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# ACCURACY STUDY OF FINITE DIFFERENCE METHODS

*by Nancy Jane Cyrus and Robert E. Fulton*

*Langley Research Center*
*Langley Station, Hampton, Va.*

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## NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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## SUMMARY

A method for studying the accuracy of finite difference approximations for linear differential equations is presented and utilized. Definitive expressions for the error in each approximation are obtained by using Taylor series to derive the differential equations which exactly represent the finite difference approximations. The resulting differential equations are accurately solved by a perturbation technique which yields the error directly.

This method is used to assess the accuracy of two alternate forms of central finite difference approximations for solving boundary value problems in structural analysis which are governed by certain equations containing variable coefficients. A "half station approximation" in which finite difference approximations are made before expanding derivatives of function products is compared with a "whole station approximation" in which derivatives of function products are expanded first for string, beam, and axisymmetric circular plate problems. An example of a square membrane is given as an application of the method to partial differential equations.

## INTRODUCTION

The differential equations governing the behavior of structural boundary value problems are often solved by approximating the derivatives by finite differences and solving the resulting algebraic equations on a digital computer. For complicated structures the number of simultaneous equations resulting from finite difference approximations can be sufficiently large to exceed the capacity of the computer or introduce round-off error. For such problems, the accuracy of the difference procedure can be a critical item in obtaining meaningful design results. In reference 1, for example, it was found that accurate answers for the stress in a shell could not be obtained by using certain finite difference approximations unless the mesh spacing was smaller than machine capacity permitted.

The most popular difference approximations are the so-called central differences which are given in textbooks on numerical methods. There are alternate formulations of central differences which can be used when odd order derivatives occur in the differential

equations. Such a situation exists in structural problems, for example, where inplane loads are not uniform (a column loaded by its own weight or a shell of revolution subjected to arbitrary loads) or where the stiffness of the structure is nonuniform (a tapered beam or a variable thickness shell).

In this paper a method for studying the accuracy of finite difference approximations is presented and utilized. As illustrative examples, the method is used to assess the accuracy of two alternate forms of central finite difference approximations used in structural problems through application to string, beam, axisymmetric circular plate, and square membrane problems. The same approach can be used to evaluate the accuracy of finite element methods.

## SYMBOLS

$B(y), L(y)$     linear differential operators

$D(y), E(y)$     linear difference operators

$f(x)$     nondimensional tension in a beam or string

$g(x)$     nondimensional stiffness of beam

$h$     finite difference spacing

$i$     any integer

$k$     superscript describing set of boundary conditions

$m, n$     Fourier wave numbers

$p(x)$     nondimensional lateral load

$q(x)$     boundary condition

$x, z$     descriptive coordinates of beam, string, plate, or membrane

$y$     deflection of beam, string, plate, or membrane

$Y$     deflection function in perturbation series

$\dot\phi$          slope of plate

$\Gamma$          boundary curve

Prime or Roman numeral with a symbol denotes differentiation with respect to x.

## METHOD OF APPROACH

The usual approach in a finite difference accuracy study is to carry out the numerical solution to a number of problems for which exact solutions can be obtained and to compare the resulting numerical answers at each station with the exact answers. Such a procedure has the liability that comparisons can only be made for each problem at specific stations and the calculations must be redone each time the mesh size changes.

The approach used in this paper is to isolate the principal finite difference error so that its magnitude and character can be evaluated. The finite difference approximations are expanded in Taylor series to give differential equations which are exactly equivalent to the finite difference approximations. Solving the resulting differential equations by a perturbation technique yields analytical expressions for the principal error term. These expressions are independent of mesh spacing and give a clear indication of the accuracy of the difference approximations not just at discrete points but over the whole domain of interest.

Consider the differential equation

$$L(y) = p \tag{1}$$

with a necessary and sufficient set of k boundary conditions, each of the form

$$B^k(y) = q^k \qquad \text{(on } \Gamma\text{)} \tag{2}$$

Equation (1) may represent either an ordinary or partial differential equation. For example, equation (1) takes the form for a string of

$$\frac{d^2y}{dx^2} = p(x)$$

and for a membrane of

$$\nabla^2 y = p(x,z)$$

where $\nabla^2$ is the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

The differential equation (1) is approximated by finite differences and is replaced by a finite difference recursion formula at the ith station of the form

$$D(y_i) = p_i \qquad\qquad (3)$$

where $D(y_i)$ is the equivalent finite difference operator for $L(y)$ and is expressed in terms of $y$ evaluated at the appropriate finite difference stations.

A similar treatment for each of the $k$ boundary conditions leads to replacing equation (2) by

$$E^k(y_i) = q_i^k \qquad\qquad (\text{on } \Gamma) \qquad (4)$$

where $E^k$ is the finite difference operator for $B^k$. Note that operators of the form of equations (3) and (4) also result for finite element problems if a continuum is approximated by finite elements and the approximate equilibrium equations and boundary conditions are obtained.

The finite difference recursion equations (3) and (4) may be expanded about the ith point by using the appropriate Taylor series expansion, such as the one-dimensional expansion

$$y_{i\pm1} = y_i \pm hy_i' + \frac{h^2 y_i''}{2!} \pm \frac{h^3 y_i'''}{3!} + \frac{h^4 y_i^{IV}}{4!} \pm \ldots$$

For any central finite difference method, the order of error of the approximation is proportional to $h^2$ and the finite difference recursion formula takes the form

$$D(y_i) = L_0(y_i) + h^2 L_1(y_i) + h^4 L_2(y_i) + \ldots \qquad (5a)$$

where $L_0$, $L_1$, and $L_2$ are differential operators which depend on the approximation method used. A similar treatment for $E^k$ leads to

$$E^k(y_i) = B_0^k(y_i) + h^2 B_1^k(y_i) + h^4 B_2^k(y_i) + \ldots \qquad (5b)$$

For other difference approximations all powers of $h$ may occur.

By use of equations (5a) and (5b), the finite difference equations (3) and (4) now take the form

$$L_0(y_i) - p_i + h^2 L_1(y_i) + h^4 L_2(y_i) + \ldots = 0 \tag{6}$$

$$B_0{}^k(y_i) - q_i{}^k + h^2 B_1{}^k(y_i) + h^4 B_2{}^k(y_i) + \ldots = 0 \qquad \text{(on } \Gamma) \tag{7}$$

Equation (6) and its $k$ boundary condition equation (7) are the differential equations which represent the finite difference recursion equations (3) and (4). As the increment $h$ goes to zero, equation (6) and equation (7) should approach equation (1) and equation (2), respectively. In fact, if the finite difference approximations used are a convergent set, then

$$L_0 = L$$

and

$$B_0 = B$$

The solution to equations (6) and (7) gives an analytical representation of the numerical finite difference answers. Unfortunately, because of the infinite number of terms in equation (6) a closed form solution does not appear feasible. However, in a practical problem where the size of the region is scaled to be of the order one, $h$ is perhaps 0.1 or 0.01 or even smaller. This small value of $h$ suggests that equation (6) may be solved by a perturbation technique with the perturbation parameter taken to be $h^2$.

Let the solution to equations (6) and (7) be taken in the form

$$y_i = Y_0 + h^2 Y_1 + h^4 Y_2 + \ldots \tag{8}$$

Substituting equation (8) into equation (6) leads to

$$L_0(Y_0) - p_i + h^2 \Big[ L_0(Y_1) + L_1(Y_0) \Big] + h^4 \Big[ L_0(Y_2) + L_1(Y_1) + L_2(Y_0) \Big] + \ldots = 0 \tag{9}$$

subject to $k$ boundary conditions of the form

$$B_0{}^k(Y_0) - q_i{}^k + h^2 \Big[ B_0{}^k(Y_1) + B_1{}^k(Y_0) \Big] + h^4 \Big[ B_0{}^k(Y_2) + B_1{}^k(Y_1) + B_2{}^k(Y_0) \Big] + \ldots = 0 \tag{10}$$

If each order of error term is solved in sequence, the following series of problems result:

$$L_0(Y_0) - p_i = 0 \qquad\qquad \Big( B_0{}^k(Y_0) - q_i{}^k = 0 \Big) \tag{11}$$

$$L_0(Y_1) + L_1(Y_0) = 0 \qquad\qquad \Big( B_0{}^k(Y_1) + B_1{}^k\, Y_0 = 0 \Big) \tag{12}$$

$$L_0(Y_2) + L_1(Y_1) + L_2(Y_0) = 0 \qquad \Big( B_0{}^k(Y_2) + B_1{}^k(Y_1) + B_2{}^k(Y_0) = 0 \Big) \tag{13}$$

$$\ldots \qquad\qquad\qquad (\ldots)$$

If the finite difference approximation is a convergent one, equation (11) is equation (1) and $Y_0$ given by equation (11) is the exact solution to equation (1).

From the form of $y_i$ in equation (8) it is seen that $Y_1$ can be interpreted as the principal error term in the finite difference results in relation to the exact answer to the problem. If two different finite difference approximations are to be considered, a comparison of error terms $Y_1$ resulting from the two different approximations indicates the relative accuracy of the two approximations.

## ILLUSTRATIVE PROBLEMS

To illustrate the procedure, the results from two finite difference methods are compared for a string with nonuniform tension. Several additional structural problems are given in the appendices as further examples of the use of the method. The examples include a beam with nonuniform stiffness, an axisymmetric plate, and a square membrane subjected to a sinusoidal loading. The beam and string problems were taken from reference 2 and are given for completeness.

### String With Nonuniform Tension

Consider a string of constant length with nonuniform tension $f(x)$ subjected to a lateral load $p(x)$. The governing differential equation is

$$(fy')' + p(x) = 0 \tag{14}$$

where primes denote differentiation with respect to $x$. The variables are nondimensionalized so that the length of the string is one and the tension at the left end is one. The boundary conditions are

$$y(x_0) = 0 \qquad y(x_0 + 1) = 0 \tag{15}$$

where $x_0$ is the coordinate at the left end of the string. Equation (14) may be solved by dividing the string into stations of equal spacing $h$. The finite difference equations are written in terms of displacements at the ith station (i = 1, 2, 3, . . .).

Two finite difference approximations to be considered are denoted, for convenience, the "half station" approximation and the "whole station" approximation. For equation (14) these two approximations lead to the following finite difference expressions:

Half station approximation

$$(fy')_i' + p_i = \frac{1}{h}\left[-(fy')_{i-\frac{1}{2}} + (fy')_{i+\frac{1}{2}}\right] + p_i$$

$$= \frac{1}{h^2}\left[f_{i-\frac{1}{2}}y_{i-1} - \left(f_{i-\frac{1}{2}} + f_{i+\frac{1}{2}}\right)y_i + f_{i+\frac{1}{2}}y_{i+1}\right] + p_i = 0 \tag{16}$$

6

or whole station approximation

$$\left(fy'' + f'y'\right)_i + p_i = \frac{f_i}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) + \frac{f_i{}'}{2h}\left(-y_{i-1} + y_{i+1}\right) + p_i$$

$$= \frac{1}{h^2}\left[\left(f_i - \frac{hf_i{}'}{2}\right)y_{i-1} - 2f_iy_i + \left(f_i + \frac{hf_i{}'}{2}\right)y_{i+1}\right] + p_i = 0 \qquad (17)$$

Note that the half station approximation is the natural result of making the finite difference approximation before expanding the derivatives, whereas the whole station approximation results from making the approximation after the expansion. The latter type of approximation is widely used. (See, for example, refs. 3 and 4.) Both of the preceding sets of finite difference approximations can be shown to be of order $h^2$ and yet they clearly lead to different coefficients for the simultaneous equations in terms of the displacements at the ith station. Of concern here are the relative magnitudes of the errors in these two different approximations.

Expand the finite difference recursion equations (16) and (17) about the ith point by using such Taylor series expansions as

$$y_{i\pm1} = y_i \pm hy_i{}' + \frac{h^2}{2!}y_i{}'' \pm \cdots$$

$$f_{i\pm1} = f_i \pm hf_i{}' + \frac{h^2}{2!}f_i{}'' \pm \cdots$$

For both the half station and whole station approximations, this procedure leads to a differential equation of the form given by equation (6) where

$$\left.\begin{aligned} L_0\!\left(y_i\right) &= \left(f_iy_i{}'\right)' \\ B_0\!\left(y_i\right) &= y_i \end{aligned}\right\} \qquad (18)$$

and for the half station approximation

$$\left.\begin{aligned} L_1\!\left(y_i\right) &= \frac{f_iy_i{}^{IV}}{12} + \frac{f_i{}'y_i{}'''}{6} + \frac{f_i{}''y_i{}''}{8} + \frac{f_i{}'''y_i{}'}{24} \\ B_1\!\left(y_i\right) &= 0 \end{aligned}\right\} \qquad (19)$$

$$\left.\begin{aligned} L_2\!\left(y_i\right) &= \frac{f_iy_i{}^{VI}}{360} + \frac{f_i{}'y_i{}^{V}}{120} + \frac{f_i{}''y_i{}^{IV}}{96} + \frac{f_i{}'''y_i{}'''}{144} + \frac{f_i{}^{IV}y_i{}''}{384} + \frac{f_i{}^{V}y_i{}'}{1920} \\ B_2\!\left(y_i\right) &= 0 \end{aligned}\right\} \qquad (20)$$

$$\cdots$$

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and for the whole station approximation

$$L_1(y_i) = \frac{f_i y_i{}^{IV}}{12} + \frac{f_i{}' y_i{}'''}{6}$$

$$B_1(y_i) = 0$$

$$\left. \right\}$$ (21)

$$L_2(y_i) = \frac{f_i y_i{}^{VI}}{360} + \frac{f_i{}' y_i{}^{V}}{120}$$

$$B_2(y_i) = 0$$

$$\left. \right\}$$ (22)

. . .

Equations (6) and (7) with equations (18) and either equations (19), (20), . . ., or equations (21), (22), . . ., are clearly differential equations and associated boundary conditions which represent exactly the two finite difference recursion equations (16) and (17) and their associated boundary conditions.

By using the method described in the previous section, the principal error functions $Y_1$ defined by equation (8) corresponding to the half station and the whole station finite difference approximations have been obtained for a family of problems. These problems are a string having a lateral load which is distributed uniformly and a tension force $f(x)$ which varies as follows:

$$f(x) = \frac{1}{x^n} \qquad\qquad (1 \leqq n \leqq 6)$$

subject to the boundary conditions

$$y(1) = 0$$

$$y(2) = 0$$

and

$$f(x) = 1 + x^n \qquad\qquad (2 \leqq n \leqq 6)$$

subject to the boundary conditions

$$y(0) = 0$$

$$y(1) = 0$$

Where $f(x)$ is linear (corresponding to $f(x) = 1$, $x$, or $1 + x$), the results for the half station and whole station finite difference approximations are exactly the same. In fact, for $f(x) = 1$, both difference answers are the exact answer. For all other cases, however, the two difference methods lead to different results. It is useful to compare the results for $f(x) = \frac{1}{x^3}$ in detail as a typical example.

For $f(x) = \dfrac{1}{x^3}$ and $y(1) = y(2) = 0$,

$$Y_0 = -\frac{x^5}{5} + \frac{31}{75}x^4 - \frac{16}{75} \tag{23}$$

and the half station approximation is

$$Y_1 = -\frac{41}{1125}x^4 + \frac{x^3}{6} - \frac{31}{150}x^2 + \frac{86}{1125} \tag{24}$$

and the whole station approximation is

$$Y_1 = -\frac{187}{450}x^4 + \frac{4}{3}x^3 - \frac{31}{30}x^2 + \frac{26}{225} \tag{25}$$

The two error terms $Y_1$ over the length of the string are presented in figure 1(a). Solutions were also obtained for the error terms in deflection for all the remaining load functions $f(x)$ noted previously. Additional results for $f(x) = 1 + x^3$ are shown in figure 1(b). The remaining solutions are not shown because figure 1 serves to illustrate the character of the results. An overall measure of the relative errors in the two methods is shown later for all solutions obtained.

Although errors in the deflections of the string are important, errors in numerically obtained derivatives should also be considered for a thorough error analysis. Therefore, results were obtained by using the finite difference answers for approximate curvatures (second derivatives). The second difference operator was applied to the difference results; Taylor and perturbation series expansions were then applied to yield

$$y_i'' = \frac{1}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) = Y_0'' + h^2 Y_1'' + \frac{h^2}{12}\left(Y_0^{IV} + h^2 Y_1^{IV} + \ldots\right) + \ldots$$

or

$$y_i'' = Y_0'' + h^2\left(Y_1'' + \frac{Y_0^{IV}}{12}\right) + \ldots \tag{26}$$

The $h^2$ error terms in the curvatures for $f(x) = \dfrac{1}{x^3}$ are as follows:
For the half station approximation,

$$Y_1'' + \frac{Y_0^{IV}}{12} = -\frac{164}{375}x^2 - x + \frac{31}{75} \tag{27}$$

and for the whole station approximation,

$$Y_1'' + \frac{Y_0^{IV}}{12} = -\frac{374}{75}x^2 + 6x - \frac{31}{25} \tag{28}$$

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The error in the curvature for each of the two approximations is also given in figure 1(a) for $f(x) = \dfrac{1}{x^3}$ and in figure 1(b) for $f(x) = 1 + x^3$. Results for the remaining load functions are shown in a subsequent section in the form of an overall measure of the relative error.

Numerical calculations were also carried out for the deflections and curvatures for the problems cited to determine whether the analytical errors adequately represented the numerical errors. The data are not included herein; however, for $h$ less than about 0.1 all analytical errors agree with calculated numerical errors within 1 percent.

### Beam, Plate, and Membrane Examples

Appendix A contains examples of a simply supported beam having a nonuniform bending stiffness and subjected to a uniformly distributed load. Figure 2 shows the distribution of deflection and curvature errors for a linearly tapered beam. Examples of a clamped circular plate and a simply supported annular plate under uniformly distributed load are given in appendix B. Appendix C contains results for a square membrane subjected to a single term Fourier load.

## RELATIVE ERRORS OF THE HALF STATION AND
## WHOLE STATION APPROXIMATIONS

Although results such as those given in figures 1 and 2 are usually sufficient to identify which of the two approximations is superior for a given problem, identification of the superior method for specific results is sometimes difficult (for example, the curvature errors of fig. 1(b)). Moreover, a quantitative measure of the relative accuracy of the approximations is desirable. Probably the fairest comparison of their overall merit can be made by examining the root-mean-square values of the errors for the whole structure; that is,

$$\sigma_{Y_1} = \sqrt{\int_{x_0}^{x_0+1} Y_1^2 \, dx}$$

for the error in deflection and

$$\sigma_{Y_1''} = \sqrt{\int_{x_0}^{x_0+1} \left( Y_1'' + \frac{Y_0^{IV}}{12} \right)^2 dx}$$

for the error in curvature, where the integration is over the (unit) length of the string, beam, or plate. Thus, to assess quantitatively the relative merits of the half station and

whole station approximations for the various problems solved, the ratios $\sigma_{Y_1,half}/\sigma_{Y_1,whole}$ and $\sigma_{Y_1'',half}/\sigma_{Y_1'',whole}$ have been calculated for each problem. The results are shown in figure 3.

## DISCUSSION OF RESULTS OF SAMPLE PROBLEMS

The results given in figure 3(a) show that for all problems studied, the error in the deflection resulting from use of the half station approximation is less than the error obtained from use of the whole station approximation, in some problems by an order of magnitude. The investigation of the accuracy of the curvature approximations gives the same result in general. Thus, the half station method is usually superior for calculation of both deflection and bending curvature for the problems studied.

Although the results clearly favor the half station approximation, one exception occurs: for the string with the load $f(x) = 1 + x^2$, the error in the curvature is 25 percent greater with the half station approximation. The difference between the two approximations is seen to be generally less in calculating the second derivatives of deflections than in calculating the deflections themselves.

The analytical representation of errors in the present paper shows the danger of using numerical data at a single station or a few points to characterize the error in a problem. An example is shown in figure 1(a) for $f(x) = \frac{1}{x^3}$. If comparisons are made of the curvature near the end $x = 1$, the whole station approximation appears much more accurate than the half station approximation; whereas figure 3(b) shows clearly that the average error with the whole station approximation is over twice as great.

The present approach to error assessment may also be useful for comparison of different finite element structural approximations. In fact, the recursion formulas given by the half station approximation (eq. (16) and eq. (A2)) are the same recursion formulas that occur for a finite element model consisting of rigid bars connected by rotational springs, which often is used to represent a physical problem such as a beam-column (for example, ref. 5). Thus, the results of the present study verify that the finite element model of reference 5 is a good representation of the behavior of the continuum problem.

A practical consideration which supports the use of the half station method is the symmetry of the matrix of coefficients in this approximation. By contrast, the matrix of coefficients associated with whole stations is not symmetric. Matrix symmetry can be of great value for many numerical procedures associated with eigenvalue routines and simultaneous equation solving routines and, in some problems, is required for an efficient numerical solution of a large order system.

The results for the square membrane example given in appendix C demonstrate the application of the method to partial differential equations and indicate the relative accuracy of two alternate patterns for the Laplacian operator. The conventional pattern having error of order $h^2$ is compared with a so-called refined pattern which can be shown to have an error of order $h^4$ if the Laplacian of the loading vanishes. It is seen that for a single Fourier loading the standard pattern is actually more accurate than the refined pattern. Definitive expressions for the error terms are presented for both approximations. These expressions give the number of finite difference stations which are required within the length of a deflection Fourier wave to restrict finite difference answers to a given percentage of error.

## CONCLUDING REMARKS

A new procedure has been developed to determine an analytical representation of the error in a finite difference solution to a specified problem. This procedure allows a direct comparison, independent of mesh size, between difference approximations. The procedure appears to have considerable merit for assessment of the relative accuracy of finite difference and finite element numerical techniques of linear structural analysis.

By using this procedure, a comparison has been made of the accuracy of two finite difference approximations for solving structural problems through applications to a spectrum of beam and string problems having the characteristics of nonuniform stiffness and inplane load and to two circular plate problems. The methods investigated were a "half station" approximation in which the finite difference approximations are made before expanding the derivatives of function products and a "whole station" approximation in which derivatives of function products are expanded first; both approximations are in use. For the same number of stations, the average error in calculated deflection resulting from use of half station difference approximations was found to be always less than the error which would result from the use of whole station difference approximations. The method was also applied to a square membrane subjected to a single Fourier type loading and a simple expression was obtained for the number of finite difference spaces required per Fourier wave length to keep finite difference results within a given percent error.

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124-08-06-29-23.

# APPENDIX A

## BEAM WITH NONUNIFORM STIFFNESS

As another example which illustrates the procedure described in this paper, consider a simply supported beam of unit length with nonuniform bending stiffness denoted by g, subjected to a uniformly distributed load of unit magnitude. The well-known differential equation governing the lateral deflection y of the beam is

$$(gy'')'' = 1 \tag{A1}$$

where primes denote differentiation with respect to x and variables are nondimensionalized to make the length of the beam, the bending stiffness at the left end, and the load each equal to one. The boundary conditions are

$$y(x_0) = 0 \qquad\qquad y(x_0 + 1) = 0$$
$$y''(x_0) = 0 \qquad\qquad y''(x_0 + 1) = 0$$

The left-hand side of equation (A1) is approximated by either the half station or whole station finite difference approximations for stations of equal spacing h. Therefore, from the half station approximation

$$(gy'')_i'' - 1 = \frac{1}{h^2}\Big[(gy'')_{i-1} - 2(gy'')_i + (gy'')_{i+1}\Big] - 1$$

$$= \frac{1}{h^4}\Big[g_{i-1}y_{i-2} - 2\big(g_{i-1} + g_i\big)y_{i-1} + \big(g_{i-1} + 4g_i + g_{i+1}\big)y_i$$

$$- 2\big(g_i + g_{i+1}\big)y_{i+1} + g_{i+1}y_{i+2}\Big] - 1 \tag{A2}$$

and from the whole station approximation

$$\Big(gy^{IV} + 2g'y''' + g''y''\Big)_i - 1 = \frac{g_i}{h^4}\big(y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}\big)$$

$$+ \frac{2g_i'}{2h^3}\big(-y_{i-2} + 2y_{i-1} - 2y_{i+1} + y_{i+2}\big) + \frac{g_i''}{h^2}\big(y_{i-1} - 2y_i + y_{i+1}\big) - 1$$

$$= \frac{1}{h^4}\Big[\big(g_i - hg_i'\big)y_{i-2} + \big(-4g_i + 2hg_i' + h^2g_i''\big)y_{i-1} + \big(6g_i - 2h^2g_i''\big)y_i$$

$$+ \big(-4g_i - 2hg_i' + h^2g_i''\big)y_{i+1} + \big(g_i + hg_i'\big)y_{i+2}\Big] - 1 \tag{A3}$$

As before, expanding $y_i$ and $g_i$ about the ith point leads to the differential equation (6) where now

$$
\left.
\begin{aligned}
L_0(y_i) &= \left[g(x)_i y_i{}''\right]'' \\
B_0{}^1(y_i) &= y_i \\
B_0{}^2(y_i) &= y_i{}''
\end{aligned}
\right\}
\tag{A4}
$$

and for the half station approximation

$$
\left.
\begin{aligned}
L_1(y_i) &= \frac{g_i y_i{}^{VI}}{6} + \frac{g_i{}' y_i{}^{V}}{2} + \frac{7}{12} g_i{}'' y_i{}^{IV} + \frac{g_i{}''' y_i{}'''}{3} + \frac{g_i{}^{IV} y_i{}''}{12} \\
\\
B_1{}^1(y_i) &= 0 \\
\\
B_1{}^2(y_i) &= \frac{1}{12} y_i{}^{IV}
\end{aligned}
\right\}
\tag{A5}
$$

$$
\left.
\begin{aligned}
L_2(y_i) &= \frac{g_i y_i{}^{VIII}}{80} + \frac{g_i{}' y_i{}^{VII}}{20} + \frac{31}{360} g_i{}'' y_i{}^{VI} + \frac{g_i{}''' y_i{}^{V}}{12} + \frac{7}{144} g_i{}^{IV} y_i{}^{IV} \\
&\quad + \frac{g_i{}^{V} y_i{}'''}{60} + \frac{g_i{}^{VI} y_i{}''}{360} \\
\\
B_2{}^1(y_i) &= 0 \\
\\
B_2{}^2(y_i) &= \frac{1}{360} y_i{}^{VI}
\end{aligned}
\right\}
\tag{A6}
$$

$$\cdots$$

and for the whole station approximation

$$
\left.
\begin{aligned}
L_1(y_i) &= \frac{g_i y_i{}^{VI}}{6} + \frac{g_i{}' y_i{}^{V}}{2} + \frac{g_i{}'' y_i{}^{IV}}{12} \\
\\
B_1{}^1(y_i) &= 0 \\
\\
B_1{}^2(y_i) &= \frac{1}{12} y_i{}^{IV}
\end{aligned}
\right\}
\tag{A7}
$$

$$L_2(y_i) = \frac{g_i y_i^{\text{VIII}}}{80} + \frac{g_i' y_i^{\text{VII}}}{20} + \frac{g_i'' y_i^{\text{VI}}}{360}$$

$$B_2^{\ 1}(y_i) = 0$$

$$B_2^{\ 2}(y_i) = \frac{1}{360} y_i^{\text{VI}}$$

(A8)

. . .

If solutions to equations (6) and (7), together with equations (A4) and either equations (A5), (A6), . . ., or equations (A7), (A8), . . ., are again taken in the form of equation (8), the series of simpler equations (11), (12), and (13) are again obtained (with $p_i = 1$). Since the beam equation is fourth order rather than second, the boundary condition of zero bending moment leads to specification of $B_1^{\ 2}$ and $B_2^{\ 2}$.

Results have been obtained for

$$g(x) = x^n$$

$(n = 2, 3, 4)$
$(1 \leq x \leq 2)$

for both the half station and whole station approximations of the derivatives. The error terms for both deflections and curvatures are shown in figure 2 for $g(x) = x^3$ corresponding to a linearly tapered beam. An overall measure of the relative error in the half and whole station approximations is given in figure 3 for all three examples. The analytical error results for both deflection and curvature also agree with numerical error calculations within 1 percent for $h$ less than about 0.1.

L

# APPENDIX B

## CIRCULAR PLATE

### Clamped Circular Plate Under Uniform Load

As another example of a second-order equation, consider the axisymmetric bending behavior of a clamped circular plate of radius 1.0 subjected to a uniformly distributed load of magnitude 2. If the plate has constant thickness and appropriate nondimensional variables are used, its behavior is governed by a second-order differential equation of the form

$$\left[\frac{1}{x}(x\phi)'\right]' = -x \tag{B1}$$

where $x$ is the radial distance from the center and where $\phi$ represents the slope of the plate. For a clamped plate the boundary conditions are $\phi = 0$ at $x = 0$ and $x = 1$.

The two finite difference patterns for equation (B1) are as follows:

For the half station approximation,

$$\left[\frac{1}{x}(x\phi)'\right]'_i + x_i = \frac{1}{h^2}\left[\frac{x_{i-1}}{x_{i-\frac{1}{2}}}\phi_{i-1} - \left(\frac{x_i}{x_{i-\frac{1}{2}}} + \frac{x_i}{x_{i+\frac{1}{2}}}\right)\phi_i + \frac{x_{i+1}}{x_{i+\frac{1}{2}}}\phi_{i+1}\right] + x_i = 0 \tag{B2}$$

and for the whole station approximation,

$$\phi_i'' + \frac{\phi_i'}{x_i} - \frac{\phi_i}{x_i^2} + x_i = \frac{1}{h^2}\left[\left(1 - \frac{h}{2x_i}\right)\phi_{i-1} - 2\left(1 - \frac{h^2}{x_i^2}\right)\phi_i + \left(1 + \frac{h}{2x_i}\right)\phi_{i+1}\right] + x_i = 0 \tag{B3}$$

The differential operators $L_0$, $L_1$, and $L_2$ in equation (6) are given by

$$\left.\begin{aligned} L_0(\phi_i) &= \left[\frac{1}{x_i}(x_i\phi_i)'\right]' \\ B_0(\phi_i) &= \phi_i \end{aligned}\right\} \tag{B4}$$

For the half station approximation,

$$\left.\begin{aligned} L_1(\phi_i) &= \frac{\phi_i^{IV}}{12} + \frac{\phi_i'''}{6x_i} - \frac{\phi_i''}{4x_i^2} + \frac{\phi_i'}{4x_i^3} - \frac{\phi_i}{4x_i^4} \\ \\ B_1(\phi_i) &= 0 \end{aligned}\right\} \tag{B5}$$

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$$L_2\left(\phi_i\right) = \frac{\phi_i{}^{VI}}{360} + \frac{\phi_i{}^{V}}{120x_i} - \frac{\phi_i{}^{IV}}{48x_i{}^2} + \frac{\phi_i{}'''}{24x_i{}^3} - \frac{\phi_i{}''}{16x_i{}^4} + \frac{\phi_i{}'}{16x_i{}^5} - \frac{\phi_i}{16x_i{}^6} \left.\vphantom{\frac{a}{b}}\right\}$$

$$B_2\left(\phi_i\right) = 0 \qquad\qquad\qquad (B6)$$

and for the whole station approximation,

$$L_1\left(\phi_i\right) = \frac{\phi_i{}^{IV}}{12} + \frac{\phi_i{}'''}{6x_i} \left.\vphantom{\frac{a}{b}}\right\}$$

$$B_1\left(\phi_i\right) = 0 \qquad\qquad (B7)$$

$$L_2\left(\phi_i\right) = \frac{\phi_i{}^{VI}}{360} + \frac{\phi_i{}^{V}}{120x_i} \left.\vphantom{\frac{a}{b}}\right\}$$

$$B_2\left(\phi_i\right) = 0 \qquad\qquad (B8)$$

Results for the average principal error terms in the slope and in the numerically obtained second derivative obtained by using the previously described technique are presented in figure 3.

### Simply Supported Annular Plate Under Uniform Load

The axisymmetric bending behavior of a circular plate is also governed by the following fourth-order equation:

$$\frac{1}{x}\left\{x\left[\frac{1}{x}(xy')'\right]'\right\}' = p \qquad\qquad (B9)$$

where $y$ represents the deflection of the plate. Results are obtained for a clamped plate annulus having an internal radius of 1 and an external radius of 2 and subjected to a uniformly distributed load of magnitude 1. The boundary conditions are $y = 0$ and $y'' = 0$ at $x = 1$ and $x = 2$, respectively.

The finite difference approximations to equation (B9) follow, and the results for the average finite difference error are given in figure 3. For the half station approximation,

$$\frac{1}{x_i}\left\{x\left[\frac{1}{x}(xy')'\right]'\right\}'_i - 1 = \frac{1}{hx_i^4}\left\{\frac{x_{i-\frac{3}{2}}x_{i-\frac{1}{2}}}{x_{i-1}}y_{i-2} - \left[\frac{x_{i-\frac{3}{2}}x_{i-\frac{1}{2}} + \left(x_{i-\frac{1}{2}}\right)^2}{x_{i-1}}\right.\right.$$

$$+ \left.\frac{\left(x_{i-\frac{1}{2}}\right)^2 + x_{i-\frac{1}{2}}x_{i+\frac{1}{2}}}{x_i}\right]y_{i-1} + \left[\frac{\left(x_{i-\frac{1}{2}}\right)^2}{x_{i-1}}\right.$$

$$+ \left.\frac{\left(x_{i-\frac{1}{2}}\right)^2 + 2x_{i-\frac{1}{2}}x_{i+\frac{1}{2}} + \left(x_{i+\frac{1}{2}}\right)^2}{x_i} + \frac{\left(x_{i+\frac{1}{2}}\right)^2}{x_{i-1}}\right]y_i$$

$$- \left[\frac{x_{i-\frac{1}{2}}x_{i+\frac{1}{2}} + \left(x_{i+\frac{1}{2}}\right)^2}{x_i} + \frac{\left(x_{i+\frac{1}{2}}\right)^2 + x_{i-\frac{1}{2}}x_{i+\frac{1}{2}}}{x_{i+1}}\right]y_{i+1}$$

$$+ \left.\frac{x_{i-\frac{1}{2}}x_{i+\frac{3}{2}}}{x_{i+1}}y_{i+2}\right\} - 1 = 0 \tag{B10}$$

and for the whole station approximation,

$$\frac{1}{x_i}\left(y^{IV} + \frac{2y'''}{x} - \frac{y''}{x^2} + \frac{y'}{x^3}\right)_i - 1 = \frac{1}{h^4}\left[\left(1 - \frac{h}{x_i}\right)y_{i-2} - \left(4 - \frac{2h}{x_i} + \frac{h^2}{x_i^2} + \frac{h^3}{2x_i^3}\right)y_{i-1}\right.$$

$$+ \left(6 + \frac{2h^2}{x_i^2}\right)y_i - \left(4 + \frac{2h}{x_i} + \frac{h^2}{x_i^2} - \frac{h^3}{2x_i^3}\right)y_{i+1}$$

$$+ \left.\left(1 + \frac{h}{x}\right)y_{i+2}\right] - 1 = 0 \tag{B11}$$

The differential operators $L_0$, $L_1$, and $L_2$ in equation (6) are given by

$$\left.\begin{aligned}
L_0(y_i) &= \frac{1}{x}\left\{x\left[-(xy')'\right]'\right\}'_i \\[2ex]
B_0^{\ 1}(y_i) &= y_i \\[2ex]
B_0^{\ 2}(y_i) &= y_i''
\end{aligned}\right\} \tag{B12}$$

For the half station approximation,

$$\left.\begin{aligned}
L_1(y_i) &= \frac{x_i y_i^{IV}}{6} + \frac{y_i^{V}}{2} - \frac{y_i^{IV}}{3x_i} + \frac{2y_i'''}{3x_i^2} - \frac{y_i''}{x_i^3} + \frac{y_i'}{x_i^4} \\[2ex]
B_1^{\ 1}(y_i) &= 0 \\[2ex]
B_1^{\ 2}(y_i) &= \frac{1}{12}\, y_i^{IV}
\end{aligned}\right\} \tag{B13}$$

$$\left.\begin{aligned}
L_2(y_i) &= \frac{x_i y_i^{VIII}}{80} + \frac{y_i^{VII}}{20} - \frac{2y_i^{VI}}{45x_i} + \frac{2y_i^{V}}{15x_i^2} - \frac{y_i^{IV}}{3x_i^3} + \frac{2y_i'''}{3x_i^4} - \frac{y_i''}{x_i^5} + \frac{y_i'}{x_i^6} \\[2ex]
B_2^{\ 1}(y_i) &= 0 \\[2ex]
B_2^{\ 2}(y_i) &= \frac{1}{360}\, y_i^{VI}
\end{aligned}\right\} \tag{B14}$$

and for the whole station approximation,

$$\left.\begin{aligned}
L_1(y_i) &= \frac{y_i^{VI}}{6} + \frac{y_i^{V}}{2x_i} - \frac{y_i^{IV}}{12x_i^2} + \frac{y_i'''}{6x_i^3} \\[2ex]
B_1^{\ 1}(y_i) &= 0 \\[2ex]
B_1^{\ 2}(y_i) &= \frac{1}{12}\, y_i^{IV}
\end{aligned}\right\} \tag{B15}$$

# APPENDIX B

$$L_2(y_i) = \frac{y_i^{VIII}}{80} + \frac{13y_i^{VII}}{960x_i} - \frac{y_i^{VI}}{360x_i^2} + \frac{y_i^V}{1920x_i^3}$$

$$B_2^{\ 1}(y_i) = 0 \qquad\qquad\qquad\qquad\qquad\qquad (B16)$$

$$B_2^{\ 2}(y_i) = \frac{1}{360}\,y_i^{VI}$$

# APPENDIX C

## DEFLECTIONS OF A MEMBRANE

As an example of the application of the method to a partial differential equation, consider a square membrane subjected to a unit sinusoidal loading and supported on all edges. The differential equation governing the membrane can be written as

$$\left. \begin{aligned} \nabla^2 y &= \sin m\pi x \, \sin n\pi z \\[2mm] y &= 0 \qquad\qquad \text{(on boundary)} \end{aligned} \right\} \qquad (C1)$$

where the length of the side is 1 and where $y$ is an appropriate nondimensional deflection. Let equation (C1) be approximated by finite differences with equal mesh spacing $h$ in both directions. Two finite difference patterns which are often used to approximate $\nabla^2 y$ are considered in this example. These operators are presented in symbolic form with their Taylor series expansions as follows:

Standard approximation

$$\frac{1}{h^2} \begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} y = \nabla^2 y + h^2 \left( y_{xxxx} + y_{zzzz} \right) + \cdots \qquad (C2)$$

"Refined" approximation

$$\frac{1}{6h^2} \begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} y = \nabla^2 y + h^2 \left( \nabla^4 y \right) + \cdots \qquad (C3)$$

where the subscripts denote partial differentiation with respect to the indicated variables. The refined approximation is denoted as such because it utilizes more node points than the standard approximation and for the special case for which the loading is linear (i.e., $\nabla^4 y = 0$) has an order of error of $h^4$.

The differential operators in equation (6) become

$$\left. \begin{aligned} L_0(y_i) &= \nabla^2 y_i \\[2mm] B_0(y_i) &= y_i \end{aligned} \right\} \qquad (C4)$$

and for the standard approximation (C2)

$$L_1(y_i) = y_{xxxx} + y_{zzzz}$$

$$B_1(y_i) = 0$$

(C5)

and for the refined approximation (C3)

$$L_1(y_i) = \nabla^4 y_i$$

$$B_1(y_i) = 0$$

(C6)

The finite difference solutions for the deflection at the ith point obtained by the perturbation method give

$$Y_0 = \frac{1}{\pi^2(m^2 + n^2)} \sin m\pi x \sin n\pi z$$

(C7)

and the principal error terms for the standard approximation

$$Y_1 = \frac{\left(1 + \dfrac{n^4}{m^4}\right)}{12\left(1 + \dfrac{n^2}{m^2}\right)^2} \sin m\pi x \sin n\pi z$$

(C8)

and the refined approximation

$$Y_1 = \frac{1}{12} \sin m\pi x \sin n\pi z$$

(C9)

Although the refined approximation given by equation (C3) might be considered to be the better approximation, the error term shows that it is, in fact, less accurate for this problem. This result holds for all finite values of $m$ and $n$; however, for $m \gg n$ the errors in the two methods become essentially the same.

Sample calculations were carried out to obtain actual numerical solutions and to compare them with the exact solution as well as with $y_i$ obtained by the perturbation method. Results were obtained for several values of $h$ and $m$ and $n$ for both approximations and substantiate the greater accuracy of the standard approximation for this problem. The results are not shown but sample calculations for the dimensionless center deflection with $h = 1/4$ and $m = n = 1$ give 0.0533 by the standard approximation and 0.0561 by the refined approximation; the exact answer is 0.0506. With $h = 1/4$

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agreement was also obtained between the numerical results and the perturbation answers to three digits. This agreement could be improved to 5 digits if the $h^4$ order error term $Y_2$ was included.

Some practical assessment of the required number of stations to give a certain percentage error is also possible if equation (8) is written for the deflection as

$$y_i = Y_0\left(1 + h^2\frac{Y_1}{Y_0} + \ldots\right)$$ (C10)

Let the principal error $e$ be denoted

$$e = h^2\frac{Y_1}{Y_0}$$ (C11)

so that, for example a maximum error of 10 percent requires that $h$ be chosen such that $e < 0.1$. For the standard approximation the principal error is

$$e = h^2\frac{\pi^2}{12}m^2\frac{\left(1 + \dfrac{n^4}{m^4}\right)}{\left(1 + \dfrac{n^2}{m^2}\right)}$$ (C12)

Since $1/m$ is the length of a displacement Fourier wave, $\dfrac{1}{mh} = N$ is the number of finite difference increments per wave length. Equation (C12) gives

$$N = \frac{1}{mh} = \sqrt{\frac{\pi^2}{12e}\frac{\left(1 + \dfrac{n^4}{m^4}\right)}{\left(1 + \dfrac{n^2}{m^2}\right)}}$$ (C13)

which is fairly insensitive to $n/m$ if $m \gtreqqless n$ and which becomes for either $m \gg n$ or $n = m$

$$N \approx \frac{\pi}{\sqrt{12e}}$$ (C14)

For example, for a finite difference error of not more than 10 percent, $N \approx 2.9$. This means that approximately three finite difference spaces are required within the smallest Fourier wave length in order to obtain a 10-percent accuracy. For a 1-percent accuracy, 9.1 spaces are required. Similar developments for the refined approximation give

$$N \approx \sqrt{\frac{\pi^2}{12e}\left(1 + \frac{n^2}{m^2}\right)}$$ (C15)

For   m = n, equation (C15) gives

$$N \approx \frac{\pi}{\sqrt{6e}} \qquad \qquad (C16)$$

or   $N \approx 4.05$   for a 10-percent error and   $N \approx 12.8$   for a 1-percent error.   For   m >> n, the error is the same as that for the standard approximation.
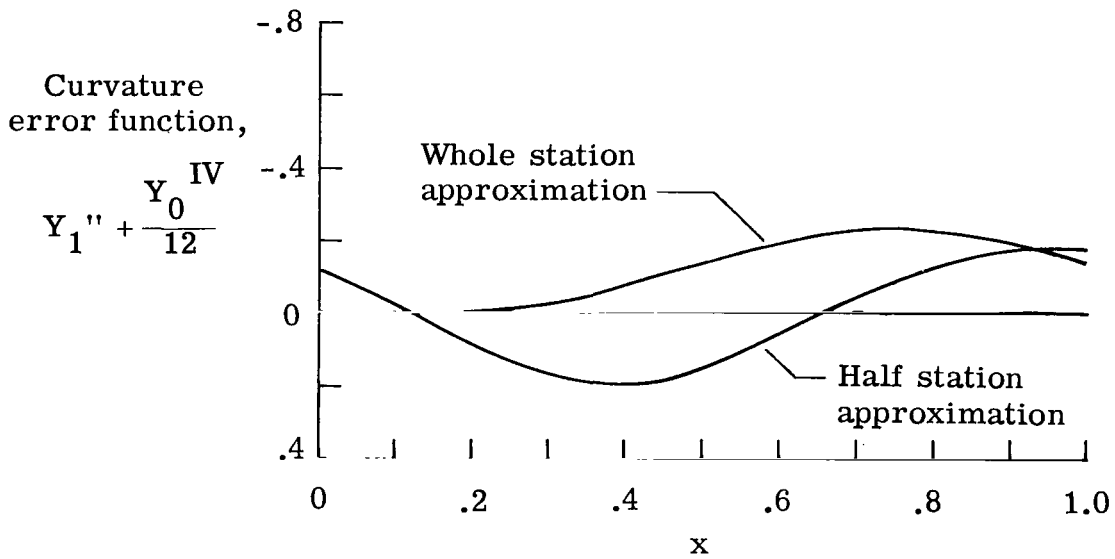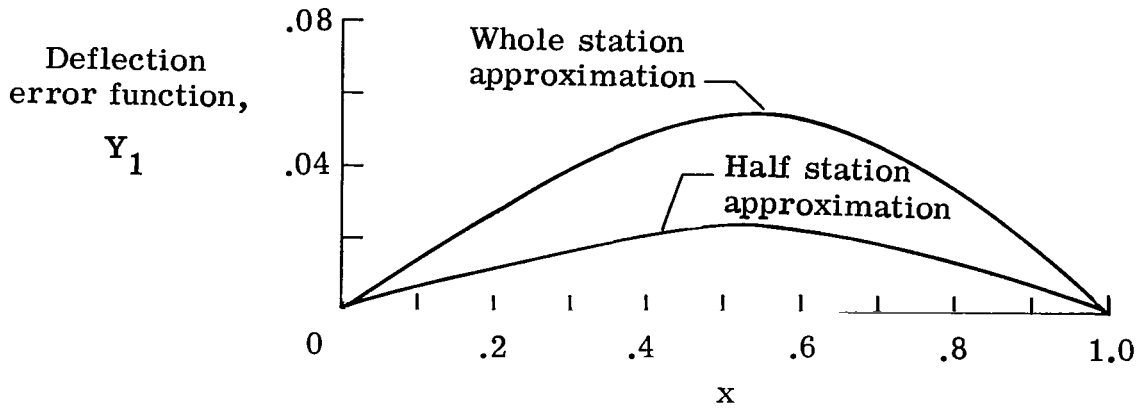
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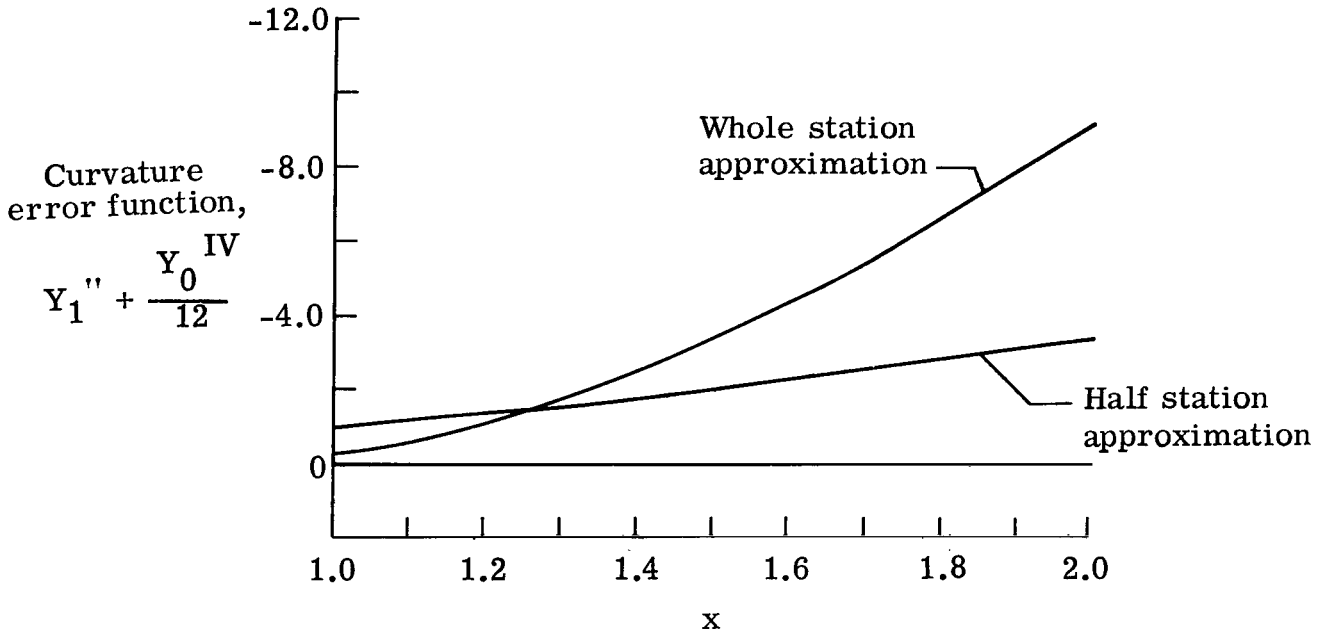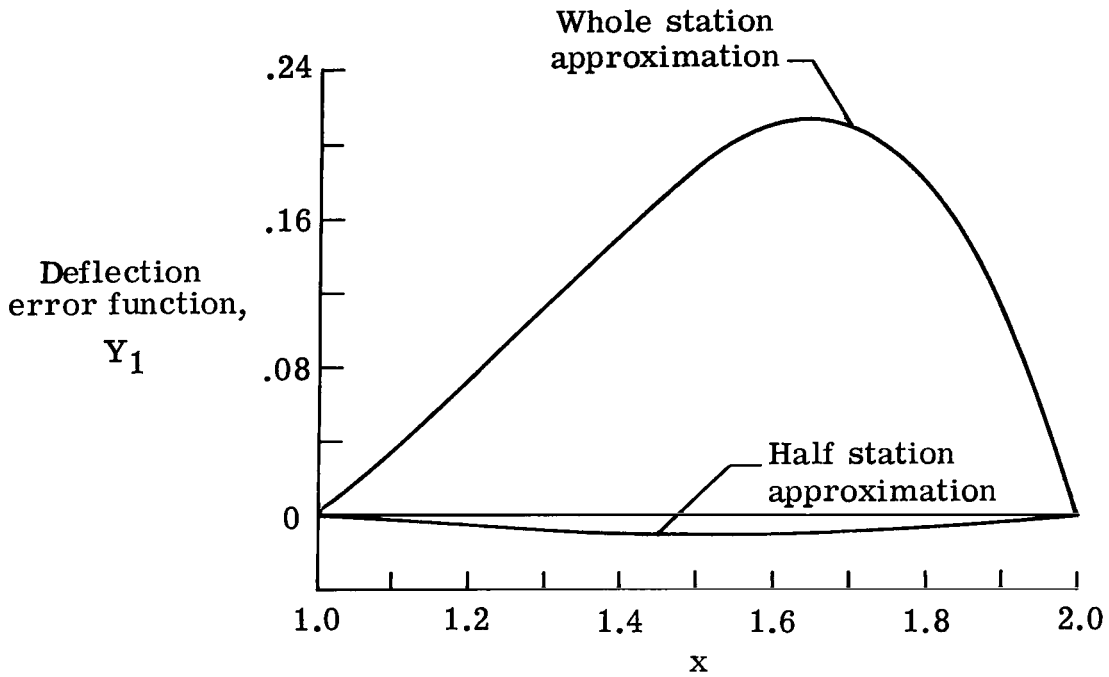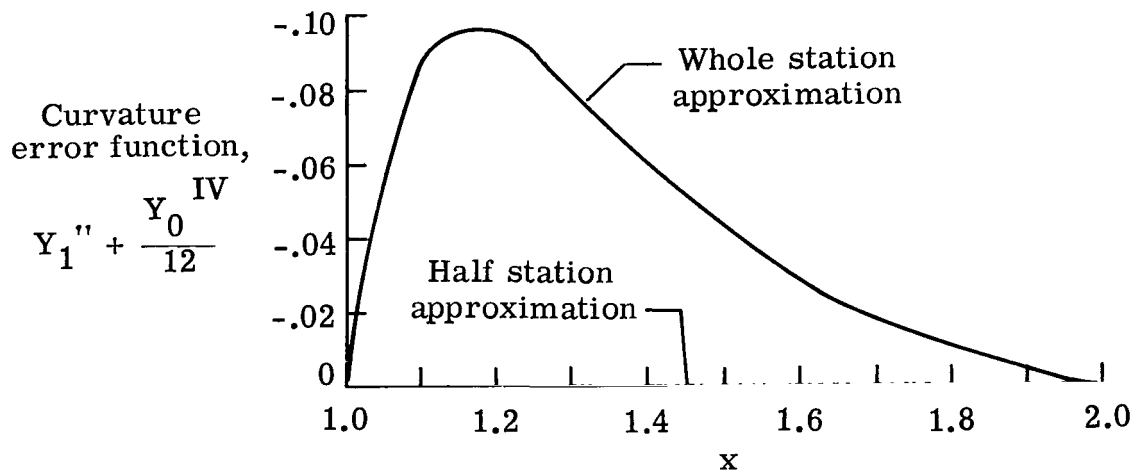
(a) $f(x) = 1/x^3$.

Figure 1.- Finite difference error in deflection and curvature for a uniformly loaded string with nonuniform tension $f(x)$.

Deflection error function, $Y_1$

Whole station approximation

Half station approximation

Curvature error function, $Y_1'' + \dfrac{Y_0^{IV}}{12}$

Whole station approximation

Half station approximation

(b) $f(x) = 1 + x^3$.

Figure 1.- Concluded.
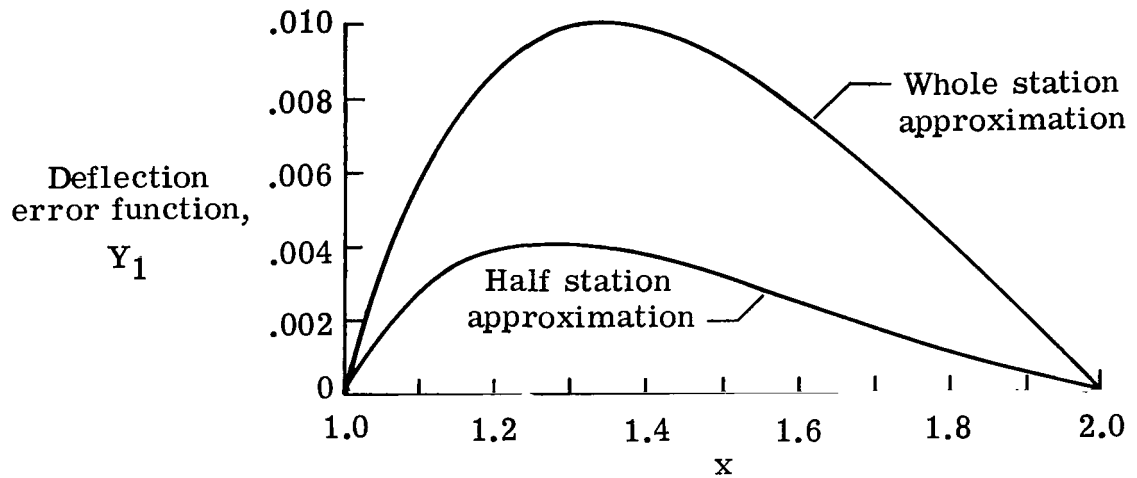
Figure 2.- Finite difference error in deflection and curvature of a uniformly loaded simply supported beam with nonuniform stiffness $g(x) = x^3$.

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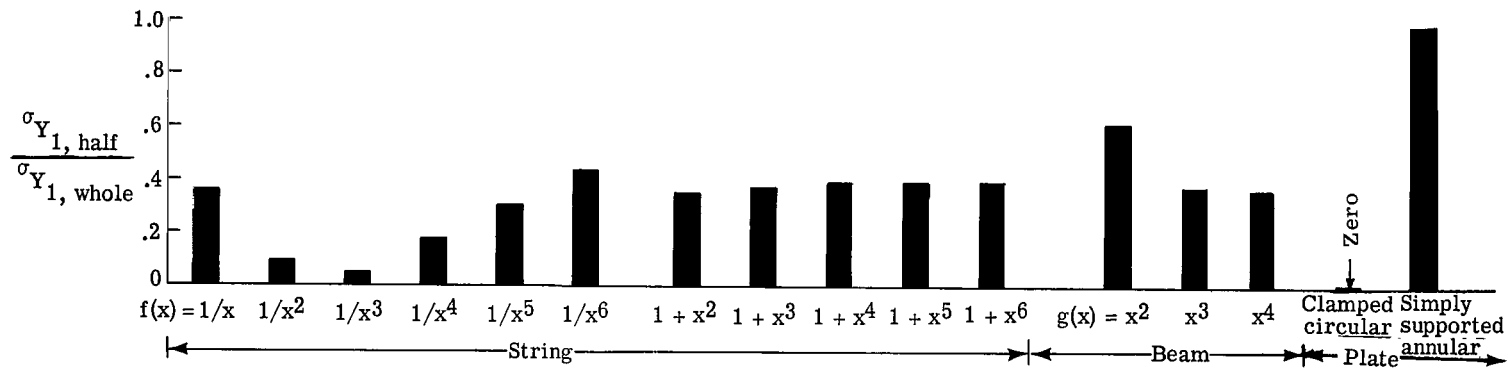
(a) Solution.
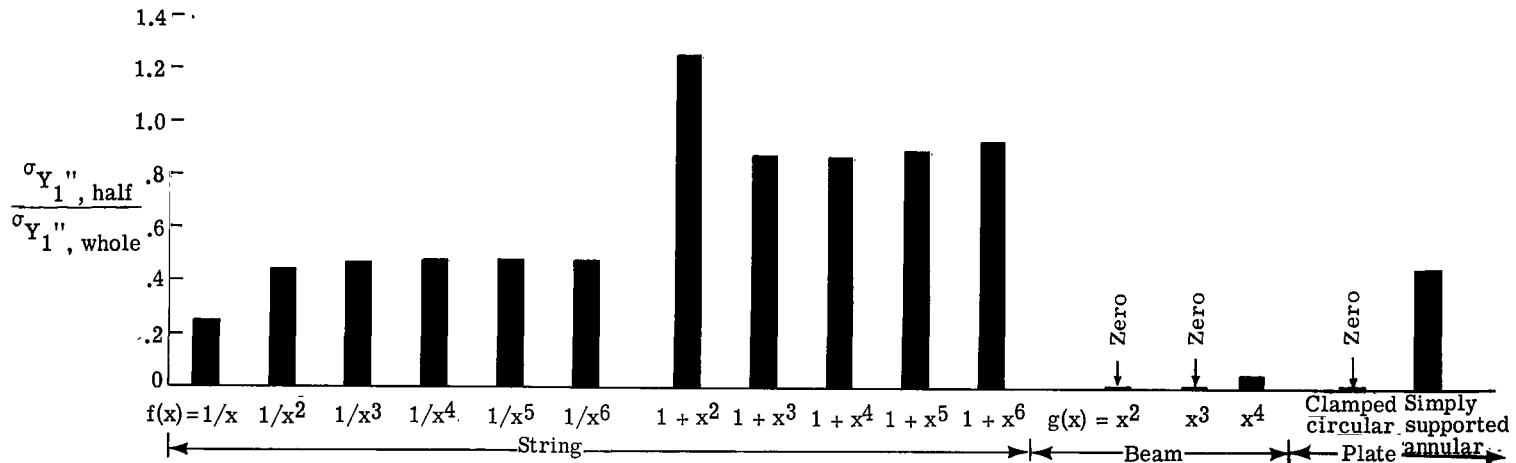
(b) Second derivative.

Figure 3.- Ratio of root-mean-square errors for half and whole station approximations.

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