Commun. Korean Math. Soc. **26** (2011), No. 1, pp. 1–12
DOI 10.4134/CKMS.2011.26.1.001

ABUNDANT SEMIGROUPS WITH QUASI-IDEAL S-ADEQUATE TRANSVERSALS

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Abstract. In this paper, the connection of the inverse transversal with the adequate transversal is explored. It is proved that if S is an abundant semigroup with an adequate transversal S^o, then S is regular if and only if S^o is an inverse semigroup. It is also shown that adequate transversals of a regular semigroup are just its inverse transversals. By means of a quasi-adequate semigroup and a right normal band, we construct an abundant semigroup containing a quasi-ideal S-adequate transversal and conversely, every such a semigroup can be constructed in this manner. It is simpler than the construction of Guo and Shum [9] through an SQ-system and the construction of El-Qallali [5] by $W(E,S)$.

Introduction

An *inverse transversal* of a regular semigroup S is an inverse subsemigroup that contains precisely one inverse of each element of S. This concept was first introduced by Blyth and McFadden in 1982 [1]. Afterwards, this class of regular semigroups attracted several authors' attention and a series of important results were obtained [1, 2, 3, 11, 12]. If T is an inverse transversal of S, then for every $x \in S$ we shall denote by x^o the unique element of $T \cap V(x)$ and write T as $S^o = \{x^o : x \in S\}$. It is well known that the set $I = \{e \in S : ee^o = e\}$ is a left regular band and $\Lambda = \{f \in S : f^o f = f\}$ is a right regular band, and they play an important role in the study of regular semigroups with inverse transversals. An analogue of an inverse transversal, which is termed an adequate transversal, was introduced for abundant semigroups by El-Qallali (see [5]). And a construction for abundant semigroups satisfying some conditions was given in [5]: S contains a multiplicative type-A transversal S^o; E generates a regular semiband $\langle E\rangle$ and E^o is a semilattice transversal of $\langle E\rangle$, where E and E^o denote the set of idempotents of S and S^o respectively. In [4], Chen introduced two important

Received January 11, 2010; Revised May 20, 2010.
2010 *Mathematics Subject Classification.* 20M10.
Key words and phrases. abundant semigroup, quasi-adequate semigroup, S-adequate transversal, quasi-ideal.
This research was partially supported by the National Natural Science Foundation of China (No. 10971086) and the Natural Science Foundation of Shandong Province (No. ZR2010AL010).

idempotent subsets I and Λ and though they play a role similar to those in the regular case, in general, they are not subsemigroups of S. Following [12], if S^o is an adequate transversal of an abundant semigroup S and I, Λ are both subsemigroups of S, then S^o is called an *S-adequate transversal* of S. According to Guo and Shum [9], a structure theorem for abundant semigroups with quasi-ideal S-adequate transversals was obtained. Besides the structure, it is an important problem to determine the connection between inverse transversals and adequate transversals, that is, whether adequate transversals of a regular semigroup are just its inverse transversals? In this paper we give a positive answer to this problem. The main purpose of this paper is to establish a construction for abundant semigroups with quasi-ideal S-adequate transversals by two components, with R a quasi-adequate semigroup and Λ a right normal band. How to describe the usual three relations by two components as well as the triple used in [9]? It is the difficulty of this paper. In Section 1, we collect some basic concepts and results from [4, 5, 6, 7, 9, 10], which are frequently used in this paper. In Section 2, we give some properties associated with adequate transversals, which demonstrate that the adequate transversal is the natural generalization of the inverse transversal in the abundant case. In Section 3, we construct an abundant semigroup with a quasi-ideal S-adequate transversal and conversely, every abundant semigroup with a quasi-ideal S-adequate transversal can be constructed in this manner.

We use the notation and terminology of [1, 6, 8]. Other undefined terms can be found in [4, 5, 10].

1. Preliminaries

We begin by recalling some results about abundant semigroups (see [6]). Let S be a semigroup and define two equivalence relations $\mathcal{R}^*$ and $\mathcal{L}^*$ on S by

$$\mathcal{R}^* = \{(a,b) \in S \times S : \ (\forall x, y \in S^1) \ \ xa = ya \Longleftrightarrow xb = yb\},$$

$$\mathcal{L}^* = \{(a,b) \in S \times S : \ (\forall x, y \in S^1) \ \ ax = ay \Longleftrightarrow bx = by\}.$$

It is evident that $\mathcal{R}^*$ is a left congruence and $\mathcal{L}^*$ is a right congruence on S. If $a, b \in RegS$, the set of regular elements of S, then $a\mathcal{R}^*b$ ($a\mathcal{L}^*b$) if and only if $a\mathcal{R}b$ ($a\mathcal{L}b$). What is more, if S is a regular semigroup, then $\mathcal{R}^* = \mathcal{R}$ and $\mathcal{L}^* = \mathcal{L}$. A semigroup is called *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains at least one idempotent. An abundant semigroup S is called *quasi-adequate* if its idempotents form a subsemigroup. An abundant semigroup is said to be *adequate* if its idempotents commute. Each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of an adequate semigroup contains a unique idempotent. For an element a of an adequate semigroup, the idempotent in the $\mathcal{L}^*$-class containing a will be denoted by a^*, the idempotent in the $\mathcal{R}^*$-class by a^+. Let S and S^o be semigroups. Throughout this paper we denote the set of idempotents of S and S^o by E and E^o respectively. We list some basic results as follows which are frequently used in this paper.

Lemma 1.1 ([6]). *Let a be an element of a semigroup S and e an idempotent of S. Then the following statements are equivalent*:
(1) $e\mathcal{R}^*a$;
(2) $ea = a$ *and for all* $x, y \in S^1$, $xa = ya$ *implies* $xe = ye$.

Lemma 1.2 ([7]). *If S is an adequate semigroup, then*
(1) *For all* $a, b \in S$, $(ab)^* = (a^*b)^*$ *and* $(ab)^+ = (ab^+)^+$.
(2) *For all* $a, b \in S$, $a\mathcal{R}^*b$ *if and only if* $a^+ = b^+$; $a\mathcal{L}^*b$ *if and only if* $a^* = b^*$.

Let S be an abundant semigroup and U an abundant subsemigroup of S. U is called a $*$-subsemigroup of S if for any $a \in U$, there exist an idempotent $e \in L^*_a(S) \cap U$ and an idempotent $f \in R^*_a(S) \cap U$. As pointed out in [5], an abundant subsemigroup U of an abundant semigroup S is a $*$-subsemigroup of S if and only if $\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U)$ and $\mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U)$.

Definition 1.3 ([5]). Let S be an abundant semigroup and S^o a $*$-adequate subsemigroup of S. S^o is called an *adequate transversal* of S, if for each $x \in S$ there exist idempotents $e, f \in S$ and a unique element $\overline{x} \in S^o$ such that $x = e\overline{x}f$, where $e\mathcal{L}\overline{x}^+$ and $f\mathcal{R}\overline{x}^*$. It can be shown that e and f are uniquely determined by x and S^o (see [5]) and therefore denoted by e_x and f_x respectively.

Let S^o be an adequate transversal of an abundant semigroup S. Then for any $x \in S, x = e_x\overline{x}f_x$, moreover, $e_x\mathcal{R}^*x\mathcal{L}^*f_x$. If $a \in S^o$, we can easily check that $e_a = a^+, f_a = a^*$ and $\overline{a} = a$. Furthermore, if $a \in E^o$, then $\overline{a} = e_a = f_a = a$. We say that the adequate transversal S^o is *multiplicative* if for any $x, y \in S, f_xe_y \in E^o$. A subsemigroup T of S is called a *quasi-ideal* of S if $TST \subseteq T$.

Let S^o be an adequate transversal of an abundant semigroup S. We write

$$I = \{e_x : x \in S\}; \quad \Lambda = \{f_x : x \in S\}.$$

Lemma 1.4 ([4]). *Let S be an abundant semigroup with an adequate transversal S^o. Then*
(1) $I \cap \Lambda = E^o$;
(2) $I = \{e \in E : (\exists l \in E^o)\ l\mathcal{L}e\}, \quad \Lambda = \{f \in E : (\exists h \in E^o)\ h\mathcal{R}f\}$;
(3) $IE^o \subseteq I,\ E^o\Lambda \subseteq \Lambda$;
(4) $x\mathcal{R}^*y$ *if and only if* $e_x = e_y$, $x\mathcal{L}^*y$ *if and only if* $f_x = f_y$.

Lemma 1.5 ([4]). *Let S be an abundant semigroup with an adequate transversal S^o. Then for every regular element x in S,* $|V(x) \cap S^o| \leqslant 1$.

Lemma 1.6 ([4]). *Let S be an abundant semigroup with a quasi-ideal adequate transversal S^o. Then for any $x, y \in S$*
(1) $\overline{xy} = \overline{x}f_xe_y\overline{y}$;
(2) $e_{xy} = e_x(\overline{x}f_xe_y)^+$;
(3) $f_{xy} = (f_xe_y\overline{y})^*f_y$.

Lemma 1.7 ([10]). *Let S be an abundant semigroup with an adequate transversal S^o. Let $R = \{x \in S : f_x \in E^o\}$ and $L = \{a \in S : e_a \in E^o\}$. Then*

$$R = \{x \in S : (\exists l \in E^o)\ x\mathcal{L}^* l\} \quad and \quad L = \{a \in S : (\exists h \in E^o)\ a\mathcal{R}^* h\}.$$

2. Some properties associated with adequate transversals

The objective in this section is to investigate some elementary properties associated with adequate transversals and of the sets R and L. It is known that R and L play an important role in the study of regular semigroups with inverse transversals [2, 3]. For any result concerning R there is a dual result for L which we list but omit its proof.

Proposition 2.1. *Let S be an abundant semigroup with an adequate transversal S^o. If S^o is a right ideal of S, then $f_x \in E^o$ for every $x \in S$ and $E = I$. Consequently, $f_x = \overline{x}^*$ and thus $x = e_x\overline{x}$.*

Dually, if S^o is a left ideal of S, then $e_a \in E^o$ for every $a \in S$ and $E = \Lambda$. Consequently, $e_a = \overline{a}^+$ and thus $a = \overline{a}f_a$.

Proof. By Definition 1.3, for every $x \in S$, $x = e_x\overline{x}f_x$, where $e_x\mathcal{L}\overline{x}^+$ and $f_x\mathcal{R}\overline{x}^*$. Since S^o is a right ideal of S, $f_x = \overline{x}^*f_x \in S^o$ and thus $f_x \in E^o$. Let $e \in E$ and e^o denote the unique inverse in S^o of e. Since $e^oe \in V(e)\cap S^o$, we have $e^oe = e^o$. Thus $ee^o = e$ for every $e \in E$. Consequently $E = I = \{e_x : x \in S\}$. □

In [10], the first author proved:

Proposition 2.2. *Let S^o be an adequate transversal of an abundant semigroup S. If $a, b \in S^o$ and $c \in S$ are such that $a\mathcal{R}^*c\mathcal{L}^*b$, then $c \in S^o$.*

By means of this proposition, we can prove:

Proposition 2.3. *Let S be an abundant semigroup with an adequate transversal S^o. Then every regular element $x \in S$ has a unique inverse in S^o, that is $|V(x) \cap S^o| = 1$.*

Proof. For every regular element x, since x, e_x and f_x are all regular, from $e_x\mathcal{R}^*x\mathcal{L}^*f_x$ we deduce that $e_x\mathcal{R}x\mathcal{L}f_x$, and so by the Miller-Clifford Theorem x has an inverse x' in $R_{f_x} \cap L_{e_x}$. Thus $\overline{x}^*\mathcal{R}f_x\mathcal{R}x'\mathcal{L}e_x\mathcal{L}\overline{x}^+$, and so by Proposition 2.2, $x' \in S^o$. It follows from Lemma 1.5 that the uniqueness of x' is obvious. □

Proposition 2.4. *Let S be an abundant semigroup with an adequate transversal S^o. Then S is regular if and only if S^o is an inverse semigroup.*

Proof. Suppose that S is a regular semigroup. Then by Proposition 2.3, every regular element $x \in S$ has a unique inverse in S^o and in particular, every element in S^o has a unique inverse in itself. Hence, S^o is an inverse semigroup.

Conversely, assume that S^o is an inverse semigroup. Then for every x in S, there exists $y \in S^o$ such that $\overline{x} = \overline{x}y\overline{x}$. Consequently,

$$x(\overline{x}^*y\overline{x}^+)x = e_x\overline{x}(f_x\overline{x}^*)y(\overline{x}^+e_x)\overline{x}f_x = e_x\overline{x}y\overline{x}f_x = e_x\overline{x}f_x = x,$$

and so x is regular in S. Thus S is regular. □

Proposition 2.5. *Let S be a regular semigroup. Then S^o is an adequate transversal of S if and only if S^o is an inverse transversal of S.*

Proof. Suppose that S is a regular semigroup and S^o is an adequate transversal of S. Then by Propositions 2.3 and 2.4, every element $x \in S$ has a unique inverse in S^o and S^o is an inverse subsemigroup of S. Consequently, S^o is an inverse transversal of S.

Conversely, let S^o be an inverse transversal of the regular semigroup S. Let x^o be the unique inverse of $x \in S$ and $x^{oo} = (x^o)^{-1}$ in S^o. Then $x = xx^ox = (xx^o)x^{oo}(x^ox)$, where $x^{oo+} = x^{oo}x^o\mathcal{L}xx^o$ and $x^{oo*} = x^ox^{oo}\mathcal{R}x^ox$. Moreover, let $\overline{x} \in S^o$ be such that $x = e\overline{x}f$, where $e, f \in E(S)$ and $e\mathcal{L}\overline{x}^+, f\mathcal{R}\overline{x}^*$. Since S^o is an inverse subsemigroup of S, $\overline{x}$ has a unique inverse $(\overline{x})^{-1}$ in S^o and so $\overline{x}^+ = \overline{x}(\overline{x})^{-1}$ and $\overline{x}^* = (\overline{x})^{-1}\overline{x}$. It is a routine matter to show that $x(\overline{x})^{-1}x = x$ and $(\overline{x})^{-1}x(\overline{x})^{-1} = (\overline{x})^{-1}$. Hence $(\overline{x})^{-1}$ is an inverse in S^o of x. Since S^o is an inverse transversal of S, we have $x^o = (\overline{x})^{-1}$ and so $x^{oo} = \overline{x}$. Consequently S^o is an adequate transversal of S. □

The connection of the inverse transversal with the adequate transversal which is explored in Propositions 2.4 and 2.5, demonstrates that the adequate transversal is the natural generalization of the inverse transversal in the abundant case.

Proposition 2.6. *Suppose that S is an abundant semigroup with a quasi-ideal S-adequate transversal S^o. Let $R = \{x \in S : f_x \in E^o\}$ and $L = \{a \in S : e_a \in E^o\}$. Then R and L are abundant semigroups sharing a common adequate transversal S^o which is a right ideal of R and a left ideal of L.*

Proof. Note that S^o is an adequate transversal of R and L if R and L are abundant semigroups. So, it suffices to prove that R and L are abundant semigroups. Indeed, we need only to show the case for R because the case for R is dual to for L. It is evident that $S^o \subseteq R$ and $S^o \subseteq L$. For any $x, y \in R$, by Lemma 1.6, we have $f_{xy} = (f_xe_y\overline{y})^*f_y = (f_xe_y\overline{y})^*\overline{y}^* \in E^o$. This shows that R is a subsemigroup of S. Let $x \in R$ and $y^o \in S^o$. Then $y^ox = y^oxf_x \in S^o$ since $f_x \in E^o$ and S^o is a quasi-ideal of S, and so S^o is a right ideal of R. □

Proposition 2.7. *Let S, R and L be described as in Proposition 2.6. Then $E(R) = \{g \in E : gg^o = g\} = I$, and $E(L) = \{h \in E : h^oh = h\} = \Lambda$, where g^o, h^o denote the inverse in S^o of g, h respectively. Moreover, for $x, y \in R$ and $g, h \in E(L)$, if $f_x = g^o, f_y = h^o$ and $xg = yh$, then $x = y$ and $g = h$.*

Proof. Clearly $E(R) = \{g \in E : f_g \in E^o\} = \{g \in E : gg^o = g\} = I$. Suppose that $x, y \in R$ and $g, h \in E(L)$ are such that $f_x = g^o, f_y = h^o$ and $xg = yh$. Then

$$\overline{xg} = \overline{x}f_xe_g\overline{g} = \overline{x}\ \overline{x}^*\overline{g}^+\overline{g} = \overline{x}\ \overline{g} = \overline{x}f_x = \overline{x}\ \overline{x}^* = \overline{x},$$

and similarly $\overline{yh} = \overline{y}$. Consequently $\overline{x} = \overline{y}$. By the assumption $xg = yh$, we have $e_x\overline{x}f_xg = e_y\overline{y}f_yh$. Premultiplying by $\overline{x}^+ = \overline{y}^+$, we obtain $\overline{x}f_xg = \overline{y}f_yh =$

$\overline{x}f_yh$. Since $\overline{x}\mathcal{L}^*\overline{x}^*$, we have $\overline{x}^*f_xg = \overline{x}^*f_yh$ and $g = \overline{x}^*f_xg = \overline{x}^*f_yh = \overline{y}^*f_yh = h$. Consequently $x = xf_x = xg^o = xgg^o = yhh^o = y$. □

Combining Propositions 2.6 and 2.7 with Lemma 1.6, we deduce that R and L are quasi-adequate semigroups sharing a common adequate transversal S^o. Moreover, $E(R) = I$ and $E(L) = \Lambda$ are left normal and right normal bands respectively sharing a common semilattice transversal E^o.

3. A structure theorem

Let S be an abundant semigroup with an adequate transversal S^o. In 2000, Chen [4] gave an example to illustrate that I and Λ need not be subsemigroups even if S^o is a quasi-ideal adequate transversal of S. Chen [4, Proposition 2.6] shown that if S satisfies the regularity condition, then I and Λ are both bands. The following example demonstrates that this is not a necessary condition.

Example. Let $S = \{i, e, o, w, f, j\}$ with the following multiplication table.

	i	e	o	w	f	j
i	i	i	o	o	o	o
e	e	e	o	o	o	o
o	o	o	o	o	o	o
w	w	w	o	o	o	o
f	o	o	o	w	f	j
j	w	o	o	w	f	j

Notice that o is the zero element of S, by routine calculation we can check the associativity. From the multiplication table we obtain

(1) $S^o = \{e, o, w, f\}$ is a $*$-adequate subsemigroup of S and $S^oSS^o \subseteq S^o$.

(2) $E(S) = \{i, e, o, f, j\}$ and $i\mathcal{L}e\mathcal{L}^*w\mathcal{R}^*f\mathcal{R}j$.

(3) It is easy to cheek that S^o is an adequate transversal of S with $I = \{i, e, o, f\}$ and $\Lambda = \{e, o, f, j\}$ subbands of S.

Therefore S^o is a quasi-ideal S-adequate transversal of S, but S does not satisfy the regularity condition since $ji \in \Lambda I \subseteq \langle E(S)\rangle$ and $ji = w \notin RegS$.

The main objective in this section is to give a structure theorem for abundant semigroups with quasi-ideal S-adequate transversals. In what follows R denotes a quasi-adequate semigroup with a right ideal adequate transversal S^o. Then by Proposition 2.1, for every $x \in R$, $f_x = \overline{x}^* \in E^o$ and $x = e_x\overline{x}$.

Theorem 3.1. *Let R be a quasi-adequate semigroup with a right ideal adequate transversal S^o and Λ a right normal band with a left ideal semilattice transversal E^o. Suppose that the set of idempotents of S^o coincides with E^o. Let $\Lambda \times R \longrightarrow S^o$ described by $(\lambda, x) \longrightarrow \lambda * x$ be a mapping, such that for any $x, y \in R$ and for any $\lambda, \mu \in \Lambda$:*

(1) *$(\lambda * x)y = \lambda * (xy)$ and $\mu(\lambda * x) = (\mu\lambda) * x$;*

(2) *if $x \in E^o$ or $\lambda \in E^o$, then $\lambda * x = \lambda x$;*

(3) *if* $x_1\mathcal{R}^*x_2$ *in* R, *then for all* $\mu_1,\mu_2\in\Lambda^1$, $y_1,y_2\in R^1$, $y_1(\mu_1*x_1)=y_2(\mu_2*x_1)$ *if and only if* $y_1(\mu_1*x_2)=y_2(\mu_2*x_2)$.

Define a multiplication on the set

$$\Gamma\equiv R|\times|\Lambda=\{(x,\lambda)\in R\times\Lambda : f_x=\lambda^o\}$$

by

$$(x,\lambda)(y,\mu)=(x(\lambda*y),f_{\lambda*y}\mu).$$

Then Γ *is an abundant semigroup with a quasi-ideal* S*-adequate transversal which is isomorphic to* S^o.

Conversely, every abundant semigroup with a quasi-ideal S*-adequate transversal can be constructed in this way.*

To prove this theorem, we give a sequence of lemmas as follows.

Lemma 3.1. *The multiplication on* Γ *is well-defined.*

Proof. We need only prove that $(x(\lambda*y),f_{\lambda*y}\mu)\in\Gamma$. Since $\lambda*y\in S^o$, clearly $x(\lambda*y)\in R$ and $f_{\lambda*y}=(\lambda*y)^*\in E^o$. Hence $f_{\lambda*y}\cdot\mu=(\lambda*y)^*\mu\in\Lambda$ and it is easy to check that $[(\lambda*y)^*\mu]^o=(\lambda*y)^*\mu^o$, where μ^o denotes the unique inverse in S^o of μ. It is evident that $\mu^o\in E^o$ and $\mu^o\mathcal{R}\mu$, and we then have

$$(\lambda*y)^*\mu^o\mathcal{L}^*(\lambda*y)\mu^o=(\lambda*y)f_y=\lambda*(yf_y)=(\lambda*y)\mathcal{L}^*(\lambda*y)^*.$$

Consequently $(\lambda*y)^*\mu^o=(\lambda*y)^*$ since each $\mathcal{L}^*$-class of an adequate semigroup contains a unique idempotent. On the other hand,

$$\begin{aligned} f_{x(\lambda*y)}&=[f_xe_{(\lambda*y)}\,\overline{\lambda*y}]^*\cdot f_{\lambda*y}=[f_x(\lambda*y)]^*\cdot(\lambda*y)^* \qquad (\text{since}\ \ \lambda*y\in S^o)\\ &=[\lambda^o(\lambda*y)]^*(\lambda*y)^*=(\lambda^o\lambda*y)^*(\lambda*y)^*=(\lambda*y)^*. \end{aligned}$$

Thus $f_{x(\lambda*y)}=(f_{\lambda*y}\mu)^o$ and therefore $(x(\lambda*y),f_{\lambda*y}\mu)\in\Gamma$. □

Lemma 3.2. Γ *is a semigroup.*

Proof. Let $(x,\lambda),(y,\mu),(z,k)\in\Gamma$. Then

$$\begin{aligned} [(x,\lambda)(y,\mu)](z,k)&=[x(\lambda*y),f_{\lambda*y}\mu](z,k)\\ &=(x(\lambda*y)[(f_{\lambda*y}\mu)*z],f_{[(\lambda*y)^*\mu]*z}k)\\ &=(x(\lambda*y)[(\lambda*y)^*(\mu*z)],[(\lambda*y)^*(\mu*z)]^*k)\\ &=(x(\lambda*y)(\mu*z),[(\lambda*y)(\mu*z)]^*k). \end{aligned}$$

On the other hand,

$$\begin{aligned} (x,\lambda)[(y,\mu)(z,k)]&=(x,\lambda)[y(\mu*z),f_{\mu*z}k]\\ &=(x([\lambda*(y(\mu*z)],[\lambda*(y(\mu*z))]^*k)\\ &=(x[(\lambda*y)(\mu*z)],[(\lambda*y)(\mu*z)]^*k)\\ &=(x(\lambda*y)(\mu*z),[(\lambda*y)(\mu*z)]^*k). \end{aligned}$$

Thus $[(x,\lambda)(y,\mu)](z,k)=(x,\lambda)[(y,\mu)(z,k)]$, and Γ is a semigroup. □

Lemma 3.3. *Let* $(x,\lambda)\in\Gamma$. *Then* $(x,\lambda)\in E(\Gamma)$ *if and only if* $\lambda*x=f_x$.

Proof. Since $(x,\lambda)(x,\lambda) = (x(\lambda * x), (\lambda * x)^*\lambda)$, $\lambda^o = f_x = \overline{x}^*$ and $\lambda^o\mathcal{R}\lambda$, we deduce that if $\lambda * x = f_x$, then $(x,\lambda) \in E(\Gamma)$. Conversely, if $(x,\lambda) \in E(\Gamma)$, then $x = x(\lambda * x)$. Thereby $f_x = f_x(\lambda * x) = \lambda^o(\lambda * x) = (\lambda^o\lambda) * x = \lambda * x$ since $x\mathcal{L}^* f_x$. □

Lemma 3.4. *Suppose that* $(x,\lambda) \in \Gamma$, *denote* $u = (e_x, \overline{x}^+)$ *and* $v = (f_x,\lambda)$. *Then* $u, v \in E(\Gamma)$ *and* $u\mathcal{R}^*(x,\lambda)\mathcal{L}^*v$.

Proof. Since $e_x = e_x \cdot \overline{x}^+ \cdot \overline{x}^+$, we have $\overline{x}^+ * e_x = \overline{x}^+ e_x = \overline{x}^+ = f_{e_x}$ and $\lambda * f_x = \lambda f_x = \lambda\lambda^o = \lambda^o = f_x$. By Lemma 3.3, $u, v \in E(\Gamma)$. Computing

$$(x,\lambda)v = (x,\lambda)(f_x,\lambda) = (x\lambda f_x, (\lambda f_x)^*\lambda) = (xf_x, f_x\lambda) = (x,\lambda),$$

since $f_x = \lambda^o \in E^o$ and $\lambda^o\mathcal{R}\lambda$.

For any $(x_1,\lambda_1),(x_2,\lambda_2) \in \Gamma^1$, if $(x,\lambda)(x_1,\lambda_1) = (x,\lambda)(x_2,\lambda_2)$, then

$$(x(\lambda * x_1), (\lambda * x_1)^*\lambda_1) = (x(\lambda * x_2), (\lambda * x_2)^*\lambda_1).$$

Thus $x(\lambda * x_1) = x(\lambda * x_2)$ and since $x\mathcal{L}^* f_x$, we have $f_x(\lambda * x_1) = f_x(\lambda * x_2)$. Therefore $(f_x,\lambda)(x_1,\lambda_1) = (f_x,\lambda)(x_2,\lambda_2)$. By the dual of Lemma 1.1, $(x,\lambda)\mathcal{L}^*v$. Also, we have

$$\begin{aligned} u(x,\lambda) &= (e_x, \overline{x}^+)\ (x,\lambda) \\ &= (e_x \cdot \overline{x}^+ \cdot x, (\overline{x}^+ x)^*\lambda) \\ &= (e_x x, \overline{x}^*\lambda) \qquad (\overline{x}^+ x = \overline{x}^+ e_x \overline{x} f_x = \overline{x}^+ \overline{x}\overline{x}^* = \overline{x}) \\ &= (x,\lambda) \qquad (\text{since } \ \overline{x}^* = \lambda^o \ \text{ and } \ \lambda^o\mathcal{R}\lambda). \end{aligned}$$

If $(x_1,\lambda_1)(x,\lambda) = (x_2,\lambda_2)(x,\lambda)$ for any $(x_1,\lambda_1),(x_2,\lambda_2) \in \Gamma^1$, then

$$(x_1(\lambda_1 * x), (\lambda_1 * x)^*\lambda) = (x_2(\lambda_2 * x), (\lambda_2 * x)^*\lambda).$$

Thus $x_1(\lambda_1 * x) = x_2(\lambda_2 * x)$, and since $x\mathcal{R}^* e_x$, we have $x_1(\lambda_1 * e_x) = x_2(\lambda_2 * e_x)$ by (4). Consequently,

$$\begin{aligned} \lambda_1 * e_x &= \lambda_1^o(\lambda_1 * e_x) = f_{x_1}(\lambda_1 * e_x)\mathcal{L}^* x_1(\lambda_1 * e_x) \\ &= x_2(\lambda_2 * e_x)\mathcal{L}^* f_{x_2}(\lambda_2 * e_x) = \lambda_2^o(\lambda_2 * e_x) = \lambda_2 * e_x. \end{aligned}$$

Therefore $(x_1,\lambda_1)(e_x,\overline{x}^+) = (x_2,\lambda_2)(e_x,\overline{x}^+)$. By Lemma 1.1, $u\mathcal{R}^*(x,\lambda)$. □

Lemma 3.5. Γ *is an abundant semigroup.*

Proof. It follows from Lemma 3.4 immediately. □

Lemma 3.6. *Let* $W = \{(x,x^*) : x \in S^o\}$. *Then* W *is an adequate* $*$-*subsemigroup of* Γ, *and* W *is isomorphic to* S^o.

Proof. Clearly $W \subseteq \Gamma$. Let $(x,x^*),(y,y^*) \in W$. It is a routine matter to prove that $(x,x^*)(y,y^*) = (xy,(xy)^*) \in W$, and W is a subsemigroup. For any $s \in S^o$, define $s\varphi = (s,s^*)$, it is easy to check that φ is an isomorphism. Thus $S^o \cong W$.

To show that W is a $*$-subsemigroup, let $(x, x^*) \in W$. By Lemmas 3.3 and 3.4, $u = (x^+, x^+) \in E(W)$ and $v = (x^*, x^*) \in E(W)$. The following is to prove that

$$v\mathcal{L}^*(\Gamma)(x, x^*)\mathcal{R}^*(\Gamma)u.$$

It is easy to see that

$$(x, x^*)v = (xx^*x^*, (x^*x^*)^*x^*) = (x, x^*)$$

and

$$u(x, x^*) = (x^+x^+x, (x^+x)^*x^*) = (x, x^*).$$

For all $(y_1, \mu_1), (y_2, \mu_2) \in \Gamma^1$, if $(x, x^*)(y_1, \mu_1) = (x, x^*)(y_2, \mu_2)$, then

$$(xx^*y_1, (x^*y_1)^*\mu_1) = (xx^*y_2, (x^*y_2)^*\mu_2).$$

This implies that $xy_1 = xy_2$ and $(x^*y_1)^*\mu_1 = (x^*y_2)^*\mu_2$. Consequently, $x^*y_1 = x^*y_2$ since $x\mathcal{L}^*x^*$. Therefore $(x^*, x^*)(y_1, \mu_1) = (x^*, x^*)(y_2, \mu_2)$. By the dual of Lemma 1.1, $v\mathcal{L}^*(\Gamma)(x, x^*)$.

Suppose that $(z_1, k_1)(x, x^*) = (z_2, k_2)(x, x^*)$ for all $(z_1, k_1), (z_2, k_2) \in \Gamma^1$, then

$$(z_1(k_1 * x), (k_1 * x)^*x^*) = (z_2(k_2 * x), (k_2 * x)^*x^*).$$

This implies that $z_1(k_1*x) = z_2(k_2*x)$ and $(k_1*x)^* = (k_1*x)^*x^* = (k_2*x)^*x^* = (k_2 * x)^*$. From (3) we deduce that $z_1(k_1 * x^+) = z_2(k_2 * x^+)$ and $z(k_1 * x^+)^* = z(k_2 * x^+)^*$ since $x\mathcal{R}^*x^+$. Therefore, $(z_1, k_1)(x^+, x^+) = (z_2, k_2)(x^+, x^+)$. By Lemma 1.1, $u\mathcal{R}^*(\Gamma)(x, x^*)$. □

Lemma 3.7. *Let* $(x_1, \lambda_1), (x_2, \lambda_2) \in \Gamma$. *Then*

(1) $(x_1, \lambda_1)\mathcal{R}^*(x_2, \lambda_2)$ *if and only if* $e_{x_1} = e_{x_2}$ *and* $\overline{x_1}^+ = \overline{x_2}^+$.

(2) $(x_1, \lambda_1)\mathcal{L}^*(x_2, \lambda_2)$ *if and only if* $f_{x_1} = f_{x_2}$ *and* $\lambda_1 = \lambda_2$.

Proof. To prove (1), by Lemma 3.4 it is equivalent to show that $(e_{x_1}, \overline{x_1}^+)\mathcal{R}^*$ $(e_{x_2}, \overline{x_2}^+)$ if and only if $e_{x_1} = e_{x_2}$. If $e_{x_1} = e_{x_2}$, then $\overline{x_1}^+ = f_{e_{x_1}} = f_{e_{x_2}} = \overline{x_2}^+$, and thus $(e_{x_1}, \overline{x_1}^+) = (e_{x_2}, \overline{x_2}^+)$.

Conversely, if $u_1 = (e_{x_1}, \overline{x}_1^+)\mathcal{R}^*(e_{x_2}, \overline{x_2}^+) = u_2$, then $u_1u_2 = u_2$ and $u_2u_1 = u_1$, this implies that

$$e_{x_1}e_{x_2} = e_{x_2}, \quad \overline{x_1}^+ \cdot \overline{x_2}^+ = \overline{x_2}^+,$$
$$e_{x_2}e_{x_1} = e_{x_1}, \quad \overline{x_2}^+ \cdot \overline{x_1}^+ = \overline{x_1}^+.$$

Hence $e_{x_1}\mathcal{R}^*e_{x_2}$ and $\overline{x_1}^+ = \overline{x_2}^+$. By the proof of Lemma 1.4 (see [4]), we have $e_{x_1} = e_{x_2}$.

(2) can be proved similarly. □

Lemma 3.8. *W is an adequate transversal of* Γ.

Proof. Given $a = (x, \lambda) \in \Gamma$, put

$$\overline{a} = (\overline{x}, \overline{x}^*) \in W, e_a = (e_x, \overline{x}^+), f_a = (f_x, \lambda),$$

it is easy to check that $a = e_a\overline{a}f_a$. Furthermore,

$$e_a, f_a \in E(\Gamma), e_a\mathcal{L}\overline{a}^+ = (\overline{x}^+, \overline{x}^+), f_a\mathcal{R}\overline{a}^* = (\overline{x}^*, \overline{x}^*).$$

Suppose that a can be written in another form $a = e'_a\overline{a}'f'_a$, where

$$e'_a = (y_1, \mu_1) \in E(\Gamma), f'_a = (y_2, \mu_2) \in E(\Gamma), \overline{a}' = (\overline{y}, \overline{y}^*) \in W$$

and

$$e'_a\mathcal{L}\overline{a}'^+ = (\overline{y}^+, \overline{y}^+), f'_a\mathcal{R}\overline{a}'^* = (\overline{y}^*, \overline{y}^*).$$

By Lemma 3.7, we have $f_{y_1} = \overline{y}^+, \mu_1 = \overline{y}^+$ and $\overline{y}^* = e_{y_2}$, $\overline{y}^* = \overline{y_2}^+$. It follows from $e_a\mathcal{R}^*a\mathcal{R}^*e'_a$ that $e_x = e_{y_1}, \overline{x}^+ = \overline{y_1}^+$. Similarly, $f_x = f_{y_2}, \lambda = \mu_2$. Therefore $e'_a = (y_1, \overline{y}^+), f'_a = (y_2, \lambda)$. Since $e'_a, f'_a \in E(\Gamma)$, we have $\overline{y}^+y_1 = f_{y_1} = \overline{y}^+$ and $\lambda * y_2 = f_{y_2}$. From $f_{y_1} \cdot y_1 = f_{y_1}$, we deduce that $y_1 \cdot y_1 = y_1$ since $y_1\mathcal{L}^*f_{y_1}$. Thus y_1 is idempotent and since $y_1 \in R$, we have $y_1 \in E(R)$ and $y_1 = e_{y_1}$. Therefore

$$y_2 = e_{y_2} \cdot \overline{y_2} \cdot f_{y_2} = \overline{y}^* \cdot \overline{y_2} \cdot \overline{y_2}^* = \overline{y_2}^+ \cdot \overline{y_2} \cdot \overline{y_2}^* = \overline{y_2} \in S^o,$$

and

$$\lambda * y_2 = \lambda * \overline{y_2} = \lambda * \overline{y_2}^+ \cdot \overline{y_2} = \lambda\overline{y_2}^+\overline{y_2} = \lambda\overline{y_2} = \lambda y_2.$$

Consequently, $\lambda y_2\lambda = f_{y_2}\lambda = \lambda^o\lambda = \lambda$ and $y_2\lambda y_2 = y_2 f_{y_2} = y_2$. Therefore $y_2 = \lambda^o = f_{y_2} = f_x$. On the other hand, we have

$$(x, \lambda) = a = e'_a\overline{a}'f'_a = (e_x, \overline{y}^+)(\overline{y}, \overline{y}^*)(f_x, \lambda) = (e_x\overline{y}f_x, (\overline{y}f_x)^*\lambda)$$

and $e_x\overline{y}f_x = x = e_x\overline{x}f_x$, therefore premultiplying by $\overline{x}^+$ and postmultiplying by $\overline{x}^*$, $\overline{x} = \overline{x}^+\overline{y}\ \overline{x}^*$. Also, $\overline{x}^* = f_x = y_2 = e_{y_2} = \overline{y}^*$ since $y_2 = \overline{x}^* \in E^o$. Similarly, $\overline{x}^+ = \overline{y_1}^+ = f_{y_1} = \mu_1^o = \mu_1 = \overline{y}^+$ since $\mu_1 = \overline{y}^+ \in E^o$. Now, we have $\overline{x} = \overline{x}^+\overline{yx}^* = \overline{y}^+\overline{yy}^* = \overline{y}$. Therefore, the uniqueness of $e_a, \overline{a}$ and f_a is obtained, by Definition 1.3, W is an adequate transversal of Γ. □

Lemma 3.9. *W is a quasi-ideal S-adequate transversal of* Γ.

Proof. For any $a = (\overline{x}, \overline{x}^*), b = (\overline{y}, \overline{y}^*) \in W$ and $c = (z, k) \in \Gamma$, since $\overline{x}z \in S^o$ and $k * \overline{y} \in S^o$, we have

$$acb = (\overline{x}, \overline{x}^*)(z, k)(\overline{y}, \overline{y}^*) \in W.$$

Let

$$E^o| \times |\Lambda = \{(x\lambda) \in \Gamma : \ x \in E^o\} \ \text{ and } \ R| \times |E^o = \{(x, \lambda) \in \Gamma : \ \lambda \in E^o\}.$$

For any $(x, \lambda) \in \Lambda(\Gamma)$, then $(x, \lambda) \in E(\Gamma)$ and there exists $(y^+, y^+) \in E(W)$ with $y^+ \in E^o$ such that $(x, \lambda)\mathcal{R}(y^+, y^+)$. Then by Lemmas 3.3 and 3.7, $\lambda * x = \lambda x = f_x$ and $e_x = y^+$. Thus $x \in R \cap L = S^o$ and consequently $f_x = \lambda * x = \lambda x$. By the proof of Lemma 3.8 we have $x \in E^o$ and so $\Lambda(\Gamma) \subseteq E^o| \times |\Lambda$. Conversely, if $(x, \lambda) \in E^o| \times |\Lambda$ with $x \in E^o$, we have $\lambda * x = \lambda x = \lambda f_x = \lambda\lambda^o = \lambda^o$, and $(x, \lambda) \in E(\Gamma)$. Certainly $(x, \lambda)\mathcal{R}(x, x) \in E(W)$ and so $E^o| \times |\Lambda \subseteq \Lambda(\Gamma)$. Therefore $\Lambda(\Gamma) = E^o| \times |\Lambda \cong \Lambda$ is a right normal band.

Similarly, by Lemma 3.7 we may show that $R(\Gamma) = R| \times |E^o \cong R$ is a quasi-adequate semigroup whose set of idempotents forms a left normal band. Consequently, W is a quasi-ideal S-adequate transversal of Γ. □

Up to now we have proved the direct half of Theorem 3.1. To prove the converse half, let S be an abundant semigroup with a quasi-ideal S-adequate transversal S^o. Let R and L be described as in Lemma 1.7. Then R and L are quasi-adequate semigroups sharing a common quasi-adequate transversal S^o, which is a right ideal of R and a left ideal of L. Also $\Lambda = E(L)$ and Λ is a right normal band with a semilattice transversal $E(S^o) = E^o$.

For every $(\lambda, x) \in \Lambda \times R$, put $\lambda * x = \lambda x$. Then $\lambda * x = \lambda x = e_\lambda \lambda x f_x \in S^o$ since $e_\lambda \in E^o, f_x \in E^o$ and S^o is a quasi-ideal of S. Clearly the map $\Lambda \times R \to S^o$ defined by $(\lambda, x) \to \lambda x$ satisfies (1), (2) and (3) since S^o is a quasi-ideal S-adequate transversal of S.

Next we prove that Γ is isomorphic to S. Define $\varphi : \Gamma \to S$ by $\varphi(x, \lambda) = x\lambda$, then certainly φ is well defined and by Proposition 2.7 φ is injective.

For any $(x, \lambda), (y, \mu) \in \Gamma$, we have

$$\begin{aligned} \varphi[(x,\lambda)(y,\mu)] &= \varphi((x\lambda y, (\lambda y)^*\mu)) = x\lambda y(\lambda y)^*\mu \\ &= x\lambda y\mu = \varphi((x,\lambda))\varphi((y,\mu)), \end{aligned}$$

and whence θ is a homomorphism.

For every $a \in S$, then there exist $\overline{a} \in S^o$ and idempotents $e_a, f_a \in S$ such that $a = e_a \overline{a} f_a$ where $e_a \mathcal{L} \overline{a}^+$ and $f_a \mathcal{R} \overline{a}^*$. Thus $e_a \overline{a} \in R$ since $f_{e_a \overline{a}} = (f_{e_a} \cdot e_{\overline{a}} \cdot \overline{a})^* f_{\overline{a}} = (\overline{a}^+ \cdot \overline{a})^* \overline{a}^* = \overline{a}^* \in E^o$ and $f_a \in \Lambda$ since $e_{f_a} = \overline{a}^* \in E^o$. Then it is easy to verify that $(fa)^o = \overline{a}^* = f_{e_a \overline{a}}$ and so $(e_a \overline{a}, f_a) \in \Gamma$. Hence $\varphi((e_a \overline{a}, f_a)) = e_a \overline{a} f_a = a$ and so θ is surjective. Therefore θ is an isomorphism.

Acknowledgements. The authors would like to express their sincere thanks to the referees for their important and constructive modifying suggestions.

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