Accuracy Analysis of General Parallel Manipulators with Joint Clearance

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Abstract— **Due to the joint clearance, parallel manipulators always exhibit some position and orientation errors at the mobile platform. This paper aims to provide a systematic framework for the error analysis problem of general parallel mechanisms influenced by the joint clearance. A novel and efficient method is proposed to evaluate the maximal pose errors of general spatial parallel manipulators with joint clearance.**

I. INTRODUCTION

Despite the fact that joint clearance simplifies the assembly and manufacturing of parallel mechanisms, the generated pose error of links, however, can not be ignored when the mechanism requires high accuracy. Compared with other error sources, such as assembly and manufacturing errors and motor actuation errors, etc, joint clearance has more significant impact on the position accuracy for both serial and parallel manipulators. Therefore, it is quite important to provide an accurate model that can predict the effects of joint clearance on the mechanism's positioning performance. As indicated in [1], contrary to the assembly and manufacturing errors, joint clearance leads to uncertain error motions at an arbitrary pose of the mechanism. Its effects are highly non-repeatable and can not be rectified with any kind of calibration.

Much research has been devoted to the problem of accuracy in parallel mechanisms. Some authors applied probabilistic analysis to determine the pose error of clearance-affected joints and moving platform [2]. Parenti-Castelli and Venanzi [3] used the virtual work method to determine the position that the moving platform reaches when a given external load is exerted on it. P.Voglewede and I.Ebert-Uphoff [4] aimed to predict precisely the mobile platform error motions for 3-RRR and some special parallel mechanisms with joint clearance. In this paper, we will provide a deeper insight into the accuracy problem of parallel mechanisms affected by joint clearance. A general and yet efficient error evaluation method is also proposed which can handle any kind of parallel mechanisms, no matter planar or spatial, overconstrained or non-overconstrained. Furthermore, its efficiency makes it possible to compute the global maximum pose errors of a clearance-affected mechanism in a prescribed workspace, other than just at a given theoretical configuration.

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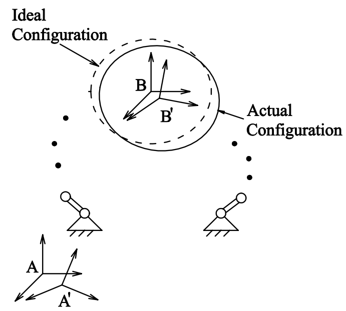

Fig. 1. Pose Error of A Rigid Body

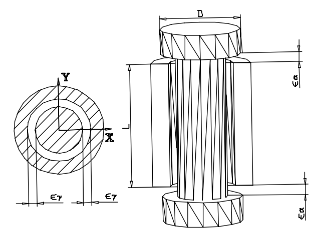

Fig. 2. Clearance-affected Revolute Joint

II. POSE ERROR ANALYSIS OF PARALLEL MANIPULATORS WITH CLEARANCE-AFFECTED PAIRS

A. End-effector and Joint Pose Error

Fig.1 shows the end-effector of a parallel mechanism with some configuration (including the position and orientation) error. Suppose that the nominal (or ideal) configuration of the end-effector (the dashed rigid body in Fig.1) is $g_0 \in SE(3)$, which is the relative transformation from the nominal body frame B to the inertial frame A. With some configuration error, the relative transformation from the actual body frame B' to the inertial frame A, however, gives rise to the real configuration g' of the end-effector. Let A' be another frame attached to the end-effector, which is chosen such that the relative transformation from B' to A' is g_0 (see Fig.1). Then it is clear that without configuration error, A' frame will co-

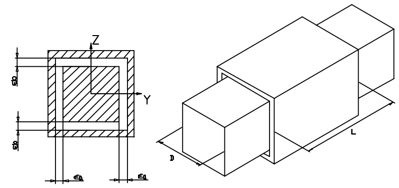

Fig. 3. Clearance-affected Prismatic Joint

incide with the inertial frame A, and B' frame will coincide with B. Hence the error rigid motion of the end-effector from the nominal configuration to the real configuration can be described either by the relative transformation from B' to B, which is $g_0^{-1}g'$, or the relative transformation from A' to A, that is $g'g_0^{-1}$. Correspondingly, the configuration error can be defined as $(g_0^{-1}g' - e)$ and $(g'g_0^{-1} - e)$, which are called the *configuration error* with respect to the body frame and the inertial frame respectively.

Normally the error rigid motions $g_0^{-1}g'$ and $g'g_0^{-1}$ are restricted within a small neighborhood of e. Hence the two expressions of the configuration error can be reasonably approximated by their first order terms, that is,

$$\begin{aligned} g_0^{-1}g' - e &= e^{\hat{e}_1\Delta x^b}e^{\hat{e}_2\Delta y^b}e^{\hat{e}_3\Delta z^b}e^{\hat{e}_4\Delta\alpha^b}e^{\hat{e}_5\Delta\beta^b}e^{\hat{e}_6\Delta\gamma^b} - e \\ &\approx (\hat{e}_1\Delta x^b + \cdots + \hat{e}_6\Delta\gamma^b) \end{aligned}$$

and

$$\begin{aligned} g'g_0^{-1} - e &= e^{\hat{e}_1\Delta x^a}e^{\hat{e}_2\Delta y^a}e^{\hat{e}_3\Delta z^a}e^{\hat{e}_4\Delta\alpha^a}e^{\hat{e}_5\Delta\beta^a}e^{\hat{e}_6\Delta\gamma^a} - e \\ &\approx (\hat{e}_1\Delta x^a + \cdots + \hat{e}_6\Delta\gamma^a) \end{aligned}$$

where $\{e_i | i = 1, \cdots, 6\}$ is the canonical basis of $\mathbb{R}^6$, and the six components $\Delta x^b, \cdots, \Delta\gamma^b$ (and $\Delta x^a, \cdots, \Delta\gamma^a$) represent the three translation and three rotation errors of the end-effector about the x, y and z axis of the B (and A) frame respectively. In the above two equations, the $\wedge$ operation identifies $\mathbb{R}^6$ with $se(3)$ (see [5] for more details). Hence if we let $\delta g^b = (\Delta x^b, \Delta y^b, \Delta z^b, \Delta\alpha^b, \Delta\beta^b, \Delta\gamma^b)^T$ and $\delta g^a = (\Delta x^a, \Delta y^a, \Delta z^a, \Delta\alpha^a, \Delta\beta^a, \Delta\gamma^a)^T$, then it is easy to see that $g_0^{-1}g' - e \approx \hat{\delta g^b}$ and $g'g_0^{-1} - e \approx \hat{\delta g^a}$. To avoid the confusion with the term *configuration error*, δg^b will be called the *pose error* with respect to body frame, and δg^a the *pose error* with respect to the initial frame. From now on, the pose error of the end-effector other than the configuration error will be mainly used and studied in this paper. As the relative transformation from B to A frame is g_0, the two expressions of the pose error are related by $\delta g^a = Ad_{g_0}\delta g^b$, where for $g = (p, R)$, the Adjoint map is defined by

$$Ad_g = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$$

The pose error of the end-effector is caused by many factors, such as assembly and manufacturing errors (occurring at the adjacencies between the links and the joints), actuation motor errors, joint clearance, etc. As the major source of the pose error comes from the error motions caused by the joint clearance [6], in this paper, we will focus on studying the impact of joint clearance on the deviations of the end-effector's configurations. Hence, we will assume that joint clearance is the unique error source. In the remain part of this subsection, we study the impact of joint clearance on the pose error of two links connected by a clearance-affected joint. Only revolute and prismatic joints are considered in this paper, but other types of joints can be analyzed in a similar manner.

A clearance-affected kinematic pair (i.e., a pair with joint clearance) actually has 6-DoFs. Thus at a given nominal configuration (the theoretical configuration of the pair), there are 6 twists associated with a pair. Some twists result in the desired motions of the joint, called *ideal twists*, whereas others give rise to the error motions caused by joint clearance. For example, for a clearance-affected revolute pair in Fig.2, if we attach a local coordinate frame C to the bearing with z-axis along the pair's theoretical axis, then e_6 generates the ideal rotational motions of the joint, whereas $e_1, \cdots, e_5$ generate error motions of the shaft with respect to the bearing. The pose error of the relative configuration of the two links with respect to the local frame C thus is given by

$$\delta\Upsilon^c = \sum_{i=1}^{5} e_i \Delta\sigma_i \tag{1}$$

where $\Delta\sigma_1, \Delta\sigma_2, \Delta\sigma_3$ represent 3-infinitesimal error translations along x, y, z axis, and $\Delta\sigma_4, \Delta\sigma_5$ represent 2-infinitesimal error rotations about x, y axis. Hereafter $\Delta\sigma_i, i = 1, \cdots, 5$ will be called the *error motions caused by the joint clearance*, with values restricted by the joint geometry and the magnitude of joint clearance. For a particular design of a clearance-affected revolute joint, it is possible to formulate explicitly the set of constraints that bound the values of $\Delta\sigma_i$. For example, for the journal bearing design (Fig.2) of a revolute joint, because of the axial symmetry, the x, y axis can be chosen freely. With origin chosen at the joint center, and the geometrical dimensions L, D and the magnitude of joint clearance ϵ_r, ϵ_a given beforehand, the values of $\Delta\sigma_i$ are constrained by a set of quadratic and second order cone constraints :

$$\begin{array}{c} (\Delta\sigma_1 + \frac{L}{2}\Delta\sigma_5)^2 + (\Delta\sigma_2 - \frac{L}{2}\Delta\sigma_4) \le \epsilon_r^2 \\ (\Delta\sigma_1 - \frac{L}{2}\Delta\sigma_5)^2 + (\Delta\sigma_2 + \frac{L}{2}\Delta\sigma_4) \le \epsilon_r^2 \\ \frac{D}{2}\sqrt{\Delta\sigma_4^2 + \Delta\sigma_5^2} - \Delta\sigma_3 \le \epsilon_a \\ \frac{D}{2}\sqrt{\Delta\sigma_4^2 + \Delta\sigma_5^2} + \Delta\sigma_3 \le \epsilon_a \end{array} \tag{2}$$

For other types of designs of clearance-affected revolute joint, though the structure may be more complicated, one can get similar constraint inequalities as (2). If there is an inertial reference frame A such that the relative transformation from C to A is g_{ac}^R, then the pose error caused by joint clearance can be written with respect to the inertial frame A by

$$\delta\Upsilon^a = Ad_{g_{ac}^R}\delta\Upsilon^c = \textstyle\sum_{i=1}^{5}(Ad_{g_{ac}^R}e_i)\Delta\sigma_i \tag{3}$$

Similarly, for a clearance-affected prismatic pair(Fig.3), if we attach a local coordinate frame C to the supporting carriage with x-axis along the pair's ideal translational axis, then the pose error of the slider with respect to the supporting carriage is given with respect to C by

$$\delta\Gamma^c = \sum_{j=2}^{6} e_j \Delta\tau_j \tag{4}$$

where $\Delta\tau_j, j = 2, \cdots, 6$ are also error motions caused by the joint clearance. If y, z axis are parallel to the two faces of the supporting carriage, and the origin lies on the geometrical center of the joint (see Fig.3), then values of the error motions are restricted by a set of linear constraints

$$\begin{array}{l} -\epsilon_a \leq +\frac{L}{2}\Delta\tau_6 - \frac{D}{2}\Delta\tau_4 + \Delta\tau_2 \leq \epsilon_a \\ -\epsilon_a \leq -\frac{L}{2}\Delta\tau_6 + \frac{D}{2}\Delta\tau_4 + \Delta\tau_2 \leq \epsilon_a \\ -\epsilon_b \leq -\frac{L}{2}\Delta\tau_5 + \frac{D}{2}\Delta\tau_4 + \Delta\tau_3 \leq \epsilon_b \\ -\epsilon_b \leq +\frac{L}{2}\Delta\tau_5 - \frac{D}{2}\Delta\tau_4 + \Delta\tau_3 \leq \epsilon_b \end{array} \quad (5)$$

Supposing that the transformation from C to the inertial frame A is g_{ac}^P, then the pose error is given with respect to frame A by

$$\delta\Gamma^a = Ad_{g_{ac}^P}\delta\Gamma^c = \sum_{j=2}^{6}(Ad_{g_{ac}^P}e_j)\Delta\tau_j \quad (6)$$

Finally, we remark that the two designs of Fig.2 and Fig.3 have ever been intensively studied in [7] and [3]. However, the constraints of error motions in those literatures are formulated as sets of general nonlinear inequalities. In this paper, we have reformulated them into sets of so-called convex constraints, as seen from (2) and (5). As we will see later, this reformulation will make the convex optimization feasible to the evaluation of the pose errors.

B. Gross Pose Error of Joints

The desired motions of an actuated joint are controlled by the motors mounted on the joint. Assuming there is no motor actuation error, then at a nominal configuration, the pose error of an actuated joint is solely raised by its joint clearance. Hence the expression of gross pose error of an actuated joint is the same as Eq.(1) and (3). However, for passive joints in a parallel mechanism, in addition to the errors motions caused by joint clearance, the idle motions about the ideal twists also play a role in the pose errors of their two links. Hence, the gross pose error of a passive revolute joint is given with respect to its local frame by

$$\delta\Upsilon^c = \sum_{i=1}^{5} e_i\Delta\sigma_i + e_6\Delta\theta \quad (7)$$

where $\Delta\theta$ is the idle rotation error of the shaft about the joint's ideal axis. With respect to the inertial frame A, the gross pose error is

$$\begin{array}{rcl} \delta\Upsilon^a = Ad_{g_{ac}^R}\delta\Upsilon^c & = & \sum_{i=1}^{5}(Ad_{g_{ac}^R}e_i)\Delta\sigma_i + (Ad_{g_{ac}^R}e_6)\Delta\theta \\ & = & \sum_{i=1}^{5}(Ad_{g_{ac}^R}e_i)\Delta\sigma_i + \xi\Delta\theta \end{array}$$

where ξ is the ideal twist of the joint when expressed in the inertial frame. Similarly, two gross pose errors of a passive prismatic joint can be written as

$$\delta\Gamma^c = \sum_{j=2}^{6} e_j\Delta\tau_j + e_1\Delta\mu \quad (8)$$

and

$$\delta\Gamma^a = Ad_{g_{ac}^P}\delta\Gamma^c = \sum_{j=2}^{6}(Ad_{g_{ac}^P}e_j)\Delta\tau_j + \eta\Delta\mu$$

respectively, where $\Delta\mu$ is the idle translational error along the joint's ideal axis, and η is the ideal twist representation of the joint in the inertial frame A.

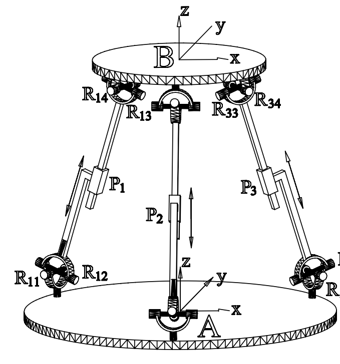

Fig. 4. Tsai's 3-UPU Manipulator

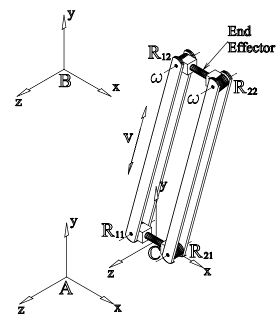

Fig. 5. A Parallelogram Joint

C. End-effector Pose Error of A Parallel Mechanism

In this subsection, we study the pose error of the end-effector of a parallel mechanism caused by the clearance of its constituent joints. We use some examples to illustrate how to get the pose error of the end-effector at a general theoretical configuration(say, g_0), subject to the error motions generated by all the clearance-affected joints.

Example *1:* **Pose Error Analysis of Tsai's Manipulator** Consider a Tsai's 3-UPU manipulator in Fig.4. Each U-joint consists of two perpendicularly connected clearance-affected revolute joints modelled by Fig.2. The three prismatic joints $P_1, \cdots, P_3$, being actuated, are also clearance-affected and modelled by Fig.3. The inertial frame A and the body frame B are located at the center of the base and the mobile platform respectively. And the local frames of $\mathcal{R}_{ij}$ (denoted C_{ij}^R), and that of P_i (denoted by C_i^P), $i = 1, \cdots, 3$, $j = 1, \cdots, 4$, are located at the center of each joint, with x,

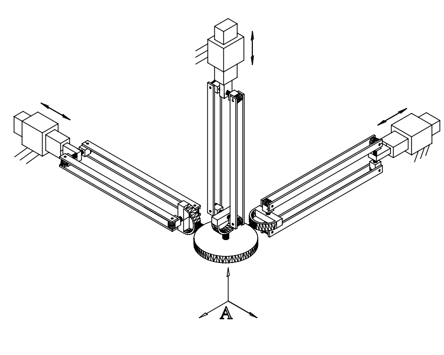

Fig. 6. Orthoglide Manipulator

y axis chosen at one's convenience. The mobile platform(or end-effector) ideally can undergo three DoF pure translations $T(3)$ under some geometrical conditions [8]. At a nominal configuration $g_0 \in T(3)$ which is not singular for the sake of simplicity, the transformation matrix from C_{ij}^R and C_i^P to A, denoted by g_{ij}^R and g_i^P respectively, can be obtained through the inverse kinematic analysis of the ideal mechanism at g_0. From the last subsection, the pose error of P_i is given with respect to the inertial frame A by

$$\Delta P_i = \textstyle\sum_{l=2}^{6}(Ad_{g_i^P} e_l)\Delta\tau_{il} \qquad i = 1, \cdots, 3$$

and the pose error of $\mathcal{R}_{ij}$ is given by

$$\Delta\mathcal{R}_{ij} = \textstyle\sum_{l=1}^{5}(Ad_{g_{ij}^R} e_l)\Delta\sigma_{ijl} + \xi_{ij}\Delta\theta_{ij}$$

where $i = 1, \cdots, 3, j = 1, \cdots, 4$, and ξ_{ij} is the ideal twist associated with $\mathcal{R}_{ij}$. The pose error of ith open subchain, $i = 1, \cdots, 3$, can thus be written by

$$\begin{aligned}\Delta E_i &= \Delta P_i + \textstyle\sum_{j=1}^{4}\Delta\mathcal{R}_{ij} \\ &= \textstyle\sum_{l=2}^{6}(Ad_{g_i^P} e_l)\Delta\tau_{il} + \sum_{j=1}^{4}\xi_{ij}\Delta\Theta_{ij} \\ &\quad + \textstyle\sum_{j=1}^{4}\sum_{l=1}^{5}(Ad_{g_{ij}^R} e_l)\Delta\sigma_{ijl} \\ &= A_i \cdot d\Gamma_i + \textstyle\sum_{j=1}^{4} B_{ij} \cdot d\Upsilon_{ij} + J_i \cdot d\Theta_i \\ &= A_i \cdot d\Gamma_i + B_i \cdot d\Upsilon_i + J_i \cdot d\Theta_i\end{aligned} \tag{9}$$

where $A_i = [Ad_{g_i^P} e_2 \ \cdots \ Ad_{g_i^P} e_6]$, $d\Gamma_i = (\Delta\tau_{i2}, \cdots, \Delta\tau_{i6})^T$, $B_{ij} = [Ad_{g_{ij}^R} e_1 \ \cdots \ Ad_{g_{ij}^R} e_5]$, $d\Upsilon_{ij} = (\Delta\sigma_{ij1}, \cdots, \Delta\sigma_{ij5})^T$, $B_i = [B_{i1} \ \cdots \ B_{i4}]$, $d\Upsilon_i = (d\Upsilon_{i1}^T, \cdots, d\Upsilon_{i4}^T)^T$, $J_i = [\xi_{i1} \ \cdots \ \xi_{i4}]$, $d\Theta_i = (\Delta\theta_{i1}, \cdots, \Delta\theta_{i4})^T$.

Equating ΔE_1 and ΔE_2, ΔE_2 and ΔE_3, we can get

$$J \cdot \begin{pmatrix} d\Theta_1 \\ d\Theta_2 \\ d\Theta_3 \end{pmatrix} = A \cdot \begin{pmatrix} d\Gamma_1 \\ d\Gamma_2 \\ d\Gamma_3 \end{pmatrix} + B \cdot \begin{pmatrix} d\Upsilon_1 \\ d\Upsilon_2 \\ d\Upsilon_3 \end{pmatrix} \tag{10}$$

where

$$J = \begin{bmatrix} J_1 & -J_2 & \\ & -J_2 & J_3 \end{bmatrix} \tag{11}$$

is a nonsingular square matrix, and

$$A = \begin{bmatrix} -A_1 & A_2 & \\ & A_2 & -A_3 \end{bmatrix}, \quad B = \begin{bmatrix} -B_1 & B_2 & \\ & B_2 & -B_3 \end{bmatrix}$$

From Eq.(10) and (11), we may see that the idle passive joint motions are uniquely determined by the error motions $\Delta\tau_{il}$ and $\Delta\sigma_{ijl}$ caused by the joint clearance. After the idle motions $d\Theta_i$ are determined, we can substitute it into Eq.(9) for any subchain to obtain the pose error of the moving platform the nominal configuration g_0. Choosing the first subchain as example, we can write the result in a concise way as follows

$$\begin{aligned}\Delta E_{g_0} &= A_1 \cdot d\Gamma_1 + B_1 \cdot d\Upsilon_1 + J_1 \cdot d\Theta_1 \\ &= A_1 \cdot d\Gamma_1 + B_1 \cdot d\Upsilon_1 + [J_1 \ \mathbf{0} \ \mathbf{0}] \begin{pmatrix} d\Theta_1 \\ d\Theta_2 \\ d\Theta_3 \end{pmatrix} \\ &= A_1 \cdot d\Gamma_1 + B_1 \cdot d\Upsilon_1 + [J_1 \ \mathbf{0} \ \mathbf{0}] \cdot J^{-1} \\ &\quad \cdot \left[A \cdot \begin{pmatrix} d\Gamma_1 \\ d\Gamma_2 \\ d\Gamma_3 \end{pmatrix} + B \cdot \begin{pmatrix} d\Upsilon_1 \\ d\Upsilon_2 \\ d\Upsilon_3 \end{pmatrix} \right] \\ &= H \cdot (d\Gamma_1^T, d\Gamma_2^T, d\Gamma_3^T, d\Upsilon_1^T, d\Upsilon_2^T, d\Upsilon_3^T)^T\end{aligned}$$

In the above equation, we get a linear transformation matrix H that maps the error motions of the clearance-affected joints, $\Delta\tau_{il}$ and $\Delta\sigma_{ijl}$, to the pose error of the moving platform ΔE_{g_0}. From Eq.(10), $\Delta\tau_{il}$ and $\Delta\sigma_{ijl}$ are free variables of the mechanism, the number of which reflects the total extra DoF of the mechanism at a non-singular nominal configuration. However, the values of $\Delta\tau_{il}$ and $\Delta\sigma_{ijl}$ are bounded within the sets of convex constraints given by inequalities (2) and (5) for each joint. Therefore, the maximal pose error of the moving platform at the nominal configuration g_0, denoted by $\Delta E_{g_0}^m$, can be obtained by six convex optimizations applied to each components of ΔE_{g_0}. If one further want to get the maximal pose error of the moving platform in a prescribed workspace W, for example, a cube centered by the home configuration, he can discretize W and calculate ΔE_g^m at any point of W. After that, the global maximal pose error in W can be obtained by

$$\Delta E^m = (\max_{g \in W} \Delta E_g^m(1), \cdots, \max_{g \in W} \Delta E_g^m(6))^T$$

where $\Delta E_g^m(i)$ is the ith component of ΔE_g^m.

For overconstrained parallel manipulators, however, not all error motions associated with joints' clearance are free variables, as seen from the next example.

Example *2:* **Pose Error Analysis of A Parallelogram Joint** In this example, we study the pose error of a parallelogram (P_a) composing of four clearance-affected revolute joints. Since P_a usually serves as an extended pair in complex parallel mechanisms, e.g., the Delta robot, we also attach a frame C, called the local frame of the parallelogram joint, to the base link of the parallelogram, as shown in Fig.5. Frame C is located at the center of the base link, with z axis parallel to the ideal rotation axis of $\mathcal{R}_{ij}$, and x axis pointing to the geometrical center of $\mathcal{R}_{21}$. Thus the ideal configuration space of the parallelogram C_M is a circle in the x-y plane of C. At each nominal configuration $g_0 \in C_M$, we define four local frames C_{ij} for $\mathcal{R}_{ij}$, $i = 1, 2$, $j = 1, 2$. As the choice of x, y axis of C_{ij} is free, for our convenience, C_{ij} may be chosen with origin lying at the geometrical center of $\mathcal{R}_{ij}$, and x, y, z axis aligned with that of C. Hence, the transformation from C_{ij} to C is given by

$$g_{ij} = \begin{bmatrix} I & p_{ij} \\ 0 & 1 \end{bmatrix}$$

where p_{ij} is the geometrical center of $\mathcal{R}_{ij}$. Assume that $\mathcal{R}_{11}$ is the actuation joint, then the pose error of four revolute joints with respect to the C frame is given by

$$\Delta\mathcal{R}_{ij} = \sum_{l=1}^{5}(Ad_{g_{ij}} e_l)\Delta\sigma_{ijl} + \xi_{ij}\Delta\theta_{ij} \quad i = 1, 2, j = 1, 2$$

where $\Delta\theta_{11} = 0$. The pose error of ith subchain, $i = 1, 2$ thus can be easily got as

$$\begin{aligned}\Delta E_i &= \Delta\mathcal{R}_{i1} + \Delta\mathcal{R}_{i2} \\ &= \textstyle\sum_{j=1}^{2}\sum_{l=1}^{5}(Ad_{g_{ij}} e_l)\Delta\sigma_{ijl} + \sum_{j=1}^{2}\xi_{ij}\Delta\Theta_{ij} \\ &= \textstyle\sum_{j=1}^{2} A_{ij} \cdot d\Upsilon_{ij} + \sum_{j=1}^{2}\xi_{ij}\Delta\Theta_{ij}\end{aligned}$$

where $A_{ij} = [Ad_{g_{ij}}e_1 \ \cdots \ Ad_{g_{ij}}e_5]$, $d\Upsilon_{ij} = (\Delta\sigma_{ij1}, \cdots, \Delta\sigma_{ij5})^T$, $i = 1,2$, $j = 1,2$. As $\Delta\theta_{11} = 0$, by equating ΔE_1 and ΔE_2, we can get

$$J \cdot \begin{pmatrix} \Delta\Theta_{12} \\ \Delta\Theta_{21} \\ \Delta\Theta_{22} \end{pmatrix} = A \cdot \begin{pmatrix} d\Upsilon_{11} \\ d\Upsilon_{12} \\ d\Upsilon_{21} \\ d\Upsilon_{22} \end{pmatrix} \tag{12}$$

where $J = [\xi_{12} \ -\xi_{21} \ -\xi_{22}]$, and $A = [-A_{11} \ -A_{12} \ A_{21} \ A_{22}]$. Clearly, J is a 6×3 full column rank matrix. J may combine with three twists $Ad_{g_{21}}e_3$, $Ad_{g_{21}}e_4$, and $Ad_{g_{21}}e_5$ to form a 6×6 non-singular square matrix $J' = [\xi_{12} \ -\xi_{21} \ -\xi_{22} \ -Ad_{g_{21}}e_3 \ -Ad_{g_{21}}e_4 \ -Ad_{g_{21}}e_5]$. Then Eq.(12) can be re-written as follows

$$J' \cdot \begin{pmatrix} \Delta\Theta_{12} \\ \Delta\Theta_{21} \\ \Delta\Theta_{22} \\ \Delta\sigma_{213} \\ \Delta\sigma_{214} \\ \Delta\sigma_{215} \end{pmatrix} = A' \cdot \begin{pmatrix} d\Upsilon_{11} \\ d\Upsilon_{12} \\ \Delta\sigma_{211} \\ \Delta\sigma_{212} \\ d\Upsilon_{22} \end{pmatrix} \tag{13}$$

where $A' = [-A_{11} \ -A_{12} \ Ad_{g_{21}}e_1 \ Ad_{g_{21}}e_2 \ A_{22}]$. Therefore, only 17 error motions $d\Upsilon_{11}$, $d\Upsilon_{12}$, $d\Upsilon_{22}$ and $\Delta\sigma_{211}$, $\Delta\sigma_{212}$ are free variables of the mechanism. The other three joint error motions $\Delta\sigma_{213}$, $\Delta\sigma_{214}$, $\Delta\sigma_{215}$, plus the idle passive joint motions $\Delta\Theta_{12}$, $\Delta\Theta_{21}$, $\Delta\Theta_{22}$, are uniquely determined by these free variables. Denote by $B = J'^{-1}A'$, $B(i)$ the ith row of B, $i = 1, \cdots, 6$, and g_{ac} the transformation from the local frame C to the inertial frame A. Then pose error of the end-effector at the nominal configuration g_0 is given with respect to the inertial frame by

$$\begin{aligned} \Delta E_{g_0} &= Ad_{g_{ac}}(\Delta\mathcal{R}_{11} + \Delta\mathcal{R}_{12}) \\ &= Ad_{g_{ac}}[A_{11} \ A_{12}] \begin{pmatrix} d\Upsilon_{11} \\ d\Upsilon_{12} \end{pmatrix} + Ad_{g_{ac}}\xi_{12}\Delta\Theta_{12} \\ &= Ad_{g_{ac}}[A_{11} \ A_{12}] \begin{pmatrix} d\Upsilon_{11} \\ d\Upsilon_{12} \end{pmatrix} \\ &\quad + Ad_{g_{ac}}\xi_{12}B(1) \begin{pmatrix} d\Upsilon_{11} \\ d\Upsilon_{12} \\ \Delta\sigma_{211} \\ \Delta\sigma_{212} \\ d\Upsilon_{22} \end{pmatrix} \\ &= H \cdot (d\Upsilon_{11}^T \ d\Upsilon_{12}^T \ \Delta\sigma_{211} \ \Delta\sigma_{212} \ d\Upsilon_{22}^T)^T \end{aligned} \tag{14}$$

subject to the constraints

$$\begin{aligned} &(\Delta\sigma_{ij1} + \tfrac{L}{2}\Delta\sigma_{ij5})^2 + (\Delta\sigma_{ij2} - \tfrac{L}{2}\Delta\sigma_{ij4}) \le \epsilon_r^2 \\ &(\Delta\sigma_{ij1} - \tfrac{L}{2}\Delta\sigma_{ij5})^2 + (\Delta\sigma_{ij2} + \tfrac{L}{2}\Delta\sigma_{ij4}) \le \epsilon_r^2 \\ &\tfrac{D}{2}\sqrt{\Delta\sigma_{ij4}^2 + \Delta\sigma_{ij5}^2} - \Delta\sigma_{ij3} \le \epsilon_a \\ &\tfrac{D}{2}\sqrt{\Delta\sigma_{ij4}^2 + \Delta\sigma_{ij5}^2} + \Delta\sigma_{ij3} \le \epsilon_a \\ &\begin{pmatrix} \Delta\sigma_{213} \\ \Delta\sigma_{214} \\ \Delta\sigma_{215} \end{pmatrix} = \begin{bmatrix} B(4) \\ B(5) \\ B(6) \end{bmatrix} \cdot \begin{pmatrix} d\Upsilon_{11} \\ d\Upsilon_{12} \\ \Delta\sigma_{211} \\ \Delta\sigma_{212} \\ d\Upsilon_{22} \end{pmatrix} \end{aligned} \tag{15}$$

where $i = 1,2$ and $j = 1,2$. Thus the calculation of maximal pose error of an overconstrained parallel manipulator at g_0 is a convex optimization problem *with linear equality constraints*. By [9], such a problem can also be quickly solved by very efficient algorithms.

P_a is often used as an extended passive pair in some common parallel mechanisms, e.g, the Delta robot. In this case, $\mathcal{R}_{11}$ is a passive revolute joint instead of an actuated one, which implies that $\Delta\theta_{11} \neq 0$. Assume that the body frame B originally coincides with A frame, then the ideal configuration space of the end-effector is given by

$$Q = \left\{ \begin{bmatrix} I & (e^{\hat{\omega}\theta_{11}} - I)\mathbf{v} \\ 0 & 1 \end{bmatrix} \mid \theta_{11} \in [0, 2\pi] \right\}$$

where ω is the ideal rotation axis of the parallelogram, and $\mathbf{v}$ the vector connecting $\mathcal{R}_{i1}$ and $\mathcal{R}_{i2}$ at the home configuration. Clearly, at the nominal configuration $g_0 = Q(\theta_{11})$, the configuration error of the end-effector caused by $\Delta\theta_{11}$ is given with respect to the inertial frame A by

$$\widehat{\Delta F}_{g_0} = \frac{dQ}{d\theta_{11}}Q^{-1}\Delta\theta_{11} = \begin{bmatrix} 0 & \omega e^{\hat{\omega}\theta_{11}}\mathbf{v} \\ 0 & 0 \end{bmatrix}\Delta\theta_{11}$$

The total pose error thus is the linear summation of ΔF_{g_0} and ΔE_{g_0}

$$\begin{aligned} \Delta T_{g_0} &= \Delta E_{g_0} + \Delta F_{g_0} \\ &= H \cdot (d\Upsilon_{11}^T \ d\Upsilon_{12}^T \ \Delta\sigma_{211} \ \Delta\sigma_{212} \ d\Upsilon_{22}^T)^T \\ &\quad + \omega e^{\hat{\omega}\theta_{11}}\mathbf{v}\Delta\theta_{11} \end{aligned} \tag{16}$$

subject to the constraints of (15). The value of $\Delta\theta_{11}$, as we will see in the next example, is determined by the loop closure equation of the parallel mechanism that contains this P_a.

Example *3:* **Pose Error Analysis of Orthoglide** The Orthoglide manipulator [10], as shown in Fig.6, consists of three $P\mathcal{R}P_a\mathcal{R}$ identical subchains. The actuated joints are the three orthogonal prismatic ones. Ideally, the manipulator is an overconstrained one, with end-effector motion a subset of $T(3)$. At a nominal configuration $g_0 \in T(3)$, using the same approach as before, we may find the pose errors of prismatic joints P_i and revolute joints $\mathcal{R}_{ij}$, $i = 1, \cdots, 3$, $j = 1,2$. As the end-effector is connected to the prismatic joints through a set of three passive parallelograms, the pose error of the parallelogram is given by Eq.(16). Thus the pose error of the ith subchain, $i = 1, \cdots, 3$, is shown to be

$$\begin{aligned} \Delta E_i &= \Delta P_i + \Delta\mathcal{R}_{i1} + \Delta T_i + \Delta\mathcal{R}_{i2} \\ &= \textstyle\sum_{l=2}^{6}(Ad_{g_i^P}e_l)\Delta\tau_{il} \\ &\quad + \textstyle\sum_{l=1}^{5}(Ad_{g_{i1}^R}e_l)\Delta\sigma_{i1l} + \xi_{i1}\Delta\Theta_{i1} \\ &\quad + H_i \cdot (d\Upsilon_{i11}^T \ d\Upsilon_{i12}^T \ \Delta\sigma_{i211} \ \Delta\sigma_{i212} \ d\Upsilon_{i22}^T)^T \\ &\quad + \omega_i e^{\hat{\omega}_i\theta_{i11}}\mathbf{v}_i\Delta\theta_{i11} \\ &\quad + \textstyle\sum_{l=1}^{5}(Ad_{g_{i2}^R}e_l)\Delta\sigma_{i2l} + \xi_{i2}\Delta\Theta_{i2} \\ &= A_i \cdot d\Gamma_i + \textstyle\sum_{j=1}^{2}B_{ij} \cdot d\Omega_{ij} + H_i \cdot d\Upsilon_i \\ &\quad + J_i \cdot d\Theta_i \\ &= A_i \cdot d\Gamma_i + B_i \cdot d\Omega_i + H_i \cdot d\Upsilon_i + J_i \cdot d\Theta_i \end{aligned}$$

where g_i^P is the transformation matrix from the local frame of the P_i to the inertial frame A, and g_{ij}^R the one from the local frame of $\mathcal{R}_{ij}$ to A. Furthermore, $A_i = [Ad_{g_i^P}e_2 \ \cdots \ Ad_{g_i^P}e_6]$, $d\Gamma_i = (\Delta\tau_{i2}, \cdots, \Delta\tau_{i6})^T$, $B_{ij} = [Ad_{g_{ij}^R}e_1 \ \cdots \ Ad_{g_{ij}^R}e_5]$, $d\Omega_{ij} = (\Delta\sigma_{ij1}, \cdots, \Delta\sigma_{ij5})^T$,

$B_i = [B_{i1} \;\; B_{i2}]$, $d\Omega_i = (d\Omega_{i1}^T, d\Omega_{i2}^T)^T$, $d\Upsilon_i = (d\Upsilon_{i11}^T \;\; d\Upsilon_{i12}^T \;\; \Delta\sigma_{i211} \;\; \Delta\sigma_{i212} \;\; d\Upsilon_{i22}^T)^T$, $J_i = [\xi_{i1} \;\; \omega_i e^{\hat{\omega}_i \theta_{i11}} \mathbf{v}_i \;\; \xi_{i2}]$, $d\Theta_i = (\Delta\theta_{i1}, \Delta\theta_{i11}, \Delta\theta_{i2})^T$, $i = 1, \cdots, 3$, $j = 1, 2$. Equating ΔE_1 and ΔE_2, ΔE_2 and ΔE_3, we can get

$$J \begin{pmatrix} d\Theta_1 \\ d\Theta_2 \\ d\Theta_3 \end{pmatrix} = A \begin{pmatrix} d\Gamma_1 \\ d\Gamma_2 \\ d\Gamma_3 \end{pmatrix} + B \begin{pmatrix} d\Omega_1 \\ d\Omega_2 \\ d\Omega_3 \end{pmatrix} + C \begin{pmatrix} d\Upsilon_1 \\ d\Upsilon_2 \\ d\Upsilon_3 \end{pmatrix}$$

where

$$J = \begin{bmatrix} J_1 & -J_2 & \\ & -J_2 & J_3 \end{bmatrix} \tag{17}$$

is a 12×9 full column rank matrix, and

$$A = \begin{bmatrix} -A_1 & A_2 & \\ & A_2 & -A_3 \end{bmatrix}, \quad B = \begin{bmatrix} -B_1 & B_2 & \\ & B_2 & -B_3 \end{bmatrix}$$

$$C = \begin{bmatrix} -H_1 & H_2 & \\ & H_2 & -H_3 \end{bmatrix}$$

Note that the rotation axis of the twist $Ad_{g_i^P} e_4$ is along the ideal translation axis of P_i, $i = 1, \cdots, 3$, hence the former equation may be re-written as

$$J' \begin{pmatrix} d\Theta_1 \\ \Delta\tau_{14} \\ d\Theta_2 \\ \Delta\tau_{24} \\ d\Theta_3 \\ \Delta\tau_{34} \end{pmatrix} = A' \begin{pmatrix} d\Gamma'_1 \\ d\Gamma'_2 \\ d\Gamma'_3 \end{pmatrix} + B \begin{pmatrix} d\Omega_1 \\ d\Omega_2 \\ d\Omega_3 \end{pmatrix} + C \begin{pmatrix} d\Upsilon_1 \\ d\Upsilon_2 \\ d\Upsilon_3 \end{pmatrix}$$

where

$$J' = \begin{bmatrix} J_1 & Ad_{g_1^P} e_4 & -J_2 & -Ad_{g_2^P} e_4 & & \\ & & -J_2 & -Ad_{g_2^P} e_4 & J_3 & Ad_{g_3^P} e_4 \end{bmatrix}$$

is a 12×12 non-singular matrix, and

$$A' = \begin{bmatrix} -A'_1 & A'_2 & \\ & A'_2 & -A'_3 \end{bmatrix},$$

$A'_i = [Ad_{g_i^P} e_2 \;\; Ad_{g_i^P} e_3 \;\; Ad_{g_i^P} e_5 \;\; Ad_{g_i^P} e_6]$, $d\Gamma'_i = (\Delta\tau_{i2}, \Delta\tau_{i3}, \Delta\tau_{i5}, \Delta\tau_{i6})^T$, $i = 1, \cdots, 3$.

From the above transformed equation, one may see that only $d\Gamma'_i$, $d\Omega_i$ and $d\Upsilon_i$, $i = 1, \cdots, 3$, are free variables of the mechanism. Three joint error motions $\Delta\tau_{i4}$, together with the idle passive joint motions $d\Theta_i$, $i = 1, \cdots, 3$, are uniquely determined by totally $4 \times 3 + 5 \times 3 + 17 \times 3 = 78$ free variables. The pose error of the end-effector thus is given by

$$\begin{aligned} \Delta E_{g_0} &= A_1 \cdot d\Gamma_1 + B_1 \cdot d\Omega_1 + H_1 \cdot d\Upsilon_1 + J_1 \cdot d\Theta_1 \\ &= A_1 \cdot d\Gamma_1 + B_1 \cdot d\Omega_1 + H_1 \cdot d\Upsilon_1 + [J_1 \; 0 \; \cdots \; 0] \cdot \\ &\quad J'^{-1} \left[A' \begin{pmatrix} d\Gamma'_1 \\ d\Gamma'_2 \\ d\Gamma'_3 \end{pmatrix} + B \begin{pmatrix} d\Omega_1 \\ d\Omega_2 \\ d\Omega_3 \end{pmatrix} + C \begin{pmatrix} d\Upsilon_1 \\ d\Upsilon_2 \\ d\Upsilon_3 \end{pmatrix} \right] \end{aligned}$$

subject to the constraint of (2) for revolute joints, (5) for prismatic joints, (15) for parallelogram, and the following linear equalities

$$\begin{pmatrix} \Delta\tau_{14} \\ \Delta\tau_{24} \\ \Delta\tau_{34} \end{pmatrix} = D \cdot (d\Gamma_1'^T, \cdots, d\Gamma_3'^T, d\Omega_1^T, \cdots, d\Omega_3^T, d\Upsilon_1^T, \cdots, d\Upsilon_3^T)^T$$

where the matrix D can be derived from Eq.(3). Hence the calculation of the maximal pose error of the Orthoglide manipulator at g_0 is a still a convex optimization problem with linear equality constraints.

III. Conclusion

In this paper, we propose a general method to evaluate the pose(position and orientation) error of the end-effectors of parallel manipulators due to the joint clearance. We show that for non-overconstrained parallel manipulators, the error motions caused by joint clearance are free variables subject to some constraints defined by the joint geometry and magnitude of clearance. However, for overconstrained parallel mechanisms, part of these joint error motions are dependent on the remaining ones. For some particular designs of common lower pairs, the pose error analysis for both non-overconstrained and overconstrained parallel manipulators can be formulated into standard convex optimization problems, which makes it possible to compute the maximum pose errors in a prescribed workspace other than just at a single configuration.

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