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ACCELERATION OF CONVERGENCE OF A TWO-LEVEL ALGORITHM BY SMOOTHING TRANSFER OPERATORS

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Summary. The technique for accelerating the convergence of the algebraic multigrid method is proposed.

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Introduction

The rate of convergence of the algebraic multigrid method is strongly dependent on the choice of the so-called transfer operators. Necessary condition for achieving high rate of convergence is that the prolongation operator p contains in its range only vectors x for which $Ax \approx 0$. Such vectors are called smooth. The algorithm described in this paper is based on: we will use any prolongation operator p and non smooth vectors from its range will be suppressed by left multiplication by the iteration operator of Jacobi method. The basic form of transfer operators can be obtained by using the so-called unknowns aggregation (see [1], [3]). The convergence analysis is carried out only for the case of a symmetric and positive definite matrix, but the algorithm can be used in the non-symmetric case as well.

1. NOTATION

Let us consider finite dimensional real spaces H^1, H^2, where $n = \dim(H^1)$, $m = \dim(H^2)$, $m < n$. Let the space H^i be equipped with an inner product $\langle \cdot, \cdot \rangle_i$ and the associated norm $\| \cdot \|_i = \langle \cdot, \cdot \rangle_i^{\frac{1}{2}}$, $i = 1, 2$. In most applications H^i, $i = 1, 2$ will be Euclidean spaces.

We are interested in numerical solution $\hat{u} \in H^1$ of the problem

$$Au = f \tag{1.1}$$

where $f \in H^1$, $u \in H^1$ and $A: H^1 \to H^1$ is a linear, symmetric and positive operator. The problem (1.1) has a unique solution for any $f \in H^1$.

Let p: $H^2 \to H^1$ be a linear injective operator called a prolongation. Adjoint operators relative to the inner products $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$ will be denoted by *.

Let us define the restriction operator $r: H^1 \to H^2$ by

$$r = p^*, \tag{1.2}$$

i.e.

$$\langle x, py \rangle_1 = \langle rx, y \rangle_2 \quad \text{for any } x \in H^1, y \in H^2 \tag{1.3}$$

For the technical details of the construction of r, p see [3].

Let us set

$$^2A = rAp. \tag{1.4}$$

It is easy to see that ^2A is a linear, symmetric and positive operator. Hence we can define other inner products

$$(\cdot, \cdot)_1 = \langle A\cdot, \cdot \rangle_1, \tag{1.5}$$

$$(\cdot, \cdot)_2 = \langle {}^2A\cdot, \cdot \rangle_2 \tag{1.6}$$

and the associated norms by

$$\|\!| \cdot |\!\|_1 = (\cdot, \cdot)_1^{\frac{1}{2}}, \tag{1.7}$$

$$\|\!| \cdot |\!\|_2 = (\cdot, \cdot)_2^{\frac{1}{2}} \tag{1.8}$$

2. Standard two-level algorithm

Let

$$\varphi(\cdot)\colon H^1 \to H^1 \tag{2.1}$$

be an iterative method for the solution of (1.1) satisfying the condition

$$\varphi(\hat{u}) = \hat{u}. \tag{2.2}$$

For any integer $\nu > 0$ we define

$$\varphi^{(\nu)}(\cdot) = \varphi(\varphi^{(\nu-1)}(\cdot)) \tag{2.3}$$

and for $\nu = 0$ we set

$$\varphi^{(0)}(\cdot) = I^1. \tag{2.4}$$

where I^1 is the identity operator on H^1.

Further, we will suppose that $\varphi(x)$ can be written in the form

$$\varphi(x) = Mx + Nf, \tag{2.5}$$

where $M, N\colon H^1 \to H^1$ are linear operators and the condition

$$I^1 = NA + M \tag{2.6}$$

is valid. Let us note that (2.6) implies (2.2).

Let $u^i \in H^1$ be an arbitrary vector, $\nu_1 \geqslant 0, \nu_2 < 0$ given integers. One iteration $(u^i \to u^{i+1})$ of the standard two-level algorithm is defined as follows:

$$\tilde{u} := \varphi^{(\nu_1)}(u^i), \quad \tilde{u} \in H^1, \tag{2.7a}$$

$$v^2 := ({}^2A)^{-1} r(A\tilde{u} - f), \quad v^2 \in H^2 \tag{2.7b}$$

$$\tilde{\tilde{u}} := \tilde{u} - pv^2, \quad \tilde{\tilde{u}} \in H^1, \tag{2.7c}$$

$$u^{i+1} := \varphi^{(\nu_2)}(\tilde{\tilde{u}}), \quad u^{i+1} \in H^1. \tag{2.7d}$$

3. Convergence of the algorithm (2.7)

Let $x \in H^1$. Let us define the error of x by

$$e(x) = x - \hat{u} \tag{3.1}$$

and the defect of x by

$$d(x) = Ax - f. \tag{3.2}$$

Let us note that

$$d(x) = Ae(x). \tag{3.3}$$

Lemma 3.1. *Let $\nu_1 = 0$,*

$$T = \mathrm{Ker}(rA), \quad S = \mathrm{Im}(p). \tag{3.4}$$

Then the following error estimate is valid:

$$\frac{\|e(u^{i+1})\|_1}{\|e(u^i)\|_1} \leqslant \sup_{x \in T \setminus \{0\}} \frac{\|M^{\nu_2} x\|_1}{\|x\|_1}. \tag{3.5}$$

Proof. It is easy to see that S, T are A-orthogonal subspaces of H^1, i.e. the equality

$$T = S^{\perp} = \{x \in H^1 \colon (x, y)_1 = 0 \text{ for any } y \in S\} \tag{3.6}$$

holds. Therefore for any $x \in H^1$ there exist unique two vectors $x_S \in S$, $x_T \in T$ such that

$$x = x_S + x_T. \tag{3.7}$$

According to (2.7), (3.1) - (3.3) we have

$$e(\tilde{u}) = [I^1 - p(rAp)^{-1} rA] e(u^i) \tag{3.8}$$

and

$$e(u^{i+1}) = M^{\nu_2} e(\tilde{u}). \tag{3.9}$$

Since $rAx_T = 0$ and $x_S = pw$ for some $w \in H^2$, we have

$$[I^1 - p(rAp)^{-1} rA] x = [I^1 - p(rAp)^{-1} rA](x_S + x_T) = x_T. \tag{3.10}$$

Of course, $\|x_T\|_1 \leqslant \|x\|_1$. Hence we may write

$$e(\tilde{\tilde{u}}) \in T, \tag{3.11}$$

$$\|e(\tilde{\tilde{u}})\|_1 \leqslant \|e(u^i)\|_1. \tag{3.12}$$

(3.5) follows from (3.9), (3.11) and (3.12). □

4. Modification

Let us consider the iterative method (2.1) in the form

$$\varphi(x) = (I^1 - \omega D^{-1} A)x + \omega D^{-1} f, \quad \omega \in (0,1), \ x \in H^1 \tag{4.1}$$

where d denotes the diagonal part of A. Therefore we have

$$M = I^1 - \omega D^{-1} A, \tag{4.2}$$

$$N = \omega D^{-1}. \tag{4.3}$$

In the sequel we will suppose that

$$\mathrm{Ker}(M) = \{0\}. \tag{4.4}$$

Let us define a new prolongation p by

$$\hat{p} = Mp. \tag{4.5}$$

Since p is injective and $\mathrm{Ker}(M) = 0$, $\hat{p}$ is injective as well. Let us set

$$\hat{r} = \hat{p}^*. \tag{4.6}$$

Further, we will consider the algorithm (2.7) with operators $\hat{r}$, $\hat{p}$ instead of r, p and with a matrix ${}^2\hat{A} = \hat{r}A\hat{p}$ instead of ${}^2A$ defined by (1.4).

Let

$$\hat{T} = \mathrm{Ker}(\hat{r}A), \qquad \hat{S} = \mathrm{Im}(\hat{p}). \tag{4.8}$$

Lemma 4.1. *The equality*

$$M^* A = AM \tag{4.9}$$

is valid.

Proof. According to (4.1) we have

$$M^* A = (I^1 - \omega D^{-1} A)^* A = (I^1 - \omega A D^{-1})A = A(I^1 - \omega D^{-1} A) = AM.$$

□

Lemma 4.2. *The equality*

$$\hat{r}A = rAM \tag{4.10}$$

holds.

Proof. The proof follows immediately from Lemma 4.1. □

Lemma 4.3.

$$e(\tilde{\tilde{u}}) \in \hat{T}, \tag{4.11}$$

$$Me(\tilde{\tilde{u}}) \in T. \tag{4.12}$$

Proof. The relation (4.11) follows immediately from (3.4), (3.11) and (4.8). We have

$$\hat{r}Ae(\tilde{\tilde{u}}) = 0.$$

According to Lemma 4.2 we obtain

$$rAMe(\tilde{\tilde{u}}) = 0, \tag{4.13}$$

which is nothing but (4.12). □

Lemma 4.4. *The inequality*

$$\|Mx\|_1^2 \leqslant \|M^2x\|_1 \cdot \|x\|_1 \tag{4.14}$$

holds for all $x \in H^1$.

Proof.

$$\|Mx\|_1^2 = \langle AMx, Mx\rangle_1 = \langle M^*AMx, x\rangle_1 .$$

Due to Lemma 4.1 we have

$$\langle M^*AMx, x\rangle_1 = \left\langle AM^2x, x\right\rangle_1 = (M^2x, x)_1 \leqslant \|M^2x\|_1 \, \|x\|_1.$$

□

Lemma 4.5. *The following inequalities hold:*

$$\sup_{x\in\hat{T}\setminus\{0\}} \frac{\|Mx\|_1}{\|x\|_1} \leqslant \sup_{x\in T\setminus\{0\}} \frac{\|Mx\|_1}{\|x\|_1}, \tag{4.15}$$

$$\sup_{x\in\hat{T}\setminus\{0\}} \frac{\|M^2x\|_1}{\|x\|_1} \leqslant \left(\sup_{x\in T\setminus\{0\}} \frac{\|Mx\|_1}{\|x\|_1}\right)^2 . \tag{4.16}$$

Proof. ad (4.15): Since $\operatorname{Ker}(M)=\{0\}$, every $x\in T$ can be written in the form

$$x = My \tag{4.17}$$

for some $y\in\hat{T}$ (see Lemma 4.2, Lemma 4.3). Therefore we can write

$$\sup_{x\in T\setminus\{0\}} \frac{\|Mx\|_1}{\|x\|_1} = \sup_{y\in\hat{T}\setminus\{0\}} \frac{\|M^2y\|_1}{\|My\|_1}. \tag{4.18}$$

According to Lemma 4.4 we get

$$\frac{\|M^2y\|_1}{\|My\|_1} \geqslant \frac{\|My\|_1}{\|y\|_1} \tag{4.19}$$

for any $y\in\hat{T}\setminus\{0\}$.

Combining (4.18), (4.19) we obtain (4.15).

ad (4.16): We write $\dfrac{\|M^2x\|_1}{\|x\|_1}$ in the form

$$\frac{\|M^2x\|_1}{\|x\|_1} = \frac{\|M^2x\|_1}{\|Mx\|_1}\cdot\frac{\|Mx\|_1}{\|x\|_1}. \tag{4.20}$$

However,

$$\frac{\|Mx\|_1}{\|x\|_1} \leqslant \frac{\|M^2x\|_1}{\|Mx\|_1} \quad \text{(see Lemma 4.4)}$$

and therefore

$$\frac{\|M^2x\|_1}{\|x\|_1} \leqslant \left(\frac{\|M^2x\|_1}{\|Mx\|_1}\right)^2. \tag{4.21}$$

(4.16) follows immediately from (4.21). □

Theorem 1. *Let $\varphi(\cdot)$ be given by (4.1), $\nu_1=\nu_2=2$. Then the following error estimate holds for the algtorithm (2.7) with operators $\hat{r}$, $\hat{p}$, ${}^2\hat{A}$ instead of r, p, ${}^2A$.*

$$\frac{\|e(u^{i+1})\|_1}{\|e(u^i)\|_1} \leqslant \left(\sup_{x\in T\setminus\{0\}} \frac{\|Mx\|_1}{\|x\|_1}\right)^4. \tag{4.22}$$

Proof. Let us set

$$\hat{Q} = I^1 - \hat{p}(\hat{r}A\hat{p})^{-1}\hat{r}A. \tag{4.23}$$

For every $x \in H^1$ there exist unique two vectors $x_{\hat{S}}$, $x_{\hat{T}}$ such that

$$x = x_{\hat{T}} + x_{\hat{S}}, \tag{4.24}$$

where $x_{\hat{T}} \in \hat{T}$, $x_{\hat{S}} \in \hat{S}$. We know that

$$\hat{Q}x = x_{\hat{T}}. \tag{4.25}$$

Hence

$$\hat{Q}x = \hat{A}^2 x = x_{\hat{T}}. \tag{4.26}$$

It is easy to see that

$$e(u^{i+1}) = M^2\hat{Q}M^2 e(u^i) = M^2\hat{Q}^2 M^2 e(u^i) \tag{4.27}$$

and

$$\|e(u^{i+1})\|_1 \leqslant \sup_{x\in H^1\setminus\{0\}} \frac{\|M^2\hat{Q}x\|_1}{\|x\|_1} \cdot \sup_{x\in H^1\setminus\{0\}} \frac{\|\hat{Q}M^2x\|_1}{\|x\|_1} \cdot \|e(u^i)\|_1. \tag{4.28}$$

Since $\|x\|_1 \geqslant \|x_{\hat{T}}\|_1$ and $\hat{Q}x \in \hat{T}$ we have

$$\sup_{x\in H^1\setminus\{0\}} \frac{\|M^2\hat{Q}x\|_1}{\|x\|_1} \leqslant \sup_{x\in \hat{T}\setminus\{0\}} \frac{\|M^2x\|_1}{\|x\|_1}. \tag{4.29}$$

From (4.16), (4.29) we obtain immediately

$$\sup_{x\in H^1\setminus\{0\}} \frac{\|M^2\hat{Q}x\|_1}{\|x\|_1} \leqslant \left(\sup_{x\in T\setminus\{0\}} \frac{\|Mx\|_1}{\|x\|_1}\right)^2. \tag{4.30}$$

Now we want to show that

$$\sup_{x\in H^1\setminus\{0\}} \frac{\|\hat{Q}M^2x\|_1}{\|x\|_1} \leqslant \left(\sup_{x\in T\setminus\{0\}} \frac{\|Mx\|_1}{\|x\|_1}\right)^2. \tag{4.31}$$

We have

$$\hat{Q}M^2x = M^2x - \hat{p}(\hat{r}A\hat{p})^{-1}\hat{r}AM^2x. \tag{4.32}$$

Since $\hat{p}(\hat{r}A\hat{p})^{-1}\hat{r}AM^2x \in \hat{S} = \mathrm{Im}(\hat{p})$ and $\hat{Q}M^2x \in \hat{T}$, we have

$$\begin{aligned}\|\hat{Q}M^2x\|_1^2 &= (M^2x - \hat{p}(\hat{r}A\hat{p})^{-1}\hat{r}AM^2x, M^2x - \hat{p}(\hat{r}A\hat{p})^{-1}\hat{r}AM^2x)_1 \\ &= (M^2x - \hat{p}(\hat{r}A\hat{p})^{-1}\hat{r}AM^2x, M^2x)_1 \\ &= \left\langle A[M^2x - \hat{p}(\hat{r}A\hat{p})^{-1}\hat{r}AM^2x], M^2x\right\rangle_1 \\ &= \left\langle (M^*)^2A[M^2x - \hat{p}(\hat{r}A\hat{p})^{-1}\hat{r}AM^2x], x\right\rangle_1.\end{aligned}$$

According to Lemma 4.1 we obtain

$$\begin{aligned}\left\langle (M^*)^2 A[M^2x - \hat p(\hat r A \hat p)^{-1}\hat r A M^2 x], x\right\rangle_1 &= \left\langle AM^2[M^2x - \hat p(\hat r A\hat p)^{-1}\hat r AM^2x], x\right\rangle_1 \\ &= \left\langle AM^2\hat Q M^2 x, x\right\rangle_1 = (M^2\hat Q M^2 x, x)_1 \\ &\leqslant \|M^2\hat Q M^2 x\|_1 \cdot \|x\|_1 \\ &\leqslant \sup_{x\in\hat T\setminus\{0\}} \frac{\|M^2x\|_1}{\|x\|_1}\cdot \|\hat Q M^2 x_1\|\cdot\|x\|_1.\end{aligned}$$

Therefore

$$\|\hat Q M^2 x\|_1 \leqslant \sup_{x\in\hat T\setminus\{0\}} \frac{\|M^2 x\|_1}{\|x\|_1}\cdot\|x\|_1. \tag{4.33}$$

(4.31) follows from (4.33) and (4.16). Using (4.28), (4.30) and (4.31) we get (4.22). □

Lemma 4.6. *Let us suppose* $D > 0$ *and let us set*

$$\|\cdot\|_D = \langle D\cdot,\cdot\rangle_1^{\frac12}, \tag{4.34}$$

Let $C_1 < 0$ *be such that for each* $x \in T$ *there exists* $w \in H^2$ *for which*

$$\|x\|_1 \geqslant C_1\|x - pw\|_D. \tag{4.35}$$

Further let $C_2 > 0$ *be a constant satisfying*

$$\|D^{-1}Ax\| \leqslant C_2\|D^{-1}Ax\|_D \tag{4.36}$$

and $C_2^2\omega < 2$. *Then*

$$\sup_{x\in T\setminus\{0\}} \frac{\|Mx\|_1^2}{\|x\|_1^2} \leqslant 1 - C_1^2(2 - C_2^2\omega). \tag{4.37}$$

Proof. For the proof see [3], Lemma 7.4. □

Theorem 2. *Let us suppose that all assumptions of Theorem 1 and Lemma 4.6 hold. Then the following error estimate is valid:*

$$\frac{\|e(u^{i+1})\|_1}{\|e(u^i)\|_1} \leqslant [1 - C_1^2(2 - \omega C_2^2)]^2. \tag{4.38}$$

Proof. The proof follows immediately from Theorem 1 and Lemma 4.6. □

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Souhrn

ZRYCHLENÍ KONVERGENCE DVOJÚROVŇOVÉHO ALGORITMU POMOCÍ ZHLAZENÍ OPERÁTORŮ PŘECHODU

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V článku je navržena technika pro zrychlení konvergence algebraického multigridu.

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