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**Abstract** Acceleration waves in nonlinear thermoelastic micropolar media are considered. We establish the kinematic and dynamic compatibility relations for a singular surface of order 2 in the media. An analogy to the Fresnel–Hadamard–Duhem theorem and an expression for the acoustic tensor are derived. The condition for acceleration wave's propagation is formulated as an algebraic spectral problem. It is shown that the condition coincides with the strong ellipticity of equilibrium equations. As an example, a quadratic form for the specific free energy is considered and the solutions of the corresponding spectral problem are presented.

**Keywords** Acceleration waves · Micropolar continuum · Cosserat continuum · Nonlinear thermoelasticity

## 0 Introduction

The propagation of nonlinear waves in solids is a complex process, analytic solutions for corresponding nonlinear problems are quite rare. However, the problem of propagation of acceleration waves is one of the exceptional cases. An acceleration wave (or wave of weak discontinuity of order 2) is a solution to the equations of motion of the medium that displays discontinuities in the second derivatives on certain surfaces that will be called *singular*. It means that the acceleration wave is described by the travelling surface which is a carrier of discontinuity jumps of the second derivatives of a solution with respect to space and time whereas the solution and its first derivatives are continuous. From the mathematical point of view, the existence of acceleration waves is closely related to the property of hyperbolicity of the equations of dynamics (or ellipticity of the equations

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of statics). The condition of existence of accelerations waves can be reduced to the problem of positivity of all eigenvalues of the algebraic spectral problem for the acoustic tensor for an arbitrary direction of propagation. From the physical point of view, the hyperbolicity of the equations of motion is a natural property of elastic media as well as ellipticity is a natural property of statics. Violation of hyperbolicity (or elipticity) means that discontinuous solutions may appear. Such solutions may model shear-bands, phase transitions, interfaces, fracture, defects, slip surfaces and other phenomena. So, the algebraic criterion for such phenomena is important in the mechanics of materials.

The investigations of acceleration waves in nonlinear elastic and thermoelastic media were performed in many works (see, e.g., the original papers [3,4,22,41,47] The main results are summarized in the classical monographs [20,49–51] where the generalization to materials with memory was also presented.

Acceleration waves in a more complex media such as porous media, fluids with complex constitutive equations, etc, have been considered in a number of papers. The acceleration waves in elastic micropolar media were considered in [24]. A generalization is presented in [32], where acceleration waves in elastic and viscoelastic micropolar media are studied. The relation between the existence of acceleration waves and the condition of strong ellipticity of the equilibrium equations is established in [11]. The derivation of acoustic tensor for micropolar media was done in [9] with application to localization phenomena in micropolar elastoplasticity. It is worth mentioning the monographs [17,21,35,40], where wave processes in micropolar continua are presented.

In a micropolar medium (also named Cosserat continuum in some publications), each particle has six degrees of freedom. From the physical point of view, every material point (or particle) of a micropolar media is equivalent phenomenologically to a rigid body. Hence rotational counteraction of the particles of the medium is taken into account. In the micropolar theory, besides ordinary stresses, couple stresses are introduced, see [6,17,25,27,34,40,42,48]. The Cosserat model is used to describe materials with a complex microstructure like granular, powder-like materials, soils, polycrystalline and composite materials, nanostructures, magnetic liquids, polymer suspensions, porous media, electromagnetic and ferromagnetic media (see, e.g., [7,8,10,18, 19,28,29,33,43]). The model has also applications to the construction of nonlinear models for beams, plates, and shells (see, e.g., [1,16,17,36,46,54,55]).

Acceleration waves were also studied in two-dimensional structures. For micropolar shells, acceleration waves are considered in [12,14]. Within the framework of the von Kármán plate theory, the propagation of acceleration waves is investigated in [52].

In nonlinear elasticity of a simple material, existence of acceleration waves is provided by assuming that the equilibrium equations are strongly elliptic. The strong ellipticity condition is one of the well-known constitutive restrictions in nonlinear elasticity [49,51]. It expresses mathematically a precise and physically intuitive restriction for constitutive equations of elastic materials. In nonlinear elasticity of simple materials, the ellipticity of the equilibrium equations is considered in a number of papers and books, see, e.g., [26,30,45,49,53]. The formulation of the strong ellipticity condition for micropolar media was given in [13].

In this paper, we extend the acceleration wave propagation analysis presented in [11,15] to the case of a thermoelastic micropolar medium. Following [17,25], in Sect. 1, we recall general relations for a micropolar continuum. In Sect. 2, we present basic definitions of two types of acceleration waves. The first one is called the homothermal acceleration wave, it will be investigated in Sect. 3. Here the acoustic tensor is constructed. The conditions of propagation of homothermal acceleration wave are reduced to the existence of the positive eigenvalues of a spectral problem. In Sect. 4, we consider homentropic (homocaloric) accelerations waves. In Sect. 5, main theorems are formulated and proved. We establish a condition for the existence of weakly discontinuous solutions to the equations of motion and heat transfer. We prove a theorem analogous to Fresnel–Hadamard–Duhem theorem in nonlinear mechanics of simple materials [39]. Under natural assumptions on the form of the heat conductivity law, we demonstrate that the condition for the existence of an acceleration wave is similar to the one for the elastic medium without temperature strains. We will show that the existence of acceleration waves is equivalent to the strong ellipticity condition of equilibrium equations of a micropolar medium. In Sect. 6, we present an example of the constitutive equations of thermoelastic micropolar media. For these, a solution of the considered spectral problems is presented.

# 1 Basic relations for micropolar media

Let us briefly recall general relations for a micropolar medium. For a comprehensive approach, we refer the reader to the books [17,25]. The symbolic (direct) tensor notation follows the one in [31]. The motion of a
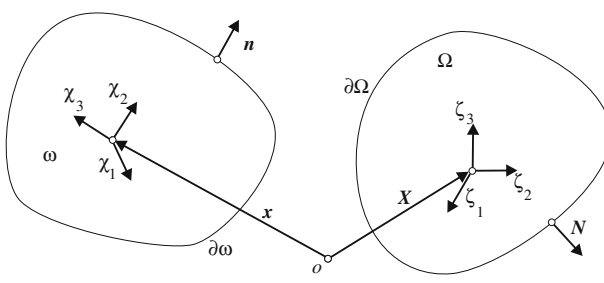
**Fig. 1** Kinematics of a micropolar medium

micropolar media is described by two sets of kinematical variables

$$x = x(X, t), \quad \chi_K = \chi_K(X, t), \quad K = 1, 2, 3, \tag{1}$$

where $x$ is the classical motion, while vectors $\chi_K$ represent the micromotion. From the physical point of view, $x$ describes the position of a material point in the actual configuration, while $X$ describes the position of the material point in the reference configuration. $\chi_K$ are called directors, they are attached to each material point. $\chi_K$ describe the orientation of the material particles in the actual configuration. For micropolar media, $\chi_K$ constitute orthonormal frame and so $\chi_K \cdot \chi_M = \delta_{KM}$.

Let us introduce three orthonormal directors $\zeta_i$ in the reference configuration. For the sake of simplicity, we choose $\zeta_K(X) \equiv \chi_K(X, 0)$. Next, we introduce the proper orthogonal tensor $\mathbf{H} \equiv \chi_K \otimes \zeta_K$ ($\det\mathbf{H} = 1$) called microrotation tensor, see [38,54,55]. $\mathbf{H}$ describes microrotation of a particle of the micropolar media. So $x$ describes the position of the particle of the medium at time $t$, whereas $\mathbf{H}$ defines its orientation. Kinematics of a micropolar medium is depicted in Fig. 1. Here $\Omega$ and $\omega$ are the domains occupied by the body in the reference and actual configurations, respectively. $N$ and $n$ are the unit normals to the boundary of the body in the reference and actual configurations, respectively.

The velocity is given by the relation $v = \dot{x}$. For brevity, we write out $\dot{(\ldots)} \equiv \frac{\mathrm{d}}{\mathrm{d}t}(\ldots)$, where $\frac{\mathrm{d}}{\mathrm{d}t}$ denotes the material derivative with respect to $t$. The angular velocity vector, called microgyration vector, is given by

$$\omega = -\frac{1}{2}\left(\mathbf{H}^T \cdot \dot{\mathbf{H}}\right)_\times, \tag{2}$$

where the dot denotes the dot (inner) product and $(\ldots)^T$—transposed. Symbol $(\ldots)_\times$ stands for the vector invariant of a second-rank tensor (cf. [30,31,55]). In particular, for a dyad $a \otimes b$ we have $(a \otimes b)_\times = a \times b$, where $\times$ is the vector (cross) product. Relation (2) means that $\omega$ is the axial vector associated with the skew-symmetric tensor $\mathbf{H}^T \cdot \dot{\mathbf{H}}$, see [31].

The equations of motion for a micropolar continuum, which represent the balance of momentum and of moment of momentum (the global balance of angular momentum) for an arbitrary part of the body in the material frame, are [17,25]

$$\mathrm{Div}\,\mathbf{T} + \rho f = \rho \dot{v}, \quad \mathrm{Div}\,\mathbf{M} + (\mathbf{F} \cdot \mathbf{T}^T)_\times + \rho m = \rho\gamma\dot{\omega}. \tag{3}$$

Here $\mathbf{T}$ is the first Piola-Kirchhoff stress tensor and $\mathbf{M}$ the couple-stress tensor [31,51], $\mathbf{F} = \mathrm{Grad}\,x$ is the gradient of the position vector, it is often called the deformation gradient, $\mathrm{Grad}$ and $\mathrm{Div}$ are the gradient and divergence operators in Lagrangian coordinates, respectively, $\rho$ is the medium density in the reference configuration, $f$ and $m$ are the vectors of mass forces and mass couples, respectively, $\rho\gamma$ is the scalar measure of the rotational inertia of a particle.

For heat transfer, dynamic equations (3) are supplemented with the heat conduction equation [17,25]

$$\rho\theta\frac{\mathrm{d}\eta}{\mathrm{d}t} = -\mathrm{Div}\,q + \rho h, \tag{4}$$

where $\theta$ is the temperature, $\eta$ is the specific entropy, $q$ is the heat flux in the reference configuration, and $h$ is the external heat source density.

The constitutive equations for a Cosserat thermoelastic continuum can be derived through the expression of the specific free energy

$$\psi = \psi(\mathbf{E}, \mathbf{K}, \theta) \tag{5}$$
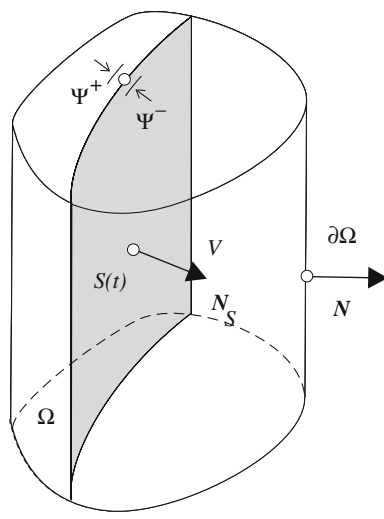
**Fig. 2** Singular surface

as follows

$$\mathbf{T} = \rho \mathbf{H} \cdot \psi_{,\mathbf{E}}, \quad \mathbf{M} = \rho \mathbf{H} \cdot \psi_{,\mathbf{K}}, \quad \eta = -\psi_{,\theta}, \quad q = q(\mathbf{E}, \mathbf{K}, \theta, \mathrm{Grad}\,\theta), \tag{6}$$

where $\mathbf{E}$ is the Cosserat deformation tensor and $\mathbf{K}$ is the wryness tensor that are metric and bending strain measures, cf. [11,38]. They are defined by the relations

$$\mathbf{E} = \mathbf{H}^T \cdot \mathbf{F}, \quad \mathbf{1} \times \mathbf{K} = (\mathbf{H}^T \cdot \mathrm{Grad}\,\mathbf{H}),$$

where $\mathbf{1}$ is the unit tensor. Detailed discussion on introduction of strain measures in the micropolar media is given in [44]. As is known, from the 2nd law of thermodynamics it follows

$$q \cdot \mathrm{Grad}\,\theta \leq 0$$

From now on, we assume $\psi$ to be a twice continuously differentiable function and vector-function $q$ to be continuously differentiable. We use the following notation:

$$\psi_{,\mathbf{E}} = \frac{\partial \psi}{\partial \mathbf{E}}, \quad \psi_{,\mathbf{K}} = \frac{\partial \psi}{\partial \mathbf{K}}, \quad \psi_{,\theta} = \frac{\partial \psi}{\partial \theta}, \ \dots$$

## 2 Acceleration waves

We consider motions of the medium when discontinuities of kinematic and dynamic quantities appear at a smooth surface $S(t)$ that is called singular (Fig. 2). For the quantities describing the motion at $S(t)$, we suppose existence of unilateral limit values. For the second derivatives of the motion, the limits from each of both sides from $S(t)$ can differ, in general. A jump for a quantity at $S(t)$ is denoted by the double square brackets, for example, $[\![\Psi]\!] = \Psi^+ - \Psi^-$.

Under these assumptions, let us derive the conditions for existence of the acceleration wave, that is the conditions under which weak discontinuous solutions can arise. On the singular surface the following balance equations must be valid (see [17,25]):

$$\rho V [\![\boldsymbol{v}]\!] = -[\![\mathbf{T}]\!] \cdot N_S, \quad \rho V [\![\boldsymbol{\omega}]\!] = -[\![\mathbf{M}]\!] \cdot N_S, \quad \rho \theta V [\![\eta]\!] = [\![q]\!] \cdot N_S, \tag{7}$$

where $V$ is the intrinsic speed of propagation of $S(t)$ in the direction $N_S$ [49–51] and $N_S$ is the unit normal to $S$.

*An acceleration wave* (or *weak discontinuity wave*, or *singular surface of the second order*) is a traveling singular surface $S(t)$ at which the second spatial and time derivatives of the position vector $x$ and of the

micro-rotation tensor $\mathbf{H}$ have jumps, while $\boldsymbol{x}$ and $\mathbf{H}$ together with all their first derivatives are continuous. So on $S(t)$ we have

$$[\![\mathbf{F}]\!] = 0, \quad [\![\mathrm{Grad}\,\mathbf{H}]\!] = 0, \quad [\![\boldsymbol{v}]\!] = 0, \quad [\![\boldsymbol{\omega}]\!] = 0. \tag{8}$$

From (7) and (8) it follows

$$[\![\mathbf{T}]\!] \cdot \boldsymbol{N}_S = 0, \quad [\![\mathbf{M}]\!] \cdot \boldsymbol{N}_S = 0.$$

Let us consider two types of acceleration waves. The first one is the *homothermal* acceleration wave when *the temperature field and its first derivatives are continuous at $S(t)$*:

$$[\![\theta]\!] = 0, \quad [\![\mathrm{Grad}\,\theta]\!] = \mathbf{0}, \quad [\![\dot\theta]\!] = 0. \tag{9}$$

The second type is the *homentropic* (or *homocaloric*) acceleration wave when *the entropy field and its first derivatives are continuous at $S(t)$*:

$$[\![\eta]\!] = 0, \quad [\![\mathrm{Grad}\,\eta]\!] = \mathbf{0}, \quad [\![\dot\eta]\!] = 0. \tag{10}$$

For the homentropic acceleration wave, the Fourier condition holds:

$$[\![\boldsymbol{q}]\!] \cdot \boldsymbol{N}_S = 0. \tag{11}$$

## 3 Homothermal acceleration waves

3.1 Transformation of the dynamic equations

First we consider the jump relations that follow from the motion equations. Equations (8) and (9) imply continuity of the measures of deformation $\mathbf{E}$ and $\mathbf{K}$ at $S(t)$

$$[\![\mathbf{E}]\!] = 0, \quad [\![\mathbf{K}]\!] = 0.$$

Hence, in view of the constitutive equations (5), (6), we establish the continuity of the tensors $\mathbf{T}$ and $\mathbf{M}$, of the entropy density, and of the heat flow vector:

$$[\![\mathbf{T}]\!] = 0, \quad [\![\mathbf{M}]\!] = \mathbf{0}, \quad [\![\eta]\!] = 0, \quad [\![\boldsymbol{q}]\!] = \mathbf{0}.$$

It follows immediately the balance equations (7) to be valid on $S(t)$.

Let us recall Maxwell's theorem [49–51]:

For a continuously differentiable field $\boldsymbol{\Psi}$ such that $[\![\boldsymbol{\Psi}]\!] = \mathbf{0}$ the following relations hold

$$[\![\dot{\boldsymbol{\Psi}}]\!] = -V\boldsymbol{\Phi}, \quad [\![\mathrm{Grad}\,\Psi]\!] = \boldsymbol{\Phi} \otimes \boldsymbol{N}_S, \tag{12}$$

where $\boldsymbol{\Phi}$ is the tensor amplitude of the jump of the first gradient of $\Psi$; the tensor amplitude is a tensor of the rank equal to the rank of $\Psi$.

Application of Maxwell's theorem to the continuous fields of $v$, $\omega$, $\mathbf{T}$, and $\mathbf{M}$ yield in the system of equations that relate the jumps in the derivatives with respect to the spatial variables and time $t$ at $S(t)$

$$[\![\dot{\boldsymbol{v}}]\!] = -V\boldsymbol{a}, \quad [\![\mathrm{Grad}\,v]\!] = \boldsymbol{a} \otimes \boldsymbol{N}_S, \quad [\![\dot{\boldsymbol{\omega}}]\!] = -V\boldsymbol{b}, \quad [\![\mathrm{Grad}\,\omega]\!] = \boldsymbol{b} \otimes \boldsymbol{N}_S,$$
$$V\,[\![\mathrm{Div}\,\mathbf{T}]\!] = -[\![\dot{\mathbf{T}}]\!] \cdot \boldsymbol{N}_S, \quad V\,[\![\mathrm{Div}\,\mathbf{M}]\!] = -[\![\dot{\mathbf{M}}]\!] \cdot \boldsymbol{N}_S,$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are the vectorial amplitudes of the jumps in the linear and angular accelerations. With this, we get the relations for the jumps in the derivatives of the strain measures with respect to $t$

$$[\![\dot{\mathbf{E}}]\!] = \mathbf{H}^T \cdot \boldsymbol{a} \otimes \boldsymbol{N}_S, \quad [\![\dot{\mathbf{K}}]\!] = \mathbf{H}^T \cdot \boldsymbol{b} \otimes \boldsymbol{N}_S. \tag{13}$$

Let the mass forces and couples be continuous. From the balance equations (3) it follows

$$[\![\mathrm{Div}\,\mathbf{T}]\!] = \rho\,[\![\dot{\boldsymbol{v}}]\!], \quad [\![\mathrm{Div}\,\mathbf{M}]\!] = \rho\gamma\,[\![\dot{\boldsymbol{\omega}}]\!]. \tag{14}$$

Using Eqs. (13) and (14) we get the relations

$$- [\![ \dot{\mathbf{T}} ]\!] \cdot \mathbf{N}_S = \rho V \, [\![ \dot{\mathbf{v}} ]\!], \quad - [\![ \dot{\mathbf{M}} ]\!] \cdot \mathbf{N}_S = \rho \gamma V \, [\![ \dot{\boldsymbol{\omega}} ]\!].$$ (15)

Differentiating the constitutive equations (6) we obtain

$$[\![ \dot{\mathbf{T}} ]\!] = \rho \psi_{,\mathbf{EE}} \cdot \cdot [\![ \dot{\mathbf{E}}^T ]\!] + \rho \psi_{,\mathbf{EK}} \cdot \cdot [\![ \dot{\mathbf{K}}^T ]\!],$$
$$[\![ \dot{\mathbf{M}} ]\!] = \rho \psi_{,\mathbf{KE}} \cdot \cdot [\![ \dot{\mathbf{E}}^T ]\!] + \rho \psi_{,\mathbf{KK}} \cdot \cdot [\![ \dot{\mathbf{K}}^T ]\!].$$

Next, with regard for (13), we transform (15) into the form containing the vectorial amplitudes $a$ and $\mathbf{b}$ only:

$$(\psi_{,\mathbf{EE}} \cdot \cdot (\mathbf{N}_S \otimes \mathbf{H}^T \cdot a)) \cdot \mathbf{N}_S + (\psi_{,\mathbf{EK}} \cdot \cdot (\mathbf{N}_S \otimes \mathbf{H}^T \cdot \mathbf{b})) \cdot \mathbf{N}_S = V^2 \mathbf{H}^T \cdot a,$$
$$(\psi_{,\mathbf{KE}} \cdot \cdot (\mathbf{N}_S \otimes \mathbf{H}^T \cdot a)) \cdot \mathbf{N}_S + (\psi_{,\mathbf{KK}} \cdot \cdot (\mathbf{N}_S \otimes \mathbf{H}^T \cdot \mathbf{b})) \cdot \mathbf{N}_S = \gamma V^2 \mathbf{H}^T \cdot \mathbf{b}.$$

Using matrix notation, we rewrite these in a more compact form:

$$\mathbb{Q}(\mathbf{N}_S) \cdot \boldsymbol{\xi} = V^2 \mathbb{B} \cdot \boldsymbol{\xi},$$ (16)

where

$$\boldsymbol{\xi} = (a', b') \in \mathbb{R}^6, \quad a' = \mathbf{H}^T \cdot a, \quad b' = \mathbf{H}^T \cdot \mathbf{b}, \quad \mathbb{Q}(\mathbf{N}_S) \equiv \begin{bmatrix} \psi_{,\mathbf{EE}}\{\mathbf{N}_S\} & \psi_{,\mathbf{EK}}\{\mathbf{N}_S\} \\ \psi_{,\mathbf{KE}}\{\mathbf{N}_S\} & \psi_{,\mathbf{KK}}\{\mathbf{N}_S\} \end{bmatrix}, \quad \mathbb{B} \equiv \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \gamma\mathbf{1} \end{bmatrix}.$$

For arbitrary 4th rank tensor $\mathbf{G}$ and vector $\mathbf{N}$ that are represented in a Cartesian basis $i_k$ ($k = 1, 2, 3$), we have used the notation

$$\mathbf{G}\{\mathbf{N}\} \equiv G_{klmn} N_l N_n i_k \otimes i_m.$$ (17)

$\mathbb{Q}(\mathbf{N}_S)$ is an analogy to the homothermal acoustic tensor for the micropolar medium. From existence of the free energy function $\psi$ it follows that $\mathbb{Q}(\mathbf{N}_S)$ is symmetric. This provides that the squared velocity of propagation for an acceleration wave in an elastic micropolar medium is real-valued. The requirement that $\mathbb{Q}(\mathbf{N}_S)$ has to be positive definite is necessary for existence of an acceleration wave. It coincides with the condition of strong ellipticity of the equilibrium equations for an elastic micropolar medium [11].

3.2 Transformation of the heat conductivity equation

Now let us consider how the temperature field affects the existence of acceleration waves: we derive some relations for the jumps. Applying Maxwell's theorem to the field of $\mathbf{q}$ and to the temperature gradient, we get

$$V \, [\![ \text{Div}\, q ]\!] = - [\![ \dot{q} ]\!] \cdot \mathbf{N}_S, \quad [\![ \text{Grad}\,\text{Grad}\,\theta ]\!] = \mathbf{g} \otimes \mathbf{N}_S, \quad [\![ \, (\text{Grad}\,\theta)\dot{} \, ]\!] = -V\mathbf{g},$$ (18)

where $\mathbf{g}$ is the vector amplitude of the jump in the second gradient of the temperature. Similar to (13), from (4) it follows that

$$[\![ \text{Div}\, q ]\!] = -\rho\theta \, [\![ \dot{\eta} ]\!].$$ (19)

From (18) and (19), we obtain

$$[\![ \dot{q} ]\!] \cdot \mathbf{N}_S = \rho\theta \, [\![ \dot{\eta} ]\!].$$ (20)

Let us restrict further discussion by the assumption that the constitutive equation for $q$ obeys Fourier's law

$$q = -\mathbf{k}(\theta) \cdot \text{Grad}\,\theta, \quad h \cdot \mathbf{k}(\theta) \cdot h > 0, \quad \forall h \neq \mathbf{0},$$ (21)

where $\mathbf{k}$ is the positive definite thermoconductivity tensor. Differentiating $(6)_3$ and (21) with respect to $t$ and using (20), we get

$$\mathbf{N}_S \cdot \mathbf{k}(\theta) \cdot \mathbf{g} = \rho\theta \left( a' \cdot \psi_{,\theta\mathbf{E}} \cdot \mathbf{N}_S + b' \cdot \psi_{,\theta\mathbf{K}} \cdot \mathbf{N}_S \right).$$ (22)

Now, using again the matrix notation, we can rewrite (16) and (22)

$$\mathbb{Q}_\theta(N_S) \cdot \boldsymbol{\zeta} = V^2 \mathbb{B}_\theta \cdot \boldsymbol{\zeta}, \tag{23}$$

where $\boldsymbol{\zeta} = (a', b', g) \in \mathbb{R}^9$, $\mathbb{Q}_\theta$, and $\mathbb{B}_\theta$ are matrices with tensor components

$$\mathbb{Q}_\theta(N_S) \equiv \begin{bmatrix} \psi_{,\mathbf{EE}}\{N_S\} & \psi_{,\mathbf{EK}}\{N_S\} & \mathbf{0} \\ \psi_{,\mathbf{KE}}\{N_S\} & \psi_{,\mathbf{KK}}\{N_S\} & \mathbf{0} \\ -\rho\theta N_S \cdot \psi_{,\theta\mathbf{E}} & -\rho\theta N_S \cdot \psi_{,\theta\mathbf{E}} & N_S \cdot \mathbf{k}(\theta) \end{bmatrix}, \quad \mathbb{B}_\theta \equiv \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{24}$$

Thus when the heat flow vector obeys Fourier's law, we reduce the problem of propagation of an acceleration wave in a thermoelastic micropolar medium to the spectral problem (23). Namely, an homothermal acceleration wave exists only if (23) has nontrivial solutions and the eigenvalues for the problem (23) are real and positive.

*Remark 1* It is easy to see that the results can be extended to the case of a nonlinear thermoconductivity law having the form

$$\boldsymbol{q} = q(\theta, \operatorname{Grad}\theta).$$

For this, it is sufficient to require that

$$\boldsymbol{g} \cdot \boldsymbol{q}_{,\operatorname{Grad}\theta} \cdot \boldsymbol{g} < 0, \quad \forall \boldsymbol{g} \neq \mathbf{0}.$$

*Remark 2* In a similar manner, we can consider a more general form of the thermoconductivity law

$$\boldsymbol{q} = \boldsymbol{q}(\theta, \operatorname{Grad}\theta, \mathbf{E}, \mathbf{K}). \tag{25}$$

Here the form of corresponding matrix $\mathbb{Q}_\theta$ becomes more complex but it does not change main properties of the spectral problem.

## 4 Homocaloric (homentropic) acceleration waves

By the definition of an homentropic acceleration wave, we have $[\![\eta]\!] = 0$. Let us suppose the temperature field to be continuous at $S(t)$, that is $[\![\theta]\!] = 0$. We can prove that this is valid if

$$\eta_{,\theta} \equiv -\psi_{,\theta\theta} \neq 0 \tag{26}$$

is true. So we suppose this to hold. A physical reason for assumption (26) is based on the fact that the second derivative of the free energy corresponds to the heat capacity [31], but a real material with zero heat capacity does not exist.

Applying Maxwell's theorem, we define a scalar thermal amplitude $\Theta$ such that

$$[\![\dot{\theta}]\!] = -V\Theta, \quad [\![\operatorname{Grad}\theta]\!] = \Theta N_S. \tag{27}$$

Differentiating constitutive equations (6), we obtain

$$[\![\dot{\mathbf{T}}]\!] = \rho\psi_{,\mathbf{EE}} \cdot \cdot [\![\dot{\mathbf{E}}^T]\!] + \rho\psi_{,\mathbf{EK}} \cdot \cdot [\![\dot{\mathbf{K}}^T]\!] + \rho\psi_{,\mathbf{E}\theta} [\![\dot{\theta}]\!],$$
$$[\![\dot{\mathbf{M}}]\!] = \rho\psi_{,\mathbf{KE}} \cdot \cdot [\![\dot{\mathbf{E}}^T]\!] + \rho\psi_{,\mathbf{KK}} \cdot \cdot [\![\dot{\mathbf{K}}^T]\!] + \rho\psi_{,\mathbf{K}\theta} [\![\dot{\theta}]\!].$$

For Fourier's law (21), from Eq. (11) we get

$$N_S \cdot \mathbf{k} \cdot [\![\operatorname{Grad}\theta]\!] = N_S \cdot \mathbf{k} \cdot N_S\Theta = 0. \tag{28}$$

For a more general thermoconductivity law (25), using $\mathbf{k} = -\boldsymbol{q}_{,\operatorname{Grad}\theta}$, we also can get (28). Relation (28) means that for any heat conductive media with a positive definite thermoconductivity tensor, the thermal amplitude is zero. Thus, for a heat conductive media we have established that the acceleration wave should be homothermal. This result was presented in [51] for simple materials.

For heat non-conductors, $\mathbf{k} = \mathbf{0}$. We can use this assumption if we neglect the heat conductivity or consider deformation processes to be very fast. Considering a heat non-conductive media, we may obtain a homocaloric acoustic tensor that differs, in general, from $\mathbb{Q}$.

# 5 Existence of acceleration waves

In what follows, we will treat heat conductive media only. Let us consider the problem of existence of positive solutions to the spectral problem (23). In general, $\mathbb{Q}_\theta$ is not symmetric and $\mathbb{B}_\theta$ is not positive definite. However, it is possible to extend the Fresnel–Hadamard–Duhem theorem to the case of a thermoelastic micropolar medium.

**Theorem 1** *For any propagation directions defined by the vector $\mathbf{N}_S$, the homothermal acoustic numbers are real.*

*Proof* The acoustic numbers are the squared speeds of propagation of acceleration waves. Spectral problem (23) splits into two problems, namely the problems (16) and (22). The components of $\mathbb{Q}$ are composed of the mixed derivatives of the free energy $\psi$ and so $\mathbb{Q}$ is symmetric. $\mathbb{B}$ also is a symmetric matrix, moreover it is positively definite. So spectral problem (16) has only real-valued solutions. Let us solve problem (16). As $\mathbf{k}$ is a nonsingular tensor, from Eq. (22) we find vector $\mathbf{g}$ that is uniquely defined.

Thus the theorem is proved under the assumptions of [11], that are supplemented by the requirement that the heat conductivity tensor $\mathbf{k}$ must be positively definite. □

This theorem does not guarantee existence of an acceleration wave as problem (16) may have zero or negative eigenvalues. For existence of acceleration waves, all eigenvalues of (16) must be positive for any $N_S$. Thus, we should impose some additional restrictions on the constitutive equations. We will use the strong ellipticity condition as such a constitutive restriction. For a micropolar media, the condition is represented by the inequality [13]

$$\frac{d^2}{d\tau^2}\,\psi(\mathbf{E} + \tau a' \otimes \mathbf{N}_S, \mathbf{K} + \tau b' \otimes \mathbf{N}_S)\big|_{\tau=0} > 0, \quad \forall N_S : |N_S| = 1, \quad a' \neq 0, \quad b' \neq 0, \tag{29}$$

Similar to [11,15], we establish the following theorem.

**Theorem 2** *The condition for existence of a homothermal acceleration wave for all directions of propagations in a micropolar thermoelastic medium is equivalent to the condition of strong ellipticity of the equilibrium equations of the medium.*

*Proof* Acceleration waves exist only if all eigenvalues of spectral problem (23) are positive for any $N_S$ defining the direction of wave propagation and it must hold $V^2 > 0$. As problem (23) is equivalent to two problems, (16) and (22), these properties of positiveness are valid if and only if $\mathbb{Q}$ is positively definite for any values of $N_S$. So by the definition of positiveness, we have

$$\xi \cdot \mathbb{Q}(N_S) \cdot \xi > 0, \quad \forall N_S : |N_S| = 1, \quad \xi \neq 0. \tag{30}$$

This is an additional restriction that is imposed on the constitutive relations of the thermoelastic micropolar medium. It is easily seen that inequality (30) coincides with (29) that completes the proof. □

*Remark* Let us note that positive definiteness of $\mathbf{k}$ implies strong ellipticity of the steady-state thermoconductivity equation. Degeneration of at least one of the quantities $\mathbb{A}$ or $\mathbf{k}$ leads to the possibility of existence of non-smooth solutions to the equilibrium equations or the steady-state thermoconductivity equation.

In general, the strong ellipticity of the equations is a property of the material in some area of deformation. For some deformations it can be fulfilled whereas for others it does not. For a number of real medias that are modeled by Cosserat continuum, arising of discontinuous solution is physically possible. For example, such medias are soils, granular and porous media. For some other materials such discontinuities are physically impossible. Thus, the strong ellipticity condition allows us "to sort" admissible types of the constitutive equations as well as to determine "dangerous" deformations.

We wish to underline that the existence of acceleration waves in all the directions and the equivalent to this condition of strong ellipticity are local as they are defined at each point of the medium. In case of non-homogeneous deformation this means, that the conditions can break or be valid in different parts of the medium. Thus the condition of strong ellipticity is, besides, a criterion to find "dangerous" domains in a body.

## 6 An example

As an example, let us consider a quadratic form as a constitutive equation for the specific free energy. Let us assume the following relation to be valid

$$\rho \psi = W_1(\mathbf{E}) + W_2(\mathbf{K}), \tag{31}$$

with

$$2W_1(\mathbf{E}) = \alpha_1 \text{tr}\left((\mathbf{E}-\mathbf{1})\cdot(\mathbf{E}-\mathbf{1})^T\right) + \alpha_2 \text{tr}((\mathbf{E}-\mathbf{1})^2) + \alpha_3 \text{tr}^{\,2}(\mathbf{E}-\mathbf{1})$$
$$+ \alpha_0(\theta - \theta_0)\text{tr}\,((\mathbf{E}-\mathbf{1})) + c(\theta-\theta_0)^2,$$
$$2W_2(\mathbf{K}) = \beta_1 \text{tr}\left(\mathbf{K}\mathbf{K}^T\right) + \beta_2 \text{tr}\,(\mathbf{K}^2) + \beta_3 \text{tr}^{\,2}(\mathbf{K}),$$

where $\alpha_k$, $\beta_k$ $(k = 1, 2, 3)$ are elastic constants, $\alpha_0$ corresponds to the thermal expansion coefficient, $c$ is the specific heat capacity, and $\theta_0$ is the reference temperature. Note we neglect the linear terms of the approximation of (31).

The structure of $\mathbb{Q}(\mathbf{N}_S)$ is

$$\mathbb{Q}(\mathbf{N}_S) \equiv \begin{bmatrix} \mathbb{Q}_1(N_S) & \mathbf{0} \\ \mathbf{0} & \mathbb{Q}_2(\mathbf{N}_S) \end{bmatrix}, \tag{32}$$

where $\rho\mathbb{Q}_1(N_S) = W_{1,\,\mathbf{EE}}\{\mathbf{N}_S\}$, $\rho\mathbb{Q}_2(N_S) = W_{2,\,\mathbf{EE}}\{\mathbf{N}_S\}$.

Thus, spectral problem (16) splits into the two problems

$$\mathbb{Q}_1(\mathbf{N}_S) \cdot a' = V^2 a', \quad \mathbb{Q}_2(\mathbf{N}_S) \cdot b' = \gamma V^2 b'. \tag{33}$$

For constitutive equations (31), inequality (30) splits into two inequalities

$$a \cdot \mathbf{Q}_1(\mathbf{N}_S) \cdot a > 0, \quad b \cdot \mathbf{Q}_2 \cdot (\mathbf{N}_S) \cdot b > 0, \quad \forall \mathbf{N}_S : |\mathbf{N}| = 1, \ a \neq 0, \ b \neq 0, \tag{34}$$

Considering (32), we see that (34) is equivalent to the following inequalities

$$\alpha_1 (a \cdot a)^2 + (\alpha_2 + \alpha_3)(a \cdot \mathbf{N}_S)^2 > 0, \quad \beta_1 (b \cdot a)^2 + (\beta_2 + \beta_3)(b \cdot \mathbf{N}_S)^2 > 0,$$

that implies the following

$$\alpha_1 > 0, \quad \alpha_1 + \alpha_2 + \alpha_3 > 0, \quad \beta_1 > 0, \quad \beta_1 + \beta_2 + \beta_3 > 0. \tag{35}$$

If (35) are valid then the system of equations for a physically linear material defined by relation (32) is strongly elliptic for any deformations. Then the solutions of (33) are given by

$$V_{1,2} = \sqrt{\frac{\alpha_1}{\rho}}, \quad \boldsymbol{\xi}_{1,2} = (e_{1,2}, 0), \quad V_3 = \sqrt{\frac{\alpha_1 + \alpha_2 + \alpha_3}{\rho}}, \quad \boldsymbol{\xi}_3 = (0, \mathbf{N}_S),$$
$$V_{4,5} = \sqrt{\frac{\beta_1}{\gamma\rho}}, \quad \boldsymbol{\xi}_{4,5} = (e_{4,5}, 0), \quad V_6 = \sqrt{\frac{\beta_1 + \beta_2 + \beta_3}{\gamma\rho}}, \quad \boldsymbol{\xi}_6 = (0, \mathbf{N}_S), \tag{36}$$

where $e_1$, $e_2$, $e_4$, $e_5$ are arbitrary unit vectors in the tangential plane to $S(t)$ such that $e_1 \cdot e_2 = e_1 \cdot \mathbf{N}_S = e_2 \cdot \mathbf{N}_S = 0$, $e_4 \cdot e_5 = e_4 \cdot N = e_5 \cdot \mathbf{N}_S = 0$.

Solutions $(36)_{1,2}$ describe transverse and longitudinal acceleration waves, while $(36)_{4,5}$ describe transverse and longitudinal acceleration waves of microrotation. The speeds from (36) coincide with the limits of the phase velocities of plane harmonic waves (acoustic waves) in linear micropolar elasticity (see [17,42,40]) when the frequency of the waves tends to infinity.

*Remark 1* For Eqs. (31) and (32), the strong ellipticity condition reduces to simple inequalities (35). They are expressed in terms of the elastic moduli and do not depend on deformation. For simple nonlinear the analogue of (31), and (32) is the semi-linear material (see [30,31]). For the latter, the strong elipticity condition depends on strains as well as the elastic moduli. When the ellipticity condition breaks down, singular solutions arise—these may correspond to infinite rotations, for example [30]. Hence it may be simpler to check the strong ellipticity condition for a micropolar material than for a simple nonlinearly elastic material.

*Remark 2* In the case of small strains, expressions (31) and (32) reduce to the equations of linear micropolar elasticity [17,40]. The condition of positive semidefiniteness of the energy reduces to the following inequalities [17]:

$$\alpha_1 + \alpha_2 \geq 0, \quad \alpha_1 - \alpha_2 \geq 0, \quad \alpha_1 + \alpha_2 + 3\alpha_3 \geq 0, \quad \beta_1 + \beta_2 \geq 0, \quad \beta_1 - \beta_2 \geq 0, \quad \beta_1 + \beta_2 + 3\beta_3 \geq 0.$$

These imply (35) but not conversely. Naturally, the strong ellipticity conditions are less restrictive than the requirement of positive semidefinitenes of the energy under infinitesimal deformations.

*Remark 3* Recently, engineering practitioners have begun to use cellular solids and metal or polymer foams [2,23]. To describe the behaviour of these complex materials, they use the model of a Cosserat continuum [8,28,29,43] and more complex models of continuum mechanics (cf., for example [5,37]). The elastic constants of such materials show a significant temperature dependence. For example, Gibson [23] proposed a linear approximation of Young's modulus with respect to temperature having the form

$$E = E_0 \left( 1 - \kappa \frac{\theta}{\theta_0} \right),$$

where $\theta_0$ is a melting temperature for metal foams or glass temperature for polymer foams, and $E_0$ and $\kappa$ are constants. If one assumes more general relations

$$\alpha_i(\theta) = \alpha_i^0 \left( 1 - \kappa_i \frac{\theta}{\theta_0} \right), \quad \beta_i(\theta) = \beta_i^0 \left( 1 - \gamma_i \frac{\theta}{\theta_0} \right),$$

then inequalities (35) are nontrivial; they depend on temperature. In the range of temperature where condition (35) is not valid, one may expect singular behavior in the strain state; shear bands can arise, for example.

# 7 Conclusion

The conditions for the existence of homothermal acceleration waves in a thermoelastic micropolar medium are established. It is shown that the conditions for the existence of acceleration waves in a thermoelastic micropolar medium do not depend on the thermoconductivity defined by Fourier's law. Hence the conditions under which the acceleration waves in a thermoelastic micropolar medium with Fourier's law of thermoconductivity propagate coincide with those for wave propagation in the corresponding elastic medium. Thus, we have shown that for analysis of acceleration waves in thermoelastic bodies in a non-homogeneous temperature field it is sufficient to consider only the equations of motion where the temperature is considered as a parameter.

The algebraic criterion for the existence of acceleration waves is formulated. This criterion is the strong ellipticity condition. An example on how to check the strong ellipticity condition is presented. The usage of the strong ellipticity condition in analyzing the behaviour of metallic or polymeric foams, when the elastic moduli are functions of temperature, is discussed.

## References

1. Antman, S.S.: Nonlinear Problems of Elasticity, 2nd edn. Springer Science Media, New York (2005)
2. Ashby, M.F., Evans, A.G., Fleck, N.A., Gibson, L.J., Hutchinson, J.W., Wadley, H.N.G.: Metal Foams: a Design Guid. Butterworth-Heinemann, Boston (2000)
3. Chen, P.J.: Growth of acceleration waves in isotropic elastic materials. J. Acoust. Soc. Am. **43**, 982–987 (1968)
4. Chen, P.J.: One dimensional acceleration waves in inhomogeneous elastic non-conductors. Acta Mech. **17**, 17–24 (1973)
5. Cielecka, I., Woźniak, M., Woźniak, C.: Elastodynamic behaviour of honeycomb cellular media. J. Elast. **60**, 1–17 (2000)
6. Cosserat, E., Cosserat, F.: Théorie des corps déformables. Herman et Flis, Paris (1909)
7. Diebels, S.: A micropolar theory of porous media: constitutive modelling. Transp. Porous Media **34**, 193–208 (1999)
8. Diebels, S., Steeb, H.: Stress and couple stress in foams. Comput. Mater. Sci. **28**, 714–722 (2003)
9. Dietsche, A., Steinmann, P., Willam, K.: Micropolar elastoplasticity and its role in localization. Int. J. Plast. **9**, 813–831 (1993)
10. Ehlers, W., Ramm, E., Diebels, S., d'Addetta, G.D.A.: From particle ensembles to Cosserat continua: Homogenization of contact forces towards stresses and couple stresses. Int. J. Solids Struct. **40**, 6681–6702 (2003)
11. Eremeyev, V.A.: Acceleration waves in micropolar elastic media. Dokl. Phys. **50**(4), 204–206 (2005)

12. Eremeyev, V.A.: Nonlinear micropolar shells: theory and applications. In: Pietraszkiewicz, W., Szymczak, C. (eds.) Shell Structures: Theory and Applications, pp. 11–18. Taylor and Francis, London (2005)
13. Eremeyev, V.A., Zubov, L.M.: On the stability of elastic bodies with couple stresses (in Russ.). Mech. Solids **3**, 181–190 (1994)
14. Eremeyev, V.A., Zubov, L.M.: On constitutive inequalities in nonlinear theory of elastic shells. ZAMM **87**(2), 94–101 (2007)
15. Eremeyev, V.A., Lebedev, L.P., Rendón, L.A.: On the propagation of acceleration waves in thermoelastic micropolar media. Revista Colombiana de Matemáticas **41**, 397–406 (2007)
16. Ericksen, J.L., Truesdell, C.: Exact tbeory of stress and strain in rods and shells. Arch. Ration. Mech. Anal. **1**(1), 295–323 (1958)
17. Eringen, A.C.: Microcontinuum Field Theory. I. Foundations and Solids. Springer, New York (1999)
18. Eringen, A.C.: Microcontinuum Field Theory. II. Fluent Media. Springer, New York (2001)
19. Eringen, A.C., Maugin, G.A.: Electrodynamics of Continua. Springer, New York (1990)
20. Eringen, A.C., Suhubi, E.S.: Elastodynamics, vol. 1. Academic Press, New York (1974)
21. Erofeev, V.I.: Wave Processes in Solids with Microstructure. World Scientific, Singapore (2003)
22. Fu, Y.B., Scott, N.H.: Acceleration wave propagation in an inhomogeneous heat conducting elastic rod of slowly varying cross section. J. Therm. Stresses **15**, 253–264 (1988)
23. Gibson, L.J., Ashby, M.F.: Cellular Solids: Structure and Properties, 2nd edn. Cambridge Solid State Science Series. Cambridge University Press, Cambridge (1997)
24. Kafadar, C.B., Eringen, A.C.: Micropolar media–I. The classical theory. Int. J. Eng. Sci. **9**, 271–305 (1971)
25. Kafadar, C.B., Eringen, A.C.: Polar field theories. In: Eringen, A.C. (ed.) Continuum Physics, vol. IV, pp. 1–75. Academic Press, New York (1976)
26. Knowles, J.K., Sternberg, E.: On the failure of ellipticity and the emergence of discontinuous deformation gradients in plane finite elastostatics. J. Elast. **10**, 255–293 (1980)
27. Koiter, W.T.: Couple–stresses in the theory of elasticity. Pt I–II. Proc. Koninkl Neterland Akad Wetensh B **67**, 17–44 (1964)
28. Lakes, R.S.: Experimental microelasticity of two porous solids. Int. J. Solids Struct. **22**, 55–63 (1986)
29. Lakes, R.S.: Experimental micro mechanics methods for conventional and negative Poisson's ratio cellular solids as Cosserat continua. Trans. ASME J. Eng. Mater. Technol. **113**, 148–155 (1991)
30. Lurie, A.I.: Nonlinear Theory of Elasticity. North-Holland, Amsterdam (1990)
31. Lurie, A.I.: Theory of Elasticity. Foundations of Engineering Mechanics. Springer, Berlin (2005)
32. Maugin, G.A.: Acceleration waves in simple and linear viscoelastic micropolar materials. Int. J. Eng. Sci. **12**, 143–157 (1974)
33. Maugin, G.A.: Continuum Mechanics of Electromagnetic Solids. Elsevier, Oxford (1988)
34. Maugin, G.A.: On the structure of the theory of polar elasticity. Philos. Trans. R. Soc. Lond. A **356**, 1367–1395 (1998)
35. Maugin, G.A.: Nonlinear Waves on Elastic Crystals. Oxford University Press, Oxford (1999)
36. Naghdi, P.: The theory of plates and shells. In: Flügge, S. (ed.) Handbuch der Physik, vol. VIa/2, pp. 425–640. Springer, Heidelberg (1972)
37. Neff, P., Forest, S.: A geometrically exact micromorphic model for elastic metallic foams accounting for affine microstructure. Modelling, existence of minimizers, identification of moduli and computational results. J. Elast. **87**, 239–276 (2007)
38. Nikitin, E., Zubov, L.M.: Conservation laws and conjugate solutions in the elasticity of simple materials and materials with couple stress. J. Elast. **51**, 1–22 (1998)
39. Noll, W.: A new mathematical theory of simple materials. Arch. Rational Mech. Anal. **48**, 1–50 (1972)
40. Nowacki, W.: Theory of Asymmetric Elasticity. Pergamon-Press, Oxford (1986)
41. Ogden, R.W.: Growth and decay of acceleration waves in incompresible elastic solids. Q. J. Mech. Appl. Math. **37**, 451–464 (1974)
42. Pal'mov, V.A.: Fundamental equations of the theory of asymmetric elasticity. J. Appl. Mech. Math. **28**(3), 496–505 (1964)
43. Park, H.C., Lakes, R.S.: Cosserat micromechanics of human bone: strain redistribution by a hydration-sensitive constituent. J. Biomech. **19**, 385–397 (1986)
44. Pietraszkiewicz, W., Eremeyev, V.A.: On natural strain measures of the non-linear micropolar continuum. Int. J. Solids Struct. **46**(3–4), 774–787 (2009)
45. Rosakis, P.: Ellipticity and deformation with discontinuous gradients in finite elastostatics. Arch. Ration. Mech. Anal. **109**, 1–37 (1990)
46. Rubin, M.B.: Cosserat Theories: Shells, Rods and Points. Kluwer, Dordrecht (2000)
47. Scott, N.H.: Acceleration waves in constrained elastic materials. Arch. Ration. Mech. Anal. **58**, 57–75 (1975)
48. Toupin, R.A.: Theories of elasticity with couple–stress. Arch. Ration. Mech. Anal. **17**, 85–112 (1964)
49. Truesdell, C.: A First Course in Rational Continuum Mechanics. Academic Press, New York (1977)
50. Truesdell, C.: Rational Thermodynamics, 2nd edn. Springer, New York (1984)
51. Truesdell, C., Noll, W.: The nonlinear field theories of mechanics. In: Flügge, S. (ed.) Handbuch der Physik, vol. III/3, pp. 1–602. Springer, Berlin (1965)
52. Vassilev, V., Djondjorov, P.: Acceleration waves in the von Kármán plate theory. In: Integral Methods in Science and Engineering, CRC Research Notes in Mathematics 418, vol. 83, pp. 131–136. Chapman & Hall, Boca Raton (2000)
53. Zee, L., Sternberg, E.: Ordinary and strong ellipticity in the equilibrium theory of incompressible hyperelastic solids. Arch. Ration. Mech. Anal. **83**, 53–90 (1983)
54. Zhilin, P.A.: Mechanics of deformable directed surfaces. Int. J. Solids Struct. **12**, 635–648 (1976)
55. Zubov, L.M.: Nonlinear Theory of Dislocations and Disclinations in Elastic Bodies. Springer, Berlin (1997)