2)}) \tag{9}$$

and

$$|x^{(2)}-x^{(1)}|\leqslant 6K^3b\sqrt{b} \tag{10}$$

and we take $z_0^{(2)}=(x^{(2)},y(x^{(2)}))$. By repeated use of this procedure we find, for each $1\leqslant n\leqslant N-1$, a unique point $z_0^{(n)}=(x^{(n)},y^{(n)})$, such that $e^{(n)}$ is tangent to W^u at $z_1^{(n)}=\varphi_a(z_0^{(n)})$ and

$$|z_0^{(n+1)}-z_0^{(n)}|\leqslant 10\sqrt{b}\,K^{(n+2)}b^n\leqslant (Kb)^n. \tag{11}$$

7B. Higher generations

Clearly, the same argument can be applied to any $C^2(b)$ segment of W^u, as long as we have a convenient (i.e. with small $|t-q|$) initial point to use in the place of $z_0^{(0)}$. One way such a starting point can be found is by relating the $C^2(b)$ segment to another one where a critical approximation is already known to exist. That is how critical approximations of higher generations are obtained. In order to explain this let $\gamma: x\mapsto z(x)=(x,y(x))$ and $\tilde\gamma: x\mapsto\tilde z(x)=(x,\tilde y(x))$ be two $C^2(b)$ segments of W^u defined for $|x-x_0|\leqslant l$. We denote $z_0=z(x_0)$ and $\zeta_0=\tilde z(x_0)$ and let ζ_1 be e^c-expanding up to some $\mu\geqslant 1$. Moreover ζ_0 is supposed to be a μth critical approximation, i.e. $\tilde t(x_0)=\tilde q^{(\mu)}(x_0)$, where $(\tilde q^{(\mu)}(x),1)$ is colinear to $e^{(\mu)}(\varphi_a(\tilde z(x)))$ and $(\tilde t(x),1)$ generates the tangent space to W^u at $\varphi_a(\tilde z(x))$. We use similar notations for γ. Fix

$$\sigma_0=\left(\frac{1}{10K^2}\right)^2 \tag{12}$$

and assume that

$$d \leqslant \frac{\sigma_0^{2\mu}}{K^2} \tag{13}$$

and

$$l \geqslant \sqrt{d} \tag{14}$$

where $d=|z_0-\zeta_0|$. In particular, by Lemma 4.3, every $\varphi_a(z(x))$ with $|x-x_0|\leqslant\sqrt{d}$ is expanding up to time μ. Observe that the $C^2(b)$ segments $\gamma, \tilde{\gamma}$ must be disjoint and this plays a crucial role: together with (14) it implies $|\dot{y}(x_0)-\dot{\tilde{y}}(x_0)|\leqslant 2\sqrt{d}$. Iterating once under $D\varphi_a$ (and using the definition of Hénon-like map) we get $|t(x_0)-\tilde{t}(x_0)|\leqslant 8K^4(\sqrt{d}+d/\sqrt{b})$. On the other hand, Lemma 6.6 gives $|q^{(\mu)}(x_0)-\tilde{q}^{(\mu)}(x_0)|\leqslant 2KK_0\sqrt{b}d$, so that

$$|t(x_0)-q^{(\mu)}(x_0)| \leqslant 10K^4\left(\sqrt{d}+\frac{d}{\sqrt{b}}\right).$$

Then, using the same procedure as above we find a μth critical approximation $z_0^{(\mu)}=z(x^{(\mu)})\in\gamma$ with

$$|x^{(\mu)}-x_0| \leqslant 20K^5\left(\sqrt{b}\sqrt{d}+d\right) \leqslant \frac{\sqrt{d}}{4} \leqslant \frac{l}{4}. \tag{15}$$

Now, if there is expansiveness up to higher iterates, we can proceed from $z_0^{(\mu)}$ to construct as before $z_0^{(\mu+1)}, z_0^{(\mu+2)}, \ldots$, with $|z_0^{(\nu+1)}-z_0^{(\nu)}|\leqslant (Kb)^\nu$, $\nu=\mu, \mu+1, \ldots$.

In particular, by this algorithm the critical approximations

$$z_0^{(i)} \in \tilde{\gamma} = G_0 \cap \{|x| \leqslant 1-\delta_0\}$$

induce critical approximations of generation 1,

$$w_0^{(i)} \in \gamma = G_1 \cap \{|x| \leqslant 1-\delta_0\}.$$

Observe that conditions (13) and (14) are satisfied if b is small enough. As we said in Section 5, for $n\leqslant N-1$ the nth critical set is defined by $\mathcal{C}_n=\{z_0^{(n-1)}, w_0^{(n-1)}\}$.

7C. The contractive fields

We end this section with the analog of the construction in Lemma 5.8 of [BC2]. This plays an important role in the binding procedure in Section 9. Let $\xi=(x,y)$ be λ-expanding up to time m and satisfy $2\delta<|x|<1-2\delta_0$. In this region the first contractive field $e^{(1)}$ is always well defined and we have shown that it is nearly vertical, as in Figure 11. In particular we can integrate the $e^{(1)}$-trajectory Γ^1 of ξ from (say) $y=-Kb^t$ to $y=Kb^t$. Note that $W^u\subset\{|y|\leqslant Kb^t\}$. For any $\eta\in\Gamma^1$ we have $|\eta-\xi|\leqslant 5Kb^t$ and $|\varphi_a(\eta)-\varphi_a(\xi)|\leqslant$

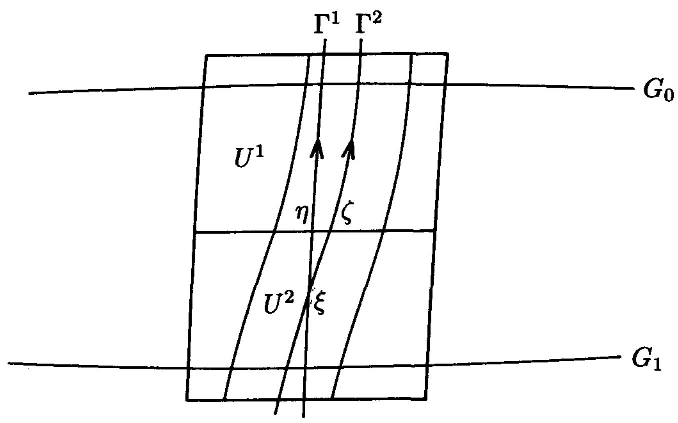

Fig. 11

const. b^t(const. b)$\leqslant(5Kb^t)^2$ as consequence of Lemma 6.1(b). It follows from Lemma 6.4(a) that η is also expanding up to time 2 and then we conclude, as in Lemma 6.3(a), that the same holds for every $\zeta\in U^1=\bigcup_{\eta\in\Gamma^1}[\eta-\sigma,\eta+\sigma]$ where $0<\sigma\leqslant(\lambda/10K^2)^2$ is fixed and $[\eta-\sigma,\eta+\sigma]$ denotes the horizontal straight segment of length 2σ centered at η. Now we integrate Γ^2, the $e^{(2)}$ trajectory of ξ, and observe that it hits $y=\pm Kb^t$ before leaving U^1. In fact, given any $\zeta\in\Gamma^2\cap[\eta-\sigma,\eta+\sigma],\eta\in\Gamma^1$, we have

$$|e^{(2)}(\zeta)-e^{(1)}(\eta)|\leqslant 2K_0\sigma$$

as a consequence of Lemmas 6.1(a) and 6.6. Hence, the maximal horizontal distance between Γ^1 and Γ^2 in $\{|y|\leqslant Kb^t\}$ is less than $10KK_0\sigma b^t\ll\sigma$. Now we define $U^2=\bigcup_{\eta\in\Gamma^2}[\eta-\sigma^2,\eta+\sigma^2]\subset U^1$ and conclude that all its points are expanding up to time 3. For $\eta\in\Gamma^2$ this follows from Lemma 6.4(a) using

$$|\eta-\xi|\leqslant 5Kb^t \quad\text{and}\quad |\varphi_a^\nu(\eta)-\varphi_a^\nu(\xi)|\leqslant \text{const.}\, b^t(\text{const.}\, b)^\nu\leqslant(5Kb^t)^{\nu+1},\ 1\leqslant\nu\leqslant 2;$$

then one extends it to arbitrary $\zeta\in U^2$ by the same calculations as in Lemma 6.3(a). In this way we eventually get to show that the $e^{(m)}$ trajectory Γ^m of ξ is an almost vertical curve crossing $\{|y|\leqslant Kb^t\}$ and so cutting G_0, G_1 inside the region $\{\delta<|x|<1-\delta_0\}$. Moreover, for $\eta\in\Gamma^m$

$$|\varphi_a^\nu(\eta)-\varphi_a^\nu(\xi)|\leqslant(5Kb^t)^{\nu+1}\quad\text{for } 0\leqslant\nu\leqslant m. \tag{16}$$

Applying Lemma 6.4 to $z_1=\xi$, $\zeta_1=\eta$ we conclude that

$$\tfrac{1}{2}\|D\varphi_a^\nu(\eta)\cdot v\|\leqslant\|D\varphi_a^\nu(\xi)\cdot u\|\leqslant 2\|D\varphi_a^\nu(\eta)\cdot v\| \tag{17}$$

$$|\mathrm{angle}(D\varphi_a^\nu(\eta)\cdot v, D\varphi_a^\nu(\xi)\cdot u)| \leqslant (Kb^{t/2})^{\nu+1} \tag{18}$$

for $1\leqslant\nu\leqslant m$ and all norm 1 vectors u, v with $|\mathrm{slope}|\leqslant\frac{1}{10}$. In particular this holds for $\eta\in\Gamma^m\cap G_i, u=(1,0)$ and $v=$ norm 1 vector tangent to G_i at η, $i=0,1$.

8. The induction: binding, folding and the splitting algorithm

Now we start the inductive step in the proof of Theorem B. This turns out to be quite long due mostly to the fact that a great deal of information concerning the previous iterates is required and must therefore be included in the induction hypothesis. The precise content of this induction hypothesis is stated in this section, where we also introduce some of the main ideas involved in the proof to be detailed later. For the time being the value of $a\in\Omega_0$ is fixed. The questions related to the variation of the parameter are treated in Section 11.

As we mentioned in Section 5, it is assumed that for each $k\leqslant n-1$ a set $\mathcal{C}_k$ of $(k-1)$st critical approximations of generation $g\leqslant\theta k$ has been constructed. Here and in what follows $\theta=\theta(b)\approx 1/\log(1/b)\ll 1$, see (9.17) for the precise definition. The approximations of critical values $z_1^{(k-1)}=\varphi_a(z_0^{(k-1)})\in\varphi_a(\mathcal{C}_k)$, are supposed to be e^c-expanding up to time k. Let $\gamma(\zeta, r)$ denote the interval in W^u of center $\zeta\in W^u$ and radius r. We assume that $\gamma(z_0^{(k-1)}, \varrho_0^{\theta k})$ is a $C^2(b)$ curve for every $z_0^{(k-1)}\in\mathcal{C}_k$, where $\varrho_0=\varrho_0(K,\delta)$ is a small fixed constant, see (9.9). Moreover, if $g=1+m\geqslant 1$ is the generation of $z_0^{(k-1)}$ then $\varphi_a^{-m}(\gamma(z_0^{(k-1)}, \varrho_0^{\theta k}))$ must be contained in $G_1\cap\{|x|\leqslant 1-\delta_0\}$, with $\delta_0=10(2-\sup\Omega_0)$ as in (7.1) and its tangent vectors must be expanded by $D\varphi_a^m$. Observe that $z_0^{(k-1)}$ is the unique element of $\mathcal{C}_k$ in $\gamma(z_0^{(k-1)}, \frac{1}{2}\varrho_0^{\theta k})$. Indeed, $\zeta_0^{(k-1)}$ be another such element. Reversing the argument of §7A we associate to $z_0^{(k-1)}$ and $\zeta_0^{(k-1)}$ uniquely determined sequences $z_0^{(k-1)}, z_0^{(k-2)}, z_0^{(k-3)}, \dots$ and $\zeta_0^{(k-1)}, \zeta_0^{(k-2)}, \zeta_0^{(k-3)}, \dots$ of increasingly coarser critical approximations. Let $l\geqslant 1$ be minimum such that $\varrho_0^{\theta k}\geqslant\sigma_0^{l+1}$ (σ_0 as in (7.12)). Then, by (7.11), the points $z_0^{(l)}, \zeta_0^{(l)}$ are defined and we have $|z_0^{(l)}-z_0^{(k-1)}|\leqslant(Kb)^l\ll\sigma_0^{l+1}\leqslant\varrho_0^{\theta k}$ and analogously for $\zeta_0^{(l)}, \zeta_0^{(k-1)}$. In particular $|z_0^{(l)}-\zeta_0^{(l)}|\leqslant\varrho_0^{\theta k}$. Now the minimality of l implies that $|z_0^{(l)}-\zeta_0^{(l)}|\leqslant\sigma_0^l$ or else $l=1$. In either case we have expansiveness up to time l for every $\varphi_a(\xi)$, $\xi\in[z_0^{(l)}, \zeta_0^{(l)}]\subset W^u$, and now §7A implies $z_0^{(l)}=\zeta_0^{(l)}$ and so $z_0^{(k-1)}=\zeta_0^{(k-1)}$. Now, from the fact that $\mathcal{C}_k\subset\bigcup_{g\leqslant\theta k}G_g$ and $\mathrm{length}(\bigcup_{g\leqslant\theta k}G_g)\leqslant K^{\theta k}\cdot\mathrm{length}(G_0)\leqslant 2K^{\theta k}$ we obtain the following bound for the number of critical points of generation $g\leqslant\theta k$

$$\#\mathcal{C}_k\leqslant 4\left(\frac{K}{\varrho_0}\right)^{\theta k}. \tag{1}$$

Now we construct $\mathcal{C}_n$ and this corresponds to

(a) replacing every $z_0^{(n-2)}\in\mathcal{C}_{n-1}$ by the associated $(n-1)$st approximation $z_0^{(n-1)}$;

(b) introducing approximations $z_0^{(n-1)}$ of critical points of generation $\theta(n-1)< g\leqslant\theta n$.

Part (a) is a simple application of the reasoning in §7A. From the inductive assumptions it follows that there is a unique $z_0^{(n-1)}\in\gamma(z_0^{(n-2)},\sigma_0^{n-1})\subset\gamma(z_0^{(n-2)},\varrho_0^{\theta(n-1)})$ such that $e^{(n-1)}(z_1^{(n-1)})$ is tangent to W^u at $z_1^{(n-1)}=\varphi_a(z_0^{(n-1)})$ and actually

$$|z_0^{(n-1)}-z_0^{(n-2)}|\leqslant (Kb)^{n-2}\leqslant\left(\frac{1}{4K}\right)^{n-1}. \tag{2}$$

We denote by C_n' the set of points $z_0^{(n-1)}$ obtained in this way. Part (b) of the definiton is through the algorithm of §7B. We let C_n'' consist of the $(n-1)$st critical approximations $z_0^{(n-1)}$ of generation $\in(\theta(n-1),\theta n]$ such that $\gamma(z_0^{(n-1)},\varrho_0^{\theta n})$ is as above and which can be obtained by applying the algorithm to a point $\zeta_0^{(n-1)}\in C_n'$ with

$$\operatorname{dist}(\zeta_0^{(n-1)},\gamma(z_0^{(n-1)},\varrho_0^{\theta n}))\leqslant b^{tg/5}\leqslant b^{t\theta(n-1)/5}. \tag{3}$$

The motivation for this comes from the binding construction of Section 9, see (9.18). In particular, by (7.15)

$$|z_0^{(n-1)}-\zeta_0^{(n-1)}|\leqslant b^{tg/10}\leqslant b^{t\theta(n-1)/10}\leqslant\left(\frac{1}{4K}\right)^{n-1}, \tag{4}$$

see also (9.19). Then we take $C_n=C_n'\cup C_n''$.

Remark. Conditions on how the expression in (3) varies with the parameter will be necessary for the proof of Lemma 11.2. These conditions are also part of our definition of C_n'' but for the sake of making the presentation easier to follow we postpone their statement to Section 11 ((11.7)) where this is used and can be better motivated.

Actually, we even assume that for each $1\leqslant k\leqslant n-1$, e^c-expansiveness up to time k has been obtained for every $\xi_1=\varphi_a(\xi_0)$ with ξ_0 bound to C_k in the sense of the following definition. We say that ξ_0 is *bound to C_k up to time p* if there is $z_0=z_0^{(k-1)}\in C_k$ such that

$$|\xi_j-z_j|\leqslant h_k e^{-\beta j}\quad\text{for all } 1\leqslant j\leqslant p,\ \text{with } h_k=2-\sum_1^{k-1}\left(\frac{e^\beta}{4}\right)^i\in(1,2). \tag{BC1}$$

If this holds for some $p\geqslant k$ then we say simply that ξ_0 *is bound to C_k*. The only purpose of introducing the coefficients h_k is to get (recall (2)–(4))

$$\xi_0 \text{ bound to } C_{k+1} \text{ (up to } p) \;\Rightarrow\; \xi_0 \text{ bound to } C_k \text{ (up to } p).$$

In particular, if ξ_0 is bound to C_n then, by induction ξ_1 is e^c-expanding up to time $(n-1)$. Now such points must be shown to be e^c-expanding at time n. This requires parameter exclusions and is done by adapting the argument of Section 3, as we explain in the sequel.

First one defines returns and binding periods for points bound to $\mathcal{C}_k$. The general idea is the same as in dimension one. One fixes $\delta>0$ small and returns $\nu\leqslant k$ of $\xi_0=(x_0,y_0)$ correspond to having $|x_\nu|<\delta$. For such a ν, a critical point ζ_0 is found with the properties (tangential position) described in Section 5. One excludes parameter values so that in the remaining set $|\xi_\nu-\zeta_0|\geqslant e^{-\alpha\nu}$. Finally the binding period associated to ν is given by $|\xi_{\nu+j}-\zeta_j|\leqslant e^{-\beta j}$, for $1\leqslant j\leqslant p$ $(\beta>\alpha)$. However, in our present setting details require some more care and in fact the definition is done by induction. For each $k\leqslant n-1$, returns, binding points and binding periods are supposed to have been defined for all $z_0\in\mathcal{C}_k$ and times $\leqslant k$. If ξ_0 is bound to $\mathcal{C}_k$, we choose $z_0\in\mathcal{C}_k$ as in (BC1) and let returns, binding points and binding periods of ξ_0 coincide with those of z_0. In particular these notions are defined for all ξ_0 bound to $\mathcal{C}_n$ and times $\leqslant n-1$. Let $z_0\in\mathcal{C}_n$ and suppose first that n belongs to the binding period introduced by some return $\nu<n$ of z_0. Take ν maximum and let ζ_0 be the binding point for z_ν. Then n is a (*bound*) *return* for z_0 iff $(n-\nu)$ is a return for ζ_0. We take the same point to bind both z_n and $\zeta_{n-\nu}$ and define the binding period for z_n to be $[n+1,n+p]$ if the binding period for $\zeta_{n-\nu}$ is $[n-\nu+1,n-\nu+p]$. Suppose now that no binding period associated to an earlier return contains n. Then n is a (*free*) *return* for z_0 iff $|x_n|<\delta$. In this case a binding point $\zeta_0\in\mathcal{C}_n$ must be found to use in the binding of z_n and this is done in Section 9. Moreover, we restrict to a-values for which holds the

$$d_n(z_0)\equiv|z_n-\zeta_0|\geqslant e^{-\alpha\nu}. \tag{BA}$$

The binding period $[n+1,n+p]$ is defined as follows. First, one lets $p_0\geqslant 1$ (the primary binding period) be given by

$$|z_{n+j}-\zeta_j|\leqslant he^{-\beta j}\quad\text{for } 1\leqslant j\leqslant p_0,\ p_0 \text{ maximum} \tag{BC2}$$

where $h=h(K,\alpha)\leqslant 1$ (to be precised later, see (10.21)) is introduced for purely technical reasons. We show in Section 10 that

$$p_0\leqslant 5\log d_n(z_0)^{-1}\leqslant 5\alpha n. \tag{5}$$

On the other hand, trivially, z_n remains bound to ζ_0 up to p_0. Therefore, we may speak of returns and binding periods during this interval of time. Now $p\geqslant 1$ is defined by:

- $(p+1)$ is a free iterate for z_n, i.e. it is not contained in any binding period of z_n;
- $1\leqslant p\leqslant p_0$ is maximum with this property.

This assures that at the end of the binding period introduced by the free return n the point z_0 is again in a free iterate. We also need to know that $p\approx p_0$. To show this, observe that if $p<p_0$ then there must be returns $\nu_1=p<\nu_2<\ldots<\nu_s<p_0$ of z_n, such that each ν_{i+1} is contained in the binding period $[\nu_i+1,\nu_i+p_i]$, $1\leqslant i\leqslant s-1$, and $[\nu_s+1,\nu_s+p_s]$ contains

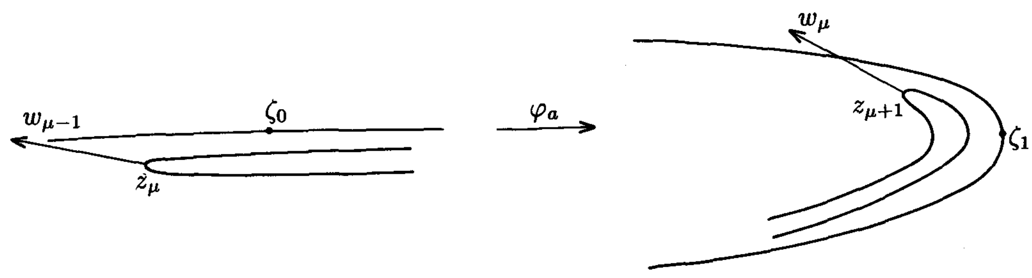

Fig. 12

p_0. It is part of our induction assumption that (5) holds for all returns $\nu \leqslant k$ of points in $\mathcal{C}_k$, $k \leqslant n$ ($d_\nu(\cdot)$ denoting always the distance to the corresponding binding point). Then $p_1 \leqslant 5\alpha p_0$ (by construction p_1 is the length of the binding period associated to a free return $\leqslant p_0$ of some element of some $\mathcal{C}_k$) and, analogously $p_{i+1} \leqslant 5\alpha p_i$ for all $1 \leqslant i \leqslant s-1$. It follows

$$p_0 - p \leqslant \sum_1^s (5\alpha)^i \cdot p_0 \leqslant 10\alpha p_0. \tag{6}$$

This also has the following useful consequence:

$$|z_{n+p+1} - \zeta_{p+1}| \geqslant |z_{n+p_0+1} - \zeta_{n+p_0+1}| \cdot K^{-10\alpha p_0} \geqslant he^{-2\beta(p+1)} \tag{7}$$

if α is sufficiently small with respect to β.

As in dimension one, it is crucial to show that the orbit of z_0 (and of every ξ_0 bound to it) has an expanding behaviour in the interval of time

$$[n, n+p] = \text{return} \cup \text{binding period}$$

(recall Lemma 3.2). However, the proof (and even the precise statement) of this, to be given in Section 10, as well as the binding algorithm of Section 9, require a more detailed description of the whole construction at stages $k \leqslant n-1$. Before we can complete the formal statement of our induction hypothesis, containing this description, we must introduce some notations and discuss another typically higher-dimensional difficulty which was already outlined in Section 5: the creation of folds in W^u.

Let us begin by making some geometric considerations. Let $\mu \leqslant k$ be a return for $z_0 \in \mathcal{C}_k$ and ζ_0 be the corresponding binding point. A typical situation is described in Figure 12: $w_{\mu-1}$ is nearly horizontal.

At the next iterate a fold of W^u is created and w_μ may have a very large slope. After that, the orbit $z_{\mu+j}$ spends some time outside $(-\delta, \delta)$ and during this period one expects to see expansion in the horizontal, direction (Lemma 7.2) and strong contraction

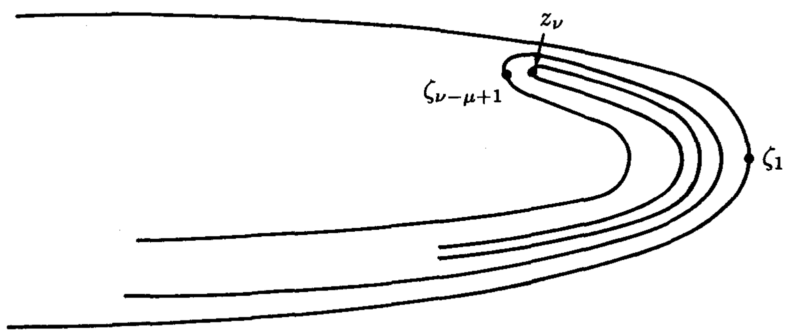

Fig. 13

in the vertical direction (Lemma 6.1). Hence the vectors $w_{\mu+j}$ should be getting more and more horizontal, so that the slope of $w_{\mu+l}$, say, is again of order $\sqrt[3]{b}$. The interval of time $[\mu+1, \mu+l]$ will be called the folding period associated to the return μ.

This geometric description is quantified by a splitting procedure as introduced in Section 5. At time $\mu+1$ we decompose $w_\mu = \omega_\mu + \sigma_\mu$ where $\omega_\mu = (u_\mu, 0)$ is horizontal and σ_μ has the direction of a convenient contractive approximation. Observe that $z_{\mu+1}$ is close to the point ζ_1 which is already known to be expanding. Now we define $\omega_{\mu+j} = D\varphi_a^j(z_{\mu+1})\cdot\omega_\mu$ and $\sigma_{\mu+j} = D\varphi_a^j(z_{\mu+1})\cdot\sigma_\mu$. As long as we stay outside $(-\delta, \delta)$ the vectors $\omega_{\mu+j}$ remain nearly horizontal (Lemma 7.2). On the other hand the $\sigma_{\mu+j}$ decrease exponentially and so for l sufficiently large we may add $\sigma_{\mu+l}$ back to the nearly horizontal component (making $w_{\mu+l} = \omega_{\mu+l}$), with no significant effect on its norm or direction. Notice that σ_μ is assumed only to be in a contractive approximation (and not in the real contractive direction, which at this stage is not yet defined) and so we have control (contraction) on its iterates during only a finite interval of time. That is why we must add this component back and restore $w_{\mu+l}$ at the end of the folding period. Of course, it may happen that a new return ν occurs still during the folding period $[\mu+1, \mu+l]$. This corresponds to a return of the binding point ζ_0 and creates a higher order fold. In such a situation again we force ω_ν to be horizontal and a new correction term $(D\varphi_a(z_\nu)\cdot\omega_{\nu-1} - \omega_\nu)$ is added to σ_ν. Actually, we may have a whole hierarchy of folds inside folds, as in Figure 13, and this corresponds to having several terms in the σ-component, see below.

The exact definition of the folding period is somewhat arbitrary and actually our choice differs (essentially by a multiplicative factor) from that in [BC2]. Again, some combinatorial care must be taken since we need the following two properties to hold for any folding period $F = [\mu+1, \mu+l]$:

(a) any folding period starting inside F must also end inside F;

(b) $\mu+l$ is not a return; moreover $z_{\mu+l+1}$ satisfies $2\delta < |x_{\mu+l+1}| < 1 - 2\delta_0$.

Recall that in the region $\delta<|x|<1-\delta_0$ the contractive vector fields are nearly vertical and G_0, G_1 are $C^2(b)$, at least if $b\ll(2-a)$, see Section 7. Hence the construction of §7C applies there and this plays an important role in the binding construction of Section 9.

We give the definition for free returns $\mu\leqslant k$ of critical points $z_0\in C_k$ and this is extended for general returns of points ξ_0 bound to C_k in precisely the same way as for the binding period. First one considers the primary folding period $[\mu+1, \mu+l_0]$ defined by

$$l_0=\frac{10\log K}{\log(1/b)}\log d_\mu(z_0)^{-1}+i \tag{8}$$

where $0\leqslant i\leqslant 4$ is chosen in such a way that property (b) holds at time $\mu+l_0+1$. Observe that l_0 is much smaller than the binding period: for $\delta>0$ sufficiently small (7) gives

$$p\geqslant\frac{\log d_\mu(z_0)^{-1}}{2(\log K+2\beta)}. \tag{9}$$

Now l is defined by:

- for any return $\nu\in[\mu+1, \mu+l]$ the (primary) folding period of ν ends at time $\leqslant\mu+l$;
- $l\geqslant l_0$ is minimum with that property.

The fact that l exists and $l\approx l_0$ is obtained by a geometric-series argument as for the binding period. We get in general

$$\frac{10\log K}{\log(1/b)}\log d_\mu(\xi_0)^{-1}\leqslant l\leqslant\frac{20\log K}{\log(1/b)}\log d_\mu(\xi_0)^{-1}+4. \tag{10}$$

We remark the following direct consequence of (10), which will be used several times in the sequel. If $e=e^{(l)}$ is the lth contractive approximation then

$$\|D\varphi_a^l\cdot e(\xi_1)\|\leqslant 4K(K^2b)^l\leqslant 4K(\sqrt{b})^l d_\mu(\xi_0)^2. \tag{11}$$

Now we give the precise definition of the splitting algorithm described above. For $0\leqslant\mu\leqslant k\leqslant n-1$ and ξ_0 bound to C_k we write $w_\mu=w_\mu(\xi_1)=\omega_\mu+\sigma_\mu$, where ω_μ and σ_μ are constructed as follows.

(i) $\omega_0=w_0=(1,0)$ and $\sigma_0=0$.

(ii) Let $\tilde\omega_\mu=D\varphi_a(\xi_\mu)\cdot\omega_{\mu-1}$ and $\tilde\sigma_\mu=D\varphi_a(\xi_\mu)\cdot\sigma_{\mu-1}$.

(iii) If μ is a return for ξ_0, split $\tilde\omega_\mu=\alpha_\mu\cdot e(\xi_{\mu+1})+\beta_\mu\cdot(1,0)$ where $e=(q,1)=(q^{(l)},1)$ has the direction of the lth contractive approximation, $l=$ length of the folding period, and then take

$$\omega_\mu=\tilde\omega_\mu-\alpha_\mu\cdot e(\xi_{\mu+1})=\beta_\mu\cdot(1,0), \quad \sigma_\mu=\tilde\sigma_\mu+\alpha_\mu\cdot e(\xi_{\mu+1}).$$

(iv) If μ is the end of a folding period, $\mu=\mu_1+l_1$, let

$$\omega_\mu=\tilde{\omega}_\mu+\alpha_{\mu_1}D\varphi_a^{l_1}(\xi_{\mu_1+1})\cdot e(\xi_{\mu_1+1}),\quad \sigma_\mu=\tilde{\sigma}_\mu-\alpha_{\mu_1}D\varphi_a^{l_1}(\xi_{\mu_1+1})\cdot e(\xi_{\mu_1+1});$$

more generally, if $s\geqslant 1$ folding periods $[\mu_1,\mu_1+l_1],\dots,[\mu_s,\mu_s+l_s]$ end at time μ, take

$$\omega_\mu=\tilde{\omega}_\mu+\sum_1^s\alpha_{\mu_i}D\varphi_a^{l_i}(\xi_{\mu_i+1})\cdot e(\xi_{\mu_i+1}),\quad \sigma_\mu=\tilde{\sigma}_\mu-\sum_1^s\alpha_{\mu_i}D\varphi_a^{l_i}(\xi_{\mu_i+1})\cdot e(\xi_{\mu_i+1}).$$

(v) If neither (iii) nor (iv) apply then take simply $\omega_\mu=\tilde{\omega}_\mu$ and $\sigma_\mu=\tilde{\sigma}_\mu$. Recall also that (iii) and (iv) never apply simultaneously.

The algorithm is designed in such a way that the ω-component corresponds essentially to the horizontal (1-dimensional) part of w_ν:

LEMMA 8.1. *For every $0\leqslant\nu\leqslant k\leqslant n-1$ and ξ_0 bound to C_k*

$$|\operatorname{slope}(\omega_\nu(\xi_1))|\leqslant b^t.$$

The proof of this requires more inductive information and is postponed to the end of the section. On the other hand, the σ-component in the splitting contains the geometric complication coming from the creation of the folds. It has the form

$$\sigma_\mu=\sum_i\alpha_{\mu_i}D\varphi_a^{\mu-\mu_i}(\xi_{\mu_i+1})\cdot e(\xi_{\mu_i+1}),$$

each term in the sum corresponding to a fold created in a previous return and still affecting ξ_0 and the w-vector at time μ. An iterate μ is said to be *fold-free* for ξ_0 if $\sigma_{\mu-1}(\xi_1)=0$, i.e. $\omega_{\mu-1}(\xi_1)=w_{\mu-1}(\xi_1)$. Fold-free iterates are quite dense in the set of times:

LEMMA 8.2. *Given $\mu\geqslant 1$ there are fold-free iterates $\mu_1\leqslant\mu\leqslant\mu_2$ with*

$$\mu_2-\mu_1\leqslant\frac{20\alpha\log K}{\log(1/b)}\mu+4.$$

Moreover, if $\nu>\mu$ is a free iterate then this may be replaced by

$$\mu_2-\mu_1\leqslant\frac{100\log^2 K}{\log(1/b)}(\nu-\mu)+4.$$

Proof. The first part, is a direct consequence of (10) and the (BA). For the second one suppose that μ belongs to a folding period $[k+1,k+l]$. The corresponding binding period must satisfy $\nu-k>p\geqslant\log d_k(\xi_0)^{-1}/3\log K$, by (9). It follows from (10) that

$l \leqslant (60\log^2 K/\log(1/b))(\nu-k)+4 \leqslant (100\log^2 K/\log(1/b))(\nu-\mu)+4$ where we also use the fact $l \ll p$. □

Now we conclude the statement of our induction. For $k \leqslant n-1$ we assume that:

• Whenever $\nu=k$ is a free return for $z_0 \in \mathcal{C}_k$ a binding critical point $\zeta_0 \in \mathcal{C}_k$ is defined for z_n. Parameter values are chosen so that

$$d_\nu(z_0) \equiv |z_\nu - \zeta_0| \geqslant e^{-\alpha\nu}. \tag{BA}$$

For every ξ_0 bound to z_0

$$|\alpha_\nu(\xi_1)| \leqslant 4K\sqrt{b}\,\|\omega_{\nu-1}(\xi_1)\| \tag{12a}$$

and

$$\frac{3a}{2}d_\nu(\xi_0) \leqslant \frac{|\beta_\nu(\xi_1)|}{\|\omega_{\nu-1}(\xi_1)\|} = \frac{\|\omega_\nu(\xi_1)\|}{\|\omega_{\nu-1}(\xi_1)\|} \leqslant \frac{5a}{2}d_\nu(\xi_0). \tag{12b}$$

Moreover, (12a), (12b) hold for every return $\nu \leqslant k$ of any point ξ_0 bound to $\mathcal{C}_k$, at least if we take there slightly worse factors $5K\sqrt{b}$ (for (12a)) and a and $3a$ (for (12b)).

• If $\nu \leqslant k$ is a return for $z_0 \in \mathcal{C}_k$ then its binding period satisfies $p \leqslant 5\alpha\nu < \nu$. Moreover, there are $\tau_1, \tau_2 > 1$ depending only on K, α and β, such that given any ξ_0 bound to z_0

$$\frac{1}{\tau_1} \leqslant \frac{\|\omega_{\nu+j}(\xi_1)\|}{|\beta_\nu(\xi_1)|\cdot\|\omega_j(\zeta_1)\|} \leqslant \tau_1 \quad \text{for } 0 \leqslant j \leqslant p-1. \tag{13}$$

and, denoting $c_1=(c+c_0)/2$,

$$\frac{\|\omega_{\nu+p}(\xi_1)\| d_\nu(\xi_0)}{\|\omega_\nu(\xi_1)\|} \geqslant \tau_2 e^{c_1(p+1)/3} \geqslant 1 \quad (\zeta_0 \text{ is the binding point of } \xi_\nu). \tag{14}$$

• Parameter values are excluded so that in the remaining set holds the

$$F_k(a; z_0) \geqslant (1-\varepsilon)k \tag{FA}$$

with F_k denoting the total number of free iterates in the interval of time $[1,k]$.

The way the (BA) and the (FA) are obtained requires some explanation which will be given in Sections 11 and 12, where we also show that a positive Lebesgue measure set of a-values remains after all the exclusions.

Here we observe that the conditions above assure the e^c-expansiveness up to time k for all $\xi_1=\varphi_a(\xi_0)$, ξ_0 bound to $\mathcal{C}_k$, $k \leqslant n-1$. We write

$$\|\omega_k(\xi_1)\| = \prod_1^k \frac{\|\omega_i(\xi_1)\|}{\|\omega_{i-1}(\xi_1)\|}.$$

Let $1<\nu_1<\nu_2<\ldots<\nu_s\leqslant k$ be the free returns of ξ_0. For each $\nu=\nu_i$, $p=p_i$,

$$\prod_{\nu}^{\nu+p}\frac{\|\omega_i(\xi_1)\|}{\|\omega_{i-1}(\xi_1)\|}=\frac{\|\omega_{\nu+p}(\xi_1)\|}{\|\omega_\nu(\xi_1)\|}d_\nu(\xi_0)\geqslant 1.$$

On the other hand, denoting $\mu=\nu_{i+1}$, $q=\mu-\nu-p-1$,

$$\prod_{\nu+p+1}^{\mu-1}\frac{\|\omega_i(\xi_1)\|}{\|\omega_{i-1}(\xi_1)\|}=\frac{\|\omega_{\mu-1}(\xi_1)\|}{\|\omega_{\nu+p}(\xi_1)\|}=\frac{\|w_{\mu-1}(\xi_1)\|}{\|w_{\nu+p}(\xi_1)\|}\geqslant e^{c_0q}$$

by Lemma 7.2 (recall also the definiton of binding period). It follows, as in (3.4),

$$\|\omega_k(\xi_1)\|\geqslant e^{c_0F_k-\alpha k}\geqslant e^{c_1k}.$$

Now we use

LEMMA 8.3. *For any* $1\leqslant\mu\leqslant k\leqslant n-1$ *and* ξ_0 *bound to* C_k

$$K^{-5}e^{-\varepsilon\mu}\|\omega_\mu(\xi_1)\|\leqslant\|w_\mu(\xi_1)\|\leqslant K^5e^{(\varepsilon+\alpha)\mu}\|\omega_\mu(\xi_1)\|.$$

Proof. Analogous to Lemma 7.7 of [BC2]. □

This gives $\|w_k(\xi_1)\|\geqslant K^{-5}e^{(c_1-\varepsilon)k}\geqslant e^{ck}$ (because we may restrict to $k\geqslant N$) and completes the argument. For the proof of Lemma 8.3 one needs the following result which will also be used in Section 9.

LEMMA 8.4. *For any* $1\leqslant\mu<\nu\leqslant k\leqslant n-1$ *and* ξ_0 *bound to* C_k

$$\|\omega_\nu(\xi_1)\|\geqslant\min_{\mu<j\leqslant\nu}\left(\frac{\|\omega_j(\xi_1)\|}{\|\omega_{j-1}(\xi_1)\|}\right)\|\omega_\mu(\xi_1)\|\geqslant\min_{\mu<j\leqslant\nu}(ad_j(\xi_0))\|\omega_\mu(\xi_1)\|$$

(*with the convention*: $d_j(\xi_0)=|x_j|$ *if* $\xi_j=(x_j,y_j)$ *is not a return iterate*).

Proof. Analogous to Lemma 7.6 of [BC2]. □

We close this section with the

Proof of Lemma 8.1. We prove by induction on ν that

$$|\operatorname{slope}(\omega_\nu(\xi_1))|\leqslant\text{const.}\sum_1^\nu\left(\sqrt{b}\right)^i\leqslant b^t \tag{15}$$

Observe that this is trivial for $\nu=0$ and for ν a return. Suppose then that $\nu\geqslant 1$ is not a return for ξ_0. Assume moreover that (15) has been proved for all iterates $\mu\leqslant\nu-1$

of every point bound to $\mathcal{C}_k, \nu \leqslant k \leqslant n-1$. If no folding period ends at time ν we write $\omega_{\nu-1}(\xi_1)=u(1,p)$, with $|p|\leqslant b^t$ and then

$$|\operatorname{slope}(\omega_\nu(\xi_1))| = |\operatorname{slope}(D\varphi_a \cdot \omega_{\nu-1}(\xi_1))| \leqslant \frac{4K}{\delta}\sqrt{b}$$

implying (15). Let now ν coincide with the end of a folding period $\mu+l$. We take such a μ minimum and then property (a) in the definition implies

$$\omega_\nu(\xi_1) = \beta_\mu(\xi_1)\omega_l(\xi_{\mu+1}) + \alpha_\mu(\xi_1) D\varphi_a^l \cdot e(\xi_{\mu+1}).$$

Now, by induction $|\operatorname{slope}(\omega_l(\xi_{\mu+1}))| \leqslant \text{const.} \sum_1^l (\sqrt{b})^i$. On the other hand the induction hypotheses give

$$\begin{aligned} \|\alpha_\mu(\xi_1) D\varphi_a^l \cdot e(\xi_{\mu+1})\| &\leqslant 5K\sqrt{b}\,\|\omega_{\mu-1}(\xi_1)\| \cdot 8K(K^2 b)^l \\ &\leqslant \text{const.} (\sqrt{b})^{l+1} \|\beta_\mu(\xi_1)\omega_l(\xi_{\mu+1})\| \end{aligned} \tag{16}$$

and so $|\operatorname{slope}(\omega_\nu(\xi_1)) - \operatorname{slope}(\omega_l(\xi_{\mu+1}))| \leqslant \text{const.} (\sqrt{b})^{l+1}$. □

9. Binding and loss of growth on returns

Let n be a free return for a point $z_0 \in \mathcal{C}_n$. We describe here how a critical point $\zeta_0 \in \mathcal{C}_n$ is found to use in the binding of z_n. As we observed before, in order that we have $\|\omega_n(z_1)\| \approx |z_n - \zeta_0| \cdot \|\omega_{n-1}(z_1)\|$ it is crucial that $(z_n, \omega_{n-1}(z_1))$ be in tangential position to a $C^2(b)$ segment γ containing ζ_0, in the sense that

$$(z_n, \gamma) \ll |z_n - \zeta_0| \quad \text{and} \quad |\operatorname{angle}(\omega_{n-1}(z_1), t(\gamma;\eta))| \ll |z_n - \zeta_0|$$

where $t(\gamma;\eta)$ denotes the tangent direction to γ at η.

The basic ingredient of the binding construction is Lemma 9.1 below. This is essentially Lemma 6.6 of [BC2] but, since our definitions differ from those in [BC2], it is stated here in a slightly different form whose proof we present in order to explicit the conditions on λ_0 in our setting. Recall that we define $d_\nu(\xi_0)=|x_\nu|$ when $\xi_\nu=(x_\nu, y_\nu)$ is a non-return iterate.

LEMMA 9.1. *Fix λ_0 sufficiently small ($\lambda_0=(\delta/2)^2$, say). If n is a free return for $z_0 \in \mathcal{C}_n$, there are $1=m_1<m_2<\ldots<m_s\leqslant n$ with $m_{i+1}\leqslant 3m_i$ for all $1\leqslant i\leqslant s-1$ and $n\leqslant 3m_s$, such that each $n-m_i$ is a favorable position for z_0, meaning that*

(a) $2\delta < |x_{n-m_i}| < 1-2\delta_0$;

(b) *$n-m_i$ is a fold-free iterate;*

(c) $d_j(z_{n-m_i}) \geqslant \lambda_0^{j+1}$ *for all* $0 \leqslant j \leqslant m_i - 1$.

Proof. Let k be the last return before n and $\mu = k+l+1$, where l is the length of the folding period of k. (Taking δ small and Ω_0 close to 2 we may assume $(n-k) \geqslant 10$, say.) Since $[\mu, n)$ contains no returns or folding times one finds easily $1 = m_1 < m_2 < \ldots < m_{r-1} < m_r = n - \mu$ favorable positions as in the statement. Observe also that by Lemma 8.2, $\mu - k \leqslant \text{const.}/\log(1/b)(n-k) + 4 \leqslant \frac{1}{2}(n-k)$. Now we suppose that $m_i \geqslant \frac{1}{2}(n-k)$ has been defined and obtain m_{i+1} as follows. Let $\mu_i \geqslant n - 3m_i$ be a fold-free iterate with $\mu_i - (n-3m_i) \leqslant \text{const.}/\log(1/b)m_i + 4 \leqslant \frac{1}{2}m_i$. We let $I = [\mu_i, n)$ and observe that $d_j(z_0) \geqslant (Ke^{\beta})^{-|I|}$ for all $j \in I$. In fact, if j is a return

$$d_j(z_0) \geqslant (Ke^{\beta})^{j-n} \geqslant (Ke^{\beta})^{-|I|}$$

(because n does not belong to the binding period of j) and otherwise

$$d_j(z_0) \geqslant \tfrac{1}{2}\delta \geqslant \tfrac{1}{4}d_k(z_0) \geqslant \tfrac{1}{4}(Ke^{\beta})^{k-n} \geqslant \tfrac{1}{4}(K^{\beta})^{-2m_i} \geqslant (Ke^{\beta})^{-|I|}.$$

This reduces the construction of m_{i+1} (and so the proof of the lemma) to proving

LEMMA 9.2. *Let* $I = [p, q)$ *be a time interval such that iterate* p *is fold-free and it is in the region* $2\delta < |x| < 1 - 2\delta_0$ *and moreover*

$$\inf_I d_j(z_0) \geqslant (Ke^{\beta})^{-|I|}. \tag{1}$$

Then there is $\nu \in [p, \frac{1}{2}(p+q))$ *a fold-free iterate in the region* $2\delta < |x| < 1 - 2\delta_0$ *such that*

$$d_{\nu+j}(z_0) \geqslant \lambda_0^{j+1} \quad \textit{for all } \nu \leqslant \nu + j < q. \tag{2}$$

Proof of Lemma 9.2. This is trivial for small intervals: if

$$|I| \leqslant \frac{\log \lambda_0^{-1}}{\log K + \beta}, \tag{3}$$

we take $\nu = p$ and (2) is an immediate consequence of (1). The proof proceeds by induction on the length of the interval. Let $m \geqslant 1$ be such that $4m - 3 \leqslant |I| \leqslant 4m$ and set $J = [p, p+2m)$. Suppose first that

$$\inf_J d_j(z_0) \geqslant (Ke^{\beta})^{-|J|}. \tag{4}$$

By induction there is a good iterate $\nu \in [p, p+m)$ with $d_{\nu+j}(z_0) \geqslant \lambda_0^{j+1}$ for all $\nu \leqslant \nu + j < p + 2m$. Also, for $p + 2m \leqslant \nu + j < q$ we have $d_{\nu+j}(z_0) \geqslant (Ke^{\beta})^{-|I|} \geqslant (Ke^{\beta})^{-4m} \geqslant (Ke^{\beta})^{-4j} \geqslant \lambda_0^{j+1}$, as long as $\delta > 0$ is small enough to imply, say

$$\lambda_0 \leqslant (Ke^{\beta})^{-10}. \tag{5}$$

This proves the lemma for I when (4) holds. Now we consider the opposite case. Let the infimum be attained at $\bar{\nu}\in J$. Clearly, we may restrict to the case where I does not satisfy (3). Then

$$(Ke^{\beta})^{-|J|} \leqslant (Ke^{\beta})^{-|I|/2} \leqslant \sqrt{\lambda_0} \leqslant \tfrac{1}{2}\delta, \tag{6}$$

implying that $\bar{\nu}$ is a return. Its binding period satisfies $3|J|/4 \geqslant 3m/2$ (because (4) does not hold). Let $\bar{\mu}=\bar{\nu}+\bar{l}+1$ where $\bar{l}$ is the length of folding period of $\bar{\nu}$ and observe that, due to (1), $\bar{\mu}-\bar{\nu}\leqslant\text{const.}/\log(1/b)m+4\leqslant m/2$. We take $L=[\bar{\mu},\bar{\mu}+m)$. Then $L\subset[\bar{\nu}+1,\bar{\nu}+\bar{p}]$ and so $d_j(z_0)\geqslant e^{-\alpha(j-\bar{\nu})}/2\geqslant(Ke^{\beta})^{-|L|}$ for all $j\in L$ (at least if $\alpha>0$ is small enough). Again by induction, there is a good $\nu\in[\bar{\mu},\bar{\mu}+m/2)$ with

$$d_{\nu+j}(z_0) \geqslant \lambda_0^{j+1} \quad \text{for } \nu \leqslant \nu+j < \bar{\mu}+m \tag{7}$$

and now one checks as before that (7) holds for all $\nu\leqslant\nu+j<q$. □

Properties (b) and (c) in the lemma have the important consequence that z_{n-m_i} is expanding up to time m_i. To show this we use Lemma 8.4 as follows. Given any $1\leqslant j\leqslant m_i$ we take, as in Lemma 8.2, a fold-free iterate $k\geqslant j$ with

$$k-j \leqslant \frac{20\log K \log \lambda_0^{-1}}{\log(1/b)} j+4.$$

Then $\|w_{n-m_i+j-1}(z_1)\|\geqslant K^{j-k}\|w_{n-m_i+k-1}(z_1)\|=K^{j-k}\|\omega_{n-m_i+k-1}(z_1)\|$ and so, by Lemma 8.4, $\|w_{n-m_i+j-1}(z_1)\|\geqslant\frac{1}{2}(\lambda_0/K)^4\lambda_0^j\|w_{n-m_i-1}(z_1)\|$ if b is small enough. Since $w_{n-m_i-1}(z_1)=\omega_{n-m_i-1}(z_1)$ is nearly horizontal (Lemma 8.1) we get from Lemma 6.2

$$\|w_j(z_{n-m_i})\| \geqslant \frac{1}{4}\left(\frac{\lambda_0}{K}\right)^4 \lambda_0^j \geqslant \left(\frac{\lambda_0}{K}\right)^{5j} \quad \text{for all } 1\leqslant j\leqslant m_i. \tag{8}$$

This means that we are in a position to apply the procedure of §7C, with $\xi=z_{n-m_i}$, $m=m_i$ and $\lambda=(\lambda_0/K)^5$, to obtain a segment Γ^{m_i} of the $e^{(m_i)}$-orbit of z_{n-m_i} cutting G_1, see Figure 14. We consider the point $\eta_0^{[i]}=\Gamma^{m_i}\cap G_1$ and the segment $\gamma_0^{[i]}=\gamma(\eta_0^{[i]},\varrho_0^{m_i})\subset G_1$, where

$$\varrho_0 = \left(\frac{(\lambda_0/K)^5}{10K^2}\right)^2 \tag{9}$$

and denote also $\eta^{[i]}=\varphi_a^{m_i}(\eta_0^{[i]})$ and $\gamma^{[i]}=\varphi_a^{m_i}(\gamma_0^{[i]})$. Observe that $\gamma_0^{[i]}$ is $C^2(b)$: using the fact that $\eta^{[i]}$ is close to $|x|=0$ one concludes in a fairly easy way that if $\eta_0^{[i]}$ is near $|x|=2\delta$ or $|x|=1-2\delta_0$ then m_i must be large, so that $\gamma_0^{[i]}$ is always contained in $\delta<|x|<1-\delta_0$.

LEMMA 9.3. *$\gamma^{[i]}$ is a $C^2(b)$ curve for all $1\leqslant i\leqslant s$.*

For the proof of this we need

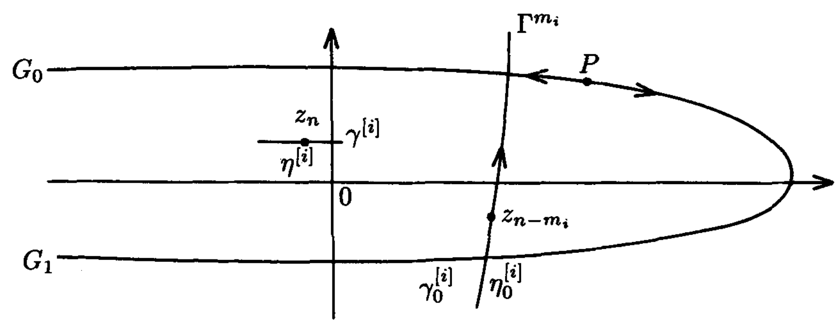

Fig. 14

LEMMA 9.4. *If $\nu \leqslant n$ is a free iterate for z_0 then*

$$\|w_{\nu-1}(\xi_1)\| \geqslant K^{-5} e^{k/10} \|w_{\nu-k-1}(\xi_1)\|$$

for every ξ_0 bound to z_0 and $1 \leqslant k \leqslant \nu-1$.

Proof. Analogous to Lemma 7.13 of [BC2]. □

Proof of Lemma 9.3. Fix $1 \leqslant i \leqslant s$. We write simply $m = m_i, \eta = \eta^{[i]}, \gamma = \gamma^{[i]}$, etc. Since $w_{n-m-1}(z_1) = \omega_{n-m-1}(z_1)$ and $t(\gamma_0; \eta_0)$ are both nearly horizontal, we have by (7.18)

$$|\text{angle}(w_{n-1}(z_1), t(\gamma; \eta))| \leqslant (K b^{t/2})^{m+1} \leqslant \text{const.}\, b^t. \tag{10}$$

On the other hand $w_{n-1}(z_1) = \omega_{n-1}(z_1)$ is also nearly horizontal, giving

$$|\operatorname{slope}(t(\gamma; \eta))| \leqslant \text{const.}\, b^t \ll b^{t/2}.$$

This reduces the proof of the lemma to showing that the curvature of γ satisfies $k(\gamma) \ll b^{t/2}$. We denote $\gamma_j = \varphi_a^j(\gamma_0)$, where γ_0 is parametrized by arc-length with $\gamma_0(0) = \eta_0$. Clearly,

$$\dot\gamma_{j+1} = D\varphi_a \cdot \dot\gamma_j \quad \text{and} \quad \ddot\gamma_{j+1} = D\varphi_a \cdot \ddot\gamma_j + D^2\varphi_a \cdot (\dot\gamma_j, \dot\gamma_j).$$

Using $k(\gamma_j) = |\det(\dot\gamma_j, \ddot\gamma_j)| / \|\dot\gamma_j\|^3$ we get the relation $k(\gamma_{j+1}) \leqslant K_j(|\det D\varphi_a| k(\gamma_j) + L_j)$ where

$$K_j = \left(\frac{\|\dot\gamma_j\|}{\|\dot\gamma_{j+1}\|}\right)^3 \quad \text{and} \quad L_j = |\det(D\varphi_a \cdot t_j, D^2\varphi_a \cdot (t_j, t_j))|, \quad t_j = \frac{\dot\gamma_j}{\|\dot\gamma_j\|}.$$

It follows that

$$k(\gamma = \gamma_m) \leqslant (Kb)^m K_{m-1} \cdot \ldots \cdot K_0 k(\gamma_0) + \sum_0^{m-1} (Kb)^{m-1-j} K_{m-1} \cdot \ldots \cdot K_j L_j. \tag{11}$$

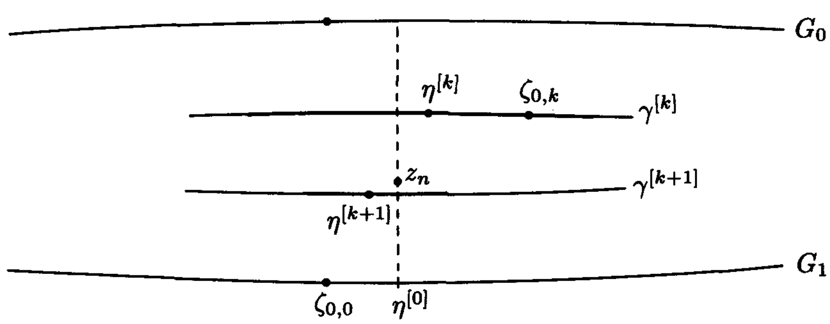

Fig. 15

The properties in Theorem 2.1 give

$$L_j \leqslant 8K^2\sqrt{b} \quad \text{for all } j. \tag{12}$$

By Lemma 6.3 and (7.17)

$$\|\dot\gamma_j\| \approx \|\dot\gamma_j(0)\| \approx \frac{\|w_{n-m+j-1}(z_1)\|}{\|w_{n-m-1}(z_1)\|} \quad \text{for every } 1 \leqslant j \leqslant m$$

and so, using also Lemma 9.4,

$$\frac{\|\dot\gamma_j\|}{\|\dot\gamma_m\|} \approx \frac{\|w_{n-m+j-1}(z_1)\|}{\|w_{n-1}(z_1)\|} \leqslant \text{const.}\, e^{-(m-j)/10}.$$

Hence,

$$K_{m-1}\cdot\ldots\cdot K_j = \left(\frac{\|\dot\gamma_j\|}{\|\dot\gamma_m\|}\right)^3 \leqslant \text{const.}\, e^{-3(m-j)/10}. \tag{13}$$

Replacing in (11) we obtain $k(\gamma) \leqslant \text{const.}\sqrt{b}$ if b is small enough. □

Observe moreover that

$$\|\dot\gamma^{[i]}\| \geqslant 1 \quad \text{for all } 1 \leqslant i \leqslant s. \tag{14}$$

This follows from $\|\dot\gamma^{[i]}\| \approx \|w_{n-1}(z_1)\|/\|w_{n-m_i-1}(z_1)\|$, invoking either Lemma 9.4 (if m_i is large) or Lemma 7.1 (otherwise). In particular, $\gamma^{[i]} \supset \gamma(\eta^{[i]}, \varrho_0^{m_i}) \supset \gamma(\eta^{[i]}, 5\varrho_0^{g_i})$, where $g_i = 1+m_i$. Clearly, $\gamma^{[i]} \subset G_{g_i}$ and $g_{i+1} \leqslant 3g_i$ and this remains true for $i=0$ if we denote $\eta^0 = G_1 \cap \{x = x_n\}, \gamma^{[0]} = \gamma(\eta^{[0]}, \frac{1}{2})$.

We are now in position to exihibit the binding point of z_n. Observe that, up to taking δ and b small enough, we may assume that $\gamma(\eta^{[0]}, \frac{1}{10})$ contains the critical approximations $w_0^{(i-1)} \in G_1 \cap \mathcal{C}_i$ constructed in §7B.

Definition. Let $k \geqslant 0$ be maximum such that $\gamma(\eta^{[k]}, \varrho_0^{g_k})$, respectively $\gamma(\eta^{[0]}, \frac{1}{10})$ if $k=0$, contains some element $\zeta_{0,k}$ of $\mathcal{C}_n$, see Figure 15. Then the binding point for z_n is $\zeta_0 = \zeta_{0,k}$.

We restrict from now on to values of the parameter for which this construction yields

$$d_n(z_0) \equiv |z_n - \zeta_0| \geqslant e^{-\alpha n}. \tag{BA}$$

This means that parameter exclusions are made and these are analysed in Section 11 and 12. For the time being we assume that (BA) holds and prove that $(z_n, \omega_{n-1}(z_1))$ is in tangential position to $\gamma^{[k]}$.

LEMMA 9.5. *Let $\eta = \eta^{[k]}$ and $\gamma = \gamma^{[k]}$. Then*

$$|z_n - \eta| \leqslant b^{3t/5} d_n(z_0) \quad \textit{and} \quad |\mathrm{angle}(\omega_{n-1}(z_1), t(\gamma;\eta))| \leqslant b^{3t/5} d_n(z_0).$$

As a consequence, there is a $C^2(b)$ curve $\tilde{\gamma}$ containing ζ_0 and z_n, tangent to γ at ζ_0 and to $\omega_{n-1}(z_1)$ at z_n.

Proof. Observe that by construction,

$$|\eta - z_n| \leqslant b^{3t/5} \cdot b^{tg_k/5} \tag{15}$$

and

$$|\mathrm{angle}(\omega_{n-1}(z_1), t(\gamma, \eta))| \leqslant b^{3t/5} \cdot b^{tg_k/5}. \tag{16}$$

In fact, (15) is a direct consequence of (7.16) $|\eta - z_n| \leqslant (5Kb^t)^{g_k}$ and the deduction of (16) is only slightly more complicated: for $g_k > 1$ it follows from (7.18)

$$|\mathrm{angle}(\omega_{n-1}(z_1), t(\gamma, \eta))| \leqslant (Kb^{t/2})^{g_k};$$

for $g_k = 1$ we use ((7.2) and Lemma 6.3)

$$|\mathrm{angle}(\omega_{n-1}(z_1), t(G_1, \eta^{[0]}))| \leqslant |\,\mathrm{slope}(\omega_{n-1}(z_1))| + |\,\mathrm{slope}(t(G_1, \eta^{[0]}))| \leqslant \mathrm{const.}\, b^t.$$

Let us consider first the case $g_k \leqslant \theta n/3$. We claim that

$$d_n(z_0) \geqslant \varrho_0^{5g_k}, \tag{C}$$

which, in view of (15) and (16), immediately implies the lemma. We prove (C) by contradiction: assuming that it does not hold we construct (starting with ζ_0) an $(n-1)$st critical approximation $\bar{\zeta}_0 \in \gamma(\eta^{[k+1]}, \varrho_0^{g_{k+1}})$ which we show to belong to $\mathcal{C}_n$; since $g_{k+1} \leqslant \theta n$ this contradicts the maximality of k. The details of this argument require the precise definition of θ which we now state:

$$\theta = \frac{10R\log(1/\sigma_0)}{t\log(1/b)}, \quad R > 1 \text{ to be given in (10.11).} \tag{17}$$

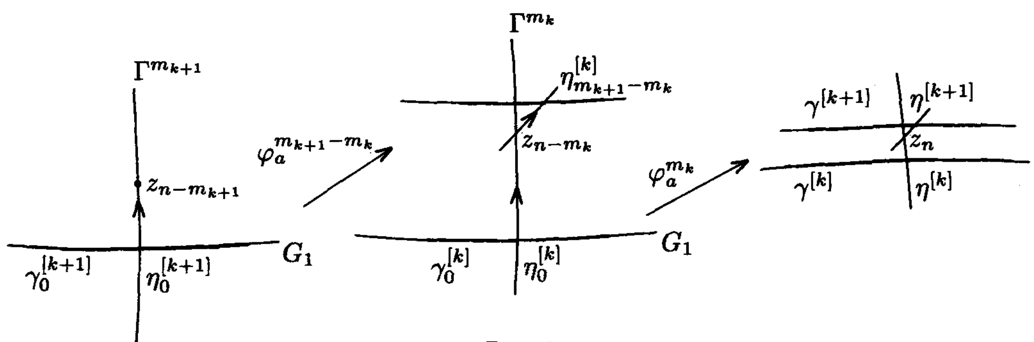

Fig. 16

Also, before describing these details, let us observe that in the case $g_k \geqslant \theta n/3$ the lemma is a simple consequence of the (BA) (and (15)–(17)):

$$\frac{t}{5} g_k \log b \leqslant n\theta \frac{t}{15} \log b \leqslant -\alpha n.$$

Hence, the lemma will be proved once we have completed the

Proof of the claim (C). Let $1 \leqslant \mu \leqslant n-1$ be such that $\theta\mu < g_{k+1} \leqslant \theta(\mu+1)$ and $\zeta_0 = \zeta_0^{(n-1)}, \zeta_0^{(n-2)}, ..., \zeta_0^{(\mu)}$ be the sequence of critical approximations obtained from ζ_0 by reversing the algorithm of §7A. Notice that for every $\mu \leqslant i \leqslant n-1$

$$\begin{aligned} |\zeta_0^{(i)} - \eta^{[k]}| &\leqslant |\zeta_0^{(i)} - \zeta_0| + |\zeta_0 - z_n| + |z_n - \eta^{[k]}| \\ &\leqslant (Kb)^i + \varrho_0^{5g_k} g_k + b^{tg_k/5} \leqslant 2\varrho_0^{5g_k} \end{aligned}$$

and $\zeta_0^{(i)} \in \mathcal{C}_{i+1}$ (because $\zeta_0 \in \mathcal{C}_n$ and $g_k \leqslant \theta\mu \leqslant \theta i$). We apply the algorithm of §7B to $\tilde{z}_0 = \zeta_0^{(\mu)}$ to find a μth critical approximation $\bar{\zeta}_0^{(\mu)} \in \bar{\gamma} = \gamma(\eta^{[k+1]}, \varrho_0^{g_{k+1}})$. The crucial estimate here is

$$d = \operatorname{dist}(\tilde{z}_0, \bar{\gamma}) \leqslant (5Kb^t)^{g_k}, \tag{18}$$

which is a consequence of the construction of §7C, see Figure 16. Together with the definition of θ this implies (7.13):

$$K^2(5Kb^t)^{g_k} \leqslant b^{tg_{k+1}/5} \leqslant b^{t\theta\mu/5} \leqslant \sigma_0^{2\mu}.$$

Moreover, we may take $l = \varrho_0^{g_{k+1}}$ and then (7.14) holds (for b small enough). Hence, there is indeed a critical approximation $\bar{\zeta}_0^{(\mu)}$ in $\gamma^{[k+1]}$. It is not difficult to check that $\bar{\zeta}_0^{(\mu)}$ belongs to $\mathcal{C}_{\mu+1}$: notice in particular that $\zeta_0^{(\mu)} \in \mathcal{C}'_{\mu+1}$ and, by (7.15),

$$|\bar{\zeta}_0^{(\mu)} - \zeta_0^{(\mu)}| \leqslant d + \tfrac{1}{4}\sqrt{d} \leqslant b^{tg_{k+1}/10}. \tag{19}$$

Then $\gamma^{[k+1]}$ contains also an element $\bar{\zeta}_0^{(n-1)}$ of $\mathcal{C}_n$ (obtained by increasing the precision of $\bar{\zeta}_0^{(\mu)}$) and actually, by (7.11), (15) and (19),

$$|\bar{\zeta}_0^{(n-1)}-\eta^{[k+1]}|\leqslant 2(Kb)^{\mu}+b^{tg_{k+1}/10}+\varrho_0^{5g_k}+2b^{tg_k/5}<\varrho_0^{g_{k+1}},$$

contradicting the choice of k. □

Now we are in position to formalize the heuristics of Section 5 to estimate the loss on the norm of the ω-vectors on returns.

LEMMA 9.6. *Let n be a free return for $z_0\in\mathcal{C}_n$. Then*

$$|\alpha_n(z_1)|\leqslant 4K\sqrt{b}\,\|\omega_{n-1}(z_1)\| \quad \textit{and} \quad \frac{3a}{2}d_n(z_0)\leqslant\frac{|\beta_n(z_1)|}{\|\omega_{n-1}(z_1)\|}\leqslant\frac{5a}{2}d_n(z_0).$$

Proof. Let $s\mapsto z(s)=(s+x_0,y(s))$ parametrize the $C^2(b)$ curve $\tilde{\gamma}$ of Lemma 9.5, with $\zeta_0=z(0)$. We split the tangent vector to $\varphi_a(\tilde{\gamma})$, $t(s)=\alpha_t(s)e(\varphi_a(z(s)))+\beta_t(s)(1,0)$, with $e=e^{(l)}=(q,1)$ colinear to the lth contractive direction. This gives

$$\alpha_t=C(z)+D(z)\dot{y} \quad \text{and} \quad \beta_t=A(z)+B(z)\dot{y}-\alpha_t q. \tag{20}$$

Since $\tilde{\gamma}$ is $C^2(b)$

$$|\alpha_t|,|\dot{\alpha}_t|\leqslant 2K\sqrt{b} \tag{21}$$

and so, using also Lemma 6.6, $|\dot{\beta}_t-A'(z)\cdot(1,\dot{y})|\leqslant 2K\sqrt{b}$. From $A'(z)\approx(-2a,0)$ (Theorem 2.1) we get $|\dot{\beta}_t+2a|\leqslant 4K\sqrt{b}$, leading to

$$\big(2a-4K\sqrt{b}\big)|s|\leqslant|\beta_t(s)-\beta_t(0)|\leqslant\big(2a+4K\sqrt{b}\big)|s|. \tag{22}$$

Since ζ_0 is an $(n-1)$st approximation, Lemma 6.1(a) gives

$$|\mathrm{angle}(t(0),e^{(l)}(\zeta_1))|\leqslant 4K(Kb)^l,$$

which implies (recall (8.11))

$$|\beta_t(0)|\leqslant 5K(Kb)^l\|t(0)\|\leqslant 10K(Kb)^l\leqslant\sqrt{b}\,d_n(z_0). \tag{23}$$

Let now $z_n=z(\sigma)$. Then, $\tilde{\gamma}$ being $C^2(b)$,

$$|\sigma|\leqslant d_n(z_0)\leqslant(1+b^{t/2})|\sigma| \tag{24}$$

and

$$\frac{\omega_{n-1}(z_1)}{\|\omega_{n-1}(z_1)\|}=\lambda t(\sigma), \quad \text{with } (1-b^{t/2})\leqslant|\lambda|\leqslant 1. \tag{25}$$

Denoting $\widehat{\alpha}_n=\alpha_n/\|\omega_{n-1}\|$ and $\widehat{\beta}_n=\beta_n/\|\omega_{n-1}\|$, it follows

$$\begin{aligned}
&|\widehat{\alpha}_n(z_1)|\leqslant|\alpha_t(\sigma)|\leqslant 2K\sqrt{b},\\
&|\widehat{\beta}_n(z_1)|\leqslant|\beta_t(\sigma)|\leqslant\left(2a+5K\sqrt{b}\right)d_n(z_0)\quad\text{and}\\
&|\widehat{\beta}_n(z_1)|\geqslant(1-b^{t/2})|\beta_t(\sigma)|\geqslant\frac{1-b^{t/2}}{1+b^{t/2}}\left(2a-5K\sqrt{b}\right)d_n(z_0).
\end{aligned}\qquad\square$$

We also need to show that the lemma holds for any point ξ_0 bound to z_0. This is easier to do using Lemma 10.2, so we postpone it to Section 10 (Corollary 10.4).

Finally, Lemma 9.6 is also (essentially) true when n is a bound return.

LEMMA 9.7. *Let n be a bound return for $z_0\in C_n$. Then*

$$|\widehat{\alpha}_n(\xi_1)|\leqslant 5K\sqrt{b}\quad\textit{and}\quad ad_n(\xi_1)\leqslant|\widehat{\beta}_n(\xi_1)|\leqslant 3ad_n(\xi_1)$$

for every ξ_0 bound to z_0 up to time n.

Proof. Take $k\geqslant 1$ minimum such that $\nu=n-k$ is a return for ξ_0 and its binding period contains n. Then k is a free return for the binding point $\tilde{\zeta}_0$ of ξ_ν and ξ_ν is bound to $\tilde{\zeta}_0$ up to time k. By induction

$$|\widehat{\alpha}_k(\xi_{\nu+1})|\leqslant 4K\sqrt{b}\quad\text{and}\quad\frac{3a}{2}d_k(\xi_\nu)\leqslant|\widehat{\beta}_k(\xi_{\nu+1})|\leqslant\frac{5a}{2}d_k(\xi_\nu).$$

If there are no folding periods $[\mu+1,\mu+l]$ with $\mu\leqslant\nu<\mu+l<n$ then

$$\omega_{n-1}(\xi_1)=\beta_\nu(\xi_1)\omega_{k-1}(\xi_{\nu+1}),$$

implying

$$\widehat{\alpha}_n(\xi_1)=\widehat{\alpha}_k(\xi_{\nu+1})\quad\text{and}\quad\widehat{\beta}_n(\xi_1)=\widehat{\beta}_k(\xi_{\nu+1}).$$

Suppose now that there are such folding periods, corresponding to returns $\mu_1<\mu_2<\ldots<\mu_s=\nu$. Let first $\mu=\mu_1$. Then

$$\omega_{n-1}(\xi_1)=\beta_\mu(\xi_1)\omega_{n-\mu-1}(\xi_{\mu+1})+\alpha_\mu(\xi_1)D\varphi_a^{n-\mu-1}e(\xi_{\mu+1}).$$

By induction

$$\begin{aligned}
\|\alpha_\mu(\xi_1)D\varphi_a^{n-\mu-1}\cdot e(\xi_{\mu+1})\|&\leqslant 5K\sqrt{b}\,\|\omega_{\mu-1}(\xi_1)\|\cdot 8K(K^2b)^lK^{n-\mu-l-1}\\
&\leqslant\text{const.}\sqrt{b}\,\|\omega_{\mu-1}(\xi_1)\|d_\mu(\xi_0)^2\left(\sqrt{b}\right)^lK^k,
\end{aligned}$$

while

$$\|\beta_\mu(\xi_1)\omega_{n-\mu-1}(\xi_{\mu+1})\|\geqslant ad_\mu(\xi_0)\|\omega_{\mu-1}(\xi_1)\|e^{c_1(n-\mu-1)}.$$

It follows that $|\text{angle}(\omega_{n-1}(\xi_1), \omega_{n-\mu-1}(\xi_{\mu+1}))| \leqslant \text{const.}\sqrt{b}K^k(\sqrt{b})^l$. Similar estimates for the contribution of each $[\mu_i+1, \mu_i+l_i]$ are obtained in precisely the same way: just replace above ξ_0 and $n-\mu$ by $\xi_{\mu_{i-1}}$ and $n-\mu_i$, respectively. Summing all this,

$$|\text{angle}(\omega_{n-1}(\xi_1), \omega_{k-1}(\xi_{\nu+1}))| \leqslant \text{const.}\sqrt{b}K^k(\sqrt{b})^l,$$

where now $l=l_s$ is the length of the folding period associated to ν. Note that $d_n(\xi_0) \approx d_k(\tilde{\zeta}_0) \geqslant e^{-\alpha k}$. On the other hand, $k \leqslant 5\log d_\nu(\xi_0)^{-1}$ (inductive assumption on the length of binding periods, recall Section 8) and the definition of l imply $K^k(\sqrt{b})^l \leqslant e^{-\alpha k}$. Therefore, $|\text{angle}(\omega_{n-1}(\xi_1), \omega_{k-1}(\xi_{\nu+1}))| \leqslant \text{const.}\sqrt{b}\, d_n(\xi_0)$. From this one gets, easily,

$$|\widehat{\alpha}_n(\xi_1) - \widehat{\alpha}_k(\xi_{\nu+1})| \leqslant \text{const.}\, b d_n(\xi_0) \leqslant K\sqrt{b}$$

and

$$|\widehat{\beta}_n(\xi_1) - \widehat{\beta}_k(\xi_{\nu+1})| \leqslant \text{const.}\, b d_n(\xi_0) \leqslant \tfrac{1}{2} a d_n(\xi_0)$$

□

10. The binding period

Let again n be a return for $z_0 \in \mathcal{C}_n$. In this section we show that $[n+1, n+p]$ satisfies the inductive properties of binding periods, recall Section 8. The global strategy is that of §§7.2, 7.3 in [BC2] but we manage to use

$$\Theta_\nu = \Theta_\nu(\eta_0, \zeta_0) = \sum_1^\nu (\sqrt[4]{b})^{\nu-s} |\eta_s - \zeta_s|$$

(instead of $\Delta_\nu = \max_{1\leqslant s\leqslant \nu} |\eta_s - \zeta_s|$, see remark after Lemma 7.8 in [BC2]) to express the estimates corresponding to time ν in terms of iterates previous to ν. This permits to obtain the property of bounded distorsion on binding periods (Corollary 10.3) as an immediate consequence and allows us to give a more direct and, we hope, more transparent form to the argument. Moreover, segments of the proof in [BC2] (e.g. the free iteration estimates, Lemma 7.9) had to be replaced by more general arguments, independent of the precise form of the family.

First we state an elementary result to be used in the sequel.

LEMMA 10.1. *Let $\xi > 0$ and $v_1, v_2, \varepsilon_1, \varepsilon_2$ be such that $\|v_i + \varepsilon_i\| \geqslant \xi \|v_i\|$ for $i=1,2$. Then*

(a) $\|v_1+\varepsilon_1\| / \|v_2+\varepsilon_2\| \leqslant (\|v_1\|/\|v_2\|)(1+\chi/\xi)$;

(b) $|\text{angle}(v_1+\varepsilon_1, v_2+\varepsilon_2)| \leqslant |\text{angle}(v_1, v_2)| + 2\chi/\xi$,

where

$$\chi = |\text{angle}(v_1, v_2)| \cdot \frac{\|\varepsilon_1\|}{\|v_1\|} + \frac{\|\varepsilon_1 - \varepsilon_2\|}{\|v_1\|} + \left|\frac{\|v_2\|}{\|v_1\|} - 1\right| \cdot \frac{\|\varepsilon_2\|}{\|v_2\|}.$$

LEMMA 10.2. *Let η_0, ζ_0 be bound up to time $q \leqslant k$ to a same element of C_s, $s \leqslant n$. Then, for all $0 \leqslant \nu \leqslant q$,*

$$\frac{\|\omega_\nu(\eta_1)\|}{\|\omega_\nu(\zeta_1)\|} \leqslant \exp\left(8K\sum_1^\nu \frac{\Theta_k}{d_k(\zeta_0)}\right) \quad \textit{and} \quad |\mathrm{angle}(\omega_\nu(\eta_1), \omega_\nu(\zeta_1))| \leqslant 2\sqrt[4]{b}\,\Theta_\nu.$$

Proof. We structure the argument in a way quite similar to that of the proof of Lemma 8.1. The lemma is contained in the following claims which are proved by induction on ν:

$$\frac{\|\omega_\nu(\eta_1)\|}{\|\omega_\nu(\zeta_1)\|} \leqslant \exp\left(4K\sum_0^\nu (\sqrt[4]{b})^i \cdot \sum_1^\nu \frac{\Theta_k}{d_k(\zeta_0)}\right) \leqslant \exp\left(8K\sum_1^\nu \frac{\Theta_k}{d_k(\zeta_0)}\right) \tag{1a}$$

$$|\mathrm{angle}(\omega_\nu(\eta_1), \omega_\nu(\zeta_1))| \leqslant \sum_1^\nu (\sqrt[4]{b})^i \cdot \Theta_\nu \leqslant 2\sqrt[4]{b}\,\Theta_\nu. \tag{1b}$$

Note that this holds, trivially, when $\nu=0$. Let $\nu \geqslant 1$ and assume that (1a), (1b) have been obtained for every iterate $\mu \leqslant \min\{k, \nu-1\}$ of each pair of points bound to a same element of C_k, $k \leqslant n$. Suppose first that ν is a return. We write, as in the proof of Lemma 9.6,

$$\frac{\tilde{\omega}_\nu(\eta_1)}{\|\omega_{\nu-1}(\eta_1)\|} = \hat{\alpha}_\nu(\eta_1)(q,1)(\eta_{\nu+1}) + \hat{\beta}_\nu(\eta_1)\cdot(1,0)$$

and analogously for ζ_1. Then we have $\hat{\alpha}_\nu = Cu + Dv$ and $\hat{\beta}_\nu = Au + Bv - \hat{\alpha}_\nu q$, where $\omega_{n-1} = \|\omega_{\nu-1}\|(u,v)$. It follows that

$$|\hat{\alpha}_\nu(\eta_1) - \hat{\alpha}_\nu(\zeta_1)| \leqslant 2K\sqrt{b}\left(|\eta_\nu - \zeta_\nu| + 2\sqrt[4]{b}\,\Theta_{\nu-1}\right) \leqslant 4K\sqrt{b}\,\Theta_\nu. \tag{2}$$

Analogously, using Lemma 6.6 $|\hat{\beta}_\nu(\eta_1) - \hat{\beta}_\nu(\zeta_1)| \leqslant 2K(|\eta_\nu - \zeta_\nu| + 2\sqrt[4]{b}\,\Theta_{\nu-1})$, implying

$$|\hat{\beta}_\nu(\eta_1) - \hat{\beta}_\nu(\zeta_1)| \leqslant 4K\frac{\Theta_\nu}{d_\nu(\zeta_0)} \cdot |\hat{\beta}_\nu(\zeta_1)|. \tag{3}$$

Clearly, this gives

$$\begin{aligned}\frac{\|\omega_\nu(\zeta_1)\|}{\|\omega_\nu(\eta_1)\|} &\leqslant \exp\left(4K\sum_0^{\nu-1} (\sqrt[4]{b})^i \cdot \sum_1^{\nu-1} \frac{\Theta_k}{d_k(\zeta_0)}\right) \cdot \left(1 + 4K\frac{\Theta_\nu}{d_\nu(\zeta_0)}\right) \\ &\leqslant \exp\left(4K\sum_0^\nu (\sqrt[4]{b})^i \cdot \sum_1^\nu \frac{\Theta_k}{d_k(\zeta_0)}\right)\end{aligned}$$

proving (1a). On the other hand (1b) is trivial on returns.

Suppose now that ν is neither a return nor the end of a folding period. Again we write $\omega_{\nu-1}=\|\omega_{\nu-1}\|(u,v)$, with (Lemma 8.1) $|v|\leqslant b^t|u|$ and so $|u|\approx 1$. Then (1b) is an immediate consequence of

$$|\text{angle}(\omega_\nu(\eta_1),\omega_\nu(\zeta_1))|\leqslant|\text{slope}(\omega_\nu(\eta_1))-\text{slope}(\omega_\nu(\zeta_1))|$$
$$\leqslant\frac{1}{\delta^2}(4\sqrt{b}K^2|\eta_\nu-\zeta_\nu|+4K^2b\cdot 2\sqrt[4]{b}\,\Theta_{\nu-1})\leqslant\sqrt[4]{b}\,\Theta_\nu.$$

Moreover,

$$\|(Au+Bv,Cu+Dv)(\eta_1)-(Au+Bv,Cu+Dv)(\zeta_1)\|\leqslant 4K\Theta_\nu,$$

while $\|(Au+Bv,Cu+Dv)(\zeta_1)\|\geqslant|x_\nu|=d_\nu(\zeta_0)$. It follows that

$$\frac{\|\omega_\nu(\eta_1)\|}{\|\omega_\nu(\zeta_1)\|}\leqslant\exp\left(4K\sum_0^{\nu-1}(\sqrt[4]{b})^i\cdot\sum_1^{\nu-1}\frac{\Theta_k}{d_k(\zeta_0)}\right)\left(1+4K\frac{\Theta_\nu}{d_\nu(\zeta_0)}\right)$$

and (1a) is proved in this case.

Finally, let there be some folding period $[\mu+1,\mu+l]$ ending at time ν. Take such μ minimum. Then, as we remarked in the proof of Lemma 8.1,

$$\omega_\nu(\eta_1)=\beta_\mu(\eta_1)\omega_l(\eta_{\mu+1})+\alpha_\mu(\eta_1)D\varphi_a^l\cdot e(\eta_{\mu+1})=\|\omega_{\mu-1}(\eta_1)\|(v(\eta_1)+\varepsilon(\eta_1)),$$

$$v(\eta_1)=\hat{\beta}_\mu(\eta_1)\omega_l(\eta_{\mu+1})\quad\text{and}\quad\varepsilon(\eta_1)=\hat{\alpha}_\mu(\eta_1)D\varphi_a^l\cdot e(\eta_{\mu+1}).$$

We introduce analogous notations for ζ_1 and want to apply Lemma 10.1. By induction (1a) and (1b) hold for $\omega_l(\zeta_{\mu+1})$ and $\omega_l(\eta_{\mu+1})$. Clearly, we also have the right to assume (2) and (3) to hold for every return previous to ν, in particular for μ. Then (recalling also $d_j(\zeta_0)\leqslant 2d_j(\eta_0)$)

$$\frac{\|v(\zeta_1)\|}{\|v(\eta_1)\|}\leqslant\left(1+8K\frac{\Theta_\mu}{d_\mu(\zeta_0)}\right)\exp\left(16K\sum_1^l\frac{\Theta_k'}{d_k(\zeta_\mu)}\right)\tag{4}$$

where $\Theta_k'=\Theta_k(\eta_\mu,\zeta_\mu)=\sum_1^k(\sqrt[4]{b})^{k-i}|\eta_{\mu+i}-\zeta_{\mu+i}|$. Observe now that

$$16K\sum_{k=1}^l\frac{\Theta_k'}{d_k(\zeta_\mu)}\leqslant 80K\sum_{k=1}^l e^{(\alpha-\beta)k}\sum_{s=1}^k(\sqrt[4]{b}\,e^\beta)^{k-s}$$
$$\leqslant 100K\sum_{k=1}^\infty e^{(\alpha-\beta)k}.$$

Hence, from (4) one gets

$$\frac{\|v(\zeta_1)\|}{\|v(\eta_1)\|} \leqslant 1+8K\frac{\Theta_\mu}{d_\mu(\zeta_0)}+\tau'\sum_1^l \frac{\Theta'_k}{d_k(\zeta_\mu)}, \tag{5}$$

with $\tau'>0$ depending only on K,α,β (and not on b). On the other hand, using Lemma 6.8,

$$\begin{aligned}\|\varepsilon(\eta_1)-\varepsilon(\zeta_1)\| &\leqslant 4K\sqrt{b}\,\Theta_\mu\cdot 4K(K^2b)^l+5K\sqrt{b}\cdot 2(K_1b)^{l-3}|\eta_\mu-\zeta_\mu| \\ &\leqslant \text{const.}\big(\sqrt{b}\big)^{l+1}d_\mu(\zeta_0)^2\Theta_\mu\end{aligned}$$

and so

$$\frac{\|\varepsilon(\eta_1)-\varepsilon(\zeta_1)\|}{\|v(\eta_1)\|} \leqslant \text{const.}\big(\sqrt{b}\big)^{l+1}d_\mu(\zeta_0)\Theta_\mu. \tag{6}$$

Again by induction

$$|\text{angle}(v(\eta_1),v(\zeta_1))| \leqslant 2\sqrt[4]{b}\,\Theta'_l. \tag{7}$$

Finally, (recall (8.16) for instance)

$$\frac{\|\varepsilon(\eta_1)\|}{\|v(\eta_1)\|} \leqslant \text{const.}\big(\sqrt{b}\big)^{l+1}d_\mu(\zeta_0)\ \left(<\tfrac{1}{2}\right) \tag{8}$$

and analogously for ζ_1. Replacing (5)–(8) we get

$$4\chi \leqslant \text{const.}\big(\sqrt{b}\big)^{l+1}\Theta_\mu+\text{const.}\big(\sqrt{b}\big)^{l+1}d_\mu(\zeta_0)\sum_{k=1}^l \frac{\Theta'_k}{d_k(\zeta_\mu)}. \tag{9}$$

Notice that, due to the (BA) and the definition of l, $d_k(\zeta_\mu)\geqslant d_\mu(\zeta_0)$ for all $1\leqslant k\leqslant l$. Hence, the last term in (9) is bounded by

$$\text{const.}\big(\sqrt{b}\big)^{l+1}\sum_{k=1}^l \Theta'_k \leqslant \text{const.}\big(\sqrt{b}\big)^{l+1}\sum_{k=1}^l\sum_{j=1}^k\big(\sqrt[4]{b}\big)^{k-j}|\eta_{\mu+j}-\zeta_{\mu+j}| \leqslant \big(\sqrt[4]{b}\big)^{l+1}\Theta'_l$$

and this gives

$$4\chi \leqslant \big(\sqrt[4]{b}\big)^{l+1}\big(\big(\sqrt[4]{b}\big)^l\Theta_\mu+\Theta'_l\big) = \big(\sqrt[4]{b}\big)^{l+1}\Theta_\nu. \tag{10}$$

Now, from Lemma 10.1(b) and (10)

$$|\text{angle}(\omega_\nu(\eta_1),\omega_\nu(\zeta_1))| \leqslant \sum_{i=1}^l\big(\sqrt[4]{b}\big)^i\Theta'_l+\big(\sqrt[4]{b}\big)^{l+1}\Theta_\nu \leqslant \sum_{i=1}^{l+1}\big(\sqrt[4]{b}\big)^i\Theta_\nu.$$

and this proves (1b). On the other hand Lemma 10.1(a) gives

$$\begin{aligned}\log\frac{\|\omega_\nu(\eta_1)\|}{\|\omega_\nu(\zeta_1)\|} &\leqslant \log\left(\frac{\|\omega_{\mu-1}(\eta_1)\|}{\|\omega_{\mu-1}(\zeta_1)\|}\cdot\frac{\|v(\eta_1)\|}{\|v(\zeta_1)\|}\right)+2\chi \\ &\leqslant 4K\sum_0^{\mu-1}\left(\sqrt[4]{b}\right)^i\cdot\sum_1^{\mu-1}\frac{\Theta_k}{d_k(\zeta_0)}+4K\frac{\Theta_\mu}{d_\mu(\zeta_0)} \\ &\quad +4K\sum_0^{l}\left(\sqrt[4]{b}\right)^i\cdot\sum_1^{l}\frac{\Theta'_k}{d_k(\zeta_\mu)}+\left(\sqrt[4]{b}\right)^{l+1}\Theta_\nu.\end{aligned}$$

Clearly $\Theta'_k \leqslant \Theta_{\mu+k}$. We add the last terms and then (1a) follows immediately. □

Suppose now that n is a return for $z_0 \in C_n$. Let ζ_0 be the binding point for z_n and p denote the length of the binding period associated to this return. Notice that $p \leqslant 5\log d_n(z_0)^{-1}$ is not yet available at this stage.

COROLLARY 10.3. *For all* $0 \leqslant k \leqslant \min\{p, 5\log d_n(z_0)^{-1}\}$

$$\tau_1^{-1} \leqslant \frac{\|\omega_{n+k}(z_1)\|}{|\beta_n(z_1)|\,\|\omega_k(\zeta_1)\|} \leqslant \tau_1$$

with $\tau_1=\tau_1(K,\alpha,\beta)$. *Moreover the same holds for any* ξ_0 *that remains bound to* z_0 *up to* $n+k$.

Proof. Let $\eta_0=\xi_n$. The lemma gives (recall the deduction of (5))

$$\frac{\|\omega_k(\eta_1)\|}{\|\omega_k(\zeta_1)\|} \leqslant \exp\left(100K\sum_1^\infty e^{j(\alpha-\beta)}\right)=\tau. \tag{11}$$

On the other hand, $\omega_{n+\nu}(\xi_1)$ differs from $\beta_n(\xi_1)\omega_\nu(\eta_1)$ only by the recomposition of fold terms corresponding to folding periods $\mu \leqslant n < \mu+l \leqslant n+k$. The effect of such terms was analysed in the proof of Lemma 9.7 and in the present situation this gives

$$\|\omega_{n+k}(\xi_1)\| \leqslant |\beta_n(\xi_1)|\,\|\omega_k(\eta_1)\|\prod_1^s\left(1+\text{const.}\sqrt{b}\,K^k\left(\sqrt{b}\right)^{l_i}\right). \tag{12}$$

The definition of folding periods implies $K^k\left(\sqrt{b}\right)^{l_i} \leqslant K^k\left(\sqrt{b}\right)^{l_s}\left(\sqrt{b}\right)^{s-i} \leqslant \left(\sqrt{b}\right)^{s-i}$ and so the upper inequality follows if we take

$$\tau_1 = 2\tau \quad \left(\geqslant \tau\prod_0^\infty\left(1+\text{const.}\left(\sqrt{b}\right)^{i+1}\right)\right). \tag{13}$$

The lower bound is obtained in the same way. □

We take a pause in the deduction of the binding period estimates, in order to extend Lemma 9.6.

COROLLARY 10.4. *Let n be a free return for $z_0 \in \mathcal{C}_n$. Then*

$$|\widehat{\alpha}_n(\xi_1)| \leqslant 4K\sqrt{b} \quad \text{and} \quad \frac{3a}{2} \leqslant |\widehat{\beta}_n(\xi_1)| \leqslant \frac{5a}{2}$$

for every point ξ_0 bound to z_0 up to time n.

Proof. Lemma 10.2 gives

$$|\text{angle}(\omega_{n-1}(\xi_1), \omega_{n-1}(z_1))| \leqslant 2\sqrt[4]{b} \sum_1^{n-1} (\sqrt[4]{b})^{n-1-i} e^{-\beta i} \leqslant 10\sqrt[4]{b}\, e^{-\beta n}.$$

On the other hand $|\xi_n - z_n| \leqslant 2e^{-\beta n} \ll e^{-\alpha n}$ (because $n \geqslant N$). It follows that

$$|\xi_n - \eta| \ll d_n(\xi_0) \quad \text{and} \quad |\text{angle}(\omega_{n-1}(\xi_1), t(\gamma; \eta))| \ll d_n(\xi_0).$$

and so we may take a nearly flat and nearly horizontal curve $\tilde{\gamma}$ tangent to γ at ζ_0 and to $\omega_{n-1}(\xi_1)$ at ξ_n. Now the proof proceeds as that of Lemma 9.6. □

We also need to obtain estimates for the distorsion of the w-vectors on bound orbits. This is now fairly easy.

LEMMA 10.5. *Let η_0 and ζ_0 be bound up to time $q \leqslant s$ to a same element of $\mathcal{C}_s$, $s \leqslant n$. Then for all $0 \leqslant \nu \leqslant q$*

$$\frac{\|w_\nu(\eta_1)\|}{\|w_\nu(\zeta_1)\|} \leqslant \exp\left(8Ke^{(\varepsilon+\alpha)\nu} \sum_1^\nu \frac{\Theta_k}{d_k(\zeta_0)}\right)$$

and

$$|\text{angle}(w_\nu(\eta_1), w_\nu(\zeta_1))| \leqslant 4\sqrt[4]{b}\, e^{(\varepsilon+\alpha)\nu} \sum_1^\nu \frac{\Theta_k}{d_k(\zeta_0)}.$$

Proof. We write $w_\nu = \omega_\nu + \sigma_\nu$ and use Lemma 10.1. Recall first that (Lemma 8.3) $\|w_\nu\| \geqslant \text{const.}\, e^{-\varepsilon\nu} \|\omega_\nu\|$. For each term in $\sigma_\nu(\eta_1)$ we have

$$\|\alpha_\mu D\varphi_a^{\nu-\mu} \cdot e(\eta_{\nu+1})\| \leqslant \text{const.} \sqrt{b}\, |\beta_\mu(\eta_1)| d_\mu(\eta_0)^{-1} (K^2 b)^{\nu-\mu}.$$

On the other hand, Corollary 10.3 gives

$$\|\omega_\nu(\eta_1)\| \geqslant \frac{1}{\tau_1} |\beta_\mu(\eta_1)| \cdot \|\omega_{\nu-\mu}(\eta_{\mu+1})\| \geqslant \frac{1}{\tau_1} |\beta_\mu(\eta_1)|.$$

It follows that

$$\frac{\|\sigma_\nu(\eta_1)\|}{\|\omega_\nu(\eta_1)\|} \leqslant \sum_\mu \text{const.} \sqrt{b}\, (K^2 b)^{\nu-\mu} d_\mu(\eta_0)^{-1} \leqslant \sqrt[3]{b}\, e^{\alpha\nu}. \tag{14}$$

Clearly, the same holds for ζ_1. On the other hand, Lemma 10.2 implies

$$|\alpha_\mu(\eta_1)-\alpha_\mu(\zeta_1)| \leqslant \text{const.}\sqrt{b}\,\|\omega_{\mu-1}(\eta_1)\| \sum_1^\mu \frac{\Theta_k}{d_k(\zeta_0)}$$

where the constant depends only on K, α and β. Using also Lemma 6.8, this leads to

$$\|\alpha_\mu D\varphi_a^{\nu-\mu}\cdot e(\eta_{\mu+1})-\alpha_\mu D\varphi_a^{\nu-\mu}(\zeta_{\mu+1})\| \leqslant \text{const.}\sqrt{b}\,\|\omega_{\mu-1}(\eta_1)\| \sum_1^\mu \frac{\Theta_k}{d_k(\zeta_0)}.$$

Therefore

$$\frac{\|\sigma_\nu(\eta_1)-\sigma_\nu(\zeta_1)\|}{\|\omega_\nu(\eta_1)\|} \leqslant \sqrt[3]{b}\, e^{\alpha\nu} \sum_1^\nu \frac{\Theta_k}{d_k(\zeta_0)}. \tag{15}$$

From (14), (15) and the estimates in Lemma 10.2 we find

$$\chi \leqslant \text{const.}\sqrt[3]{b}\, e^{\alpha\nu} \sum_1^\nu \frac{\Theta_k}{d_k(\zeta_0)}.$$

Now Lemma 10.1 gives

$$\frac{\|w_\nu(\eta_1)\|}{\|w_\nu(\zeta_1)\|} \leqslant \frac{\|\omega_\nu(\eta_1)\|}{\|\omega_\nu(\zeta_1)\|} \exp(\text{const.}\, e^{\varepsilon\nu}\chi) \leqslant \exp\left((8K+\sqrt[4]{b}\, e^{(\varepsilon+\alpha)\nu}) \sum_1^\nu \frac{\Theta_k}{d_k(\zeta_0)}\right)$$

and $|\text{angle}(w_\nu(\eta_1), w_\nu(\zeta_1))| \leqslant 2\sqrt[4]{b}\,(\Theta_\nu + e^{(\varepsilon+\alpha)\nu}\sum_1^\nu \Theta_k/d_k(\zeta_0))$ and the lemma follows immediately. □

LEMMA 10.6. *Let* $s \mapsto \eta_0(s) = (x_0 \pm s, y(x_0 \pm s))$ *be a* $C^2(b)$ *curve, with* $\zeta_0 = \eta_0(0) \in \mathcal{C}_k$, $k \leqslant n$. *Let* $q \leqslant k-1$ *and* $0 < \sigma < 2\delta$ *be such that*

$$|\eta_\nu(s)-\zeta_\nu| \leqslant he^{-\beta\nu} \quad \textit{for all } 1 \leqslant \nu \leqslant q \textit{ and } 0 \leqslant s \leqslant \sigma. \tag{BC2}$$

(a) *Then* $\|w_\nu(\zeta_1)\|\sigma^2 \leqslant e^{-\beta\nu}$ *for all* $0 \leqslant \nu \leqslant q-1$.

(b) *If moreover* $\|w_q(\zeta_1)\|\sigma^2 \leqslant h^2 e^{-2\beta q}$ *then* $|\eta_{q+1}(s)-\zeta_{q+1}| \leqslant he^{-2\beta(q+1)}$.

Proof. Part (a) is proved by induction on ν. Notice that case $\nu=0$ is trivial. We split the tangent vector to $\varphi_a(\eta_0)$, $t_0(s) = \alpha_0(s)e(s) + \beta_0(s)(1,0)$, where $e(s) = (\bar{e}(s), 1)$ has the direction of the qth contractive approximation at $\eta_1(s)$. As in (9.22), (9.23), $|\beta_0(0)| \leqslant \text{const.}(Kb)^q$ and $as \leqslant |\beta_0(s)-\beta_0(0)| \leqslant 3as$.

Let now $\nu \geqslant 1$ and assume that (a) has been proved for $\nu-1$. We have $t_\nu(s) = \alpha_0(s)D\varphi_a^\nu \cdot e(s) + \beta_0(s)w_\nu(s)$ (denoting $w_\nu(s) = w_\nu(\eta_1(s))$. We write

$$w_\nu(s) = \lambda(s)(w_\nu(0)+\varepsilon_\nu(s)), \quad \lambda(s) = \frac{\|w_\nu(s)\|}{\|w_\nu(0)\|}$$

and want to estimate $\lambda(s)$ and $\|\varepsilon_\nu(s)\|$. For this we use Lemma 10.5 but first we must bound $e^{(\varepsilon+\alpha)\nu}\sum_1^\nu \Theta_k(s)/d_k(\zeta_0)\leqslant 4\sum_1^\nu e^{(\varepsilon+\alpha)\nu+\alpha j}|\eta_j(s)-\zeta_j|$. We assume $(\varepsilon+\alpha)\ll\beta$, so that

$$\frac{(\varepsilon+\alpha)}{\beta-(\varepsilon+\alpha)}\leqslant\frac{(\beta+c)-2(\varepsilon+\alpha)}{2(\log K+(\varepsilon+\alpha))} \tag{16}$$

and distinguish two cases in the sum above. For $j\geqslant((\varepsilon+\alpha)/(\beta-(\varepsilon+\alpha)))\nu$ we use the binding condition to get $e^{(\varepsilon+\alpha)\nu+\alpha j}|\eta_j(s)-\zeta_j|\leqslant e^{-\varepsilon j}$. In the opposite case argue we as follows. By induction $e^{c(\nu-1)}s^2\leqslant\|w_{\nu-1}(0)\|\sigma^2\leqslant e^{-\beta(\nu-1)}$ and so

$$|\eta_j(s)-\zeta_j|\leqslant 2K^j s\leqslant 5K^j e^{-(\beta+c)\nu/2}.$$

In view of (16) this gives $e^{(\varepsilon+\alpha)\nu+\alpha j}|\eta_j(s)-\zeta_j|\leqslant 5e^{-\varepsilon j}$ in this case. Now Lemma 10.5 implies

$$\tau^{-1}\leqslant\lambda(s)\leqslant\tau \tag{17}$$

$$\|\varepsilon(s)\|\leqslant 2\sqrt[4]{b}\,\tau\|w_\nu(0)\|\leqslant\tfrac{1}{2}\|w_\nu(0)\|. \tag{18}$$

with

$$\tau=\exp\left(50K\sum_1^\infty e^{-\varepsilon j}\right). \tag{19}$$

Let us now write $\eta_{\nu+1}(\sigma)-\zeta_{\nu+1}=\int_0^\sigma t_\nu(s)\,ds$ in the form

$$\begin{aligned}(\eta_{\nu+1}(\sigma)-\zeta_{\nu+1})-\int_0^\sigma \alpha_0(s)D\varphi_a^\nu\cdot e(s)\,ds-\int_0^\sigma \beta_0(0)w_\nu(s)\,ds\\ =w_\nu(0)\int_0^\sigma\lambda(s)(\beta_0(s)-\beta_0(0))\,ds+\int_0^\sigma\lambda(s)(\beta_0(s)-\beta_0(0))\varepsilon(s)\,ds.\end{aligned} \tag{20}$$

Clearly, $\|\int_0^\sigma\alpha_0(s)D\varphi_a^\nu\cdot e(s)\,ds\|\leqslant 5K\sqrt{b}\cdot 8K(K^2b)^\nu\sigma\leqslant(\sqrt{b})^\nu$, and

$$\left\|\int_0^\sigma\beta_0(0)w_\nu(s)\,ds\right\|\leqslant(Kb)^q K^\nu\sigma\leqslant(\sqrt{b})^\nu.$$

On the other hand

$$\left\|\int_0^\sigma\lambda(s)(\beta_0(s)-\beta_0(0))\varepsilon(s)\,ds\right\|\leqslant\frac{1}{2}\left\|w_\nu(0)\int_0^\sigma\lambda(s)(\beta_0(s)-\beta_0(0))\,ds\right\|,$$

as a consequence of (18). Moreover

$$\tau^{-1}\frac{a\sigma^2}{2}\leqslant\left|\int_0^\sigma\lambda(s)(\beta_0(s)-\beta_0(0))\,ds\right|\leqslant\tau\frac{3a\sigma^2}{2}.$$

Hence, from (20) $\frac{1}{2}\|w_\nu(0)\|a\sigma^2/2\tau\leqslant he^{-\beta\nu}+(2\sqrt{b})^\nu\leqslant 2he^{-\beta\nu}$. We define

$$h=(10\tau)^{-1}=\frac{1}{10}\exp\left(-50K\sum_1^\infty e^{-\varepsilon j}\right) \tag{21}$$

and (a) follows. Finally the same argument also gives

$$|\eta_{q+1}(s)-\zeta_{q+1}|\leqslant 5\tau\|w_q(0)\|\sigma^2+\frac{h}{2}e^{-2\beta q}$$

and (b) is now an easy consequence. □

COROLLARY 10.7. *Let n be a return for $z_0 \in C_n$ and p be the length of its binding period. Then* $p \leqslant (2/c) \log d_n(z_0)^{-1} \leqslant 5\alpha n$. *Moreover, for some* $\tau_2 = \tau_2(K, \alpha, \beta)$ *we have*

$$\|\omega_{n+p}(\xi_1)\| d_n(\xi_0) \geqslant \tau_2 e^{c_1(p+1)/3} \|\omega_n(\xi_1)\| \geqslant \|\omega_n(\xi_1)\|$$

for all ξ_0 that remain bound to z_0 up to time $n+p$.

Proof. Let ζ_0 be the binding point for ξ_n and take $s \mapsto \eta_0(s)$, as before a $C^2(b)$ curve with $\xi_n = \eta_0(\sigma)$. For $\nu > (2/c) \log d_n(\xi_0)^{-1}$ we have $\|w_\nu(\zeta_1)\| \sigma^2 \geqslant \frac{9}{10} e^{c\nu} d_n(\xi_0)^2 \geqslant e^{-\beta\nu}$ and so (BC2) can no longer hold at time ν, by Lemma 10.6(a). This proves the first part of the corollary. On the other hand (8.7) and Lemma 10.6(b) imply

$$\|w_p(\zeta_1)\| \sigma^2 \geqslant h^2 e^{-2\beta p}.$$

Recall also that $p+1$ is a free iterate for ξ_n, by definition of binding period. It follows that

$$\|\omega_p(\xi_{n+1})\| \sigma \geqslant \frac{h}{\tau_1} e^{(c/2-\beta)p} \geqslant \frac{h}{\tau_1} e^{c(p+1)/3}.$$

Finally, as in the proof of Corollary 10.3 ((12))

$$\|\omega_{n+p}(\xi_1)\| \geqslant \tfrac{1}{2} |\beta_n(\xi_1)| \, \|\omega_p(\xi_{n+1})\| \geqslant \frac{h}{2\tau_1} e^{c(p+1)/3} \|\omega_n(\xi_1)\|.$$

and the corollary follows by taking $\tau_2 = h/2\tau_1$. □

11. Dependence on the parameter. Partitions

Now we establish the tools (partitions, uniformity of a-derivatives) required to prove that the (BA) and the (FA) are satisfied by all critical points and at all times, for a positive measure set E_k of a-values. In brief terms, what one does is to apply to each critical point z_0 the argument of Section 3, constructing partitions $\mathcal{P}_k(z_0)$ and sets $E_k(z_0)$ of *good* parameter values for z_0 at time k. The exclusions corresponding to each z_0 have estimates analogous to (3.12) and then one uses (8.1) to bound the total measure of the excluded set and prove that $E = \bigcap_{k \geqslant 1} \bigcap_{z_0 \in C_k} E_k(z_0)$ has positive Lebesgue measure.

Naturally, this requires some explanation: critical points depend on the parameter and are defined only for special values of $a \in \Omega_0$ and so it makes no sense to speak of the *same critical point* for different values of the parameter, at least globally. On the other hand, critical approximations are, by definition, solutions of equations

$$e(a; \varphi(a; z)) \text{ colinear to } t(W^u(a), \varphi(a; z))$$

and so one may use the implicit function theorem to find analytic continuations of them for nearby parameter values. This permits us to bypass the difficulty above in the following way. Whenever introducing a new critical approximation z_0 we take ζ_0 a lower order approximation to which z_0 has remained bound and show that z_0 admits an analytic continuation to an interval ω of a convenient partition associated to ζ_0. Then partitions and exclusions for z_0 at further iterates are done inside ω. Note that some care must be taken in the choice of ω so that the escape argument applies to $z_0\colon \omega\to\mathbf{R}^2$.

Formally, the whole construction is still part of the induction developed in the preceding sections. We start with the interval Ω_0 and the critical approximation $z_0^{(i)}\in G_0$ and $w_0^{(i)}\in G_1$ introduced in Section 7.

LEMMA 11.1. *For all* $1\leqslant i\leqslant N-1$, $z_0^{(i)}$ *and* $w_0^{(i)}$ *are defined on all* Ω_0. *Moreover*

$$\|\dot z_0^{(i)}(a)\|, \|\dot w_0^{(i)}(a)\|\leqslant b^{t/2}\quad \textit{for all } a\in\Omega_0.$$

Proof. Let $x\to z(a,x)=(x,y(a,x))$ parametrize $G_0(a)$ (respectively $G_1(a)$) in $|x|\leqslant\frac{1}{2}$, say. The critical approximations are given by $\zeta(a)=(x(a),y(a,x(a)))$, where $x(a)$ is the implicit solution of

$$F(a,x)=(A(a;z)+B(a;z)\partial_x y(a,x))-q(a;\varphi(a;z))(C(a;z)+D(a;z)\partial_x y(a,x))=0.$$

Here $(q,1)$ is colinear to the corresponding contractive approximation and we denote $z=z(a,x)$. Now, all the terms in $(\partial_a F)(a,x(a))$ are $\leqslant$const. b^t, as a consequence of Theorem 2.1, Lemma 6.6, (6.9) and (7.2). Observe in particular that $|\partial_a A+\partial_z A\cdot\partial_a z|\leqslant$ const. b^t because $|x(a)|\leqslant$const.$\sqrt{b}$, recall §7A. Hence $|(\partial_a F)(a,x(a))|\leqslant$const. b^t. On the other hand, the same is still true for the terms in $(\partial_x F)(a,x(a))$, except for $|\partial_z A\cdot\partial_x z|\geqslant 2$. Therefore $|(\partial_x F)(a,x(a))|\geqslant 1$. It follows that $|\dot x(a)|\leqslant$const. b^t and then, using (7.2) once more, $\|\dot\zeta(a)\|\leqslant b^{t/2}$. □

For $k\leqslant N-1$ we set simply $E_k=\Omega_0$. Clearly, we may choose $\Omega_0\subset(1,2)$ so that time N is an *escape situation* for $z_0^{(N-2)}\colon\Omega_0\to\mathbf{R}^2$, (i.e. length$(z_N^{(N-2)}(\Omega_0))>\sqrt{\delta}$) and also for $w_0^{(N-2)}$. At stage $n\geqslant N$ we assume that E_k has been constructed for $k\leqslant n-1$ in such a way that, given any $a_0\in E_k$ and $\zeta_0=\zeta_0^{(k-1)}\in\mathcal{C}_k$ there are $m\in\left[\frac{1}{2}(k+1),k+1\right]$ and $\omega\subset\Omega_0$ an interval with

$$K^{-3m/2}\leqslant \text{length}(\omega)\leqslant e^{-2m/3}, \tag{1}$$

such that $\zeta_0^{(m-2)}$ admits analytic continuation to ω with:

- $\zeta_0^{(m-2)}(a)$ satisfying the properties of Section 8 ((BA), (FA), expansiveness, binding, etc.) for all $a\in\omega$ and times $\leqslant m-1$;

• time m being an escape iterate (i.e. belonging to an escape period) of $\zeta_0^{(m-2)}$: $\omega\to\mathbf{R}^2$.

Let now $a_0\in E_{n-1}$ and $\tilde z_0=z_0^{(n-1)}\in\mathcal{C}_n(a_0)$. By definition, there is $\tilde\zeta_0=\zeta_0^{(n-2)}\in\mathcal{C}_{n-1}(a_0)$ such that $\tilde z_0$ remains bound to $\tilde\zeta_0$ up to time $n-1$. Let $m\in\left[\frac{1}{2}n,n\right]$ and $\omega\subset\Omega_0$ be the interval associated to $\tilde\zeta_0$ in the sense of the inductive assumptions above. We denote $z_0=z_0^{(m-2)}$ and $\zeta_0=\zeta_0^{(m-2)}$.

Lemma 11.2. *z_0 admits analytic continuation defined on ω. Moreover $\|\dot z_0(a)\|\leqslant b^\tau$ where τ is a small positive constant.*

Proof. The statement on the derivative is proved by induction on the generation. We assume that

$$\dot\zeta_0(a)=\sum_1^{\bar g}\tfrac{1}{2}b^{\tau i}\;\;(\leqslant b^\tau), \tag{2}$$

$\bar g=$ generation of ζ_0, and prove that the same holds for z_0. Clearly, we may suppose that the generation of z_0 is $g=[\theta n]>\bar g$ since in case $g\leqslant\theta(n-1)$ (corresponding to part (a) of the definition of $\mathcal{C}_n$, recall Section 8) we have simply $z_0=\zeta_0$ and so the lemma is immediate. Let $s\mapsto\xi(a,s)=(s,\eta(a,s))\in G_1(a)$ be a smooth parametrization and denote $z(a,s)=(x(a,s),y(a,s))=\varphi^{g-1}(a;\xi(a,s))$. Here $a\in\omega$ and the domain $S\subset\{|s|\leqslant1-\delta_0\}$ of s-values is fixed in such a way that $z(a_0,S)$ coincides with $\gamma(z_0,\varrho_0^{\theta n})$, recall Section 8. We denote $\gamma(a)=z(a,S)$ and, in a similar way, define $\tilde\gamma(a)$ continuation of $\gamma(\zeta_0,\varrho_0^{\theta n})$ for $a\in\omega$. Our purpose is to show that the algorithm of §7B applies to $\gamma(a),\tilde\gamma(a)$ for all $a\in\omega$. Clearly, for any $r\geqslant1$

$$\|z\|_{C^r(a,s)}\leqslant K_2^g,\quad\text{with } K_2=K_2(K,r). \tag{3}$$

By definition of critical point (recall Section 8 and (9.14))

$$\frac{\|\partial_s y\|}{\|\partial_s x\|}(a_0,s)\leqslant b^{t/2}\quad\text{and}\quad\|\partial_s z(a_0,s)\|\geqslant1,$$

implying $\|\partial_s x(a_0,s)\|\geqslant\frac{1}{2}$. This (essentially) persists for all $a\in\omega$, at least if b (and so θ) is small and N is large:

$$\|\partial_s x(a,s)\|\geqslant\|\partial_s x(a_0,s)\|-K_2^g|a-a_0|\geqslant\tfrac{1}{2}-\text{const.}(K_2^\theta e^{-c/3})^n\geqslant\tfrac{1}{4}. \tag{4}$$

We denote $t(a,s)=(\partial_s y/\partial_s x)(a,s)$, the slope of the tangent to $\gamma(a)$, and then (3), (4) give

$$\|t\|_{C^2(a,s)}\leqslant K_3^g,\quad\text{for some } K_3=K_3(K). \tag{5}$$

Recall that $\gamma(a_0)=\gamma(z_0,\varrho_0^{\theta n})$ is $C^2(b)$, by definition. For general $a\in\omega$ we have at least $|t(a,s)|\leqslant b^{t/2}+\text{const.}(K_3^\theta e^{-c/3})^n\leqslant\frac{1}{100}$ and analogously $|\partial_s t(a,s)|\leqslant\frac{1}{100}$ and this is perfectly sufficient for algorithm §7B. Naturally, similar properties hold for $\tilde\gamma(a)$. The problem of checking (7.13) is somewhat delicate and actually requires some additional information. We write $\gamma(a)$ and $\tilde\gamma(a)$ as the graphs of $x\mapsto\bar y(a,x)$ and $x\mapsto\bar\eta(a,x)$, respectively. Then (8.3) means $|\bar y(a_0,x)-\bar\eta(a_0,x)|\leqslant b^{t\theta(n-1)/5}$ $(\leqslant\sigma_0^{2n})$. We suppose $r\in\mathbf{N}$ fixed in such a way that

$$e^{-r/10}\leqslant\sigma_0^2 \tag{6}$$

and *include in the definition* of C_n'' the condition that

$$|\partial_a^i(\bar y-\bar\eta)|_{a=a_0}|\leqslant\sigma_0^{2n},\quad\text{for all } 1\leqslant i\leqslant r-1. \tag{7}$$

On the other hand (3) and (4) give, in a straightforward way,

$$\|\bar y\|_{C^r(a,x)}\leqslant K_4^g,\quad\text{where } K_4=K_4(K,r), \tag{8}$$

and analogously for $\bar\eta$. Hence, for all $a\in\omega$

$$\begin{aligned}|\bar y(a,x)-\bar\eta(a,x)|&\leqslant\sum_0^{r-1}\frac{\sigma_0^{2n}}{i!}e^{-2cmi/3}+\frac{2K_4^g}{r!}e^{-2cmr/3}\\&\leqslant 2\sigma_0^{2n}+(K_4^\theta e^{-cr/3})^n\leqslant\frac{1}{K^2}\sigma_1^{2(m-2)}\end{aligned} \tag{9}$$

as we pretended. Finally, (7.14) follows easily from this and the remark that the length of $\gamma(a)$ and $\tilde\gamma(a)$ does not implode, recall (4). Therefore the algorithm of §7B applies on all $a\in\omega$ and it yields a uniquely defined extension $z_0:\omega\ni a\to z_0(a)$ of z_0. We write $z_0(a)=(x_0(a),\bar y(a,x_0(a)))$ and $\zeta_0(a)=(\xi_0(a),\bar\eta(a,\xi_0(a)))$ and then (9) and (7.15) give $|x_0(a)-\xi_0(a)|\leqslant\sigma_0^n$. On the other hand, $x_0(a)$ is given implicitly by

$$F(a,x)=(A(a;z)+B(a;z)\partial_x\bar y(a,x))-q(a;\varphi(a;z))(C(a;z)+D(a;z)\partial_x\bar y(a,x))=0,$$

$z=(x,\bar y(a,x))$, and so by (5), (8) and Lemma 6.7, $|\ddot x_0(a)|\leqslant K_5^g$, for some $K_5=K_5(K)$. Clearly, the same argument and conclusion hold for ξ_0. Since we also have the lower bound in (1) for the length of ω, we are in position to apply Hadamard's lemma (see [BC2, Lemma 8.7]) to get $|\dot x_0(a)-\dot\xi_0(a)|\leqslant 2\sigma_0^{n/2}K_5^g$. A similar argument works for $y_0(a)=\bar y(a,x_0(a))$ and $\eta_0(a)=\bar\eta(a,\xi_0(a))$ and we get $\|\dot z_0(a)-\dot\zeta_0(a)\|\leqslant\text{const.}(K_6^\theta\sigma_0)^{n/2}\leqslant\sigma_0^{n/4}\leqslant\frac12 b^{\tau g}$, as long as b is small and we take, say

$$\tau=\frac{t}{10R}\quad\text{(see (11) below).} \tag{10}$$

This gives (2) for z_0 and completes the proof of the lemma. □

Remark 1. It is straightforward to check that in the definition of σ_0 one may replace K by 5 ($\geqslant\|D\varphi_a\|$), getting in this way $\sigma_0=\left(\frac{1}{250}\right)^2$. Then (6) holds if

$$r > 40\log 250.$$

Of course, this explicit bound on r is far from being the best possible.

Remark 2. We must also go back to the proof of the claim (C) in Lemma 9.5 and check that $\bar{\zeta}_0^{(\mu)}$ satisfies (7), in order to conclude that it belongs to $\mathcal{C}_{\mu+1}$. This can be done as follows. Let $z_0=z_0(a_0)$ be the point being bound at time n. It extends to $z_0:\omega\to\mathbf{R}^2$ with $\text{length}(\omega)\geqslant e^{-2cn/3}$. In this interval the binding construction is uniform and we have (9.18) $\phi(a)=\text{dist}(\gamma^{[k+1]}(a),\gamma^{[k]}(a))\leqslant b^{tg/4}, g=g_{k+1}$, for all $a\in\omega$. On the other hand, as in (8) $|\partial_a^i\phi|\leqslant K_7^g$ for all $1\leqslant i\leqslant r$, with $K_7=K_7(K,r)$. Then we may apply Hadamard's lemma to $\partial_a^{i-1}\phi$ and $\partial_a^{i+1}\phi$, successively for $i=1,2,\dots,r-1$, to obtain $|\partial_a^i\phi|\leqslant b^{tg/4\cdot 3^i}$ for all $1\leqslant i\leqslant r-1$. In order to have this imply (7) we just have to take in the definition of θ ((9.17))

$$R=3^r. \tag{11}$$

We let from now on $z_0:\omega\to\mathbf{R}^2$ be as above and describe the construction of the partitions $\mathcal{P}_\nu(z_0)$ and the sets of good parameter values $E_\nu(z_0)\subset\omega$ for iterates $m-1\leqslant\nu\leqslant n$. This is quite analogous to the one-dimensional procedure so we just recall the main ideas. First we set $E_{m-1}(z_0)=\omega$ and $\mathcal{P}_{m-1}(z_0)=\{\omega\}$. Then, given $m\leqslant\nu\leqslant n$ and $\bar{\omega}\in\mathcal{P}_{\nu-1}(z_0)$, $\bar{\omega}\subset E_{\nu-1}(z_0)$ we distinguish the following cases. If ν is not a return situation for $\bar{\omega}$, i.e. if $z_\nu(\bar{\omega})\cap\{|x|<\delta\}=\varnothing$ then, by definition, $\bar{\omega}\in\mathcal{P}_\nu(z_0)$. If ν is a bound return situation, again we take $\bar{\omega}$ to be in $\mathcal{P}_\nu(z_0)$. Suppose now that n is a free return situation. We take $\zeta_0=\zeta_0(a_0)=(\xi_0,\eta_0)$ to be the binding point of $z_\nu(a_0), a_0\in\bar{\omega}$, and say that n is essential or inessential, according to whether $\{x_n(a)-\xi_0: a\in\bar{\omega}\}$ contains some $I_{r,i}$ (Section 3) or not. In the inessential case still $\bar{\omega}\in\mathcal{P}_\nu(z_0)$. On the other hand, if ν is an essential situation then we define $\omega''\subset\bar{\omega}$ and $\omega_{r,i}\subset\omega'=\bar{\omega}\setminus\omega''$ by

$$a\in\omega''\Rightarrow|x_n(a)-\xi_0|>\delta \quad\text{and}\quad a\in\omega_{r,i}\Rightarrow(x_n(a)-\xi_0)\in I_{r,i}, 1\leqslant i\leqslant r^2, |r|\geqslant\Delta.$$

By definition the $\omega_{r,i}$ and the connected components of ω'' are the elements of $\mathcal{P}_\nu(z_0)$ inside $\bar{\omega}$. It follows from Lemma 11.3 below that on free iterates $z_\nu(\bar{\omega})$ is a nearly straight and nearly horizontal curve so the situation can be described by Figure 17.

Corresponding to the basic assumption, parameter values $a\in\bigcup\{\omega_{r,i}:|r|>\alpha\nu\}$ are excluded: they do not belong to $E_\nu(z_0)$. Observe that (Lemma 11.2) $|\xi_0(a)-\xi_0|\leqslant b^\tau e^{-c\nu/3}\ll e^{-\alpha\nu}$ and so $d_\nu(z_0(a))\geqslant|x_\nu(a)-\xi_0(a)|\approx|x_\nu(a)-\xi_0|$. Incidentally, this shows

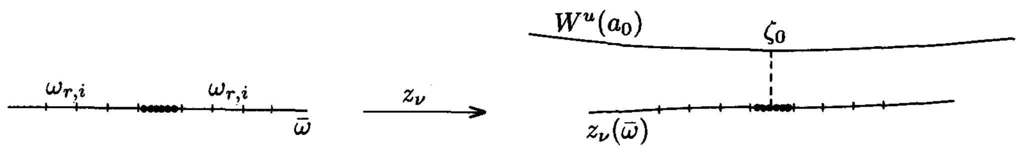

Fig. 17

that the particular choice of a_0 above is irrelevant. We also apply the escape argument and exclude parameters so that the number of free iterates of z_0 in $[1,\nu]$ satisfies $F_{[1,\nu]}(a;z_0)\geqslant(1-\varepsilon)\nu$ for every parameter value in the remaining set $E_\nu(z_0)$. It is convenient at this point to assume that all returns and binding periods of $z_0(a)$ in $[1,\nu)$ coincide for all $a\in\bar{\omega}$. Since, by construction, $\log d_j(z_0(a))$, $1\leqslant j\leqslant\nu-1$, is nearly constant on $\bar{\omega}$ this can be obtained just by slightly adjusting the definitions, without affecting the arguments of Sections 8–10.

In order to estimate these exclusions we now need to show that on free return situations the curve $z_\nu(\bar{\omega})$ is nearly horizontal and its velocity has bounded distorsion. This corresponds to Lemma 3.3 in the one-dimensional setting. For the sake of simplifying the notations we let $w_\nu(a)=w_\nu(z_1(a))$ and $\omega_\nu(a)=\omega_\nu(z_1(a))$.

First we need a higher-dimensional version of Lemma 3.4.

LEMMA 11.3. *For all* $2\leqslant\nu\leqslant n$ *and* $a\in\bar{\omega}\in\mathcal{P}_{\nu-1}(z_0)$, $\bar{\omega}\subset E_{\nu-1}(z_0)$ *we have*

$$\frac{1}{100}\leqslant\frac{\|\dot{z}_\nu(a)\|}{\|w_{\nu-1}(a)\|}\leqslant 100.$$

Moreover, if ν *is a free iterate then* $|\mathrm{angle}(\dot{z}_\nu(a),w_{\nu-1}(a))|\leqslant b^{t/2}$.

This is proved in the same way as for the Hénon case in [BC2, Lemmas 8.1, 8.4], with $M_\nu=D\varphi_a(z_\nu)$ and $\varphi_\nu=\partial_a\varphi_a(z_\nu)$. Observe that, although our statement on the angle is somewhat stronger, it actually follows from the proof of [BC2, Lemma 8.1] together with the easy remark that both $w_{\nu-1}$ and $\varphi_{\nu-1}\approx(-x_{\nu-1}^2,0)$ are nearly horizontal if ν is a free iterate.

COROLLARY 11.4. *For every* $m\leqslant\nu\leqslant n$ *and* $\tilde{\omega}\in\mathcal{P}_\nu(z_0),\tilde{\omega}\subset E_{\nu-1}(z_0)$

$$K^{-3\nu/2}\leqslant\mathrm{length}(\tilde{\omega})\leqslant e^{-2c\nu/3}.$$

Proof. Take $\mu\in\left[\frac{9}{10}\nu,\nu\right]$ a free iterate (recall Corollary 10.7). Note that $z_\mu(\tilde{\omega})$ is nearly straight, by Lemmas 8.1 and 11.3. Hence

$$\mathrm{length}(\tilde{\omega})\cdot\inf\|w_{\mu-1}(a)\|\leqslant\mathrm{const.\,length}(z_\mu(\tilde{\omega}))\leqslant\mathrm{const.}$$

and so $\operatorname{length}(\tilde{\omega}) \leqslant \text{const.}\, e^{-c(\mu-1)} \leqslant e^{-2c\nu/3}$. Let now $m \leqslant \lambda \leqslant \nu$ be the moment at which $\tilde{\omega}$ was created. By construction

$$\operatorname{length}(\tilde{\omega}) \cdot \sup \|w_{\lambda-1}(a)\| \geqslant \text{const.} \operatorname{length}(z_\lambda(\tilde{\omega})) \geqslant \text{const.} \frac{e^{-\alpha\lambda}}{(\alpha\lambda)^2}$$

and so $\operatorname{length}(\tilde{\omega}) \geqslant \text{const.}\, K^{-\lambda} e^{-\alpha\lambda}/(\alpha\lambda)^2 \geqslant K^{-3\nu/2}$. □

LEMMA 11.5. *There is $\tau_3 = \tau_3(K, \alpha, \beta, \delta) > 0$ such that if ν is a free return situation for $\bar{\omega} \in \mathcal{P}_{\nu-1}(z_0)$, $\bar{\omega} \subset E_{\nu-1}(z_0)$, then for all $a, a' \in \bar{\omega}$*

$$\frac{\|w_{\nu-1}(a')\|}{\|w_{\nu-1}(a)\|} \leqslant \tau_3 \quad \textit{and} \quad |\operatorname{angle}(w_{\nu-1}(a), w_{\nu-1}(a'))| \leqslant 5\sqrt[4]{b}.$$

Proof. We denote

$$T_k = T_k(a, a') = \sum_1^k \left(\sqrt[4]{b}\right)^{k-j} (|a-a'| + |z_j(a) - z_j(a')|).$$

The lemma is an immediate consequence of the following facts:

$$\sum_1^{\nu-1} \frac{T_k}{d_k(z_0(a))} \leqslant \tau(K, \alpha, \beta, \delta), \tag{12}$$

$$\frac{\|\omega_{\nu-1}(a')\|}{\|\omega_{\nu-1}(a)\|} \leqslant \exp\left(8K \sum_1^{\nu-1} \frac{T_k}{d_k(z_0(a))}\right) \tag{13a}$$

and

$$|\operatorname{angle}(\omega_{\nu-1}(a), \omega_{\nu-1}(a'))| \leqslant 2\sqrt[4]{b}\, T_{\nu-1}. \tag{13b}$$

Here we only need to derive (12) since then (13a), (13b) follow in precisely the same way as Lemma 10.2. Clearly,

$$\sum_1^{\nu-1} (d_k(z_0(a)))^{-1} \sum_1^k \left(\sqrt[4]{b}\right)^{k-j} |a-a'| \leqslant \text{const.}\, e^{-c\nu/2} \sum_1^{\nu-1} e^{\alpha k} \sum_1^k \left(\sqrt[4]{b}\right)^{k-j}$$

is uniformly bounded, so we are left to show that

$$\sum_1^{\nu-1} \frac{\Theta_k}{d_k(z_0(a))} \leqslant \tau'(K, \alpha, \beta, \delta)$$

with $\Theta_k = \Theta_k(a, a') = \sum_1^k \left(\sqrt[4]{b}\right)^{k-j} |z_j(a) - z_j(a')|$. For this we adapt the one-dimensional argument, [BC1, Lemma 5], [BC2, §2.2]. Let $N = \nu_1 < \nu_2 < \ldots < \nu_s = \nu$ be the free return situations for $\bar{\omega}$. Suppose first $j \leqslant N$. Lemmas 7.2 and 11.3 (and the fact that both $z_j(\bar{\omega})$ and $z_N(\bar{\omega})$ are nearly straight) give $|z_j(a)-z_j(a')| \leqslant \text{const.}\, e^{-c_0(N-j)}|z_N(a)-z_N(a')|$. It follows that

$$\Theta_k \leqslant \text{const.} e^{-c_0(N-k)}|z_N(a)-z_N(a')|$$

for all $k \leqslant N$. Let now $i \geqslant 1$, $\mu = \nu_i$, $\lambda = \nu_{i+1}$ and p be the length of the binding period associated to μ. For $\mu+1 \leqslant j \leqslant \mu+p$ we have, by Lemma 11.3,

$$\frac{\|\dot{z}_j\|}{\|\dot{z}_\mu\|} \leqslant \text{const.} \frac{\|w_{j-1}\|}{\|w_{\mu-1}\|} = \text{const.} \left\| D\varphi_a^{j-\mu-1}\left(\frac{w_\mu}{\|w_{\mu-1}\|}\right)\right\|.$$

We write

$$\frac{w_\mu}{\|w_{\mu-1}\|} = \widehat{\alpha}_\mu e + \widehat{\beta}_\mu (1,0)$$

where $e=(q,1)$ has the direction of the pth contractive approximation. Then

$$\begin{aligned}\frac{\|\dot{z}_j\|}{\|\dot{z}_\mu\|} &\leqslant \text{const.}(|\widehat{\alpha}_\mu|(Kb)^{j-\mu-1} + |\widehat{\beta}_\mu| \cdot \|w_{j-\mu-1}(z_{\mu+1})\|) \\ &\leqslant \text{const.}((Kb)^{j-\mu-1} + e^{-\beta(j-\mu-1)} d_\mu(z_0)^{-1}) \\ &\leqslant \text{const.}(e^{-\beta(j-\mu-1)} d_\mu(z_0)^{-1}),\end{aligned}$$

by Lemmas 9.6 and 10.6. Therefore (again because $z_\mu(\bar{\omega})$ is nearly straight)

$$|z_j(a)-z_j(a')| \leqslant \text{const.}\, e^{-\beta(j-\mu-1)} \frac{|z_\mu(a)-z_\mu(a')|}{d_\mu(z_0(a))}.$$

Observe also that we may assume (inductively) that

$$\Theta_\mu \leqslant \text{const.}|z_\mu(a)-z_\mu(a')|. \tag{14}$$

Then, given any $\mu+1 \leqslant k \leqslant \mu+p$,

$$\Theta_k \leqslant \text{const.}\left(\left(\sqrt[4]{b}\right)^{k-\mu} |z_\mu(a)-z_\mu(a')| + e^{-\beta(k-\mu)} \frac{|z_\mu(a)-z_\mu(a')|}{d_\mu(z_0(a))}\right).$$

Hence, by the (BA)

$$\frac{\Theta_k}{d_k(z_0(a))} \leqslant \text{const.}\, e^{-(\beta-\alpha)(k-\mu)} \frac{|z_\mu(a)-z_\mu(a')|}{d_\mu(z_0(a))}. \tag{15}$$

Suppose now $\mu+p < k \leqslant \lambda$. By Lemmas 9.4 and 11.3 we have

$$|z_j(a)-z_j(a')| \leqslant \text{const.}\, e^{(j-\lambda)/10}|z_\lambda(a)-z_\lambda(a')| \tag{16}$$

for all $j \leqslant \lambda$. Then

$$\Theta_k \leqslant \left(\text{const.}\left(\sqrt[4]{b}\right)^{k-\mu} e^{-(\lambda-k)/10} + \text{const.}\, e^{-(\lambda-k)/10}\right) |z_\lambda(a) - z_\lambda(a')|.$$

Hence

$$\Theta_k \leqslant \text{const.}\, e^{-(\lambda-k)/10} |z_\lambda(a) - z_\lambda(a')|. \tag{17}$$

and we also we recover (14) for time λ (with the same constant). Altogether this gives

$$\sum_1^{\nu-1} \frac{\Theta_k}{d_k(z_0(a))} \leqslant \text{const.} \sum_1^{s} \frac{|z_{\nu_i}(a) - z_{\nu_i}(a')|}{d_{\nu_i}(z_0(a))}.$$

Moreover, if δ is small then $\nu_{i+1} - \nu_i$ is large and (16) gives $|z_{\nu_{i+1}}(a) - z_{\nu_{i+1}}(a')| \geqslant 2|z_{\nu_i}(a) - z_{\nu_i}(a')|$ (compare Lemma 3.5). In particular we may write for each fixed r

$$\sum_{r_i = r} \frac{|z_{\nu_i}(a) - z_{\nu_i}(a')|}{d_{\nu_i}(z_0(a))} \leqslant \text{const.} \frac{|z_{\mu_r}(a) - z_{\mu_r}(a')|}{d_{\mu_r}(z_0(a))}$$

where the sum is taken over the free returns ν_i for which $z_{\nu_i}(\bar{\omega}) \subset I_r$ and we denote by μ_r the maximum of such ν_i. Finally, by construction of $\bar{\omega}$,

$$\sum_r \frac{|z_{\mu_r}(a) - z_{\mu_r}(a')|}{d_{\mu_r}(z_0(a))} \leqslant \text{const.} \sum \frac{1}{r^2} < \infty. \qquad \square$$

Remark. Observe that we even proved (compare [BC2, Lemma 8.8])

$$|\text{angle}(w_{\nu-1}(a), w_{\nu-1}(a'))| \leqslant \text{const.} \sqrt[4]{b}\, |z_\nu(a) - z_\nu(a')|.$$

Now the uniformity of the a-derivatives on free returns is an immediate consequence of Lemmas 11.3 and 11.5.

COROLLARY 11.6. *There is $\tau_4 = \tau_4(K, \alpha, \beta, \delta)$ such that if ν is a free return situation for $\bar{\omega} \in \mathcal{P}_{\nu-1}(z_0)$, $\bar{\omega} \subset E_{\nu-1}(z_0)$, then for all $a, a' \in \bar{\omega}$*

$$\frac{\|\dot{z}_\nu(a')\|}{\|\dot{z}_\nu(a)\|} \leqslant \tau_4 \quad \textit{and} \quad |\text{angle}(\dot{z}_\nu(a), \dot{z}_\nu(a'))| \leqslant 10\sqrt[4]{b}.$$

12. Conclusion

Now we have completely recovered the formalism of the one-dimensional situation. Exclusions of parameter values $a \in \omega$ determined by the (BA) and (FA) during $[m, n]$ are made as described before. It follows from the (FA) that a parameter $\alpha \in \omega$ in $E_n(z_0)$ must have an escape iterate $\tilde{m} \in \left[\frac{1}{2}(n+1), n\right]$. This assures that z_0 satisfies the inductive assumptions of Section 11 and so that whenever it is used, in a forthcoming iterate, as the binding point of some higher-generational $\tilde{z}_0$, we have convenient $\tilde{m}$ and $\tilde{\omega}$ to start the construction for $\tilde{z}_0$. Actually we must also check that $z_0^{(\tilde{m}-2)}$ satisfies Lemma 11.2 but this presents no difficulty: the fact that it is defined on all $\tilde{\omega}$ follows simply from the expansiveness and the statement on the derivative may be easily derived from $|z_0^{(\tilde{m}-2)} - z_0^{(m-2)}| \leqslant (Kb)^{m-2}$ by another use of Hadamard's lemma.

The total excluded measure is (cf. (3.12), see also [BC2, Section 2])

$$m(\omega \setminus E_n(z_0)) \leqslant B_0 e^{-\alpha_0 n} m(\omega) \tag{1}$$

where B_0 and α_0 depend on K, α, β and δ but not on N or b.

We define $E_n = E_{n-1} \setminus \left(\bigcup_{z_0} (\omega \setminus E_n(z_0)) \right)$ and then (1) and (8.1) give

$$m(E_{n-1} \setminus E_n) \leqslant 4B_0((K/\varrho_0)^\theta e^{-\alpha_0})^n.$$

By taking $\theta = \theta(b)$ small enough we may replace this by $m(E_{n-1} \setminus E_n) \leqslant 4B_0 e^{-\alpha_0 n/2}$ and now the proof that $\tilde{E} = \bigcap_{n \geqslant N} E_n$ has positive Lebesgue measure follows in precisely the same way as for the one-dimensional case in Section 3.

This completes the induction argument started in Section 7. For $a \in \tilde{E}$ *true* critical points are defined and their images are expanding for all times. We may take, say, $z_0 = \lim_i z_0^{(i)}$ the critical point of generation zero and then $z_1 = \varphi_a(z_0)$ satisfies (b)(ii) in Theorem B. Moreover, for almost every $a \in \tilde{E}$ the positive orbit of z_1 is dense in Λ. This is shown in precisely the same way as in the Hénon case, [BC2, Section 10], so we do not detail it here. The proof of Theorem B is complete.

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