Теорія Ймовір. та Матем. Статист.
Вип. 71, 2004

Theor. Probability and Math. Statist.
No. 71, 2005, Pages 105–117
S 0094-9000(05)00651-4
Article electronically published on December 28, 2005

ACCURACY AND RELIABILITY OF MODELS OF STOCHASTIC PROCESSES OF THE SPACE $\mathrm{Sub}_\varphi(\Omega)$

UDC 519.21

YU. V. KOZACHENKO AND I. V. ROZORA

Abstract. Stochastic processes of the space $\mathrm{Sub}_\varphi(\Omega)$ are considered in the paper. We prove upper bounds for large deviation probabilities and construct models of stochastic processes in the space $C[0,1]$ with a given accuracy and reliability. Strongly sub-Gaussian processes are also considered as a particular case.

1. Introduction

We consider stationary stochastic processes of the space $\mathrm{Sub}_\varphi(\Omega)$ and construct models of these processes with a given accuracy and reliability. Similar problems for models of some stochastic processes are considered in [1]–[3]. In the proofs below we follow the method of [1].

Section 2 of this paper contains basic definitions and properties of stochastic processes of the space $\mathrm{Sub}_\varphi(\Omega)$. More detail can be found in [4]. Stationary stochastic processes with discrete spectrum are considered in Section 3. Section 4 is devoted to models of stationary processes of the space $\mathrm{Sub}_\varphi(\Omega)$. We obtain results for models approximating a stochastic process with a given accuracy and reliability in the Banach space $C[0,1]$. A particular case of sub-Gaussian processes is considered in Section 5.

2. Stochastic processes of the space $\mathrm{Sub}_\varphi(\Omega)$

Definition 2.1 ([4]). A convex even continuous function $\varphi(x)$ such that $\varphi(0)=0$ is called an N-function if $\varphi(x)>0$ for $x\neq 0$, $\varphi(x)/x\to 0$ as $x\to 0$, and $\varphi(x)/x\to\infty$ as $x\to\infty$.

Lemma 2.1 ([4]). *A function $\varphi(x)$, $x\in\mathbb{R}$, is an N-function if and only if*

$$\varphi(x)=\int_0^{|x|} l(u)\,du, \qquad x\in\mathbb{R},$$

where the density $l(u)$, $u\geq 0$, is a nondecreasing right continuous function such that $l(0)=0$, $l(x)>0$ for $x\neq 0$, and $l(x)\to\infty$ as $x\to\infty$.

Definition 2.2. Let $f(x)$, $x\in\mathbb{R}$, be a real function. A function $f^*(x)$, $x\in\mathbb{R}$, is called the Young–Fenchel transform of the function f or the conjugate function to f if

$$f^*(x)=\sup_{y\in\mathbb{R}}(xy-f(y)).$$

2000 *Mathematics Subject Classification.* Primary 68U20; Secondary 60G10.

Supported in part by NATO grant PST.CLG.980408.

Lemma 2.2 ([4])**.** *If $f(x)$, $x \in \mathbb{R}$, is an N-function, then $f^*(x)$, $x \in \mathbb{R}$, is also an N-function. Moreover*

$$f^*(x) = xy_0 - f(y_0)$$

for all $x > 0$, where $y_0 = l^{(-1)}(x)$ and $l^{(-1)}(x)$ is the generalized inverse function for the density $l(x)$.

Now we give the definition of the space $\mathrm{Sub}_\varphi(\Omega)$ and stochastic processes of the space $\mathrm{Sub}_\varphi(\Omega)$.

Let $\varphi(x)$ be an N-function and let there exist constants x_0 and $c > 0$ such that $\varphi(x) = c \cdot x^2$ for $|x| < x_0$.

By (Ω, B, P) we denote the standard probability space.

Definition 2.3. The space of centered random variables ξ such that for all $\lambda \in \mathbb{R}$ there exists a constant $r \geq 0$ for which

$$\mathsf{E}\exp\{\lambda\xi\} \leq \exp\{\varphi(\lambda r)\}$$

is called the space $\mathrm{Sub}_\varphi(\Omega)$ of random variables.

Theorem 2.1 ([4])**.** *The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm*

$$\tau_\varphi(\xi) = \sup_{\lambda > 0} \frac{\varphi^{(-1)}(\ln \mathsf{E}\exp\{\lambda\xi\})}{\lambda} \tag{1}$$

where $\varphi^{(-1)}$ is the generalized inverse function to φ. Then the norm $\tau_\varphi(\xi)$ is such that

$$\mathsf{E}\exp\{\lambda\xi\} \leq \exp\{\varphi(\lambda\tau_\varphi(\xi))\}$$

for all $\lambda \in \mathbb{R}$. Moreover there exists a constant $c > 0$ such that

$$\left(\mathsf{E}\left(\xi^2\right)\right)^{1/2} < c\tau_\varphi(\xi).$$

Theorem 2.2 ([4])**.** *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent random variables of the space $\mathrm{Sub}_\varphi(\Omega)$. If $\varphi(\sqrt{x})$ is a convex function, then*

$$\tau_\varphi^2\left(\sum_{i=1}^n \xi_i\right) \leq \sum_{i=1}^n \tau_\varphi^2(\xi_i).$$

If $\varphi(x) = x^2/2$, then random variables ξ of the space $\mathrm{Sub}_\varphi(\Omega)$ are called sub-Gaussian. We denote the space of sub-Gaussian random variables by

$$\mathrm{Sub}_{x^2/2}(\Omega) = \mathrm{Sub}(\Omega).$$

If $\mathsf{E}\xi^2 = \tau^2(\xi)$, then the random variable ξ is called *strongly sub-Gaussian.* The family of all strongly sub-Gaussian random variables defined on the standard probability space is denoted by $\mathrm{SSub}(\Omega)$.

Definition 2.4. We say that a stochastic process $\xi(t)$, $t \in [0,T]$, belongs to the space $\mathrm{Sub}_\varphi(\Omega)$ if

$$\xi(t) \in \mathrm{Sub}_\varphi(\Omega)$$

for any fixed $t \in [0,T]$ and $\sup_{t\in[0,T]} \tau_\varphi(\xi(t)) < \infty$.

In the space $\mathrm{Sub}_\varphi(\Omega)$ equipped with the norm τ_φ, consider a stochastic process

$$\xi = \{\xi(t), t \in [0,1]\},$$

where $\varphi(x)$ is an N-function such that $\varphi(\sqrt{x})$ is convex. The following result is proved in [5].

Lemma 2.3. *Let $\xi = \{\xi(t), t \in [0,1]\}$ be a separable stochastic process of the space* $\mathrm{Sub}_\varphi(\Omega)$. *Assume that there exists an increasing continuous function $\sigma(h)$, $h \geq 0$, such that $\sigma(h) \to 0$ as $h \to 0$ and*

$$\sup_{|t-s|<h} (\tau_\varphi(\xi(t) - \xi(s))) < \sigma(h).$$

Let $\gamma_0 = \sup_{t\in[0,1]} \tau_\varphi(\xi(t))$, $\beta \leq \sigma(\frac{1}{2})$, and let $\{r(u), u \geq 1\}$ be a nondecreasing continuous function such that $r(u) \geq 0$ for $u \geq 1$, $r(1) = 0$, and the function $r(\exp\{u\})$ is convex. Assume that

$$\int_0^\beta \theta_\varphi(u)\, du < \infty,$$

where

$$\theta_\varphi(u) = \frac{r\left(N\left(\sigma^{(-1)}(u)\right)\right)}{\varphi^{(-1)}\left(\ln N\left(\sigma^{(-1)}(u)\right)\right)},$$

$N(u)$ is the metric massiveness, that is, the minimal number of closed balls of radius u that cover the interval $[0,1]$. Then

$$\begin{aligned}\mathsf{E}\exp\left\{\lambda \sup_{t\in[0,1]} |\xi(t)|\right\} &\leq 2\exp\left\{\varphi\left(\frac{\lambda\gamma_0}{1-p}\right)(1-p) + \varphi\left(\frac{\lambda\beta}{1-p}\right)p\right\} \\ &\quad\times\left(r^{(-1)}\left(\lambda\gamma_0\theta_\varphi(p\beta) + \frac{\lambda}{(1-p)p}\int_0^{\beta p^2}\theta_\varphi(u)\,du\right)\right)^2\end{aligned} \tag{2}$$

for all $\lambda \in \mathbb{R}$ and $p \in (0,1)$.

We prove the following result.

Theorem 2.3. *Let the assumptions of Lemma 2.3 hold and let $\beta \leq \min\{\gamma_0, \sigma(\frac{1}{2})\}$. If $x > 0$ is such that*

$$\gamma_0 < x,$$

$$\beta\gamma_0 < x\sigma\left(\frac{1}{2(\exp\{\varphi(1)\} - 1)}\right),$$

then

$$\begin{aligned}&\mathsf{P}\left\{\sup_{t\in[0,1]} |\xi(t)| > x\right\} \\ &\quad\leq 2\exp\left\{-\varphi^*\left(\frac{x}{\gamma_0} - 1\right)\right\} \\ &\quad\times\left(r^{(-1)}\left(\frac{x\cdot l^{(-1)}(x/\gamma_0 - 1)}{\beta\gamma_0}\cdot\int_0^{\beta\gamma_0/x} r\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right)du\right)\right)^2.\end{aligned}$$

Proof. We apply Lemma 2.3.

The assumption of Theorem 2.3 implies that $\beta \leq \min\{\gamma_0, \sigma(\frac{1}{2})\} \leq \gamma_0$. Thus

$$\varphi\left(\frac{\lambda\gamma_0}{1-p}\right)(1-p) + \varphi\left(\frac{\lambda\beta}{1-p}\right)p \leq \varphi\left(\frac{\lambda\gamma_0}{1-p}\right),$$

whence we obtain by the Chebyshev inequality and (2) that

$$\mathsf{P}\left\{\sup_{t\in[0,1]} |\xi(t)| > x\right\} \leq \exp\left\{-\lambda x + \varphi\left(\frac{\lambda\gamma_0}{1-p}\right)\right\}\cdot 2I_r, \tag{3}$$

where

$$I_r = \left(r^{(-1)} \left(\lambda\gamma_0\theta_\varphi(p\beta) + \frac{\lambda}{(1-p)p} \int_0^{\beta p^2} \theta_\varphi(u)\,du \right) \right)^2 .$$

It follows from Lemma 2.2 that $xy = \varphi(x) + \varphi^*(y)$ for $x = l^{(-1)}(y)$, where $\varphi^*(y)$ is the Young–Fenchel transform and $l^{(-1)}(y)$ is the inverse function to the density $l(x)$. Using this result, we obtain that

$$\lambda x - \varphi\left(\frac{\lambda\gamma_0}{1-p}\right) = \frac{\lambda\gamma_0}{1-p} \cdot \frac{x(1-p)}{\gamma_0} - \varphi\left(\frac{\lambda\gamma_0}{1-p}\right) = \varphi^*\left(\frac{x(1-p)}{\gamma_0}\right) \tag{4}$$

for $\lambda\gamma_0/(1-p) = l^{(-1)}(x(1-p)/\gamma_0)$. Thus

$$\lambda = \frac{1-p}{\gamma_0} l^{(-1)}\left(\frac{x(1-p)}{\gamma_0}\right). \tag{5}$$

Note that the function

$$\theta_\varphi(u) = \frac{r\left(N\left(\sigma^{(-1)}(u)\right)\right)}{\varphi^{(-1)}\left(\ln N\left(\sigma^{(-1)}(u)\right)\right)}, \qquad u \geq 0,$$

decreases in u. Hence

$$\theta_\varphi(p\beta) \leq \frac{1}{\beta p(1-p)} \int_{\beta p^2}^{\beta p} \theta_\varphi(u)\,du$$

and

$$\lambda\gamma_0\theta_\varphi(p\beta) \leq \frac{\lambda\gamma_0}{\beta p(1-p)} \int_{\beta p^2}^{\beta p} \theta_\varphi(u)\,du.$$

Since $\gamma_0/\beta \geq 1$, we have

$$\lambda\gamma_0\theta_\varphi(p\beta) + \frac{\lambda}{p(1-p)} \int_0^{\beta p^2} \theta_\varphi(u)\,du \leq \frac{\gamma_0}{\beta} \frac{\lambda}{p(1-p)} \int_0^{\beta p} \theta_\varphi(u)\,du. \tag{6}$$

Now we apply (6) for I_r with λ defined in (5) and get

$$(I_r)^{1/2} \leq r^{(-1)}\left(\frac{l^{(-1)}\left(x(1-p)/\gamma_0\right)}{\beta p} \int_0^{\beta p} \theta_\varphi(u)\,du\right). \tag{7}$$

Note that $N(u) \leq 1/(2u) + 1$, whence

$$\theta_\varphi(u) = \frac{r\left(N\left(\sigma^{(-1)}(u)\right)\right)}{\varphi^{(-1)}\left(\ln N\left(\sigma^{(-1)}(u)\right)\right)} \leq \frac{r\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right)}{\varphi^{(-1)}\left(\ln\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right)\right)}.$$

Since the function

$$\varphi^{(-1)}\left(\ln\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right)\right)$$

decreases in u, $u \in (0, \beta p)$, and

$$\beta\gamma_0 < x\sigma\left(\frac{1}{2(\exp\{\varphi(1)\} - 1)}\right),$$

we obtain, by the assumptions of the theorem, that

$$\varphi^{(-1)}\left(\ln\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right)\right) > 1$$

for $u \in (0, \beta p)$. Thus

$$\int_0^{\beta p} \theta_\varphi(u)\,du \le \int_0^{\beta p} \frac{r\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right)}{\varphi^{(-1)}\left(\ln\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right)\right)} \le \int_0^{\beta p} r\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right) du.$$

The latter relation together with (3), (4), and (7) implies that

(8)

$$\mathsf{P}\left\{\sup_{t\in[0,1]} |\xi(t)| > x\right\} \le 2\exp\left\{-\varphi^*\left(\frac{x(1-p)}{\gamma_0}\right)\right\} \times \left(r^{(-1)}\left(\frac{l^{(-1)}\big(x(1-p)/\gamma_0\big)}{\beta p} \int_0^{\beta p} r\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right) du\right)\right)^2.$$

Since $p \in (0,1)$ is arbitrary, one can substitute $p = \gamma_0/x < 1$ in (8). This will complete the proof of the theorem. □

Example 2.1. If $\varphi(x) = x^2/2$, that is, if we deal with the space of sub-Gaussian random variables, then the Young–Fenchel transform and the density for φ are given by $\varphi^*(y) = y^2/2$ and $l(x) = x$, respectively.

Theorem 2.3 implies the following result.

Theorem 2.4. *Let $\xi = \{\xi(t), t \in [0,1]\}$ be a separable stochastic process of the space* $\mathrm{Sub}(\Omega)$ *of sub-Gaussian random variables; let a function $\sigma(h)$ satisfy the assumptions of Lemma 2.3, $\gamma_0 = \sup_{t\in[0,1]} \tau(\xi(t))$, $\beta = \min\{\sigma(\frac{1}{2}), \gamma_0\}$; and let $\{r(u), u \ge 1\}$ be a nondecreasing continuous function satisfying all the assumptions of Lemma 2.3. If $x > 0$ is such that*

$$\gamma_0 < x,$$

$$\beta\gamma_0 < x\sigma\left(\frac{1}{2(\exp\{1/2\} - 1)}\right),$$

then

$$\mathsf{P}\left\{\sup_{t\in[0,1]} |\xi(t)| > x\right\} \le 2\exp\left\{-\frac{1}{2}\left(\frac{x}{\gamma_0} - 1\right)^2\right\} \times \left(r^{(-1)}\left(\frac{x\cdot(x-\gamma_0)}{\beta\gamma_0^2} \cdot \int_0^{\beta\gamma_0/x} r\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right) du\right)\right)^2.$$

3. Stationary processes with discrete spectrum

Let $\{\xi(t), t \in [0,1]\}$ be a stationary stochastic process such that $\mathsf{E}\xi(t) = 0$, $t \in [0,1]$, and $\mathsf{E}\xi(t+h)\xi(t) = B(h)$.

Definition 3.1. A stationary process $\xi(t)$ is called a process with discrete spectrum if its correlation function $B(h)$ can be represented in the form

$$B(h) = \sum_{k=0}^{\infty} b_k^2 \cos\lambda_k h,$$

where $b_k^2 > 0$, $\sum_{k=0}^{\infty} b_k^2 < \infty$, and λ_k are some numbers such that $0 \le \lambda_k \le \lambda_{k+1}$ and $\lambda_k \to \infty$ as $k \to \infty$.

The latter definition and Karhunen theorem imply that a stationary stochastic process $\xi = \{\xi(t), t \in [0,1]\}$ with discrete spectrum can be represented in the form of the series

$$\xi(t) = \sum_{k=0}^{\infty} (\xi_k b_k \cos \lambda_k t + \eta_k b_k \sin \lambda_k t), \tag{9}$$

where ξ_k and η_k are independent random variables such that

$$\mathsf{E}\xi_k = \mathsf{E}\eta_k = \mathsf{E}\xi_k\eta_l = 0, \qquad \mathsf{E}\xi_k\xi_l = \mathsf{E}\eta_k\eta_l = \delta_k^l, \qquad k = 0,1,2\dots, \ l = 0,1,2\dots,$$

and series (9) converges in the mean square sense.

If the random variables ξ_k and η_k, $k = 0,1,2\dots$, belong to the space $\mathrm{Sub}_\varphi(\Omega)$, then $\xi(t) \in \mathrm{Sub}_\varphi(\Omega)$, too. Similarly, if ξ_k and η_k, $k = 0,1,2\dots$, are strongly sub-Gaussian, then $\xi(t) \in \mathrm{SSub}(\Omega)$.

4. Models of stochastic processes of the space $\mathrm{Sub}_\varphi(\Omega)$

We construct a model $\tilde{\xi}_N(t)$ of a process $\xi(t)$ such that $\tilde{\xi}_N(t)$ approximates $\xi(t)$ with a given accuracy and reliability in the Banach space $C[0,1]$.

Consider a stationary process ξ with a discrete spectrum and assume that the process belongs to the space $\mathrm{Sub}_\varphi(\Omega)$. In what follows we assume that $\tau_\varphi(\xi_k) = \tau_\varphi(\eta_k) = d > 0$. Let the numbers b_k be unknown, but we know their approximate values $\tilde{b}_k$. We also know that

$$|b_k - \tilde{b}_k| \le \gamma_k \tag{10}$$

for some known constants γ_k.

Definition 4.1. A process $\tilde{\xi}_N(t)$ is called a model of $\xi(t)$ if

$$\tilde{\xi}_N(t) = \sum_{k=0}^{N} \tilde{b}_k (\xi_k \cos \lambda_k t + \eta_k \sin \lambda_k t),$$

where the numbers $\tilde{b}_k$ satisfy inequality (10) and ξ_k and η_k are independent random variables of the space $\mathrm{Sub}_\varphi(\Omega)$ such that

$$\mathsf{E}\xi_k = \mathsf{E}\eta_k = \mathsf{E}\xi_k\eta_l = 0, \qquad \mathsf{E}\xi_k\xi_l = \mathsf{E}\eta_k\eta_l = \delta_k^l, \qquad k = 0,\dots,N, \ l = 0,\dots,N.$$

Definition 4.2. We say that a model $\tilde{\xi}_N(t)$ approximates a process $\xi(t)$ with a given reliability $1-\nu$, $\nu \in (0,1)$, and accuracy $\delta > 0$ in the space $C([0,1])$ if

$$\mathsf{P}\left\{ \sup_{t \in [0,1]} \left| \xi(t) - \tilde{\xi}_N(t) \right| > \delta \right\} \le \nu.$$

It is easy to see that the error of the model is

$$\begin{aligned}
\Delta(t,N) &= \xi(t) - \tilde{\xi}_N(t) \\
&= \sum_{k=0}^{N} (b_k - \tilde{b}_k)(\xi_k \cos \lambda_k t + \eta_k \sin \lambda_k t) + \sum_{k=N+1}^{\infty} b_k (\xi_k \cos \lambda_k t + \eta_k \sin \lambda_k t) \\
&:= \Delta_1(t,N) + \Delta_2(t,N).
\end{aligned}$$

Next we find upper estimates for $\tau_\varphi(\Delta(t,N))$, $t \in [0,1]$, and $\tau_\varphi(\Delta(t,N) - \Delta(s,N))$, $t,s \in [0,1]$.

Theorem 2.2 implies that $\tau_\varphi^2(\Delta(t,N)) \le \tau_\varphi^2(\Delta_1(t,N)) + \tau_\varphi^2(\Delta_2(t,N))$ and

$$\begin{aligned} \tau_\varphi^2(\Delta_1(t,N)) &\le \sum_{k=0}^{N} \left(b_k - \tilde{b}_k\right)^2 \left(\tau_\varphi^2(\xi_k)\cos^2\lambda_k t + \tau_\varphi^2(\eta_k)\sin^2\lambda_k t\right) \\ &\le d^2 \sum_{k=0}^{N} \gamma_k^2 := A_N. \end{aligned} \tag{11}$$

Recall that we consider the case where $\tau_\varphi(\xi_k) = \tau_\varphi(\eta_k) = d > 0$. Similarly to the case of Δ_1 we estimate

$$\tau_\varphi^2(\Delta_2(t,N)) \le d^2 \sum_{k=N+1}^{\infty} b_k^2 := B_N. \tag{12}$$

To estimate $\tau_\varphi(\Delta(t,N) - \Delta(s,N))$, we use the following auxiliary result proved in [3].

Lemma 4.1. *Let $\psi(u)$, $u \ge 0$, be a continuous increasing function such that $\psi(0) = 0$. Assume that the function $u/\psi(u)$ is nondecreasing for $u > u_0$ where $u_0 \ge 0$ is some constant. Then*

$$\left|\sin\frac{u}{v}\right| \le \frac{\psi(u+u_0)}{\psi(v+u_0)}$$

for all $u \ge 0$ and $v > 0$.

Example 4.1. The function $\psi(u) = u^\alpha$, $\alpha \in (0,1]$, satisfies the assumptions of Lemma 4.1 for $u_0 = 0$. Thus

$$\left|\sin\frac{u}{v}\right| \le \frac{u^\alpha}{v^\alpha} \quad \text{for } u, v > 0.$$

Example 4.2. Another example for Lemma 4.1 is the function $\psi(u) = \ln^\alpha(u+1)$, $\alpha > 0$, and $u_0 = e^\alpha - 1$. We have in this case

$$\left|\sin\frac{u}{v}\right| \le \left(\frac{\ln(e^\alpha + u)}{\ln(e^\alpha + v)}\right)^\alpha.$$

Using Lemma 4.1, we get the estimate

$$\begin{aligned} &\tau_\varphi^2(\Delta_1(t,N) - \Delta_1(s,N)) \\ &\quad\le d^2 \sum_{k=0}^{N} (b_k - \tilde{b}_k)^2 \left[(\cos\lambda_k t - \cos\lambda_k s)^2 + (\sin\lambda_k t - \sin\lambda_k s)^2\right] \\ &\quad\le 2d^2 \sum_{k=0}^{N} \gamma_k^2 (1 - \cos\lambda_k(t-s)) = 4d^2 \sum_{k=0}^{N} \gamma_k^2 \left(\sin^2\frac{\lambda_k(t-s)}{2}\right) \\ &\quad\le 2^2 d^2 \sum_{k=0}^{N} \gamma_k^2 \frac{\psi^2\left(\lambda_k/2 + u_0\right)}{\psi^2\left(|t-s|^{-1} + u_0\right)} \\ &\quad:= C_N \frac{1}{\psi^2\left(|t-s|^{-1} + u_0\right)}. \end{aligned} \tag{13}$$

Assume that $\sum_{k=1}^{\infty} b_k^2 \psi^2(\lambda_k/2 + u_0) < \infty$. Following the same idea, one can prove that

$$\begin{aligned} \tau_\varphi^2\bigl(\Delta_2(t,N) - \Delta_2(s,N)\bigr) &\le 2^2 d^2 \sum_{k=N+1}^{\infty} b_k^2 \frac{\psi^2\left(\lambda_k/2 + u_0\right)}{\psi^2\left(|t-s|^{-1} + u_0\right)} \\ &:= D_N \frac{1}{\psi^2\left(|t-s|^{-1} + u_0\right)}. \end{aligned} \tag{14}$$

Therefore relations (11)–(14) and Theorem 2.2 imply that

$$\tau_\varphi^2(\Delta(t,N)) \le A_N + B_N, \tag{15}$$

$$\tau_\varphi^2\big(\Delta(t,N) - \Delta(s,N)\big) \le (C_N + D_N)\frac{1}{\psi^2\left(|t-s|^{-1} + u_0\right)}. \tag{16}$$

In what follows we assume that the function $\psi(u)$ in Lemma 4.1 is such that $\psi(u) \to \infty$ as $u \to \infty$. Then it follows from (15) and (16) that

$$\gamma_0 = \sqrt{A_N + B_N} := \gamma_0(N), \qquad \sigma(h) = \frac{\sqrt{C_N + D_N}}{\psi\left(1/h + u_0\right)} := \frac{L(N)}{\psi\left(1/h + u_0\right)} = \sigma_N(h), \tag{17}$$

where γ_0 and $\sigma(h)$ satisfy all the assumptions of Lemma 2.3.

Theorem 4.1. *Let $\sum_{k=1}^{\infty} b_k^2 \psi^2(\lambda_k/2 + u_0) < \infty$, let the function $\psi(u)$ satisfy the assumptions of Lemma 4.1, and moreover let $\psi(u) \to \infty$ as $u \to \infty$. Let there exist a nondecreasing continuous function $\{r(u), u \ge 1\}$ such that $r(u) \ge 0$ for $u \ge 1$, $r(1) = 0$, and $r(\exp\{u\})$ is convex. We also assume that $\int_0^\beta r(\theta(u))\,du < \infty$, where $\theta(u) = \psi^{(-1)}(L(N)/u)$.*

A stochastic process $\tilde{\xi}_N(t)$ is a model approximating a separable process $\xi(t)$ of the space $\operatorname{Sub}_\varphi(\Omega)$ with a given reliability $1-\nu$, $\nu \in (0,1)$, and accuracy $\delta > 0$ in the space $C([0,1])$ if N is such that

$$\gamma_0(N) < \delta,$$

$$\beta_N \gamma_0(N) < \frac{\delta L(N)}{\psi\left(2(\exp\{\varphi(1)\} - 1) + u_0\right)}, \tag{18}$$

$$2\exp\left\{-\varphi^*\left(\frac{\delta}{\gamma_0(N)} - 1\right)\right\} \times \left(r^{(-1)}\left(\frac{\delta \cdot l^{(-1)}\left(\delta/\gamma_0(N) - 1\right)}{\beta_N \gamma_0(N)} \int_0^{\beta_N \gamma_0(N)/\delta} r(\theta(u))\,du\right)\right)^2 < \nu, \tag{19}$$

where $\beta_N = \min\{\gamma_0(N), L(N)/\psi(2+u_0)\}$ and constants $\gamma_0(N)$ and $L(N)$ are defined in (17).

Proof. Since $\xi(t), \tilde{\xi}_N(t) \in \operatorname{Sub}_\varphi(\Omega)$, their difference $\xi(t) - \tilde{\xi}_N(t)$ is also a process of the space $\operatorname{Sub}_\varphi(\Omega)$. We proved above that

$$\gamma_0 = \gamma_0(N) = \sqrt{A_N + B_N}, \qquad \sigma(h) = \sigma_N(h) = L(N)\frac{1}{\psi\left(1/h + u_0\right)},$$

whence

$$\sigma^{(-1)}(u) = \left(\psi^{(-1)}(L(N)/u) - u_0\right)^{-1}.$$

Thus

$$r\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right) = r\left(\frac{1}{2}\left(\psi^{(-1)}\left(\frac{L(N)}{u}\right) - u_0\right) + 1\right) \le r\left(\frac{1}{2}\psi^{(-1)}\left(\frac{L(N)}{u}\right) + 1\right).$$

Now we prove that

$$\frac{1}{2}\psi^{(-1)}(L(N)/u) > 1$$

for all u of the interval $(0, \beta_N \gamma_0(N)/\delta)$. Since the function $\psi^{(-1)}(L(N)/u)$ decreases in u and $\gamma_0(N)/\delta < 1$ by the assumptions of the theorem, it remains to prove that

$$\frac{1}{2}\psi^{(-1)}(L(N)/\beta_N) > 1.$$

The latter inequality is equivalent to the relation

$$L(N)/\psi(2) > \beta_N$$

that is obviously true, since

$$\beta_N = \min\left\{\gamma_0(N), \sigma_N\left(\frac{1}{2}\right)\right\} \le \sigma_N\left(\frac{1}{2}\right) \le L(N)/\psi(2).$$

Hence

$$r\left(\frac{1}{2\sigma^{(-1)}(u)} + 1\right) \le r\left(\psi^{(-1)}\left(\frac{L(N)}{u}\right)\right).$$

It is easy to see that

$$\sigma\left(\frac{1}{2(\exp\{\varphi(1)\} - 1)}\right) = \frac{L(N)}{\psi\left(2(\exp\{\varphi(1)\} - 1) + u_0\right)}.$$

Substituting the upper bound obtained above into the inequality of Theorem 2.3, we complete the proof of Theorem 4.1. □

Consider the case of the function $\psi(u) = u^\alpha$, $\alpha \in (0,1]$, and $u_0 = 0$. In this case we obtain the following result.

Theorem 4.2. *Let $\sum_{k=1}^\infty b_k^2 \lambda_k^{2\alpha} < \infty$, $\alpha \in (0,1]$. A stochastic process $\tilde{\xi}_N(t)$ is a model approximating a separable process $\xi(t) \in \mathrm{Sub}_\varphi(\Omega)$ with a given reliability $1-\nu$, $\nu \in (0,1)$, and accuracy $\delta > 0$ in the space $C([0,1])$ if N is such that*

$$\gamma_0(N) < \delta,$$

$$\beta_N \gamma_0(N) < \frac{\delta L(N)}{2^\alpha(\exp\{\varphi(1)\} - 1)^\alpha},$$

$$2\exp\left\{-\varphi^*\left(\frac{\delta}{\gamma_0(N)} - 1\right)\right\}\left(\left(\frac{\alpha}{\alpha - b}\left(\frac{\delta L(N)}{\beta_N \gamma_0(N)}\right)^{b/\alpha} - 1\right) l^{(-1)}\left(\frac{\delta}{\gamma_0(N)} - 1\right) + 1\right)^{2/b} < \nu,$$

where $\gamma_0(N)$ and $L(N)$ are defined in (17), *$\beta_N = \min\{\gamma_0(N), L(N)/2^\alpha\}$, and $0 < b < \alpha$.*

Proof. Recall that we deal with the case of $\psi(u) = u^\alpha$, $\alpha \in (0,1]$, and $u_0 = 0$. Then the inequalities of Theorem 4.1 imply the first two inequalities of Theorem 4.2. It remains to prove that the third inequality of Theorem 4.2 holds. We apply relation (19) with $r(u) = u^b - 1$, $b \in (0, \alpha)$. Note that this function $r(u)$ satisfies all the assumptions of Theorem 4.1 and that $\psi^{(-1)}(u) = u^{1/\alpha}$, $\alpha \in (0,1]$. We have

$$\begin{aligned}\int_0^{\beta_N\gamma_0(N)/\delta} r(\theta(u))\,du &= \int_0^{\beta_N\gamma_0(N)/\delta} \left(\left(\psi^{(-1)}\left(\frac{L(N)}{u}\right)\right)^b - 1\right) du \\ &= \int_0^{\beta_N\gamma_0(N)/\delta} \left(\left(\frac{L(N)}{u}\right)^{b/\alpha} - 1\right) du \\ &= \frac{\alpha (L(N))^{b/\alpha}}{\alpha - b}\left(\frac{\beta_N\gamma_0(N)}{\delta}\right)^{1-b/\alpha} - \frac{\beta_N\gamma_0(N)}{\delta}.\end{aligned}$$

Since $r^{(-1)}(u)=(u+1)^{1/b}$ for $b\in(0,\alpha)$, we conclude that

$$r^{(-1)}\left(\frac{\delta\cdot l^{(-1)}\left(\delta/\gamma_0(N)-1\right)}{\beta_N\gamma_0(N)}\int_0^{\beta_N\gamma_0(N)/\delta} r(\theta(u))\,du\right)$$
$$=\left(\left(\frac{\alpha}{\alpha-b}\left(\frac{\delta L(N)}{\beta_N\gamma_0(N)}\right)^{b/\alpha}-1\right)l^{(1)}\left(\frac{\delta}{\gamma_0(N)}-1\right)+1\right)^{1/b}.$$

Substituting the right-hand side of the latter equality into (19), we complete the proof of the theorem. □

5. Models of sub-Gaussian stochastic processes

We consider sub-Gaussian stochastic processes in this section; that is, we deal with the function $\varphi(u)=u^2/2$. Then the Young–Fenchel transform and the density for φ are $\varphi^*(x)=x^2/2$ and $l(x)=x$, respectively. The following result follows from Theorem 4.1

Theorem 5.1. *Let the assumptions of Theorem 4.1 hold. A model $\tilde{\xi}_N(t)$ approximates a process $\xi(t)$ with a given reliability $1-\nu$, $\nu\in(0,1)$, and accuracy $\delta>0$ in the space $C([0,1])$ if N is such that*

$$\gamma_0(N)<\delta, \tag{20}$$
$$2\exp\left\{-\frac{1}{2}\left(\frac{\delta}{\gamma_0(N)}-1\right)^2\right\}\left(r^{(-1)}\left(\frac{\delta(\delta-\gamma_0(N))}{\beta_N\gamma_0^2(N)}\int_0^{\beta_N\gamma_0(N)/\delta} r(\theta(u))\,du\right)\right)^2<\nu.$$

Proof. Theorem 5.1 follows from Theorem 4.1. We prove only that relation (20) implies inequality (18) in the case of sub-Gaussian processes; that is, we prove that

$$\gamma_0(N)<\delta \quad\Longrightarrow\quad \beta_N\gamma_0(N)<\frac{\delta L(N)}{\psi\left(2(\exp\{1/2\}-1)+u_0\right)}.$$

Indeed,

$$\frac{L(N)}{\psi\left(2(\exp\{1/2\}-1)+u_0\right)}=\sigma\left(\frac{1}{2(\exp\{1/2\}-1)}\right)\approx\sigma(0.7).$$

Note that $\beta_N=\min\{\gamma_0(N),\sigma(1/2)\}\le\sigma(1/2)$. The function $\sigma(u)$ is increasing by definition. Thus

$$\beta_N\le\delta L(N)/\psi\big(2(\exp\{1/2\}-1)+u_0\big).$$

The theorem is proved. □

If $\psi(u)=u^\alpha$, $\alpha\in(0,1]$, and $u_0=0$, then Theorems 4.1 and 5.1 yield the following result.

Theorem 5.2. *Let*

$$\sum_{k=1}^{\infty}b_k^2\lambda_k^{2\alpha}<\infty,\qquad \alpha\in(0,1].$$

A stochastic process $\tilde{\xi}_N(t)$ is a model approximating a separable sub-Gaussian process $\xi(t)$ with a given reliability $1-\nu$, $\nu\in(0,1)$, and accuracy $\delta>0$ in the space $C([0,1])$ if N is *such that*

$$\gamma_0(N) < \delta, \tag{21}$$

$$2\exp\left\{-\frac{1}{2}\left(\frac{\delta}{\gamma_0(N)}-1\right)^2\right\} \times \left(\frac{\alpha}{\alpha-b}\left(\left(\frac{\delta L(N)}{\beta_N\gamma_0(N)}\right)^{b/\alpha}-1\right)\left(\frac{\delta}{\gamma_0(N)}-1\right)+1\right)^{2/b} < \nu \tag{22}$$

for some $b\in(0,\alpha)$, $\alpha\in(0,1]$, *where* $\beta=\min\{\gamma_0(N),L(N)/2^\alpha\}$ *and* $\gamma_0(N)$ *and* $L(N)$ *are defined in* (17).

Example 5.1. Consider the case of $\psi(u)=u^\alpha$, $\alpha\in(0,1]$, and $u_0=0$. Let the process $\xi(t)$ in (9) be sub-Gaussian. This means that the random variables ξ_k and η_k, $k\ge 0$, are strongly sub-Gaussian random variables; that is,

$$d^2=\tau^2(\xi_k)=\mathsf{E}\xi_k^2,\qquad d^2=\tau^2(\eta_k)=\mathsf{E}\eta_k^2.$$

Thus $d=1$.

Let $b_k=k^{-a}$, $a>1$, $\lambda_k=\sqrt{k}$, and let the errors of approximation of the numbers b_k be the constants $\gamma_k=\gamma$ for all $k\ge 1$. Then

$$\sum_{k=0}^\infty b_k^2\lambda_k^2=\sum_{k=0}^\infty\frac{1}{k^{2a-1}}<\infty$$

and the numbers A_N, B_N, and $\gamma_0(N)=\sqrt{A_N+B_N}$ are such that

$$A_N=d^2\sum_{k=0}^N\gamma_k^2=(N+1)\gamma,$$

$$B_N=d^2\sum_{k=N+1}^\infty\frac{1}{b_k^2}=\sum_{k=N+1}^\infty\frac{1}{k^{2a}}=\sum_{k=N+1}^\infty\int_{k-1}^k\frac{1}{k^{2a}}\,dx\le\int_N^\infty\frac{1}{x^{2a}}\,dx=\frac{1}{(2a-1)N^{2a-1}},$$

$$\gamma_0^2(N)\le(N+1)\gamma^2+\frac{1}{(2a-1)N^{2a-1}}.$$

Now we find the point of minimum and the minimum value of the right-hand side of the latter inequality over all real numbers $N>0$.

It is obvious that

$$N_{\min}=\left(\frac{1}{\gamma}\right)^{1/(a-1)}$$

is the point of minimum in this case. Then

$$\min\gamma_0(N)=\left(\gamma^{2-1/a}+\gamma^2+\frac{1}{2a-1}\gamma^{2-1/a}\right)^{1/2}\le\gamma^{1-1/(2a)}\sqrt{\frac{4a-1}{2a-1}}$$
$$:=\gamma_0(\gamma),\qquad a>1.$$

To construct the model, we take $N=[N_{\min}]+1$, where $[c]$ stands for the integer part of a number c.

Theorem 5.2 implies that $\gamma_0(N)$ satisfies condition (21). All possible $\gamma<1$ in this case are determined by the condition

$$\gamma^{1-1/(2a)}\sqrt{\frac{4a-1}{2a-1}}<\delta\Longrightarrow\gamma<\left(\delta\sqrt{\frac{2a-1}{4a-1}}\right)^{2a/(2a-1)}.$$

Thus

$$\gamma \in \left(0, \min\left\{1, \left(\delta\sqrt{\frac{2a-1}{4a-1}}\right)^{2a/(2a-1)}\right\}\right). \tag{23}$$

Remark. When we apply Theorem 4.1, the numbers δ, ν, a, α, and $b \in (0,\alpha)$ as well as γ in (23) are known. Thus the problem of constructing a model of a process is reduced to the problem of finding the minimal number N that satisfies conditions (21)–(22).

Now we evaluate $L(N) = \sqrt{C_N + D_N}$ for $b_k = k^{-a}$, $a > 1$, $\lambda_k = \sqrt{k}$, and $\gamma_k = \gamma$. First,

$$C_N = 2^{2-2\alpha} d^2 \sum_{k=0}^{N} \gamma_k^2 \lambda_k^{2\alpha} = 2^{2-2\alpha}\gamma^2 \sum_{k=0}^{N} \int_k^{k+1} k^\alpha\, dx \le \frac{2^{2-2\alpha}\gamma^2(N+1)^{\alpha+1}}{\alpha+1},$$

$$D_N = 2^{2-2\alpha} d^2 \sum_{k=N+1}^{\infty} b_k^2 \lambda_k^{2\alpha} = 2^{2-2\alpha} \sum_{k=N+1}^{\infty} \frac{1}{k^{2a-\alpha}} \le \frac{2^{2-2\alpha}}{(2a-\alpha-1)N^{2a-\alpha-1}}.$$

Then

$$L^2(N) \le 2^{2-2\alpha}\left(\frac{\gamma^2(N+1)^{\alpha+1}}{\alpha+1} + \frac{1}{(2a-\alpha-1)N^{2a-\alpha-1}}\right).$$

Substituting $N_{\min}$ into $L(N)$, we get

$$L(N_{\min}) = 2^{1-\alpha}\gamma^{1-(\alpha-1)/(2a-2)}\left(\frac{\left(1+\gamma^{1/(a-1)}\right)^{\alpha+1}}{\gamma^2(\alpha+1)} + \frac{1}{(2a-\alpha-1)}\right)^{1/2} := L(\gamma).$$

Thus Theorem 5.2 can be rewritten as follows.

Theorem 5.3. *A stochastic process*

$$\tilde{\xi}_N(t) = \sum_{k=0}^{N} \frac{1}{\tilde{b}_k}\left(\xi_k \cos\sqrt{k}t + \eta_k \sin\sqrt{k}t\right), \qquad a > 1,\ N = \left[\left(\frac{1}{\gamma}\right)^{1/a}\right] + 1,$$

is a model approximating a separable process

$$\xi(t) = \sum_{k=0}^{\infty} \frac{1}{k^a}\left(\xi_k \cos\sqrt{k}t + \eta_k \sin\sqrt{k}t\right), \qquad \left|\frac{1}{k^a} - \tilde{b}_k\right| < \gamma,\ 0 \le k \le N,$$

with a given reliability $1-\nu$, $\nu \in (0,1)$, *and accuracy* $\delta > 0$ *in the space* $C([0,1])$ *if for* $b \in (0,\alpha)$, $\alpha \in (0,1]$, *there exists a number* γ *in the interval*

$$\left(0, \min\left\{1, \left(\delta\sqrt{\frac{2a-1}{4a-1}}\right)^{2a/(2a-1)}\right\}\right)$$

such that

$$2\exp\left\{-\frac{1}{2}\left(\frac{\delta}{\gamma_0(\gamma)} - 1\right)^2\right\}\left(\frac{\alpha}{\alpha-b}\left(\left(\frac{\delta L(\gamma)}{\beta_\gamma \gamma_0(N)}\right)^{b/\alpha} - 1\right)\left(\frac{\delta}{\gamma_0(\gamma)} - 1\right) + 1\right)^{2/b} < \nu,$$

where $\beta_\gamma = \min\{\gamma_0(\gamma), L(\gamma)/2^\alpha\}$ *and*

$$\gamma_0(\gamma) = \gamma^{1-1/(2a)}\sqrt{\frac{4a-1}{2a-1}},$$

$$L(\gamma) = 2^{1-\alpha}\gamma^{1-(\alpha-1)/(2a-2)}\left(\frac{\left(1+\gamma^{1/(a-1)}\right)^{\alpha+1}}{\gamma^2(\alpha+1)} + \frac{1}{(2a-\alpha-1)}\right)^{1/2}.$$

Conclusion

Above we obtained some results for models of separable stationary processes belonging to the space $\mathrm{Sub}_\varphi(\Omega)$. We found conditions under which a model approximates a process with a given accuracy and reliability in the Banach space $C[0,1]$. The results can be extended to the case of other Banach spaces.

Bibliography

1. Yu. Kozachenko, T. Sottinen, and O. Vasylyk, *Simulation of Weakly Self-Similar Stationary Increment* $\mathrm{Sub}_\varphi(\Omega)$*-Processes: a Series Expansion Approach*, Reports of the Department of Mathematics, Preprint 398, University of Helsinki, October 2004.
2. Yu. V. Kozachenko and O. A. Pashko, *Models of Stochastic Processes*, Kyiv University, Kyiv, 1999. (Ukrainian)
3. Yu. Kozachenko and I. Rozora, *Simulation of stochastic Gaussian processes*, Random Operators Stoch. Equations **11** (2003), no. 3, 275–296. MR2009187 (2004i:60050)
4. V. V. Buldygin and Yu. V. Kozachenko, *Metric Characterization of Random Variables and Random Processes*, TViMS, Kiev, 1998; English transl. AMS, Providence, 2000. MR1743716 (2001g:60089)
5. Yu. V. Kozachenko and O. I. Vasylyk, *On the distribution of suprema of* $\mathrm{Sub}_\varphi(\Omega)$ *random processes*, Theory Stoch. Processes **4**(**20**) (1998), no. 1–2, 147–160. MR2026624 (2004k:60094)

Department of Probability Theory and Mathematical Statistics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine

E-mail address: yvkuniv.kiev.ua

Department of Probability Theory and Mathematical Statistics, Faculty for Mechanics and Mathematics, National Taras Shevchenko University, Academician Glushkov Avenue 6, Kyiv 03127, Ukraine

E-mail address: irozora@bigmir.net

Received 27/FEB/2004

Translated by OLEG KLESOV