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ACCUMULATION OF COMPLEX EIGENVALUES OF AN INDEFINITE STURM—LIOUVILLE OPERATOR WITH A SHIFTED COULOMB POTENTIAL

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Abstract. For a particular family of long-range potentials V, we prove that the eigenvalues of the indefinite Sturm–Liouville operator $A = \mathrm{sign}(x)(-\Delta + V(x))$ accumulate to zero asymptotically along specific curves in the complex plane. Additionally, we relate the asymptotics of complex eigenvalues to the two-term asymptotics of the eigenvalues of associated self-adjoint operators.

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Paper OaM-10-14

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