MATHEMATICS OF COMPUTATION
Volume 83, Number 285, January 2014, Pages 475–512
S 0025-5718(2013)02701-9
Article electronically published on April 26, 2013

ACCELERATING DOUGALL'S ${}_5F_4$-SUM AND INFINITE SERIES INVOLVING π

WENCHANG CHU AND WENLONG ZHANG

Abstract. The modified Abel lemma on summation by parts is employed to investigate the partial sum of Dougall's ${}_5H_5$-series. Several unusual transformation formulae into fast convergent series are established. They lead surprisingly to numerous infinite series identities involving π, $\zeta(3)$ and the Catalan constant, including several important ones discovered by Ramanujan (1914) and recently by Guillera.

1. Introduction

As one of the most important mathematical constants, π appears virtually in the entire history of mathematics and plays a fundamental role in innumerable proofs and calculations in mathematics, physics and the applied sciences. The first eight digits of π are 3.1415926, while it is sufficient to approximate π as 3.14 in most computations. Several great mathematicians, including Archimedes, Newton, Euler and Gauss, have made significant contributions to computing π more accurately. In the twentieth century, remarkable progress has been made in representing π by rapidly converging series discovered mainly by Ramanujan [49, 1914] (for his life and work, refer, for example, to the twelve wonderful lectures by Hardy [44]), Jonathan and Peter Borwein [22, 1987], and David and Gregory Chudnovsky [30, 1987]. There are several marvelous monographs presenting the history and computation methods for π:

- Borweins [22, Chapters 5 and 11]: Modular equations.
- Berndt [15, Chapter 9]: Apéry series.
- Berndt [16, Chapter 29]: Ramanujan's series for $1/\pi$.
- Borwein–Bailey [20, Chapter 3]: π and its friends.
- Borwein–Bailey–Girgensohn [21, §1.7]: Apéry series.
- Bailey–Borwein–Calkin–Girgensohn–Luke–Moll [8, §2.4, §2.7 and §3.6].
- Boros–Moll [19, Chapter 6]: π and integrals of rational functions.
- Berggren–Borweins [14]: π-source book.

Further introductory information on π can be found in the following excellent papers by Borweins [23], Bailey, Borweins and Plouffe [7], Bailey–Borwein [5, 6], Baruah, Berndt and Chan [12] and Guillera [40].

Received by the editor December 9, 2011 and, in revised form, March 27, 2012.
2010 *Mathematics Subject Classification.* Primary 33D15, Secondary 05A15.
Key words and phrases. Abel's lemma on summation by parts, classical hypergeometric series, partial sum of Dougall's ${}_5H_5$-series, acceleration of convergent series, π-formulae of Ramanujan and Guillera.

Among many known infinite series about π in mathematical literature, three classical ones are (see ($\star$) below for the notation used here)

$$\frac{4}{\pi}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac12, & \frac12, & \frac12\\ 1, & 1, & 1\end{matrix}\right]_k\frac{1+6k}{4^k}, \qquad \text{(Example 9)}; \tag{1}$$

$$\frac{8}{\pi}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac12, & \frac14, & \frac34\\ 1, & 1, & 1\end{matrix}\right]_k\frac{3+20k}{(-4)^k}, \qquad \text{(Example 10)}; \tag{2}$$

$$\frac{16}{\pi}=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac12, & \frac12, & \frac12\\ 1, & 1, & 1\end{matrix}\right]_k\frac{5+42k}{64^k}, \qquad \text{(Example 116)}. \tag{3}$$

They are instances from the 17 formulae recorded by Ramanujan [49, 1914] (see Berndt [16, Chapter 29] also). Although Ramanujan himself did not explain how he arrived at his series, he did indicate that they belong to what is now called as "the theories of elliptic functions to alternative bases" mainly developed by Borweins [24] and Berndt, Bhargava and Garvan [18] (cf. Berndt [17, Chapter 33] also). The first rigorous mathematical proofs of Ramanujan's identities and generalizations of them were given by the Borwein brothers [22, §5] and the Chudnovsky brothers [30].

The earliest formula of Ramanujan–type is (see also Hardy [43, Equation 2])

$$\frac{2}{\pi}=\sum_{k=0}^{\infty}(-1)^k\left[\begin{matrix}\frac12, \frac12, \frac12\\ 1,\ 1,\ 1\end{matrix}\right]_k\Big\{1+4k\Big\}.$$

Bauer [13] discovered this in 1859 by computing the Fourier–Legendre coefficients for the algebraic function $1/\sqrt{1-a^2x^2}$. However, Glaisher [32] was probably the first before Ramanujan, who had made a systematic investigation of π-formulae through elliptic functions. Observe that the last formula can also be deduced by letting $a=b=c=1/2$ and $d\to-\infty$ in the following theorem for the well–poised ${}_5F_4$-series due to Dougall [31] (cf. Bailey [10, §4.4] and Slater [50, §2.3] also)

$$\begin{aligned}&{}_5F_4\left[\begin{matrix}a, & 1+a/2, & b, & c, & d\\ & a/2, & 1+a-b, & 1+a-c, & 1+a-d\end{matrix}\,\middle|\,1\right]\\ &=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}\end{aligned}$$

provided that $\Re(1+a-b-c-d)>0$ for convergence. This is not an isolated case. In fact, all 31 principal series for $1/\pi$ and $1/\pi^2$, including Bauer's, collected by Glaisher [32] can be derived from this formula published by Dougall [31] in 1907, two years after Glaisher's paper. Following the same approach of Bauer, Levrie [45] recently derived two further infinite series expressions for $1/\pi$ and $1/\pi^2$; but both of them result again from special cases of Dougall's summation theorem.

Wilf and Zeilberger [53] introduced a new and powerful method, based on Gosper's indefinite summation algorithm [33], for proving identities for hypergeometric series (cf. Petkovšek, Wilf and Zeilberger [46] also). This approach has further been developed intensively by Guillera [39], who reviewed systematically the formulae discovered by Ramanujan [49], found further π-formulae of Ramanujan–type and conjectured experimentally the following beautiful and challenging series representations for $1/\pi^2$ (cf. [34, 35, 36] for the published version):

(4) $$\frac{8}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \begin{bmatrix} \frac12, \frac12, \frac12, \frac12, \frac12 \\ 1, 1, 1, 1, 1 \end{bmatrix}_n \Big\{1+8n+20n^2\Big\}, \qquad \text{(Example 5);}$$

(5) $$\frac{32}{\pi^2} = \sum_{n=0}^{\infty} \frac{1}{2^{4n}} \begin{bmatrix} \frac12, \frac12, \frac12, \frac14, \frac34 \\ 1, 1, 1, 1, 1 \end{bmatrix}_n \Big\{3+34n+120n^2\Big\}, \qquad \text{(Example 16);}$$

(6) $$\frac{48}{\pi^2} = \sum_{n=0}^{\infty} \Big(\frac{27}{64}\Big)^n \begin{bmatrix} \frac12, \frac12, \frac12, \frac13, \frac23 \\ 1, 1, 1, 1, 1 \end{bmatrix}_n \Big\{3+27n+74n^2\Big\}, \qquad \text{(Example 53);}$$

(7) $$\frac{128}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \begin{bmatrix} \frac12, \frac12, \frac12, \frac12, \frac12 \\ 1, 1, 1, 1, 1 \end{bmatrix}_n \Big\{13+180n+820n^2\Big\}, \quad \text{(Example 64);}$$

(8) $$\frac{48}{\pi^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{48^n} \begin{bmatrix} \frac12, \frac13, \frac23, \frac14, \frac34 \\ 1, 1, 1, 1, 1 \end{bmatrix}_n \Big\{5+63n+252n^2\Big\},$$

(9) $$\frac{256}{\pi^2\sqrt3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{10n}} \begin{bmatrix} \frac12, \frac14, \frac34, \frac16, \frac56 \\ 1, 1, 1, 1, 1 \end{bmatrix}_n \Big\{15+278n+1640n^2\Big\},$$

(10) $$\frac{640}{\pi^2\sqrt5} = \sum_{n=0}^{\infty} \frac{(-1)^n}{80^{3n}} \begin{bmatrix} \frac12, \frac13, \frac23, \frac16, \frac56 \\ 1, 1, 1, 1, 1 \end{bmatrix}_n \Big\{29+693n+5418n^2\Big\},$$

(11) $$\frac{392}{\pi^2\sqrt7} = \sum_{n=0}^{\infty} \frac{1}{7^{4n}} \begin{bmatrix} \frac12, \frac18, \frac38, \frac58, \frac78 \\ 1, 1, 1, 1, 1 \end{bmatrix}_n \Big\{15+304n+1920n^2\Big\},$$

$$\frac{32}{\pi^3} = \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \begin{bmatrix} \frac12 \\ 1 \end{bmatrix}_n^7 \Big\{1+14n+76n^2+168n^3\Big\}, \qquad \text{(Boris Gourévitch).}$$

For the first four formulae, Guillera [34, 38, 41, 42] himself found a WZ–style proof via *Computer Algebra*. The other five formulae just displayed remain unproven. By combining certain representations for Eisenstein series and identities for hypergeometric series with the values of singular moduli, Baruah and Berndt [11] derived many new infinite series expressions for $1/\pi^2$ besides (4), (5) and (7). A more thorough experimental search for similar identities was attempted in [8, §2.7]. Zudilin [54, 55] discussed and illustrated how to deal with these series. Further formulae of similar type may be found in [41].

The purpose of this work is to present a hypergeometric series approach to infinite series identities involving π, $\zeta(3)$ and the Catalan constant G. In the classical analysis, Abel's lemma on summation by parts plays an important role in determining convergence conditions for infinite series. Recently, a modified version of this lemma has also been found to be quite useful to evaluate infinite series. In fact, this approach has successfully been employed by Chu [25, 26, 27, 28] and Chu and Jia [29] to review several summation and transformation formulae of classical, basic and elliptic hypergeometric series. In the present work, the modified Abel's lemma on summation by parts will be utilized further to investigate the following partial sum of the bilateral well–poised ${}_5H_5$-series due to Dougall [31]

$$\boxed{\Omega(a;b,c,d,e) \;:=\; \sum_{k=0}^{\infty}(a+2k)\begin{bmatrix} b, & c, & d, & e \\ 1+a-b, & 1-a-c, & 1-a-d, & 1+a-e\end{bmatrix}_k}$$

provided that $\Re(1+2a-b-c-d-e)>0$ for convergence. We shall show that the series $\Omega(a;b,c,d,e)$ can be expressed in terms of another shifted one, $\Omega(a+n_a;b+n_b,c+n_c,d+n_d,e+n_e)$, with $\{n_a,n_b,n_c,n_d,n_e\}$ being integers, that will be called an *iteration pattern* $[n_a\ n_b\ n_c\ n_d\ n_e]$. Three fundamental recurrence relations will be established. They will be iterated to derive several transformation formulae which express $\Omega(a;b,c,d,e)$ in terms of fast convergent series. Surprisingly, these transformations will further lead to numerous π-formulae of infinite series, including several important ones discovered by Ramanujan [49] (1914) and more recently by Guillera [35, 36, 37, 38, 39, 40, 41, 42].

For instance, four formulae due to Guillera (4), (5), (6) and (7) will be shown to be very particular cases of the transformations with the iteration patterns [20111], [20101], [31113] and [31111], respectively. Further series from the other iteration patterns for $\pi^{\pm1}$, $\pi^{\pm2}$, $\zeta(3)$ and G (the Catalan constant) can be exemplified as follows (where the formulae displayed in Examples 2, 73, 88 and 114 have been derived also by Guillera [41]):

- Example 2
$$\frac{7}{2}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}1, & 1, & 1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\Big\{5+14k+10k^2\Big\}.$$
- Example 118
$$144\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{1}{729}\Big)^k\begin{bmatrix}1, & 1, & 1, & 1\\ \frac{4}{3}, & \frac{4}{3}, & \frac{5}{3}, & \frac{5}{3}\end{bmatrix}_k\frac{173+501k+364k^2}{1+2k}.$$
- Example 73
$$\frac{\pi^2}{4}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\begin{bmatrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\Big\{2+3k\Big\}.$$
- Example 114
$$\frac{4\pi^2}{3}=\sum_{k=0}^{\infty}\Big(\frac{1}{64}\Big)^k\begin{bmatrix}1, & 1, & 1\\ \frac{3}{2}, & \frac{3}{2}, & \frac{3}{2}\end{bmatrix}_k\Big\{13+21k\Big\}.$$
- Example 47
$$\frac{15\pi}{4}=\sum_{k=0}^{\infty}\Big(\frac{16}{27}\Big)^k\begin{bmatrix}1, & \frac{1}{4}, & \frac{3}{4}\\ \frac{1}{2}, & \frac{7}{6}, & \frac{11}{6}\end{bmatrix}_k\Big\{8+11k\Big\}.$$
- Example 105
$$\frac{81\sqrt{3}}{2\pi}=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}\frac{1}{3}, & \frac{2}{3}, & \frac{1}{6}, & \frac{5}{6}\\ 1, & 1, & 1, & \frac{3}{2}\end{bmatrix}_k\Big\{20+243k+414k^2\Big\}.$$
- Example 59
$$\frac{64}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}\frac{1}{4}, & \frac{3}{4}, & \frac{1}{6}, & \frac{5}{6}\\ 1, & 1, & 1, & \frac{3}{2}\end{bmatrix}_k\Big\{15+182k+296k^2\Big\}.$$

- Example 88

$$2G=\sum_{k=0}^{\infty}\Big(\frac{-1}{8}\Big)^k\begin{bmatrix}1,\ 1,\ 1\\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2}\end{bmatrix}_k\Big\{2+3k\Big\}.$$

- Example 26

$$\frac{256}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4}\\ 1,\ 1,\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k \times\Big\{27+492k+3376k^2+8832k^3+7168k^4\Big\}.$$

- Example 45

$$\frac{64}{\pi^2}=\sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4}\\ 1,\ 1,\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k \times\Big\{9+138k+784k^2+1808k^3+1376k^4\Big\}.$$

Our study shows that these formulae together with (4), (5), (6) and (7) form only the tip of the iceberg, with each of them representing, in fact, one of the simplest expressions among a very large class of infinite series identities involving π, G and $\zeta(3)$ with the same convergence rate.

This paper will be organized as follows. In the next section, Abel's lemma on summation by parts will be utilized to establish three fundamental recurrence relations concerning the partial Dougall sum $\Omega(a;b,c,d,e)$. From Section 3 to Section 9, these recursions will be iterated to derive transformations of $\Omega(a;b,c,d,e)$ into fast convergent series with their convergence rates equal to $\pm\frac{1}{4}$, $\frac{1}{16}$, $-\frac{1}{27}$, $\pm\frac{4}{27}$, $\pm\frac{16}{27}$, $\frac{27}{64}$, $-\frac{1}{1024}$, respectively. These transformations will lead in turn to numerous infinite series identities involving π, G and $\zeta(3)$ with the convergence rates just displayed. Among each class of infinite series, ten representative formulae will be displayed. Finally, the paper will end up with 55 miscellaneous infinite series identities of particular interest that are carefully selected from thousands of similar formulae.

Even though there exist no technical difficulties to derive several thousands of formulae involving π, G and $\zeta(3)$, we have restricted ourselves to making a careful selection of 125 representatives, due to the space limitations. The exhibited identities concentrate primarily on infinite series of Ramanujan–type. However, there are several formulae covering different infinite series of other forms as well (for example, Apéry series and BBP–type formulae). The authors believe that the exhibited infinite series identities will provide a compelling resource for the reader for further insight and understanding of the number π.

Throughout the paper, we shall utilize the following notation for shifted factorial and hypergeometric series as well as three relations about the Γ-function. First we define the shifted factorial by

$$(x)_0\equiv 1 \quad\text{and}\quad (x)_n=x(x+1)\cdots(x+n-1)\quad\text{for}\quad n\in\mathbb{N}$$

with its product and quotient forms being abbreviated respectively as

$$(\star)\qquad \begin{aligned}&[A,\ B,\ \ldots,\ C]_n=(A)_n(B)_n\ldots(C)_n,\\ &\begin{bmatrix}\alpha,\ \beta,\ \cdots,\ \gamma\\ A,\ B,\ \cdots,\ C\end{bmatrix}_n=\frac{(\alpha)_n(\beta)_n\cdots(\gamma)_n}{(A)_n(B)_n\cdots(C)_n}.\end{aligned}$$

Then, following Bailey [10], the generalized hypergeometric series is given by

$$ {}_{1+p}F_q\left[\begin{matrix} a_0,\, a_1,\, a_2,\, \cdots,\, a_p \\ b_1,\, b_2,\, \cdots,\, b_q \end{matrix} \,\Big|\, z\right] = \sum_{n=0}^{\infty} \frac{(a_0)_n(a_1)_n(a_2)_n\cdots(a_p)_n}{n!(b_1)_n(b_2)_n\cdots(b_q)_n} z^n. $$

Most of the classical results for hypergeometric series concern the case $p = q$. When the parameters satisfy the condition $1+a_0+a_1+\cdots+a_p = b_1+\cdots+b_p$, the series is said to be *balanced*, while when $1+a_0 = a_1+b_1 = \cdots = a_p+b_p$, the corresponding series is *well–poised*.

The shifted factorial can also be expressed in terms of the Γ-function quotient

$$ (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{where} \quad \Gamma(x) = \int_0^\infty u^{x-1}e^{-u}\mathrm{d}u \quad \text{with} \quad \Re(x) > 0. $$

In fact, the Γ-function is one of the well–known classical functions which possesses many useful properties (cf. Rainville [48, Chapter 2]). Three important ones among them read as the reciprocal relations

$$ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \quad \text{and} \quad \Gamma(\tfrac{1}{2}+x)\Gamma(\tfrac{1}{2}-x) = \frac{\pi}{\cos \pi x}, $$

the multiplication formula with $n \in \mathbb{N}$,

$$ \Gamma(nx) = (2\pi)^{\frac{1-n}{2}} n^{nx-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma(x+\tfrac{k}{n}), $$

as well as the asymptotic formula as $m \to \infty$,

$$ \Gamma(x+m) \approx m^x\,(m-1)!\,\Gamma(x), $$

where the last one plays the same role as the Stirling formula in this work.

2. Three recurrence relations for the Ω-sum

As the preliminaries for the subsequent development, this section will establish three fundamental relations for the Ω-sum defined by the well–poised series

$$ \Omega(a;b,c,d,e) := \sum_{k=0}^{\infty}(a+2k)\left[\begin{matrix} b, & c, & d, & e \\ 1+a-b, & 1-a-c, & 1-a-d, & 1+a-e \end{matrix}\right]_k $$

provided that $\Re(1+2a-b-c-d-e) > 0$ for convergence. In order to facilitate applications, we reproduce Abel's lemma on summation by parts as follows. For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and Δ, respectively, by

$$ \nabla\tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \Delta\tau_k = \tau_k - \tau_{k+1}, $$

where Δ differs from the usual operator Δ only in the minus sign. Then **Abel's lemma** on summation by parts may be reformulated as

$$ \boxed{\sum_{k=0}^{\infty} V_k \nabla U_k = [UV]_+ - U_{-1}V_0 + \sum_{k=0}^{\infty} U_k \Delta V_k} $$

provided that the limit $[UV]_+ := \lim_{n\to\infty} U_n V_{n+1}$ exists and one of the two series just displayed is convergent.

Proof. According to the definition of the backward difference, we have

$$\sum_{k=0}^{m} V_k \nabla U_k = \sum_{k=0}^{m} V_k\{U_k - U_{k-1}\} = \sum_{k=0}^{m} U_k V_k - \sum_{k=0}^{m} U_{k-1} V_k.$$

Replacing k by $k+1$ for the last sum results in the expression

$$\begin{aligned}\sum_{k=0}^{m} V_k \nabla U_k &= U_m V_{m+1} - U_{-1} V_0 + \sum_{k=0}^{m} U_k\{V_k - V_{k+1}\}\\ &= U_m V_{m+1} - U_{-1} V_0 + \sum_{k=0}^{m} U_k \Delta V_k.\end{aligned}$$

Letting $m \to \infty$, we get the desired formula displayed in Abel's lemma. □

§2.1. **[10001]: The first recursion.** For the two sequences given by

$$\begin{aligned}A_k &= \begin{bmatrix} b+c+d-a, & 1+e \\ 2+2a-b-c-d, & 1+a-e \end{bmatrix}_k,\\ B_k &= \begin{bmatrix} b, & c, & d, & 2+2a-b-c-d \\ 1+a-b, & 1+a-c, & 1+a-d, & b+c+d-1-a \end{bmatrix}_k;\end{aligned}$$

it is routine to verify the relations

$$[AB]_+ = 0 \quad \text{and} \quad A_{-1}B_0 = \frac{(e-a)(1+2a-b-c-d)}{e(1+a-b-c-d)}$$

as well as the finite differences

$$\begin{aligned}\nabla A_k &= \frac{(a+2k)(1+2a-b-c-d-e)}{e(1+a-b-c-d)} \begin{bmatrix} b+c+d-1-a, & e \\ 2+2a-b-c-d, & 1+a-e \end{bmatrix}_k,\\ \Delta B_k &= \begin{bmatrix} b, & c, & d, & 2+2a-b-c-d \\ 2+a-b, & 2+a-c, & 2+a-d, & b+c+d-a \end{bmatrix}_k\\ &\times \frac{(1+a+2k)(1+a-b-c)(1+a-b-d)(1+a-c-d)}{(1+a-b)(1+a-c)(1+a-d)(1+a-b-c-d)}.\end{aligned}$$

According to Abel's lemma on summation by parts, the nonterminating well–poised Ω-sum can be manipulated as

$$\begin{aligned}\Omega(a;b,c,d,e) &= \frac{e(1+a-b-c-d)}{1+2a-b-c-d-e} \sum_{k\geq 0} B_k \nabla A_k\\ &= \frac{e(1+a-b-c-d)}{1+2a-b-c-d-e}\Big\{[AB]_+ - A_{-1}B_0 + \sum_{k\geq 0} A_k \Delta B_k\Big\}.\end{aligned}$$

Observing that the last sum can be expressed as

$$\sum_{k\geq 0} A_k \Delta B_k = \frac{(1+a-b-c)(1+a-b-d)(1+a-c-d)\Omega(1+a;b,c,d,1+e)}{(1+a-b)(1+a-c)(1+a-d)(1+a-b-c-d)}$$

we establish the following functional equation.

Lemma 1 (Recurrence relation **[10001]**: $\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e)=\frac{(1+2a-b-c-d)(a-e)}{1+2a-b-c-d-e}+\Omega(1+a;b,c,d,1+e)$$
$$\times\frac{e(1+a-b-c)(1+a-b-d)(1+a-c-d)}{(1+a-b)(1+a-c)(1+a-d)(1+2a-b-c-d-e)}.$$

§2.2. **[10101]: The second recursion.** Alternatively, recall the difference pairs devised in [28, §3]

$$C_k=\begin{bmatrix}1+c, & 1+e\\ 1+a-c, & 1+a-e\end{bmatrix}_k \quad\text{and}\quad D_k=\begin{bmatrix}b, & d\\ 1+a-b, & 1+a-d\end{bmatrix}_k.$$

It is not hard to check the relations

$$[CD]_+=0 \quad\text{and}\quad C_{-1}D_0=\frac{(a-c)(a-e)}{ce}$$

as well as the finite differences

$$\nabla C_k = \begin{bmatrix}c, & e\\ 1+a-c, & 1+a-e\end{bmatrix}_k \frac{(a+2k)(c+e-a)}{ce},$$
$$\Delta D_k = \begin{bmatrix}b, & d\\ 2+a-b, & 2+a-d\end{bmatrix}_k \frac{(1+a+2k)(1+a-b-d)}{(1+a-b)(1+a-d)}.$$

Then applying Abel's lemma on summation by parts, we can reformulate the nonterminating well–poised Ω-sum as follows:

$$\Omega(a;b,c,d,e) = \frac{ce}{c+e-a}\sum_{k\geq 0}D_k\nabla C_k$$
$$=\frac{ce}{c+e-a}\Big\{[CD]_+-C_{-1}D_0+\sum_{k\geq 0}C_k\Delta D_k\Big\}.$$

Writing the last sum explicitly

$$\sum_{k\geq 0}C_k\Delta D_k = \frac{1+a-b-d}{(1+a-b)(1+a-d)}$$
$$\times\sum_{k\geq 0}(1+a+2k)\begin{bmatrix}b, & 1+c, & d, & 1+e\\ 2+a-b, & 1+a-c, & 2+a-d, & 1+a-e\end{bmatrix}_k$$

we find the following functional equation.

Lemma 2 (Recurrence relation **[10101]**: $\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e)=\frac{(a-c)(a-e)}{a-c-e}+\Omega(1+a;b,1+c,d,1+e)$$
$$\times\frac{ce(1+a-b-d)}{(1+a-b)(1+a-d)(c+e-a)}.$$

§2.3. **[11110]: The third recursion.** Finally, define the two sequences by

$$E_k = \begin{bmatrix}1+b, & 1+c, & 1+d, & 1+2a-b-c-d\\ 1+a-b, & 1+a-c, & 1+a-d, & 1+b+c+d-a\end{bmatrix}_k,$$
$$F_k = \begin{bmatrix}1+b+c+d-a, & e\\ 2a-b-c-d, & 1+a-e\end{bmatrix}_k.$$

Under the condition $\Re(2a-b-c-d-e)>0$, it is not difficult to evaluate the limiting relations

$$[EF]_+ = 0 \quad \text{and} \quad E_{-1}F_0 = \frac{(a-b)(a-c)(a-d)(b+c+d-a)}{bcd(2a-b-c-d)}$$

and to factor the following differences:

$$\begin{aligned}\nabla E_k &= \begin{bmatrix} b, & c, & d, & 2a-b-c-d \\ 1+a-b, & 1+a-c, & 1+a-d, & 1+b+c+d-a \end{bmatrix}_k \\ &\times \frac{(a+2k)(a-b-c)(a-b-d)(a-c-d)}{bcd(2a-b-c-d)}, \\ \Delta F_k &= \begin{bmatrix} 1+b+c+d-a, & e \\ 1+2a-b-c-d, & 2+a-e \end{bmatrix}_k \\ &\times \frac{(1+a+2k)(2a-b-c-d-e)}{(1+a-e)(2a-b-c-d)}.\end{aligned}$$

By means of Abel's lemma on summation by parts, the Ω-sum can be expressed as follows:

$$\begin{aligned}\Omega(a;b,c,d,e) &= \frac{bcd(2a-b-c-d)}{(a-b-c)(a-b-d)(a-c-d)} \sum_{k\geq 0} F_k \nabla E_k \\ &= \frac{bcd(2a-b-c-d)}{(a-b-c)(a-b-d)(a-c-d)} \Big\{ [EF]_+ - E_{-1}F_0 + \sum_{k\geq 0} E_k \Delta F_k \Big\}.\end{aligned}$$

Keeping in mind that

$$\begin{aligned}\sum_{k\geq 0} E_k \Delta F_k &= \frac{2a-b-c-d-e}{(2a-b-c-d)(1+a-e)} \\ &\times \sum_{k\geq 0} (1+a+2k) \begin{bmatrix} 1+b, & 1+c, & 1+d, & e \\ 1+a-b, & 1+a-c, & 1+a-d, & 2+a-e \end{bmatrix}_k\end{aligned}$$

we derive the following functional equation.

Lemma 3 (Recurrence relation **[11110]**: $\Re(2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e) =& \frac{(a-b)(a-c)(a-d)(a-b-c-d)}{(a-b-c)(a-b-d)(a-c-d)} \\ &+ \frac{bcd(2a-b-c-d-e)\Omega(1+a;1+b,1+c,1+d,e)}{(a-b-c)(a-b-d)(a-c-d)(1+a-e)}.\end{aligned}$$

§2.4. **[10001]: Reciprocal relation.** Iterating m-times the equation displayed in Lemma 1, we get the following expression

$$\begin{aligned}\Omega(a;b,c,d,e) =& \Omega(a+m;b,c,d,e+m) \times \begin{bmatrix} 1+a-b-c, \ 1+a-b-d, \ 1+a-c-d, \ e \\ 1+a-b, 1+a-c, 1+a-d, 1+2a-b-c-d-e \end{bmatrix}_m \\ &+ \sum_{k=0}^{m-1} \frac{(a-e)(1+2a-b-c-d+2k)}{1+2a-b-c-d-e+k} \begin{bmatrix} 1+a-b-c, \ 1+a-b-d, \ 1+a-c-d, \ e \\ 1+a-b, 1+a-c, 1+a-d, 1+2a-b-c-d-e \end{bmatrix}_k\end{aligned}$$

which is equivalent to the transformation between two well–poised series.

Proposition 4 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e) =\ & \Omega(a+m;b,c,d,e+m)\left[\begin{matrix}1{+}a{-}b{-}c,\ 1{+}a{-}b{-}d,\ 1{+}a{-}c{-}d,\ e\\ 1{+}a{-}b, 1{+}a{-}c, 1{+}a{-}d, 1{+}2a{-}b{-}c{-}d{-}e\end{matrix}\right]_m\\ &+\sum_{k=0}^{m-1}\frac{(a-e)(1+2a-b-c-d+2k)}{1+2a-b-c-d-e}\left[\begin{matrix}1{+}a{-}b{-}c,\ 1{+}a{-}b{-}d,\ 1{+}a{-}c{-}d,\ e\\ 1{+}a{-}b, 1{+}a{-}c, 1{+}a{-}d, 2{+}2a{-}b{-}c{-}d{-}e\end{matrix}\right]_k.\end{aligned}$$

By applying the Weierstrass M-test on uniformly convergent series (cf. Stromberg [51, p. 141]), it is trivial to verify the asymptotic relation

$$\Omega(a+m;b,c,d,e+m)\ \approx\ a+m \quad \text{as} \quad m\to\infty.$$

Furthermore, when $m\to\infty$, the sum with respect to k in Proposition 4 becomes a convergent series under the condition $\Re(a-e)>0$. Consequently, we have established the following interesting reciprocal formula.

Theorem 5 ($\Re(1+2a-b-c-d-e)>0$ and $\Re(a-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e)\ =\ & \frac{a-e}{1+2a-b-c-d-e}\\ & \times\Omega(1{+}2a{-}b{-}c{-}d;1{+}a{-}b{-}c,1{+}a{-}b{-}d,1{+}a{-}c{-}d,e).\end{aligned}$$

We remark that this formula can also be deduced by letting $c=1$ in a more general relation between two well–poised ${}_7F_6$-series due to Bailey [10, Page 62: Equation 1]. The transformation in Theorem 5 is said to be self–reciprocal because it turns back to the original $\Omega(a;b,c,d,e)$ if we apply it again to the series on the right-hand side of the equation.

In particular, when $1+2a=b+c+d+e$, we can evaluate

$$\Omega(1+2a-b-c-d;1+a-b-c,1+a-b-d,1+a-c-d,e)$$

by Dougall's theorem on nonterminating ${}_5F_4$-series. Then the transformation displayed in Theorem 5 leads to the following useful limiting relation.

Corollary 6 (Limiting relation: $\Re(1+a-b-c-d)<0$)**.**

$$\begin{aligned}&\lim_{1+2a\rightleftharpoons b+c+d+e}(1+2a-b-c-d-e)\times\Omega(a;b,c,d,e)\\ &=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)}{\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(e)}\\ &=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(b+c+d-a)}{\Gamma(b)\Gamma(c)\Gamma(d)\Gamma(1+2a-b-c-d)}.\end{aligned}$$

§2.5. **Special values of** $\Omega(a;b,c,d,e)$**.** By means of compositions of the recurrence relations displayed in Lemmas 1, 2 and 3, we shall establish several transformation formulae for the series $\Omega(a;b,c,d,e)$. Their particular cases will lead to numerous infinite series expressions for $\pi^{\pm1}$, $\pi^{\pm2}$, $\zeta(3)$ and G, the Catalan constant. This will be fulfilled by utilizing mainly Dougall's theorem for the nonterminating well–poised ${}_5F_4$-series

$$\Omega(a;b,c,d,a)=\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-b-c-d)}{\Gamma(a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}$$

as well as its limiting case $c\to-\infty$ for alternating series

$$\Omega(a;b,a,d,-\infty)=\frac{\Gamma(1+a-b)\Gamma(1+a-d)}{\Gamma(a)\Gamma(1+a-b-d)}$$

subject to the convergence conditions $\Re(1+a-b-c-d) > 0$ and $\Re(1+\frac{a}{2}-b-d) > 0$, respectively. Furthermore, the following particular values for $\Omega(a; b, c, d, e)$ will also be useful in deriving infinite series identities.

- According to the definition of ζ-function, it is trivial to see that
$$\Omega(2; 1, 1, 1, 1) = \sum_{n=1}^{\infty} \frac{2}{n^3} = 2\zeta(3).$$
- Analogously, we have also another well–known relation:
$$\Omega(1; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = \sum_{n=0}^{\infty} \frac{1}{(1+2n)^3} = \frac{7}{8}\zeta(3).$$
- In view of Theorem 5, we can deduce the following unusual expression:
$$\Omega(\tfrac{3}{2}; 1, 1, 1, \tfrac{1}{2}) = 2\Omega(1; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) = \tfrac{7}{4}\zeta(3).$$
- Euler's classical and famous formula for $\zeta(2)$ can be reproduced as:
$$\Omega(2; 1, 1, 1, \tfrac{3}{2}) = \sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{3}.$$
- Applying Theorem 5 to the last series, we get another expression:
$$\Omega(\tfrac{3}{2}; 1, 1, \tfrac{1}{2}, \tfrac{1}{2}) = \frac{1}{2}\Omega(2; 1, 1, 1, \tfrac{3}{2}) = \frac{\pi^2}{6}.$$
- There is also the following limiting relation:
$$\Omega(2; 1, 1, 1, -\infty) = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{(1+n)^2} = \frac{\pi^2}{6}.$$
- The Catalan constant is given by the following series:
$$\Omega(1; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\infty) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+2n)^2} = G.$$
- The last series can be transformed by Theorem 5 into another one:
$$\Omega(\tfrac{3}{2}; 1, 1, 1, -\infty) = \Omega(1; \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\infty) = G.$$

3. Infinite series with convergence rate $\pm\frac{1}{4}$

In this section, we shall establish a general transformation formula that expresses the Ω-sum in terms of fast convergent series with convergence rate equal to $\pm\frac{1}{4}$. As consequences, several infinite series formulae related to π and $\zeta(3)$ will be derived.

Applying Lemma 2 to the shifted Ω-sum
$$\Omega(1+a; b, c, d, 1+e) = \Omega(1+a; b, c, 1+e, d)$$
we get the equality
$$\Omega(1+a; b, c, d, 1+e) = \frac{(1+a-c)(1+a-d)}{1+a-c-d} - \Omega(2+a; b, 1+c, 1+d, 1+e) \times \frac{cd(1+a-b-e)}{(2+a-b)(1+a-e)(1+a-c-d)}.$$
Substituting this expression into Lemma 1 yields the three–term relation.

Lemma 7 (Recurrence relation **[20111]**: $\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e) = &\left[\begin{matrix}1+2a-b-c-d, a-e\\ 1+2a-b-c-d-e\end{matrix}\right]_1 + \left[\begin{matrix}e, 1+a-b-c, 1+a-b-d\\ 1+a-b, 1+2a-b-c-d-e\end{matrix}\right]_1\\ &- \frac{\Omega(2+a;b,1+c,1+d,1+e)}{(1+a-b)_2}\\ &\times \left[\begin{matrix}c,d,e,1+a-b-c,1+a-b-d,1+a-b-e\\ 1+a-c,1+a-d,1+a-e,1+2a-b-c-d-e\end{matrix}\right]_1.\end{aligned}$$

Iterating this relation m-times and then denoting by α_k the weight factor

$$\begin{aligned}\alpha_k \ :=\ \alpha_k(a;b,c,d,e) \ &=\ \frac{(1+2a-b-c-d+2k)(a-e+k)}{1+2a-b-c-d-e+k}\\ &+\ \frac{(1+a-b-c+k)(1+a-b-d+k)(e+k)}{(1+a-b+2k)(1+2a-b-c-d-e+k)},\end{aligned}$$

we can simplify the resulting equation to the transformation formula.

Proposition 8 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e) \ &=\ \frac{(-1)^m}{(1+a-b)_{2m}}\Omega(a+2m;b,c+m,d+m,e+m)\\ &\times\ \frac{[c,d,e,1+a-b-c,1+a-b-d,1+a-b-e]_m}{[1+a-c,1+a-d,1+a-e,1+2a-b-c-d-e]_m}\\ &+\ \sum_{k=0}^{m-1}\frac{(-1)^k}{(1+a-b)_{2k}}\alpha_k(a;b,c,d,e)\\ &\times\ \frac{[c,d,e,1+a-b-c,1+a-b-d,1+a-b-e]_k}{[1+a-c,1+a-d,1+a-e,1+2a-b-c-d-e]_k}.\end{aligned}$$

When $m\to\infty$, the first term on the right-hand side in the equation just displayed vanishes. We establish, consequently, the following unusual transformation formula.

Theorem 9 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e) \ &=\ \sum_{k=0}^{\infty}\frac{(-1)^k}{(1+a-b)_{2k}}\alpha_k(a;b,c,d,e)\\ &\times\ \frac{[c,d,e,1+a-b-c,1+a-b-d,1+a-b-e]_k}{[1+a-c,1+a-d,1+a-e,1+2a-b-c-d-e]_k}.\end{aligned}$$

The transformation displayed in the last theorem can be reformulated in a more symmetric expression. In fact, recalling Theorem 5 and then Theorem 9, we have, respectively,

$$\begin{aligned}\Omega(a;b,c,d,e) = &\frac{a-b}{1+2a-b-c-d-e}\\ &\times \Omega(1+2a-c-d-e;b,1+a-c-d,1+a-c-e,1+a-d-e)\end{aligned}$$

and

$$\begin{aligned}&\Omega(1+2a-c-d-e;b,1+a-c-d,1+a-c-e,1+a-d-e)\\&=\sum_{k=0}^{\infty}(-1)^k\frac{[1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-c-e,1+a-d-e]_k}{[a-b,1+a-c,1+a-d,1+a-e]_k}\\&\quad\times\frac{\alpha_k(1+2a-c-d-e;b,1+a-d-e,1+a-c-e,1+a-c-d)}{(2+2a-b-c-d-e)_{2k}}.\end{aligned}$$

Substituting the latter into the former and then simplifying the resulting equation, we get the following symmetric transformation.

Theorem 10 ($\Re(a-b)>0$ and $\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e)=&\sum_{k=0}^{\infty}(-1)^k\frac{\left\{\begin{matrix}(1+2a-b-c-d+2k)(2+2a-b-c-d-e+2k)(a-e+k)\\+(1+a-b-c+k)(1+a-b-d+k)(1+a-c-d+k)\end{matrix}\right\}}{(1+2a-b-c-d-e)_{2+2k}}\\&\times\frac{[1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-c-e,1+a-d-e]_k}{[1+a-b,1+a-c,1+a-d,1+a-e]_k}.\end{aligned}$$

Now we examine the particular case of the last theorem specified with $a=\frac{1}{2}+x$ and $b=c=d=e=\frac{1}{2}$. According to definition, we have

$$\Omega(x+\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})=\sum_{n=0}^{\infty}\left[\begin{matrix}\frac{1}{2}\\1+x\end{matrix}\right]_n^4\left\{\tfrac{1}{2}+x+2n\right\}.$$

Then factoring the summand on the right-hand side in Theorem 10, we reach the following identity discovered by Guillera via the WZ–method.

Corollary 11 (Guillera [41, Identity 8: Equation 13])**.**

$$\begin{aligned}&8x\sum_{n=0}^{\infty}\left[\begin{matrix}\frac{1}{2}\\1+x\end{matrix}\right]_n^4\left\{1+2x+4n\right\}\\&=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}\frac{1}{2}+x\\1+x\end{matrix}\right]_k^5\left\{1+8(x+k)+20(x+k)^2\right\}.\end{aligned}$$

When one of the parameters b,c,d,e is equal to a, the corresponding Ω-sum in Theorem 9 can be evaluated by Dougall's formula for the well–poised ${}_5F_4$-series. By specifying the parameters a,b,c,d,e further in Theorem 9 such that the corresponding Ω-sum can be evaluated by those displayed in §2.5, we can establish numerous infinite series formulae for $\zeta(3)$, $\pi^{\pm1}$ and $\pi^{\pm2}$ with convergence rate equal to $\frac{1}{4}$ or $-\frac{1}{4}$. Ten selected formulae are exhibited below as examples.

Example 1 (Theorem 9: $a=1,\ b=c=d=e=\frac{1}{2}$)**.**

$$\frac{21}{2}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}1,&1\\\frac{5}{4},&\frac{7}{4}\end{matrix}\right]_k\frac{13+32k+20k^2}{(1+k)(1+2k)^2}.$$

Example 2 (Theorem 9: $a=\frac{3}{2},\ b=\frac{1}{2},\ c=d=e=1$; Guillera [41, §3.1: $a=\frac{1}{2}$])**.**

$$\frac{7}{2}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}1,&1,&1,&1,&1\\\frac{3}{2},&\frac{3}{2},&\frac{3}{2},&\frac{3}{2},&\frac{3}{2}\end{matrix}\right]_k\left\{5+14k+10k^2\right\}.$$

Example 3 (Theorem 9: $a=b=1,\ c=d=e=\frac{1}{2}$).

$$\frac{\pi^2}{10}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ \frac{3}{2},\ \frac{3}{2}\end{bmatrix}_k.$$

Example 4 (Theorem 9: $a=b=2,\ c=d=e=\frac{1}{2}$).

$$\frac{81\pi^2}{32}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ \frac{5}{2},\ \frac{5}{2},\ \frac{5}{2},\ \frac{5}{2}\end{bmatrix}_k\Big\{25+44k+20k^2\Big\}.$$

Example 5 (Theorem 9: $a=b=c=d=e=\frac{1}{2}$).

$$\frac{8}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ 1,\ 1,\ 1,\ 1\end{bmatrix}_k\Big\{1+8k+20k^2\Big\}.$$

Example 6 (Theorem 9: $a=b=\frac{3}{2},\ c=d=e=\frac{1}{2}$).

$$\frac{128}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ 2,\ 2,\ 2,\ 2\end{bmatrix}_k\Big\{13+32k+20k^2\Big\}.$$

Example 7 (Theorem 9: $a=c=1,\ e=\frac{1}{2},\ b=\frac{3}{4},\ d=\frac{1}{4}$).

$$\frac{5\pi}{8}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}1,\ \frac{1}{4},\ \frac{1}{4}\\ \frac{3}{2},\ \frac{9}{8},\ \frac{13}{8}\end{bmatrix}_k\Big\{2+5k\Big\}.$$

Example 8 (Theorem 9: $a=c=1,\ e=\frac{1}{2},\ b=\frac{1}{4},\ d=\frac{3}{4}$).

$$\frac{21\pi}{8}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}1,\ \frac{3}{4},\ \frac{3}{4}\\ \frac{3}{2},\ \frac{11}{8},\ \frac{15}{8}\end{bmatrix}_k\Big\{10+29k+20k^2\Big\}.$$

Example 9 (Theorem 9: $a=b=c=d=\frac{1}{2},\ e\to-\infty$).

$$\frac{4}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ 1,\ 1\end{bmatrix}_k\Big\{1+6k\Big\}.$$

Example 10 (Theorem 9: $a=b=c=\frac{1}{2},\ d=\frac{1}{4},\ e=\frac{3}{4}$).

$$\frac{8}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4}\\ 1,\ 1,\ 1\end{bmatrix}_k\Big\{3+20k\Big\}.$$

4. Infinite series with convergence rate $\frac{1}{16}$

In this section, we shall establish a general transformation formula that expresses the Ω-sum in terms of fast convergent series with convergence rate equal to $\frac{1}{16}$. As consequences, several infinite series formulae related to π and $\zeta(3)$ will be derived.

Applying Lemma 1 to the shifted Ω-sum

$$\Omega(1+a;b,c,d,1+e)=\Omega(1+a;b,1+e,d,c)$$

we have the equality

$$\begin{aligned}\Omega(1+a;b,c,d,1+e)=&\frac{(1+a-c)(2+2a-b-d-e)}{2+2a-b-c-d-e}+\Omega(2+a;b,1+c,d,1+e)\\ &\times\frac{c(2+a-b-d)(1+a-b-e)(1+a-d-e)}{(2+a-b)(2+a-d)(1+a-e)(2+2a-b-c-d-e)}.\end{aligned}$$

Substituting the last expression into Lemma 1 and then simplifying the result, we get the following three–term relation.

Lemma 12 (Recurrence relation **[20101]**: $\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e)=\frac{(1+2a-b-c-d)(a-e)}{1+2a-b-c-d-e}+\Omega(2+a;b,1+c,d,1+e)$$
$$\times\frac{ce(1+a-b-c)(1+a-b-e)(1+a-c-d)(1+a-d-e)(1+a-b-d)_2}{(1+a-c)(1+a-e)[1+a-b,1+a-d,1+2a-b-c-d-e]_2}$$
$$+\frac{e(1+a-b-c)(1+a-b-d)(1+a-c-d)(2+2a-b-d-e)}{(1+a-b)(1+a-d)(1+2a-b-c-d-e)_2}.$$

Iterating this relation m times and then denoting by β_k the weight factor

$$\beta_k:=\beta_k(a;b,c,d,e)\;=\;\frac{(1+2a-b-c-d+3k)(a-e+k)}{1+2a-b-c-d-e+2k}$$
$$+\;\frac{(e+k)(1+a-b-c+k)(1+a-b-d+2k)(1+a-c-d+k)(2+2a-b-d-e+3k)}{(1+a-b+2k)(1+a-d+2k)(1+2a-b-c-d-e+2k)_2}$$

we derive the following transformation formula.

Proposition 13 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e)=\Omega(a+2m;b,c+m,d,e+m)\frac{(1+a-b-d)_{2m}}{[1+a-c,1+a-e]_m}$$
$$\times\frac{[c,e,1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e]_m}{[1+a-b,1+a-d,1+2a-b-c-d-e]_{2m}}$$
$$+\sum_{k=0}^{m-1}\beta_k(a;b,c,d,e)\frac{(1+a-b-d)_{2k}}{[1+a-c,1+a-e]_k}$$
$$\times\frac{[c,e,1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e]_k}{[1+a-b,1+a-d,1+2a-b-c-d-e]_{2k}}.$$

The first term on the right-hand side in the last equation tends to zero as $m\to-\infty$. We therefore find the following limiting transformation formula.

Theorem 14 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e)=\sum_{k=0}^{\infty}\beta_k(a;b,c,d,e)\frac{(1+a-b-d)_{2k}}{[1+a-c,1+a-e]_k}$$
$$\times\frac{[c,e,1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e]_k}{[1+a-b,1+a-d,1+2a-b-c-d-e]_{2k}}.$$

When $a=\frac{1}{2}+2x$, $b=d=\frac{1}{2}$ and $c=e=\frac{1}{2}+x$, we have

$$\Omega(\tfrac{1}{2}+2x;\tfrac{1}{2},\tfrac{1}{2}+x,\tfrac{1}{2},\tfrac{1}{2}+x)=\sum_{n=0}^{\infty}\begin{bmatrix}\frac{1}{2}, & \frac{1}{2}+x\\ 1+x, & 1+2x\end{bmatrix}_n^2\Big\{\tfrac{1}{2}+2x+2n\Big\}.$$

Then simplifying the expression on the right-hand side displayed in Theorem 14, we recover the identity due to Guillera, who finds it by using the WZ–method.

Corollary 15 (Guillera [41, Identity 10: Equation 19]).

$$32x\sum_{n=0}^{\infty}\left[\begin{matrix}\frac12, & \frac12+x\\ 1+x, & 1+2x\end{matrix}\right]_n^2\Big\{1+4x+4n\Big\}$$
$$=\sum_{k=0}^{\infty}\left[\begin{matrix}\frac12+x,\frac12+x,\frac12+x,\frac14+x,\frac34+x\\ 1+x,1+x,1+x,1+x,1+x\end{matrix}\right]_k\frac{3+34(x+k)+120(x+k)^2}{16^k}.$$

In view of Dougall's formula of the well–poised ${}_5F_4$-series, we can deduce, by specifying the parameters a,b,c,d,e in Theorem 14, numerous infinite series identities with convergence rate equal to $\frac{1}{16}$. Ten formulae are given in the sequel as representative examples.

Example 11 (Theorem 14: $a=2,\ b=c=d=e=1$).

$$16\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}1, & 1\\ \frac32, & \frac32\end{matrix}\right]_k\frac{19+30k}{(1+k)(1+2k)}.$$

Example 12 (Theorem 14: $a=1,\ b=c=d=e=\frac12$).

$$\frac{63}{2}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}1, & 1, & 1, & 1\\ \frac54, & \frac54, & \frac74, & \frac74\end{matrix}\right]_k\frac{37+94k+60k^2}{1+2k}.$$

Example 13 (Theorem 14: $a=2,\ b=c=e=1,\ d=\frac32$).

$$2\pi^2=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}1, & \frac12\\ \frac54, & \frac74\end{matrix}\right]_k\frac{59+170k+120k^2}{(1+k)(1+4k)(3+4k)}.$$

Example 14 (Theorem 14: $a=\frac32,\ b=d=\frac12,\ c=e=1$; Guillera [41, §3.3: $a=\frac12$]).

$$\frac{8\pi^2}{3}=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}1, & 1, & 1, & \frac34, & \frac54\\ \frac32, & \frac32, & \frac32, & \frac32, & \frac32\end{matrix}\right]_k\Big\{25+77k+60k^2\Big\}.$$

Example 15 (Theorem 14: $a=d=1,\ b=c=e=\frac12$).

$$\frac{3\pi^2}{8}=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}1, & \frac12\\ \frac54, & \frac74\end{matrix}\right]_k\frac{11+64k+111k^2+60k^3}{(1+2k)(1+4k)(3+4k)}.$$

Example 16 (Theorem 14: $a=b=c=d=e=\frac12$).

$$\frac{32}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}\frac12, & \frac12, & \frac12, & \frac14, & \frac34\\ 1, & 1, & 1, & 1, & 1\end{matrix}\right]_k\Big\{3+34k+120k^2\Big\}.$$

Example 17 (Theorem 14: $a=b=d=\frac32,\ c=e=\frac12$).

$$\frac{128}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}\frac12, & \frac12, & \frac12, & -\frac14, & \frac14\\ 1, & 1, & 1, & 2, & 2\end{matrix}\right]_k\Big\{13+118k+120k^2\Big\}.$$

Example 18 (Theorem 14: $a=b=\frac56,\ d=\frac12,\ c=e=\frac13$).

$$\frac{20\pi}{3\sqrt3}=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}1, & \frac13, & \frac23, & \frac14, & \frac34\\ \frac32, & \frac32, & \frac32, & \frac76, & \frac{11}{6}\end{matrix}\right]_k\Big\{\big(3+5k\big)\big(4+15k+12k^2\big)\Big\}.$$

Example 19 (Theorem 14: $a=b=c=\frac{1}{2}$, $d=\frac{2}{3}$, $e=\frac{1}{3}$).

$$\frac{15\sqrt{3}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}\frac{1}{2}, & \frac{1}{3}, & \frac{1}{3}, & \frac{2}{3}, & \frac{2}{3}\\ 1, & 1, & 1, & \frac{11}{12}, & \frac{17}{12}\end{matrix}\right]_k\Big\{8+75k+135k^2\Big\}.$$

Example 20 (Theorem 14: $a=b=c=\frac{1}{2}$, $d=\frac{1}{6}$, $e=\frac{5}{6}$).

$$\frac{128\sqrt{3}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{1}{16}\Big)^k\left[\begin{matrix}\frac{1}{2}, & \frac{5}{6}, & \frac{5}{6}, & \frac{5}{12}, & \frac{11}{12}\\ 1, & 1, & 1, & \frac{2}{3}, & \frac{5}{3}\end{matrix}\right]_k\Big\{55+798k+1080k^2\Big\}.$$

5. Infinite series with convergence rate $-\frac{1}{27}$

In this section, we shall establish a general transformation formula that expresses the Ω-sum in terms of fast convergent series with convergence rate equal to $-\frac{1}{27}$. As a consequence, several infinite series formulae related to π, G and $\zeta(3)$ will be derived.

Applying Lemma 2 to the shifted Ω-sum

$$\Omega(2+a;b,1+c,d,1+e)=\Omega(2+a;b,d,1+c,1+e)$$

we get the equality

$$\begin{aligned}\Omega(2+a;b,1+c,d,1+e)=&\frac{(2+a-d)(1+a-e)}{1+a-d-e}-\Omega(3+a;b,1+c,1+d,2+e)\\ &\times\frac{d(1+e)(2+a-b-c)}{(3+a-b)(2+a-c)(1+a-d-e)}.\end{aligned}$$

Substituting this expression into Lemma 12 gives the four–term relation.

Lemma 16 (Recurrence relation **[30112]**: $\Re(1+2a-b-c-d-e)>0$).

$$\begin{aligned}\Omega(a;b,c,d,e)=&\frac{(1+2a-b-c-d)(a-e)}{1+2a-b-c-d-e}-\Omega(3+a;b,1+c,1+d,2+e)\\ &\times\frac{[c,d,1+a-b-e,1+a-c-d]_1[e,1+a-b-c,1+a-b-d]_2}{(1+a-e)_1[1+a-c,1+a-d,1+2a-b-c-d-e]_2(1+a-b)_3}\\ &+\frac{[c,e,1+a-b-c,1+a-b-e,1+a-c-d]_1(1+a-b-d)_2}{[1+a-c,1+a-d]_1[1+a-b,1+2a-b-c-d-e]_2}\\ &+\frac{[e,1+a-b-c,1+a-b-d,1+a-c-d,2+2a-b-d-e]_1}{[1+a-b,1+a-d]_1(1+2a-b-c-d-e)_2}.\end{aligned}$$

Iterating the last relation m times and then denoting by γ_k the weight factor

$$\begin{aligned}\gamma_k:=\gamma_k(a;b,c,d,e)\;=\;&\frac{(1+2a-b-c-d+4k)(a-e+k)}{1+2a-b-c-d-e+2k}\\ &+\frac{[e+2k,1+a-b-c+2k,1+a-b-d+2k,1+a-c-d+k,2+2a-b-d-e+3k]_1}{[1+a-b+3k,1+a-d+2k]_1(1+2a-b-c-d-e+2k)_2}\\ &+\frac{[c+k,e+2k,1+a-b-c+2k,1+a-b-e+k,1+a-c-d+k]_1(1+a-b-d+2k)_2}{[1+a-c+2k,1+a-d+2k]_1[1+a-b+3k,1+2a-b-c-d-e+2k]_2}\end{aligned}$$

we derive the following transformation formula.

Proposition 17 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e) &= \frac{(-1)^m}{(1+a-b)_{3m}}\Omega(a+3m;b,c+m,d+m,e+2m)\\ &\times \frac{[c,d,1+a-b-e,1+a-c-d]_m[e,1+a-b-c,1+a-b-d]_{2m}}{(1+a-e)_m[1+a-c,1+a-d,1+2a-b-c-d-e]_{2m}}\\ &+ \sum_{k=0}^{m-1}\gamma_k(a;b,c,d,e)\frac{(-1)^k}{(1+a-b)_{3k}}\\ &\times \frac{[c,d,1+a-b-e,1+a-c-d]_k[e,1+a-b-c,1+a-b-d]_{2k}}{(1+a-e)_k[1+a-c,1+a-d,1+2a-b-c-d-e]_{2k}}.\end{aligned}$$

When $m\to\infty$, observing that the first term on the right-hand side in the last equation vanishes, we find the following limiting transformation.

Theorem 18 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e) &= \sum_{k=0}^{\infty}\gamma_k(a;b,c,d,e)\frac{(-1)^k}{(1+a-b)_{3k}}\\ &\times \frac{[c,d,1+a-b-e,1+a-c-d]_k[e,1+a-b-c,1+a-b-d]_{2k}}{(1+a-e)_k[1+a-c,1+a-d,1+2a-b-c-d-e]_{2k}}.\end{aligned}$$

By specifying the parameters a,b,c,d,e in Theorem 18 such that the corresponding Ω-sum can be evaluated by Dougall's formula for well–poised ${}_5F_4$-series, we can establish numerous infinite series formulae involving π, G and $\zeta(3)$. Ten of them are selected here as examples.

Example 21 (Theorem 18: $a=2,\ b=c=d=e=1$; Amdelberhan [2] and Wilf [52, Equation 8])**.**

$$24\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1,\ 1\\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\frac{29+80k+56k^2}{(1+k)(1+2k)^2}.$$

Example 22 (Theorem 18: $a=\frac{3}{2},\ b=c=d=1,\ e=\frac{1}{2}$)**.**

$$\frac{105}{2}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1,\ 1\\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\frac{569+2928k+5824k^2+5248k^3+1792k^4}{(1+k)(1+4k)^2(3+4k)^2}.$$

Example 23 (Theorem 18: $a=1,\ b=c=d=e=\frac{1}{2}$)**.**

$$\frac{315}{2}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ \frac{1}{2}\\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\frac{571+3934k+9464k^2+9664k^3+3584k^4}{(1+2k)(1+4k)(3+4k)}.$$

Example 24 (Theorem 18: $a=2,\ b=c=d=1,\ e=\frac{3}{2}$)**.**

$$\frac{\pi^2}{2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1,\ 1\\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\frac{5+7k}{1+2k}.$$

Example 25 (Theorem 18: $a=c=1,\ b=d=e=\frac{1}{2}$)**.**

$$\frac{15\pi^2}{16}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1,\ 1\\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\frac{28+139k+220k^2+112k^3}{(1+2k)(1+4k)(3+4k)}.$$

Example 26 (Theorem 18: $a=b=c=d=e=\frac{1}{2}$).

$$\frac{256}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4}\\ 1,\ 1,\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\Big\{27+492k+3376k^2+8832k^3+7168k^4\Big\}.$$

Example 27 (Theorem 18: $a=e=1,\ b=\frac{1}{2},\ c=\frac{1}{4},\ d=\frac{3}{4}$).

$$\frac{15\pi}{8}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1,\ \frac{1}{2}\\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\Big\{6+7k\Big\}.$$

Example 28 (Theorem 18: $a=c=e=\frac{1}{2},\ b=\frac{3}{4},\ d=\frac{1}{4}$).

$$\frac{84}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4}\\ 1,\ 1,\ 1,\ \frac{3}{2},\ \frac{11}{12},\ \frac{19}{12}\end{bmatrix}_k\Big\{27+372k+1088k^2+896k^3\Big\}.$$

Example 29 (Theorem 18: $a=\frac{3}{2},\ b=c=d=1,\ e\to-\infty$).

$$30G=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1,\ 1\\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\frac{83+192k+112k^2}{(1+4k)(3+4k)}.$$

Example 30 (Theorem 18: $a=1,\ b=c=d=\frac{1}{2},\ e\to-\infty$).

$$270G=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ \frac{5}{4},\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{4},\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\Big\{253+1790k+4432k^2+4672k^3+1792k^4\Big\}.$$

6. Infinite series with convergence rate $\pm\frac{4}{27}$

In this section, we shall establish a general transformation formula that expresses the Ω-sum in terms of fast convergent series with convergence rate equal to $\pm\frac{4}{27}$. As a consequence, several infinite series formulae related to π, G and $\zeta(3)$ will be derived.

Applying Lemma 2 to $\Omega(2+a;b,1+c,d,1+e)$, we have the equation

$$\Omega(2+a;b,1+c,d,1+e)=\frac{(1+a-c)(1+a-e)}{a-c-e}$$
$$+\frac{(1+c)(1+e)(3+a-b-d)\,\Omega(3+a;b,2+c,d,2+e)}{(c+e-a)(3+a-b)(3+a-d)}.$$

Substituting this expression into Lemma 12 leads to the following relation.

Lemma 19 (Recurrence relation **[30202]**: $\Re(1+2a-b-c-d-e)>0$).

$$\Omega(a;b,c,d,e)=\frac{(1+2a-b-c-d)(a-e)}{1+2a-b-c-d-e}$$
$$+\frac{[e,1+a-b-c,1+a-b-d,1+a-c-d,2+2a-b-d-e]_1}{[1+a-b,1+a-d]_1(1+2a-b-c-d-e)_2}$$
$$-\frac{ce[1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e]_1(1+a-b-d)_2}{(c+e-a)[1+a-b,1+a-d,1+2a-b-c-d-e]_2}$$
$$+\frac{[1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e]_1}{[1+a-c,1+a-e,c+e-a]_1(1+2a-b-c-d-e)_2}$$
$$\times\Omega(3+a;b,2+c,d,2+e)\frac{[c,e]_2(1+a-b-d)_3}{[1+a-b,1+a-d]_3}.$$

Iterating the last relation m times and then denoting by λ_k the weight factor

$$\lambda_k := \lambda_k(a;b,c,d,e) \;=\; \frac{(1+2a-b-c-d+4k)(a-e+k)}{1+2a-b-c-d-e+2k}$$
$$+\;\frac{[e+2k,1+a-b-c+k,1+a-b-d+3k,1+a-c-d+k,2+2a-b-d-e+4k]_1}{[1+a-b+3k,1+a-d+3k]_1(1+2a-b-c-d-e+2k)_2}$$
$$-\;\frac{[c+2k,e+2k,1+a-b-c+k,1+a-b-e+k,1+a-c-d+k,1+a-d-e+k]_1(1+a-b-d+3k)_2}{(c+e-a+k)[1+a-b+3k,1+a-d+3k,1+2a-b-c-d-e+2k]_2}$$

we establish the following transformation formula.

Proposition 20 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e)=\Omega(a+3m;b,c+2m,d,e+2m)\frac{[c,e]_{2m}(1+a-b-d)_{3m}}{[1+a-b,1+a-d]_{3m}}$$
$$\times\frac{[1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e]_m}{[1+a-c,1+a-e,c+e-a]_m(1+2a-b-c-d-e)_{2m}}$$
$$+\sum_{k=0}^{m-1}\lambda_k(a;b,c,d,e)\frac{[c,e]_{2k}(1+a-b-d)_{3k}}{[1+a-b,1+a-d]_{3k}}$$
$$\times\frac{[1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e]_k}{[1+a-c,1+a-e,c+e-a]_k(1+2a-b-c-d-e)_{2k}}.$$

When $m\to\infty$, the last equation results in the limiting transformation formula.

Theorem 21 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e) \;=\; \sum_{k=0}^{\infty}\lambda_k(a;b,c,d,e)\frac{[c,e]_{2k}(1+a-b-d)_{3k}}{[1+a-b,1+a-d]_{3k}}$$
$$\times\frac{[1+a-b-c,1+a-b-e,1+a-c-d,1+a-d-e]_k}{[1+a-c,1+a-e,c+e-a]_k(1+2a-b-c-d-e)_{2k}}.$$

By specifying the parameters a,b,c,d,e in Theorem 21 such that the corresponding Ω-sum has closed expressions as those displayed in §2.5, the following infinite series formulae with convergence rate equal to $\frac{4}{27}$ or $-\frac{4}{27}$ can be established.

Example 31 (Theorem 21: $a=\frac{3}{2},\ b=\frac{1}{2},\ c=d=e=1$)**.**

$$\frac{945}{2}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ 1\\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\frac{1078+7227k+17282k^2+17712k^3+6624k^4}{(1+2k)(1+3k)(2+3k)}.$$

Example 32 (Theorem 21: $a=\frac{3}{2},\ b=d=\frac{1}{2},\ c=e=1$)**.**

$$4\pi^2=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ \frac{5}{6},\ \frac{7}{6}\\ \frac{3}{2},\ \frac{4}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{5}{3}\end{bmatrix}_k\Big\{35+98k+69k^2\Big\}.$$

Example 33 (Theorem 21: $a=2,\ b=c=d=1,\ e\to-\infty$)**.**

$$3\pi^2=\sum_{k=0}^{\infty}\Big(\frac{-4}{27}\Big)^k\begin{bmatrix}1,\ \frac{1}{2}\\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\frac{61+302k+504k^2+279k^3}{(1+k)(1+3k)(2+3k)}.$$

Example 34 (Theorem 21: $a=e=1,\ b=c=d=\frac{1}{2}$)**.**

$$\frac{225\pi^2}{8}=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ \frac{1}{3},\ \frac{2}{3}\\ \frac{3}{2},\ \frac{7}{6},\ \frac{7}{6},\ \frac{11}{6},\ \frac{11}{6}\end{bmatrix}_k\Big\{256+1171k+1728k^2+828k^3\Big\}.$$

Example 35 (Theorem 21: $a=c=1,\ b=d=\frac{1}{2},\ e\to-\infty$).

$$\frac{225\pi}{4}=\sum_{k=0}^{\infty}\Big(\frac{-4}{27}\Big)^k\begin{bmatrix}1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3}\\ \frac{7}{6},\ \frac{7}{6},\ \frac{11}{6},\ \frac{11}{6}\end{bmatrix}_k\Big\{192+1253k+2160k^2+1116k^3\Big\}.$$

Example 36 (Theorem 21: $a=\frac{3}{2},\ c=e=1,\ b=\frac{3}{4},\ d=\frac{5}{4}$).

$$\frac{1155\pi}{16}=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}1,\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{6},\ \frac{5}{6}\\ \frac{3}{2},\ \frac{13}{12},\ \frac{17}{12},\ \frac{19}{12},\ \frac{23}{12}\end{bmatrix}_k\Big\{224+1263k+2128k^2+1104k^3\Big\}.$$

Example 37 (Theorem 21: $a=b=c=d=\frac{1}{3},\ e\to-\infty$).

$$\frac{243\sqrt{3}}{2\pi}=\sum_{k=0}^{\infty}\Big(\frac{-4}{27}\Big)^k\begin{bmatrix}\frac{2}{3},\ \frac{1}{6},\ \frac{2}{9},\ \frac{5}{9},\ \frac{8}{9}\\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{4}{3}\end{bmatrix}_k\Big\{80+1281k+5697k^2+7533k^3\Big\}.$$

Example 38 (Theorem 21: $a=b=c=d=\frac{2}{3},\ e\to-\infty$).

$$\frac{486\sqrt{3}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-4}{27}\Big)^k\begin{bmatrix}\frac{1}{3},\ \frac{5}{6},\ \frac{1}{9},\ \frac{4}{9},\ \frac{7}{9}\\ 1,\ 1,\ 1,\ \frac{5}{3},\ \frac{5}{3}\end{bmatrix}_k\Big\{280+4314k+10908k^2+7533k^3\Big\}.$$

Example 39 (Theorem 21: $a=c=\frac{1}{2},\ b=\frac{1}{4},\ d=\frac{3}{4},\ e\to-\infty$).

$$\frac{105}{2\sqrt{2}}=\sum_{k=0}^{\infty}\Big(\frac{-4}{27}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{6},\ \frac{5}{6}\\ 1,\ \frac{11}{12},\ \frac{13}{12},\ \frac{17}{12},\ \frac{19}{12}\end{bmatrix}_k\Big\{40+517k+1640k^2+1488k^3\Big\}.$$

Example 40 (Theorem 21: $a=\frac{3}{2},\ b=c=d=1,\ e\to-\infty$).

$$90G=\sum_{k=0}^{\infty}\Big(\frac{-4}{27}\Big)^k\begin{bmatrix}1,\ \frac{1}{2}\\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\frac{419+2610k+4404k^2+2232k^3}{(1+2k)(1+6k)(5+6k)}.$$

7. Infinite series with convergence rate $\pm\frac{16}{27}$

In this section, we shall establish a general transformation formula that expresses the Ω-sum in terms of fast convergent series with convergence rate equal to $\pm\frac{16}{27}$. As a consequence, several infinite series formulae related to π, G and $\zeta(3)$ will be derived.

Applying Lemma 2 to the shifted Ω-sum

$$\Omega(2+a;b,1+c,1+d,1+e)=\Omega(2+a;b,1+d,1+c,1+e)$$

we get the equality

$$\Omega(2+a;b,1+c,1+d,1+e)=\frac{(1+a-d)(1+a-e)}{a-d-e}$$
$$+\Omega(3+a;b,1+c,2+d,2+e)\frac{(1+d)(1+e)(2+a-b-c)}{(d+e-a)(3+a-b)(2+a-c)}.$$

Substituting the last expression into Lemma 7 yields the four–term relation.

Lemma 22 (Recurrence relation **[30122]**: $\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e) &= \begin{bmatrix}1+2a-b-c-d,a-e\\ 1+2a-b-c-d-e\end{bmatrix}_1\\ &+\begin{bmatrix}e,\ 1+a-b-c,\ 1+a-b-d\\ 1+a-b,1+2a-b-c-d-e\end{bmatrix}_1\\ &+\frac{[c,d,e,1+a-b-c,1+a-b-d,1+a-b-e]_1}{[d+e-a,1+a-c,1+2a-b-c-d-e]_1(1+a-b)_2}\\ &-\frac{[c,1+a-b-d,1+a-b-e]_1}{[d+e-a,1+a-d,1+a-e,1+2a-b-c-d-e]_1}\\ &\times\frac{[d,e,1+a-b-c]_2}{(1+a-c)_2(1+a-b)_3}\Omega(3+a;b,1+c,2+d,2+e).\end{aligned}$$

Iterating this relation m times and then denoting by μ_k the weight factor

$$\begin{aligned}\mu_k:=\mu_k(a;b,c,d,e) \ &= \ \frac{(1+2a-b-c-d+3k)(a-e+k)}{1+2a-b-c-d-e+k}\\ &+\ \frac{[e+2k,1+a-b-c+2k,1+a-b-d+k]_1}{[1+a-b+3k,1+2a-b-c-d-e+k]_1}\\ &+\ \frac{[c+k,d+2k,e+2k,1+a-b-c+2k,1+a-b-d+k,1+a-b-e+k]_1}{[d+e-a+k,1+a-c+2k,1+2a-b-c-d-e+k]_1(1+a-b+3k)_2}\end{aligned}$$

we get the following transformation formula.

Proposition 23 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}&\Omega(a;b,c,d,e)=\Omega(a+3m;b,c+m,d+2m,e+2m)\frac{(-1)^m}{(1+a-b)_{3m}}\\ &\times\frac{[c,1+a-b-d,1+a-b-e]_m[d,e,1+a-b-c]_{2m}}{[d+e-a,1+a-d,1+a-e,1+2a-b-c-d-e]_m(1+a-c)_{2m}}\\ &+\sum_{k=0}^{m-1}\mu_k(a;b,c,d,e)\frac{(-1)^k}{(1+a-b)_{3k}}\\ &\times\frac{[c,1+a-b-d,1+a-b-e]_k[d,e,1+a-b-c]_{2k}}{[d+e-a,1+a-d,1+a-e,1+2a-b-c-d-e]_k(1+a-c)_{2k}}.\end{aligned}$$

When $m\to\infty$, the last equation yields the following transformation formula.

Theorem 24 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}&\Omega(a;b,c,d,e)=\sum_{k=0}^{\infty}\mu_k(a;b,c,d,e)\frac{(-1)^k}{(1+a-b)_{3k}}\\ &\times\frac{[c,1+a-b-d,1+a-b-e]_k[d,e,1+a-b-c]_{2k}}{[d+e-a,1+a-d,1+a-e,1+2a-b-c-d-e]_k(1+a-c)_{2k}}.\end{aligned}$$

By specifying the parameters a,b,c,d,e in Theorem 24 such that the corresponding Ω-sum can be evaluated by Dougall's formula for well–poised ${}_5F_4$-series, we can further establish numerous infinite series identities involving π, G and $\zeta(3)$ with convergence rate equal to $\pm\frac{16}{27}$. These are exemplified by the following ten formulae.

Example 41 (Theorem 24: $a = \frac{3}{2},\ c = \frac{1}{2},\ b = d = e = 1$).

$$\frac{105}{2}\zeta(3) = \sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k \begin{bmatrix} 1, & 1 \\ \frac{7}{6}, & \frac{11}{6} \end{bmatrix}_k \frac{67+151k+86k^2}{(1+k)(1+2k)^2}.$$

Example 42 (Theorem 24: $a = \frac{3}{2},\ b = \frac{1}{2},\ c = d = e = 1$).

$$63\zeta(3) = \sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k \begin{bmatrix} 1, & 1, & 1, & 1, & 1 \\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3}, & \frac{5}{4}, & \frac{7}{4} \end{bmatrix}_k \Big\{106+269k+172k^2\Big\}.$$

Example 43 (Theorem 24: $a = 2,\ b = c = d = 1,\ e = \frac{3}{2}$).

$$4\pi^2 = \sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k \begin{bmatrix} 1, & \frac{3}{4}, & \frac{5}{4} \\ \frac{3}{2}, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_k \frac{45+124k+86k^2}{(1+k)(1+2k)}.$$

Example 44 (Theorem 24: $a = e = 1,\ b = c = d = \frac{1}{2}$).

$$\frac{15\pi^2}{8} = \sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k \begin{bmatrix} 1, & 1 \\ \frac{7}{6}, & \frac{11}{6} \end{bmatrix}_k \frac{64+375k+646k^2+344k^3}{(1+2k)(1+4k)(3+4k)}.$$

Example 45 (Theorem 24: $a = b = c = d = e = \frac{1}{2}$).

$$\frac{64}{\pi^2} = \sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k \begin{bmatrix} \frac{1}{2}, & \frac{1}{4}, & \frac{1}{4}, & \frac{1}{4}, & \frac{3}{4}, & \frac{3}{4}, & \frac{3}{4} \\ 1, & 1, & 1, & 1, & 1, & \frac{4}{3}, & \frac{5}{3} \end{bmatrix}_k \Big\{9+138k+784k^2+1808k^3+1376k^4\Big\}.$$

Example 46 (Theorem 24: $a = d = e = \frac{3}{2},\ b = -\frac{1}{2},\ c = \frac{1}{2}$).

$$\frac{8192}{\pi^2} = \sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k \begin{bmatrix} \frac{3}{2}, & \frac{3}{4}, & \frac{3}{4}, & \frac{5}{4}, & \frac{5}{4}, & \frac{5}{4}, & \frac{7}{4} \\ 1, & 1, & 2, & 2, & 2, & \frac{5}{3}, & \frac{7}{3} \end{bmatrix}_k$$
$$\times\Big\{1575+6150k+8960k^2+5760k^3+1376k^4\Big\}.$$

Example 47 (Theorem 24: $a = e = 1,\ b = d = \frac{1}{2},\ c \to -\infty$).

$$\frac{15\pi}{4} = \sum_{k=0}^{\infty}\Big(\frac{16}{27}\Big)^k \begin{bmatrix} 1, & \frac{1}{4}, & \frac{3}{4} \\ \frac{1}{2}, & \frac{7}{6}, & \frac{11}{6} \end{bmatrix}_k \Big\{8+11k\Big\}.$$

Example 48 (Theorem 24: $a = e = 1,\ d = \frac{1}{2},\ b = \frac{2}{3},\ c = \frac{1}{3}$).

$$\frac{56\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k \begin{bmatrix} 1, & \frac{1}{3}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{3}{2}, & \frac{10}{9}, & \frac{13}{9}, & \frac{16}{9} \end{bmatrix}_k \Big\{36+143k+129k^2\Big\}.$$

Example 49 (Theorem 24: $a = b = c = \frac{1}{2},\ d = \frac{1}{3},\ e = \frac{2}{3}$).

$$\frac{27\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k \begin{bmatrix} \frac{1}{3}, & \frac{2}{3}, & \frac{1}{4}, & \frac{3}{4} \\ 1, & 1, & 1, & \frac{3}{2} \end{bmatrix}_k \Big\{20+167k+258k^2\Big\}.$$

Example 50 (Theorem 24: $a = \frac{3}{2},\ b = d = e = 1,\ c \to -\infty$).

$$30G = \sum_{k=0}^{\infty}\Big(\frac{16}{27}\Big)^k \begin{bmatrix} 1, & 1 \\ \frac{7}{6}, & \frac{11}{6} \end{bmatrix}_k \frac{21+22k}{1+2k}.$$

8. Infinite series with convergence rate $\frac{27}{64}$

In this section, we shall establish a general transformation formula that expresses the Ω-sum in terms of fast convergent series with convergence rate equal to $\frac{27}{64}$. As a consequence, several infinite series formulae related to π will be derived.

First applying Lemma 2 to the shifted Ω-sum

$$\Omega(1+a;b,1+c,d,1+e)=\Omega(1+a;1+c,b,d,1+e)$$

yields the equation

$$\begin{aligned}\Omega(1+a;b,1+c,d,1+e)=\frac{(1+a-b)(a-e)}{a-b-e}&+\Omega(2+a;1+b,1+c,d,2+e)\\ &\times\frac{b(1+e)(1+a-c-d)}{(1+a-c)(2+a-d)(b+e-a)}.\end{aligned}$$

Then the last shifted Ω-sum

$$\Omega(2+a;1+b,1+c,d,2+e)=\Omega(2+a;1+b,d,1+c,2+e)$$

can further be reformulated by the same lemma as

$$\begin{aligned}\Omega(2+a;1+b,1+c,d,2+e)=&\frac{(2+a-d)(a-e)}{a-d-e}\\ &+\Omega(3+a;1+b,1+c,1+d,3+e)\\ &\times\frac{d(2+e)(1+a-b-c)}{(2+a-b)(2+a-c)(d+e-a)}.\end{aligned}$$

Substituting these two expressions successively into Lemma 2 and then simplifying the result, we get the following four–term relation.

Lemma 25 (Recurrence relation **[31113]**: $\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e)=&\frac{(a-c)(a-e)}{a-c-e}-\frac{ce(a-e)(1+a-b-d)}{(1+a-d)(a-b-e)(a-c-e)}\\ &+\frac{bce(1+e)(a-e)(1+a-b-d)(1+a-c-d)}{(1+a-b)(1+a-c)(1+a-d)(a-b-e)(a-c-e)(a-d-e)}\\ &+\Omega(3+a;1+b,1+c,1+d,3+e)\\ &\times\frac{bcd(e)_3[1+a-b-c,1+a-b-d,1+a-c-d]_1}{[1+a-b,1+a-c,1+a-d]_2[b+e-a,c+e-a,d+e-a]_1}.\end{aligned}$$

Iterating the last relation m times and then denoting by ν_k the weight factor

$$\begin{aligned}\nu_k:=\nu_k(a;b,c,d,e)\ =&\ \frac{(a-c+2k)(a-e)}{a-c-e-k}\\ &-\frac{(c+k)(e+3k)(a-e)(1+a-b-d+k)}{(1+a-d+2k)(a-b-e-k)(a-c-e-k)}\\ &+\frac{(b+k)(c+k)(e+3k)(1+e+3k)(a-e)(1+a-b-d+k)(1+a-c-d+k)}{(1+a-b+2k)(1+a-c+2k)(1+a-d+2k)(a-b-e-k)(a-c-e-k)(a-d-e-k)}\end{aligned}$$

we derive the following transformation formula.

Proposition 26 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\begin{aligned}\Omega(a;b,c,d,e) = &\frac{(e)_{3m}\,\Omega(a+3m;b+m,c+m,d+m,e+3m)}{[1{+}a{-}b,1{+}a{-}c,1{+}a{-}d]_{2m}}\\ &\times\begin{bmatrix}b,c,d,1+a-b-c,1+a-b-d,1+a-c-d\\ b+e-a,\ c+e-a,\ d+e-a\end{bmatrix}_m\\ &+\sum_{k=0}^{m-1}\frac{(e)_{3k}\,\nu_k(a;b,c,d,e)}{[1{+}a{-}b,1{+}a{-}c,1{+}a{-}d]_{2k}}\\ &\times\begin{bmatrix}b,c,d,1+a-b-c,1+a-b-d,1+a-c-d\\ b+e-a,\ c+e-a,\ d+e-a\end{bmatrix}_k.\end{aligned}$$

In order to examine the limiting case $m\to\infty$, recall Bailey's transformation on nonterminating well–poised $_7F_6$-series (Bailey [10, §4.4])

$$\begin{aligned}&{}_7F_6\left[\begin{matrix}\lambda,1+\frac{\lambda}{2}, & a, & b, & c, & d, & e\\ \frac{\lambda}{2}, & 1+\lambda-a, & 1+\lambda-b, & 1+\lambda-c, & 1+\lambda-d, & 1+\lambda-e\end{matrix}\,\middle|\,1\right]\\ &={}_4F_3\left[\begin{matrix}a,\ c,\ e,\ 1+\lambda-b-d\\ 1+\lambda-b,1+\lambda-d,a+c+e-\lambda\end{matrix}\,\middle|\,1\right]\\ &\times\Gamma\begin{bmatrix}1+\lambda-a,1+\lambda-c,1+\lambda-e,1+\lambda-a-c-e\\ 1+\lambda,1+\lambda-a-c,1+\lambda-a-e,1+\lambda-c-e\end{bmatrix}\\ &+{}_4F_3\left[\begin{matrix}1{+}\lambda{-}a{-}c,\ 1{+}\lambda{-}a{-}e,\ 1{+}\lambda{-}c{-}e,\ 2{+}2\lambda{-}a{-}b{-}c{-}d{-}e\\ 2{+}\lambda{-}a{-}c{-}e,\ 2{+}2\lambda{-}a{-}b{-}c{-}e,\ 2{+}2\lambda{-}a{-}c{-}d{-}e\end{matrix}\,\middle|\,1\right]\\ &\times\Gamma\begin{bmatrix}1{+}\lambda{-}a,1{+}\lambda{-}b,1{+}\lambda{-}c,1{+}\lambda{-}d,1{+}\lambda{-}e,a{+}c{+}e{-}1{-}\lambda,2{+}2\lambda{-}a{-}b{-}c{-}d{-}e\\ 1{+}\lambda,\ a,\ c,\ e,\ 1{+}\lambda{-}b{-}d,\ 2{+}2\lambda{-}a{-}b{-}c{-}e,\ 2{+}2\lambda{-}a{-}c{-}d{-}e\end{bmatrix}.\end{aligned}$$

Letting $\lambda\to a+3m$, $a\to 1$ and $b\to b+m$, $c\to c+m$, $d\to d+m$, $e\to e+3m$, we can reformulate the last equation as

$$\begin{aligned}\Omega(a+3m;b+m,c+m,d+m,e+3m)\ =\ &\frac{(a-c+2m)(a-e)}{a-c-e-m}\\ \times\,{}_4F_3&\left[\begin{matrix}1,\ c+m,\ e+3m,\ 1+a-b-d+m\\ 1+a-b+2m,1+a-d+2m,1+c+e-a+m\end{matrix}\,\middle|\,1\right]\\ +\,{}_3F_2&\left[\begin{matrix}a-c+2m,\ a-e,\ 1+2a-b-c-d-e\\ 1+2a-b-c-e+m,\ 1+2a-c-d-e+m\end{matrix}\,\middle|\,1\right]\\ \times\,\Gamma&\begin{bmatrix}1{+}a{-}b{+}2m,1{+}a{-}c{+}2m,1{+}a{-}d{+}2m,1{+}a{-}e,c{+}e{-}a{+}m,1{+}2a{-}b{-}c{-}d{-}e\\ 1{+}a{-}b{-}d{+}m,\ c{+}m,\ e{+}3m,\ 1{+}2a{-}b{-}c{-}e{+}m,\ 1{+}2a{-}c{-}d{-}e{+}m\end{bmatrix}.\end{aligned}$$

According to this transformation, we can show the following asymptotic relation

$$\begin{aligned}\lim_{m\to\infty}&\frac{(e)_{3m}\,\Omega(a+3m;b+m,c+m,d+m,e+3m)}{[1{+}a{-}b,1{+}a{-}c,1{+}a{-}d]_{2m}}\\ &\times\begin{bmatrix}b,c,d,1+a-b-c,1+a-b-d,1+a-c-d\\ b+e-a,\ c+e-a,\ d+e-a\end{bmatrix}_m\\ &=\ \Gamma\begin{bmatrix}1{+}a{-}b,1{+}a{-}c,1{+}a{-}d,1{+}a{-}e,b{+}e{-}a,c{+}e{-}a,d{+}e{-}a,1{+}2a{-}b{-}c{-}d{-}e\\ b,\ c,\ d,\ e,\ 1+a-b-c,\ 1+a-b-d,\ 1+a-c-d\end{bmatrix}.\end{aligned}$$

Finally, letting $m\to\infty$ in the equation displayed in Proposition 26, we derive consequently the following transformation formula.

Theorem 27 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e)-\Gamma\begin{bmatrix}1{+}a{-}b,1{+}a{-}c,1{+}a{-}d,1{+}a{-}e,b{+}e{-}a,c{+}e{-}a,d{+}e{-}a,1{+}2a{-}b{-}c{-}d{-}e\\ b,\ c,\ d,\ e,\ 1+a-b-c,\ 1+a-b-d,\ 1+a-c-d\end{bmatrix}$$
$$=\sum_{k=0}^{\infty}\frac{(e)_{3k}\,\nu_k(a;b,c,d,e)}{[1{+}a{-}b,1{+}a{-}c,1{+}a{-}d]_{2k}}$$
$$\times\begin{bmatrix}b,c,d,1+a-b-c,1+a-b-d,1+a-c-d\\ b+e-a,\ c+e-a,\ d+e-a\end{bmatrix}_k.$$

The following special case corresponding to $c=a$ is particularly useful.

Corollary 28 ($\Re(1+a-b-d-e)>0$)**.**

$$\Gamma\begin{bmatrix}1{+}a{-}b,1{+}a{-}d,1{+}a{-}e,1{+}a{-}b{-}d{-}e\\ a,\ 1{+}a{-}b{-}d,1{+}a{-}b{-}e,1{+}a{-}d{-}e\end{bmatrix}\left\{1-\frac{\sin(b\pi)\,\sin(d\pi)}{\sin(a{-}b{-}e)\pi\,\sin(a{-}d{-}e)\pi}\right\}$$
$$=\sum_{k=0}^{\infty}\frac{(e)_{3k}\,\nu_k(a;b,a,d,e)}{[1,1{+}a{-}b,1{+}a{-}d]_{2k}}\begin{bmatrix}a,b,d,1-b,1-d,1+a-b-d\\ e,\ b+e-a,\ d+e-a\end{bmatrix}_k.$$

Infinite series identities involving π with convergence rate equal to $\frac{27}{64}$ can be derived by specifying the parameters a,b,c,d,e in Theorem 27 such that the corresponding Ω-sum can be evaluated by Dougall's formula for well–poised ${}_5F_4$-series. Ten of them are shown here as examples.

Example 51 (Theorem 27: $a=\frac{3}{2},\ b=\frac{1}{2},\ c=d=e=1$)**.**

$$\frac{9\pi^2}{4}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ \frac{1}{3},\ \frac{2}{3}\\ \frac{3}{2},\ \frac{5}{4},\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{4}\end{bmatrix}_k\Big\{16+83k+138k^2+74k^3\Big\}.$$

Example 52 (Theorem 27: $a=2,\ b=c=d=1,\ e=\frac{3}{2}$; Guillera [42, Equation 18])**.**

$$\frac{16\pi^2}{3}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ \frac{5}{6},\ \frac{7}{6}\\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2}\end{bmatrix}_k\Big\{35+101k+74k^2\Big\}.$$

Example 53 (Theorem 27: $a=b=c=d=\frac{1}{2},\ e=0$)**.**

$$\frac{48}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3}\\ 1,\ 1,\ 1,\ 1,\ 1\end{bmatrix}_k\Big\{3+27k+74k^2\Big\}.$$

Example 54 (Theorem 27: $a=b=c=d=\frac{1}{2},\ e=1$)**.**

$$\frac{64}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3}\\ 1,\ 1,\ 2,\ 2,\ 2\end{bmatrix}_k\Big\{6+69k+135k^2+74k^3\Big\}.$$

Example 55 (Theorem 27: $a=b=c=d=\frac{1}{2},\ e=2$)**.**

$$\frac{2048}{9\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{2}{3},\ \frac{4}{3}\\ 1,\ 1,\ 3,\ 3,\ 3\end{bmatrix}_k\Big\{22+219k+243k^2+74k^3\Big\}.$$

Example 56 (Theorem 27: $a=b=c=d=\frac{1}{2},\ e=3$)**.**

$$\frac{12288}{25\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{4}{3},\ \frac{5}{3}\\ 1,\ 1,\ 4,\ 4,\ 4\end{bmatrix}_k\Big\{48+453k+351k^2+74k^3\Big\}.$$

Encouraged by the last three examples, we have further a more general formula with an extra free integer parameter $n > 0$.

Example 57 (Theorem 27: $a=b=c=d=\frac{1}{2},\ e=n$)**.**

$$\frac{16n}{\pi^2}\begin{bmatrix}1,\ 1\\ \frac{1}{2},\ \frac{1}{2}\end{bmatrix}_n=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{n}{3},\ \frac{n+1}{3},\ \frac{n+2}{3}\\ 1,\ 1,\ 1,\ n+1,\ n+1,\ n+1\end{bmatrix}_k$$
$$\times\Big\{n+5n^2+3k+24nk+42n^2k+27k^2+108nk^2+74k^3\Big\}.$$

Example 58 (Theorem 27: $a=c=1,\ b=\frac{1}{3},\ d=\frac{2}{3},\ e=\frac{1}{2}$)**.**

$$\frac{80\pi}{3\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}1,\ \frac{1}{3},\ \frac{2}{3}\\ \frac{3}{2},\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\Big\{36+133k+111k^2\Big\}.$$

Example 59 (Theorem 27: $a=c=\frac{3}{4},\ b=d=\frac{1}{4},\ e=\frac{1}{2}$)**.**

$$\frac{64}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}\frac{1}{4},\ \frac{3}{4},\ \frac{1}{6},\ \frac{5}{6}\\ 1,\ 1,\ 1,\ \frac{3}{2}\end{bmatrix}_k\Big\{15+182k+296k^2\Big\}.$$

Example 60 (Theorem 27: $a=b=c=d=e=\frac{1}{2}$)**.**

$$8=\sum_{k=0}^{\infty}\Big(\frac{27}{64}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{6},\ \frac{5}{6}\\ 1,\ 1,\ 1\end{bmatrix}_k\frac{7+74k}{1+2k}.$$

9. Infinite series with convergence rate $-\frac{1}{1024}$

In this section, we shall establish a general transformation formula that expresses the Ω-sum in terms of fast convergent series with convergence rate equal to $-\frac{1}{1024}$. As a consequence, several infinite series formulae related to π and $\zeta(3)$ will be derived.

Applying Lemma 2 to the shifted Ω-sum

$$\Omega(2+a;b,1+c,d,1+e)=\Omega(2+a;1+c,b,1+e,d)$$

we get the following equality:

$$\Omega(2+a;b,1+c,d,1+e)=\frac{(2+a-b)(2+a-d)}{2+a-b-d}-\Omega(3+a;1+b,1+c,1+d,1+e)$$
$$\times\frac{bd(1+a-c-e)}{(2+a-c)(2+a-e)(2+a-b-d)}.$$

Substituting this expression into Lemma 12 gives the four–term relation.

Lemma 29 (Recurrence relation **[31111]**: $\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e)=\frac{(1+2a-b-c-d)(a-e)}{1+2a-b-c-d-e}-\Omega(3+a;1+b,1+c,1+d,1+e)$$
$$\times\frac{[b,c,d,e,1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-c-e,1+a-d-e]_1}{[1+a-b,1+a-c,1+a-d,1+a-e,1+2a-b-c-d-e]_2}$$
$$+\frac{[e,1+a-b-c,1+a-b-d,1+a-c-d,2+2a-b-d-e]_1}{[1+a-b,1+a-d]_1(1+2a-b-c-d-e)_2}$$
$$+\frac{[c,e,1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-d-e]_1}{[1+a-b,1+a-c,1+a-d,1+a-e]_1(1+2a-b-c-d-e)_2}.$$

Iterating the last relation m times and then denoting by σ_k the weight factor

$$\sigma_k := \sigma_k(a;b,c,d,e) \;=\; \frac{(1+2a-b-c-d+3k)(a-e+2k)}{1+2a-b-c-d-e+2k}$$
$$+\;\frac{[e+k,1+a-b-c+k,1+a-b-d+k,1+a-c-d+k,2+2a-b-d-e+3k]_1}{[1+a-b+2k,1+a-d+2k]_1(1+2a-b-c-d-e+2k)_2}$$
$$+\;\frac{[c+k,e+k,1+a-b-c+k,1+a-b-d+k,1+a-b-e+k,1+a-c-d+k,1+a-d-e+k]_1}{[1+a-b+2k,1+a-c+2k,1+a-d+2k,1+a-e+2k]_1(1+2a-b-c-d-e+2k)_2}$$

we obtain the following transformation formula.

Proposition 30 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e) = (-1)^m\Omega(a+3m;b+m,c+m,d+m,e+m)$$
$$\times\;\frac{[b,c,d,e,1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-c-e,1+a-d-e]_m}{[1+a-b,1+a-c,1+a-d,1+a-e,1+2a-b-c-d-e]_{2m}}$$
$$+\sum_{k=0}^{m-1}(-1)^k\frac{[b,c,d,e]_k}{(1+2a-b-c-d-e)_{2k}}\sigma_k(a;b,c,d,e)$$
$$\times\;\frac{[1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-c-e,1+a-d-e]_k}{[1+a-b,1+a-c,1+a-d,1+a-e]_{2k}}.$$

Letting $m\to\infty$ in the last equation yields the following transformation formula.

Theorem 31 ($\Re(1+2a-b-c-d-e)>0$)**.**

$$\Omega(a;b,c,d,e) = \sum_{k=0}^{\infty}(-1)^k\frac{[b,c,d,e]_k}{(1+2a-b-c-d-e)_{2k}}\sigma_k(a;b,c,d,e)$$
$$\times\;\frac{[1+a-b-c,1+a-b-d,1+a-b-e,1+a-c-d,1+a-c-e,1+a-d-e]_k}{[1+a-b,1+a-c,1+a-d,1+a-e]_{2k}}.$$

When $a=\frac{1}{2}+3x$ and $b=c=d=e=\frac{1}{2}+x$, we have the simplified Ω-sum

$$\Omega(\tfrac{1}{2}+3x;\tfrac{1}{2}+x,\tfrac{1}{2}+x,\tfrac{1}{2}+x,\tfrac{1}{2}+x) = \sum_{n=0}^{\infty}\begin{bmatrix}\frac{1}{2}+x\\ 1+2x\end{bmatrix}_n^4\Big\{\tfrac{1}{2}+3x+2n\Big\}.$$

Then factoring the weight factor $\sigma_k(a;b,c,d,e)$ appearing in Theorem 31 leads us to the following identity due to Guillera, who gets it by using the WZ–method.

Corollary 32 (Guillera [41, Identity 9: Equation 16])**.**

$$128x\sum_{n=0}^{\infty}\begin{bmatrix}\frac{1}{2}+x\\ 1+2x\end{bmatrix}_n^4\Big\{1+6x+4n\Big\}$$
$$=\;\sum_{k=0}^{\infty}\Big(\frac{-1}{1024}\Big)^k\begin{bmatrix}\frac{1}{2}+x\\ 1+x\end{bmatrix}_k^5\Big\{13+180(x+k)+820(x+k)^2\Big\}.$$

There exist other infinite series identities involving π and $\zeta(3)$ with convergence rate equal to $-\frac{1}{1024}$. They can be deduced from Theorem 31 by specifying the parameters a,b,c,d,e such that the corresponding Ω-sum can be evaluated by Dougall's formula for well–poised ${}_5F_4$-series. We present ten of them as examples.

Example 61 (Theorem 31: $a=2,\ b=c=d=e=1$)**.**

$$64\zeta(3) = \sum_{k=0}^{\infty}\Big(\frac{-1}{1024}\Big)^k\begin{bmatrix}1,&1,&1,&1,&1\\ \frac{3}{2},&\frac{3}{2},&\frac{3}{2},&\frac{3}{2},&\frac{3}{2}\end{bmatrix}_k\Big\{77+250k+205k^2\Big\}.$$

This formula was discovered by Amdeberhan and Zeilberger [3], and was used by Greg Fee and Simon Plouffe to obtain 520,000 decimal places of $\zeta(3)$ in 1996.

Example 62 (Theorem 31: $a = 1,\ b = c = d = e = \frac{1}{2}$).

$$\frac{567}{2}\zeta(3) = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} 1,\ 1,\ 1,\ 1,\ 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2} \\ \frac{5}{4},\ \frac{5}{4},\ \frac{5}{4},\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{4},\ \frac{7}{4},\ \frac{7}{4} \end{bmatrix}_k \times \left\{341 + 2844k + 9122k^2 + 14236k^3 + 10888k^4 + 3280k^5\right\}.$$

Example 63 (Theorem 31: $a = 2,\ b = c = d = 1,\ e = \frac{3}{2}$).

$$24\pi^2 = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} 1,\ 1,\ 1,\ \frac{1}{2} \\ \frac{5}{4},\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{4} \end{bmatrix}_k \left\{237 + 623k + 410k^2\right\}.$$

Example 64 (Theorem 31: $a = b = c = d = e = \frac{1}{2}$).

$$\frac{128}{\pi^2} = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2} \\ 1,\ 1,\ 1,\ 1,\ 1 \end{bmatrix}_k \left\{13 + 180k + 820k^2\right\}.$$

Example 65 (Theorem 31: $a = c = d = e = \frac{1}{2},\ b = -\frac{1}{2}$).

$$\frac{2048}{\pi^2} = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} -\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{3}{2},\ \frac{3}{2} \\ 1,\ 1,\ 1,\ 2,\ 2 \end{bmatrix}_k \left\{207 + 2046k + 3476k^2 + 1640k^3\right\}.$$

Example 66 (Theorem 31: $a = b = \frac{3}{2},\ c = d = \frac{1}{2},\ e = -\frac{1}{2}$).

$$\frac{131072}{9\pi^2} = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} -\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{5}{2},\ \frac{5}{2} \\ 1,\ 2,\ 2,\ 2,\ 2 \end{bmatrix}_k \left\{1475 + 4614k + 4788k^2 + 1640k^3\right\}.$$

Example 67 (Theorem 31: $a = b = 1,\ c = \frac{1}{2},\ d = \frac{1}{4},\ e = \frac{3}{4}$).

$$\frac{105\pi}{2} = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{3}{4} \\ \frac{9}{8},\ \frac{11}{8},\ \frac{13}{8},\ \frac{15}{8} \end{bmatrix}_k \left\{165 + 902k + 1533k^2 + 820k^3\right\}.$$

Example 68 (Theorem 31: $a = b = 1,\ c = \frac{1}{2},\ d = \frac{1}{3},\ e = \frac{2}{3}$).

$$36\sqrt{3}\pi = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3} \\ \frac{5}{4},\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{4} \end{bmatrix}_k \left\{196 + 1217k + 2197k^2 + 1230k^3\right\}.$$

Example 69 (Theorem 31: $a = b = c = \frac{1}{2},\ d = \frac{1}{6},\ e = \frac{5}{6}$).

$$\frac{3072}{\pi\sqrt{3}} = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} \frac{1}{2},\ \frac{1}{6},\ \frac{1}{6},\ \frac{5}{6},\ \frac{5}{6} \\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{5}{3} \end{bmatrix}_k \left\{565 + 12234k + 46620k^2 + 44280k^3\right\}.$$

Example 70 (Theorem 31: $a = b = c = \frac{1}{2},\ d = \frac{1}{4},\ e = \frac{3}{4}$).

$$\frac{480}{\pi} = \sum_{k=0}^{\infty}\left(\frac{-1}{1024}\right)^k \begin{bmatrix} \frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4} \\ 1,\ 1,\ 1,\ \frac{7}{8},\ \frac{9}{8},\ \frac{11}{8},\ \frac{13}{8} \end{bmatrix}_k \left\{153 + 3360k + 24480k^2 + 64000k^3 + 52480k^4\right\}.$$

10. Miscellaneous infinite series

According to the acceleration theorems established in the last seven sections, one can derive numerous infinite series expressions for $\pi^{\pm 1}$, $\pi^{\pm 2}$, G and $\zeta(3)$. The ten displayed examples for each section are only the tip of the iceberg, which represent, in fact, the simplest formulae among a very large class of infinite series with the same convergence rate. Here we collect further 55 examples of particular interest (e.g., Apéry–series and BBP–series) to illustrate their diversity and variety.

Example 71 (Theorem 9: $a=2,\ b=c=d=e=1$; Apéry series [4, 47]).

$$\frac{4}{5}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\frac{(1)_k}{(\frac{3}{2})_k(1+k)^2}\quad\rightleftharpoons\quad\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}.$$

Example 72 (Theorem 14: $a=\frac{3}{2},\ b=c=d=1,\ e=\frac{1}{2}$).

$$\frac{7}{4}\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\begin{bmatrix}1,\ 1\\ \frac{3}{2},\ \frac{3}{2}\end{bmatrix}_k\frac{2+3k}{(1+k)(1+2k)}.$$

Example 73 (Theorem 14: $a=c=1,\ b=d=e=\frac{1}{2}$; Guillera [41, §2.1: $a=\frac{1}{2}$]).

$$\frac{\pi^2}{4}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\begin{bmatrix}1,\ 1,\ 1\\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2}\end{bmatrix}_k\Big\{2+3k\Big\}.$$

Example 74 (Theorem 9: $a=2,\ b=c=d=1,\ e\to-\infty$; Apéry series [4, 47]).

$$\frac{\pi^2}{9}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\frac{(1)_k}{(\frac{3}{2})_k(1+k)}\quad\rightleftharpoons\quad\frac{\pi^2}{18}=\sum_{n=1}^{\infty}\frac{1}{n^2\binom{2n}{n}}.$$

Example 75 (Theorem 9: $a=c=\frac{3}{2},\ b=d=e=\frac{1}{2}$).

$$\frac{256}{\pi^2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{3}{2},\ \frac{3}{2}\\ 1,\ 2,\ 2,\ 2,\ 2\end{bmatrix}_k\Big\{27+94k+108k^2+40k^3\Big\}.$$

Example 76 (Theorem 9: $a=c=1,\ b=\frac{1}{2},\ d=\frac{1}{4},\ e=\frac{3}{4}$).

$$\frac{3\pi}{2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4}\\ \frac{3}{2},\ \frac{5}{4},\ \frac{7}{4}\end{bmatrix}_k\Big\{5+21k+20k^2\Big\}.$$

This is equivalent to the formula of BBP–type due to Adamchik and Wagon [1].

$$\pi=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\Big\{\frac{2}{1+4k}+\frac{2}{2+4k}+\frac{1}{3+4k}\Big\}.$$

Example 77 (Theorem 9: $a=c=1,\ b=\frac{2}{3},\ d=\frac{1}{3},\ e\to-\infty$).

$$\frac{16\pi}{9\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\begin{bmatrix}1,\ \frac{1}{3},\ \frac{1}{3}\\ \frac{5}{3},\ \frac{5}{3},\ \frac{7}{6}\end{bmatrix}_k\Big\{3+10k+9k^2\Big\}.$$

Example 78 (Theorem 9: $a=c=1,\ b=\frac{1}{3},\ d=\frac{2}{3},\ e\to-\infty$).

$$\frac{20\pi}{9\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\begin{bmatrix}1,\ \frac{2}{3},\ \frac{2}{3}\\ \frac{4}{3},\ \frac{4}{3},\ \frac{11}{6}\end{bmatrix}_k\Big\{3+11k+9k^2\Big\}.$$

Example 79 (Theorem 9: $a=c=e=\frac{1}{2},\ b=\frac{3}{4},\ d=\frac{1}{4}$)**.**

$$\frac{3}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{4}\\ 1,\ 1,\ 1,\ \frac{7}{8},\ \frac{11}{8}\end{matrix}\right]_k\Big\{1+10k+20k^2\Big\}.$$

Example 80 (Theorem 9: $a=b=c=\frac{1}{2},\ d=\frac{1}{3},\ e=\frac{2}{3}$)**.**

$$\frac{3\sqrt{3}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}\frac{1}{2},\ \frac{1}{3},\ \frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\\ 1,\ 1,\ 1,\ \frac{5}{6},\ \frac{7}{6}\end{matrix}\right]_k\Big\{2+18k+45k^2\Big\}.$$

Example 81 (Theorem 9: $a=b=c=d=\frac{1}{3},\ e\to-\infty$)**.**

$$\frac{9\sqrt{3}}{2\pi}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\left[\begin{matrix}\frac{1}{3},\ \frac{1}{3},\ \frac{2}{3},\ \frac{2}{3}\\ 1,\ 1,\ 1,\ \frac{3}{2}\end{matrix}\right]_k\Big\{2+18k+27k^2\Big\}.$$

Example 82 (Theorem 9: $a=b=c=d=\frac{1}{4},\ e\to-\infty$)**.**

$$\frac{8\sqrt{2}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\left[\begin{matrix}\frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4}\\ 1,\ 1,\ 1,\ \frac{3}{2}\end{matrix}\right]_k\Big\{3+32k+48k^2\Big\}.$$

Example 83 (Theorem 9: $a=b=c=d=\frac{1}{6},\ e\to-\infty$)**.**

$$\frac{18}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\left[\begin{matrix}\frac{1}{6},\ \frac{1}{6},\ \frac{5}{6},\ \frac{5}{6}\\ 1,\ 1,\ 1,\ \frac{3}{2}\end{matrix}\right]_k\Big\{5+72k+108k^2\Big\}.$$

Example 84 (Theorem 9: $a=1,\ b=c=d=\frac{1}{2},\ e\to-\infty$)**.**

$$6G=\sum_{k=0}^{\infty}\Big(\frac{1}{4}\Big)^k\left[\begin{matrix}1,\ 1\\ \frac{5}{4},\ \frac{7}{4}\end{matrix}\right]_k\frac{5+6k}{1+2k}.$$

Example 85 (Theorem 14: $a=1,\ b=c=d=\frac{1}{2},\ e\to-\infty$)**.**

$$18G=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}1,\ 1,\ 1,\ \frac{1}{2}\\ \frac{5}{4},\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{4}\end{matrix}\right]_k\Big\{19+56k+40k^2\Big\}.$$

Example 86 (Theorem 14: $a=\frac{3}{2},\ b=c=d=1,\ e\to-\infty$)**.**

$$6G=\sum_{k=0}^{\infty}\Big(\frac{-1}{4}\Big)^k\left[\begin{matrix}1,\ \frac{1}{2}\\ \frac{5}{4},\ \frac{7}{4}\end{matrix}\right]_k\frac{17+94k+156k^2+80k^3}{(1+2k)(1+4k)(3+4k)}.$$

Example 87 (Theorem 31: $a=b=c=d=\frac{1}{4},\ e\to-\infty$; See Guillera [38, Table 2:3])**.**

$$\frac{2\sqrt{2}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{8}\Big)^k\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ 1,\ 1\end{matrix}\right]_k\Big\{1+6k\Big\}.$$

Example 88 (Theorem 27: $a=1,\ b=c=d=\frac{1}{2},\ e\to-\infty$ (bisection series); Guillera [41, §2.3: $a=\frac{1}{2}$])**.**

$$2G=\sum_{k=0}^{\infty}\Big(\frac{-1}{8}\Big)^k\left[\begin{matrix}1,\ 1,\ 1\\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2}\end{matrix}\right]_k\Big\{2+3k\Big\}.$$

Example 89 (Theorem 14: $a = c = 1,\ e = \frac{1}{2},\ b = \frac{1}{3},\ d = \frac{2}{3}$).

$$\frac{4\pi}{\sqrt{3}} = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \begin{bmatrix} 1,\ \frac{1}{2} \\ \frac{5}{4},\ \frac{7}{4} \end{bmatrix}_k \Big\{7 + 10k\Big\}.$$

This is the bisection series of the following Apéry series [4, 47]:

$$\frac{2\pi}{\sqrt{3}} = 3\sum_{k=0}^{\infty} \frac{(1)_k}{4^k(\frac{3}{2})_k} \quad \rightleftharpoons \quad \frac{\pi}{3\sqrt{3}} = \sum_{n=1}^{\infty} \frac{1}{n\binom{2n}{n}}.$$

Example 90 (Theorem 14: $a = b = 1,\ c = \frac{1}{2},\ d = \frac{3}{4},\ e = \frac{1}{4}$).

$$15\pi = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \begin{bmatrix} \frac{1}{2},\ \frac{3}{4},\ \frac{1}{8},\ \frac{5}{8} \\ \frac{3}{2},\ \frac{7}{4},\ \frac{9}{8},\ \frac{13}{8} \end{bmatrix}_k \Big\{47 + 151k + 120k^2\Big\}.$$

It is equivalent to the following celebrated BBP–formula due to Bailey, Borwein and Plouffe [9, Theorem 1] *(cf.* [8, §2.4] *also) which serves, amazingly, as a digit–extraction algorithm for* π*:*

$$\pi = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \Big\{\frac{4}{1+8k} - \frac{2}{4+8k} - \frac{1}{5+8k} - \frac{1}{6+8k}\Big\}.$$

Example 91 (Theorem 14: $a = b = 1,\ c = \frac{1}{2},\ d = \frac{1}{4},\ e = \frac{3}{4}$).

$$21\pi = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \begin{bmatrix} \frac{1}{2},\ \frac{1}{4},\ \frac{3}{8},\ \frac{7}{8} \\ \frac{3}{2},\ \frac{5}{4},\ \frac{11}{8},\ \frac{15}{8} \end{bmatrix}_k \Big\{65 + 413k + 812k^2 + 480k^3\Big\}.$$

It is equivalent to the following BBP–formula due to Adamchik and Wagon [1]*:*

$$2\pi = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \Big\{\frac{8}{2+8k} + \frac{4}{3+8k} + \frac{4}{4+8k} - \frac{1}{7+8k}\Big\}.$$

Example 92 (Theorem 14: $a = b = c = \frac{1}{2},\ d = \frac{3}{4},\ e = \frac{1}{4}$).

$$\frac{48}{\pi} = \sum_{k=0}^{\infty} \Big(\frac{1}{16}\Big)^k \begin{bmatrix} \frac{1}{2},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{8},\ \frac{5}{8} \\ 1,\ 1,\ 1,\ \frac{7}{8},\ \frac{11}{8} \end{bmatrix}_k \Big\{15 + 212k + 480k^2\Big\}.$$

Example 93 (Theorem 18: $a = \frac{3}{2},\ b = e = 1,\ c = d = \frac{1}{2}$).

$$10\pi^2 = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \begin{bmatrix} \frac{1}{2},\ \frac{1}{2} \\ \frac{7}{6},\ \frac{11}{6} \end{bmatrix}_k \frac{99 + 314k + 328k^2 + 112k^3}{(1+k)^2(1+2k)}.$$

Example 94 (Theorem 18: $a = c = 1,\ e = \frac{1}{2},\ b = \frac{2}{3},\ d = \frac{1}{3}$).

$$\frac{28\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \begin{bmatrix} 1,\ \frac{2}{3},\ \frac{2}{3},\ \frac{1}{6} \\ \frac{3}{2},\ \frac{10}{9},\ \frac{13}{9},\ \frac{16}{9} \end{bmatrix}_k \Big\{17 + 54k + 42k^2\Big\}.$$

Example 95 (Theorem 18: $a = c = 1,\ e = \frac{1}{2},\ b = \frac{1}{3},\ d = \frac{2}{3}$).

$$\frac{80\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \Big(\frac{-1}{27}\Big)^k \begin{bmatrix} 1,\ \frac{1}{3},\ \frac{1}{3},\ \frac{5}{6} \\ \frac{3}{2},\ \frac{11}{9},\ \frac{14}{9},\ \frac{17}{9} \end{bmatrix}_k \Big\{49 + 261k + 456k^2 + 252k^3\Big\}.$$

Example 96 (Theorem 18: $a=b=1,\ c=\frac{1}{3},\ d=\frac{2}{3},\ e\to-\infty$).

$$\frac{40\pi}{\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}\frac{1}{3},\ \frac{2}{3},\ \frac{1}{6},\ \frac{5}{6}\\ \frac{4}{3},\ \frac{5}{3},\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\Big\{73+460k+864k^2+504k^3\Big\}.$$

This is equivalent to the following formula of BBP–type:

$$2\sqrt{3}\pi=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\Big\{\frac{9}{1+6k}+\frac{3}{2+6k}+\frac{1}{4+6k}+\frac{1}{5+6k}\Big\}.$$

Example 97 (Theorem 18: $a=b=e=\frac{1}{2},\ c=\frac{1}{4},\ d=\frac{3}{4}$).

$$\frac{64}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4}\\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\Big\{21+296k+992k^2+896k^3\Big\}.$$

Example 98 (Theorem 18: $a=b=c=d=\frac{2}{3},\ e\to-\infty$).

$$\frac{243\sqrt{3}}{8\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3},\ \frac{1}{6},\ \frac{1}{6}\\ 1,\ 1,\ 1,\ \frac{3}{2},\ \frac{3}{2}\end{bmatrix}_k\Big\{17+279k+864k^2+756k^3\Big\}.$$

Example 99 (Theorem 18: $a=b=c=d=\frac{1}{3},\ e\to-\infty$).

$$\frac{243\sqrt{3}}{4\pi}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ \frac{5}{6},\ \frac{5}{6}\\ 1,\ 1,\ 1,\ \frac{3}{2},\ \frac{3}{2}\end{bmatrix}_k\Big\{35+504k+1620k^2+1512k^3\Big\}.$$

Example 100 (Theorem 18: $a=b=\frac{1}{2},\ c=\frac{1}{4},\ d=\frac{3}{4},\ e\to-\infty$).

$$16\sqrt{2}=\sum_{k=0}^{\infty}\Big(\frac{-1}{27}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{3}{4}\\ 1,\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\Big\{23+110k+112k^2\Big\}.$$

Example 101 (Theorem 18: $a=1,\ b=c=e=\frac{1}{2},\ d\to-\infty$).

$$30G=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}1,\ 1\\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\frac{79+428k+708k^2+368k^3}{(1+2k)(1+4k)(3+4k)}.$$

Example 102 (Theorem 18: $a=2,\ b=c=e=1,\ d\to-\infty$).

$$2\pi^2=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}1,\ \frac{1}{2}\\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\frac{19+58k+46k^2}{(1+k)(1+2k)}.$$

Example 103 (Theorem 18: $a=b=1,\ c=\frac{2}{3},\ e=\frac{1}{3},\ d\to-\infty$).

$$4\pi\sqrt{3}=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}\frac{2}{3},\ \frac{1}{6}\\ 1,\ \frac{4}{3}\end{bmatrix}_k\frac{43+246k+414k^2}{(2+3k)(1+6k)}.$$

Example 104 (Theorem 18: $a=b=1,\ c=\frac{1}{3},\ e=\frac{2}{3},\ d\to-\infty$).

$$8\pi\sqrt{3}=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}\frac{1}{3},\ \frac{5}{6}\\ 1,\ \frac{5}{3}\end{bmatrix}_k\frac{214+591k+414k^2}{(1+3k)(5+6k)}.$$

Example 105 (Theorem 18: $a=b=c=e=\frac{1}{3},\ d\to-\infty$).

$$\frac{81\sqrt{3}}{2\pi}=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}\frac{1}{3},\ \frac{2}{3},\ \frac{1}{6},\ \frac{5}{6}\\ 1,\ 1,\ 1,\ \frac{3}{2}\end{bmatrix}_k\Big\{20+243k+414k^2\Big\}.$$

Example 106 (Theorem 18: $a=b=c=e=\frac{1}{2}$, $d\to-\infty$).

$$\frac{32}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4}\\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\Big\{9+118k+400k^2+368k^3\Big\}.$$

Example 107 (Theorem 18: $a=e=\frac{1}{2}$, $b=\frac{1}{4}$, $c=\frac{3}{4}$, $d\to-\infty$).

$$\frac{45}{2\sqrt{2}}=\sum_{k=0}^{\infty}\Big(\frac{4}{27}\Big)^k\begin{bmatrix}\frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4},\ \frac{3}{4}\\ 1,\ \frac{7}{8},\ \frac{11}{8},\ \frac{13}{12},\ \frac{17}{12}\end{bmatrix}_k\Big\{12+169k+644k^2+736k^3\Big\}.$$

Example 108 (Theorem 24: $a=d=1$, $c=e=-\frac{1}{2}$, $b=\frac{1}{2}$).

$$\frac{3375\pi^2}{256}=\sum_{k=0}^{\infty}\Big(-\frac{16}{27}\Big)^k\begin{bmatrix}1,\ 2,\ \frac{3}{2},\ -\frac{1}{4},\ \frac{1}{4}\\ \frac{7}{2},\ \frac{7}{4},\ \frac{9}{4},\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\Big\{128+303k+172k^2\Big\}.$$

Example 109 (Theorem 21: $a=c=1$, $b=\frac{1}{3}$, $e=\frac{2}{3}$, $d\to-\infty$).

$$\frac{80\pi}{9\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{16}{27}\Big)^k\begin{bmatrix}1,\ \frac{1}{2},\ \frac{5}{3},\ \frac{5}{6}\\ \frac{2}{3},\ \frac{11}{9},\ \frac{14}{9},\ \frac{17}{9}\end{bmatrix}_k\Big\{9+11k\Big\}.$$

Example 110 (Theorem 21: $a=b=c=e=\frac{1}{2}$, $d\to-\infty$).

$$\frac{16}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{16}{27}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{4},\ \frac{1}{4},\ \frac{3}{4},\ \frac{3}{4}\\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\Big\{3+36k+108k^2+88k^3\Big\}.$$

Example 111 (Theorem 21: $a=b=c=e=\frac{1}{3}$, $d\to-\infty$).

$$\frac{27\sqrt{3}}{2\pi}=\sum_{k=0}^{\infty}\Big(\frac{16}{27}\Big)^k\begin{bmatrix}\frac{2}{3},\ \frac{2}{3},\ \frac{2}{3},\ \frac{1}{6},\ \frac{1}{6}\\ 1,\ 1,\ 1,\ \frac{4}{3},\ \frac{4}{3}\end{bmatrix}_k\Big\{4+63k+243k^2+297k^3\Big\}.$$

Example 112 (Theorem 21: $a=b=c=e=\frac{2}{3}$, $d\to-\infty$).

$$\frac{54\sqrt{3}}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{16}{27}\Big)^k\begin{bmatrix}\frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ \frac{5}{6},\ \frac{5}{6}\\ 1,\ 1,\ 1,\ \frac{5}{3},\ \frac{5}{3}\end{bmatrix}_k\Big\{20+216k+486k^2+297k^3\Big\}.$$

Example 113 (Theorem 21: $a=c=\frac{1}{2}$, $b=\frac{3}{4}$, $e=\frac{1}{4}$, $d\to-\infty$).

$$\frac{21}{2\sqrt{2}}=\sum_{k=0}^{\infty}\Big(\frac{16}{27}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{3}{4},\ \frac{1}{8},\ \frac{5}{8}\\ 1,\ \frac{5}{4},\ \frac{11}{12},\ \frac{19}{12}\end{bmatrix}_k\Big\{6+37k+44k^2\Big\}.$$

Example 114 (Theorem 31: $a=2$, $b=c=d=1$, $e\to-\infty$; Guillera [41, §2.2: $a=\frac{1}{2}$]).

$$\frac{4\pi^2}{3}=\sum_{k=0}^{\infty}\Big(\frac{1}{64}\Big)^k\begin{bmatrix}1,\ 1,\ 1\\ \frac{3}{2},\ \frac{3}{2},\ \frac{3}{2}\end{bmatrix}_k\Big\{13+21k\Big\}.$$

Example 115 (Theorem 31: $a=b=1$, $c=\frac{1}{3}$, $d=\frac{2}{3}$, $e\to-\infty$).

$$\frac{40\pi}{9\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{1}{64}\Big)^k\begin{bmatrix}1,\ \frac{1}{3},\ \frac{2}{3}\\ \frac{3}{2},\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\Big\{8+27k+21k^2\Big\}.$$

Example 116 (Theorem 31: $a=b=c=d=\frac{1}{2}$, $e\to-\infty$).

$$\frac{16}{\pi}=\sum_{k=0}^{\infty}\Big(\frac{1}{64}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ 1,\ 1\end{bmatrix}_k\Big\{5+42k\Big\}.$$

Example 117 (Theorem 31: $a=b=\frac{1}{2}$, $c=\frac{1}{6}$, $d=\frac{5}{6}$, $e\to-\infty$)**.**

$$\frac{256}{3\sqrt{3}}=\sum_{k=0}^{\infty}\Big(\frac{1}{64}\Big)^k\begin{bmatrix}\frac{1}{2},\ \frac{1}{6},\ \frac{5}{6}\\ 1,\ \frac{4}{3},\ \frac{5}{3}\end{bmatrix}_k\Big\{49+240k+252k^2\Big\}.$$

In addition, there exist other iteration patterns from which some extraordinary infinite series may come out. We limit ourselves to producing just eight spectacular examples for $\zeta(3)$ and π^2 with better convergence rates, where the five boldface numbers in headers indicate the iteration pattern.

Example 118 ([**30111**]: $a=2,\ b=c=d=e=1$)**.**

$$144\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{1}{729}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ 1\\ \frac{4}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{5}{3}\end{bmatrix}_k\frac{173+501k+364k^2}{1+2k}.$$

Example 119 ([**30111**] with $a=\frac{3}{2},\ b=c=\frac{1}{2},\ d=e=1$)**.**

$$36\pi^2=\sum_{k=0}^{\infty}\Big(\frac{1}{729}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ \frac{4}{3},\ \frac{4}{3},\ \frac{5}{3},\ \frac{5}{3},\ \frac{5}{4},\ \frac{7}{4}\end{bmatrix}_k\Big\{355+2494k+6381k^2+7100k^3+2912k^4\Big\}.$$

Example 120 ([**40112**]: $a=2,\ b=c=d=e=1$)**.**

$$1728\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{1}{4096}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ 1\\ \frac{5}{4},\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{4}\end{bmatrix}_k\frac{4154+26427k+62152k^2+64161k^3+24570k^4}{(1+2k)(1+3k)(2+3k)}.$$

Example 121 ([**50222**]: $a=2,\ b=c=d=e=1$)**.**

$$720\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{12500}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ 1,\ \frac{1}{2},\ \frac{1}{2}\\ \frac{5}{4},\ \frac{7}{4},\ \frac{6}{5},\ \frac{7}{5},\ \frac{8}{5},\ \frac{9}{5}\end{bmatrix}_k\times\frac{1731+14012k+44625k^2+70060k^3+54324k^4+16668k^5}{(1+3k)(2+3k)}.$$

Example 122 ([**50222**]: $a=\frac{3}{2},\ b=c=\frac{1}{2},\ d=e=1$)**.**

$$400\pi^2=\sum_{k=0}^{\infty}\Big(\frac{-1}{12500}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3}\\ \frac{6}{5},\ \frac{7}{5},\ \frac{8}{5},\ \frac{9}{5},\ \frac{7}{6},\ \frac{11}{6}\end{bmatrix}_k\times\Big\{3948+26275k+63270k^2+65745k^3+25002k^4\Big\}.$$

Example 123 ([**51122**]: $a=2,\ b=c=d=e=1$)**.**

$$10368\zeta(3)=\sum_{k=0}^{\infty}\Big(\frac{-1}{110592}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ 1,\ 1,\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ \frac{4}{3},\ \frac{5}{3},\ \frac{5}{4},\ \frac{5}{4},\ \frac{5}{4},\ \frac{7}{4},\ \frac{7}{4},\ \frac{7}{4}\end{bmatrix}_k\times\Big\{12463+104000k+336367k^2+531578k^3+412708k^4+126392k^5\Big\}.$$

Example 124 ([**51122**]: $a=2,\ b=c=d=1,\ e=\frac{3}{2}$)**.**

$$8400\pi^2=\sum_{k=0}^{\infty}\Big(\frac{-1}{110592}\Big)^k\begin{bmatrix}1,\ 1,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{3}{4}\\ \frac{7}{6},\ \frac{11}{6},\ \frac{9}{8},\ \frac{11}{8},\ \frac{13}{8},\ \frac{15}{8}\end{bmatrix}_k\times\Big\{82905+557539k+1333522k^2+1361936k^3+505568k^4\Big\}.$$

Example 125 ([**82233**]: $a = 2,\ b = c = d = e = 1$)**.**

$$1036800000\zeta(3) = \sum_{k=0}^{\infty} \frac{P(k)}{156250000^k} \left[\begin{matrix} 1, & 1, & 1, & 1, & 1, & \frac{1}{2}, & \frac{1}{3}, & \frac{1}{3}, & \frac{1}{3}, & \frac{2}{3}, & \frac{2}{3}, & \frac{2}{3}, & \frac{1}{4}, & \frac{3}{4} \\ \frac{6}{5}, & \frac{6}{5}, & \frac{7}{5}, & \frac{7}{5}, & \frac{8}{5}, & \frac{8}{5}, & \frac{9}{5}, & \frac{9}{5}, & \frac{7}{6}, & \frac{7}{6}, & \frac{7}{6}, & \frac{11}{6}, & \frac{11}{6}, & \frac{11}{6} \end{matrix}\right]_k$$

where

$$\begin{aligned} P(k) = & 1246292592 + 29319141784k + 303380460778k^2 + 1825595213885k^3 + 7113793975559k^4 \\ & + 18891214749816k^5 + 34963711340436k^6 + 45193377663801k^7 + 40057491691179k^8 \\ & + 23227624872090k^9 + 7942499953056k^{10} + 1214999992224k^{11}. \end{aligned}$$

The convergence rate of this last series can be highlighted as follows:

$$\frac{1}{156250000} = \frac{1}{2^4 \times 5^{10}} = 6.4 \times 10^{-9}.$$

Acknowledgments

Two anonymous referees have carefully read and corrected the manuscript. The authors are grateful to them for their valuable comments and suggestions.

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School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, People's Republic of China

E-mail address: chu.wenchang@unisalento.it

Current address: Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Via Arnesano P. O. Box 193, 73100 Lecce, Italia

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, People's Republic of China

E-mail address: wenlong.dlut@yahoo.com.cn