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ACCURATE SOLUTION OF BETHE-SALPETER EQUATIONS FOR TIGHTLY BOUND FERMION-ANTIFERMION SYSTEMS

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A B S T R A C T

The Bethe-Salpeter equation for a fermion-antifermion system for the dynamical generation of a bound state via the exchange of a pseudoscalar particle leads to a set of four coupled integral equations for the state $J^P = 0^-$ and a system of eight for 1^-. It is argued that physically sensible solutions exist only after the introduction of a cut-off. These equations are solved with high accuracy by expanding the invariant amplitudes in terms of hyperspherical harmonics and then retaining the lowest n contribution only. It is found that for weakly bound systems the 1^- state is always lighter than the 0^- while for strong couplings the situation is reversed. Considering this solution as a relativistic quark model to form mesons as quark-antiquark bound states, the ratio 4:1 between the masses of the vector and the pseudoscalar mesons occurs for quark masses of about two nucleon masses. For the case of scalar particles we obtain an algebraic perturbation formula which is very accurate in the limit of small exchange masses.

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1. INTRODUCTION

There are quite a number of situations in current elementary particle physics where it is desirable to have at hand a numerically accurate solution of the realistic Bethe-Salpeter [1] equation in the bound state region. By realistic we mean at present inclusion of the spin of the constituents and non-zero mass for both the exchanged particle and the generated bound state.

Such a B-S equation but restricted to scalar particles has recently been discussed and solved in the bound state region by several authors [2,3]. The more general equations for the bound state of two spin $\frac{1}{2}$ fermions via the exchange of a pseudoscalar particle have been written down and studied by Gourdin [4], Goldstein [5] and Kummer [6]. They have been discussed further in connection with specific physical problems by Swift and Lee [7], Harte [8] and by Delbourgo, Salam and Strathdee [9]. We will refer to these papers for most of the basic formalism. The latter authors do not attempt a numerical discussion of the spin equations. Goldstein and Kummer discuss the zero mass cases only. Gourdin has obtained some solutions of reasonable accuracy by employing approximations which are good only in the weak binding limit of the deuteron.

In the present note we are more interested in tightly bound systems - a situation we encounter, for instance, in the quark models [10] for the elementary particles. It was shown earlier [3] that a simple exchange potential is not able to reproduce the observed particle spectrum on a Chew-Frautschi plot. Now we are asking how this situation is changed by the inclusion of spin. To be specific, we consider the B-S equation for a fermion-antifermion pair which is bound via the exchange of a pseudoscalar meson and generates a meson bound state. The inclusion of the spin 1/2 character of the fermions leads to four coupled integral equations for the generation of $J^P = 0^-$ state and to a set of eight for a 1^- state. Due to the fact that we now get coupled equations, most of the previous methods would exceed computer capacity. We find it therefore advantageous to expand the invariant functions in terms of four-dimensional Euclidean harmonics. This is possible after doing the Wick rotation [11]. The use of this method has been encouraged by our finding that retaining only the leading n term which leads to one-dimensional integral equations produces a four to five place accuracy in the scalar case. The resulting set of coupled one-dimensional integral equations is solved on the computer by direct matrix inversion or evaluation of the Fredholm determinant.

In Section 2 we give a brief review of the situation for scalar particles. We put special emphasis on the expansion in hyperspherical harmonics and to the accuracy which can be expected. For very small mass of the exchanged particle ($\mu/m \ll 1$), the potential becomes nearly singular. In this situation, however, we can obtain a perturbation formula for the eigenvalue. This allows us to obtain very accurate results for the coupling constant and serves as a check on the computer results in the domain where both methods are valid. This perturbation formula is slightly more general in that it also allows us to find the variation of the coupling constant for unequal masses of the two particles. In Section 3 we present the B-S equation and our solution for a spin $1/2$ fermion antifermion pair. We obtain the eigenvalues (coupling constants) and bound state masses for the states $J^P = 0^-$ and 1^- after introducing a high mass cut-off. These states are the ones of primary physical interest. Others can be obtained by the same method. Our main observation, discussed in Section 4, is that for the pseudoscalar exchange potential which we are using, the triplet state is more tightly bound than the singlet for weak couplings only. This is the same situation as one observes for the deuteron. For strong couplings the situation is reversed in that the 0^- state becomes less massive than the 1^-, a situation which is observed in the elementary particle spectrum where the vector mesons are heavier than the pseudoscalar ones.

We discuss some of the implications of these results on the relativistic quark model and show that the mass ratio of 4:1 of vector mesons to pseudoscalar mesons results for a quark mass of the order of two nucleon masses.

2. SCALAR PARTICLES

In this section we will review and generalize an earlier solution [3] of the B-S equation for scalar particles. The main reason for doing this is that it allows us to present the fundamental method and especially the formulae for the expansion of the amplitudes in hyperspherical harmonics in a simple situation. It also provides an excellent test for the approximation we introduce, of retaining the lowest n term only. For the special case of very small exchange masses ($\mu/m \ll 1$) the matrix approximation on the computer fails due to the fact that the potential becomes nearly singular. For this case we derive an accurate and convenient perturbation formula for the eigenvalue. We work in momentum space throughout although similar calculations could also be carried out in x space [2].

We write the homogeneous B-S equation for two spinless particles of mass m which are bound via the exchange of a similar particle of mass μ as

$$\phi(p,q) = ig^2(2\pi)^{-4}\left[(q+p)^2-m^2\right]^{-1}\left[(q-p)^2-m^2\right]^{-1}\int d^4k\,\frac{\phi(p,k)}{(q-k)^2-\mu^2} \tag{1}$$

A partial wave expansion of this equation results in a two-dimensional integral equation. Numerical solutions of such an equation arising from the zeroes of the Fredholm determinant have been discussed elsewhere [3]. Here we shall employ an expansion in hyperspherical harmonics with the view to reduce the equation to an approximate but accurate one-dimensional integral equation. This method can be extended to the case of particles with spin.

As a first step, we perform a Wick rotation [11] of the contour of integration and go over to the Euclidean metric for the momenta p, q and k. The amplitude can then be expanded as

$$\phi(p,q) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} \sum_{m=-\ell}^{\ell} \phi_{n\ell m}(pq)\, Y_{n\ell m}(\Omega_{pq}) \tag{2}$$

in terms of the harmonics which are normalized by [4]

$$Y_{n\ell m}(\theta;\phi\psi) = \mathcal{C}_n^{\ell}(\theta)\, \mathcal{Y}_{\ell m}(\phi\psi)\,, \tag{3}$$

$$\mathcal{C}_n^{\ell}(\theta) = \left[\frac{2^{2\ell+1}}{\pi} \frac{(n+1)\,(n-\ell)!}{(n+\ell+1)\,!}\right]^{1/2} \ell!\,(\sin\theta)^{\ell}\, C_{n-\ell}^{\ell+1}(\cos\theta) \tag{4}$$

where $\mathcal{Y}_{\ell m}(\phi\psi)$ are the usual spherical harmonics and $C_N^L(\cos\theta)$ are the Gegenbauer functions. The observation that the exchange propagator has the expansion

$$\frac{1}{(k-q)^2+\mu^2} = \frac{8\pi^2}{(r+s)^2} \sum_{n'\ell'm'} \frac{1}{(n'+1)} \left(\frac{r-s}{r+s}\right)^{n'} Y^{*}_{n'\ell'm'}(\Omega_k)\, Y_{n'\ell'm'}(\Omega_q) \tag{5}$$

and the use of orthonormality properties of the harmonics enable us to obtain the integral equation satisfied by the partial wave amplitude $\phi_{n\ell m}(p,q)$.

In Eq. (5), we have denoted

$$r = \sqrt{(k+q)^2+\mu^2} \quad , \quad s = \sqrt{(k-q)^2+\mu^2}$$

where k and q now stand for the Euclidean lengths. Thus, we obtain

$$\phi_{n\ell m}(p,q) = \frac{ig^2}{2\pi^2}\int k^3 dk \sum_{n'=0}^{\infty} \Delta_{n'}(k,q)\, E^{\ell}_{nn'}(p,q)\, \phi_{n'\ell m}(p,k) \tag{6}$$

where

$$\Delta_{n'}(kq) = \frac{1}{(r+s)^2}\frac{1}{(n'+1)}\left(\frac{r-s}{r+s}\right)^{n'} \tag{7}$$

and

$$E^{\ell}_{nn'}(pq) = \int_0^{\pi} \sin^2\theta_{pq}\, d\theta_{pq}\, \frac{C^{\ell}_{n}(\theta_{pq})\, C^{\ell}_{n'}(\theta_{pq})}{(p^2+q^2+m^2)^2 - 4p^2q^2\cos^2\theta_{pq}} \tag{8}$$

We can now rotate the fourth component of p back to a Minkowski metric by replacing p_0 by ip_0. The angular integration over the Gegenbauer functions involved in Eq (8) is in general complicated [12),13)] and the closed form so obtained is not amenable to numerical evaluation, whereas it is relatively easier if we specify the value of ℓ. For $\ell = 0$, we obtain

$$E^{0}_{nn'}(pq) = \frac{1}{2QR}(-1)^{\frac{n+n'}{2}}\left[1+(-1)^{n+n'}\right] \times \left\{\left(\frac{Q-R}{2pq}\right)^{n+n'+2} + (-1)^{n'}\left(\frac{Q-R}{2pq}\right)^{|n-n'|}\right\} \tag{9}$$

where

$$Q = \sqrt{(q^2-p^2+m^2)^2 + 4p^2q^2}\,,$$

$$R = (q^2-p^2+m^2) \tag{10}$$

and p,q denote $\sqrt{p^2} = \sqrt{p_0^2-\bar{p}^2}$, $\sqrt{q^2}$ respectively. Equation (6) together with Eqs. (7) and (9) describes a coupled and infinite set (corresponding to $n,n' = 0,1,2\ \ldots$) of one-dimensional integral equations.

The kernel of Eq. (6) contains the functions Δ_n [Eq. (7)] and $E^0_{nn'}$ [Eq. (9)]. In the bound state region $p^2 < m^2$ and therefore, especially for tightly bound systems, $p^2 \ll m^2 + q^2$. This implies that $Q \approx R$ in Eq. (9) and therefore $E^0_{nn'}$ will be small unless $n = n'$. We may further take the lowest value of n because for all higher n the function Δ_n (7) is reduced by a considerable factor. So we try to approximate the infinite system (6) by taking only $n = n' = 0$, in which case we obtain a single one-dimensional integral equation for each partial wave. Computer capacity actually allows us to retain several n values which we have done to test the accuracy. It was found that using more than one n value did not alter the fifth significant figure in the coupling constant. Therefore this method is quite accurate and other errors introduced by the use of the B-S equation in ladder approximation are certainly of much greater weight.

We have solved the resulting integral equation by direct matrix inversion where we have used Gaussian integration and have taken 6, 12 and 24 supporting points for each of the variables q, k. The results agree well with earlier ones [2),3)] and are accurate to at least four places. It was also found that the one-dimensional formulation is more stable against variations of the mesh size than the full two-dimensional problem. In Table 1 we have plotted the values of the coupling constant $\lambda m^2 = g^2(4\pi)^{-2}$, obtained as zeros of the Fredholm denominator, against the square of the bound state mass, $S = m_B^2$ for different values of the ratio $\alpha = m/\mu$. We have selected as the physical solution the smallest positive λ which gives a bound state. To the coupling constant which for a given α gives a zero mass bound state we refer as strong coupling. If the coupling is increased beyond this limit, the bound state moves to negative S, or imaginary mass. For $\alpha = \infty$, we have $\lambda_{\text{strong}} = 2$.

One can solve Eq. (6) also by a so-called variational ansatz as Schwartz [2)] did. We have opted for direct matrix inversion because the variational method works only for a positive (or negative) definite kernel. This property is quite easy to prove for the scalar case but is not true [14)] for the coupled equations in the spin $\frac{1}{2}$ problem, to which we want to apply our method. Furthermore, the amount of computer time to be saved [15)] is nearly negligible while the analytic manipulations become very complicated.

If the mass of the exchanged particle is very small ($\alpha \gtrsim 10$), the potential in Eq. (1) becomes nearly singular and can no longer be described well by a 12 or even 24 point matrix approximation. For this situation, we have therefore derived a perturbative formula by making use of the fact that for $p^2 = 0$, $\mu^2 = 0$, Eq. (1) is solved exactly by Wick's ground state solution

$$\phi_0(q) = (q^2 - m^2)^{-3} \tag{11}$$

and

$$\lambda_0 = 2 \tag{12}$$

The relation we shall derive below is algebraic and describes very accurately the connection between the coupling constant and the mass of the bound state. Moreover, since it does not use the Wick rotation, it can be generalized to the case where the two parallel propagators carry slightly different masses: $M_1 = m + m_1$, $M_2 = m + m_2$. This latter situation is of special interest to quark models with different quark masses. Accordingly we generalize Eq. (1) to

$$[(p+q)^2 - M_1^2][(p-q)^2 - M_2^2]\,\phi(q) = \frac{i\lambda}{\pi^2}\int \frac{d^4k}{(q-k)^2 - \mu^2}\,\phi(k) , \tag{13}$$

where we assume that p_0, μ, m_1 and m_2 are all much smaller than m. For $p_0 = \mu = m_1 = m_2 = 0$. Wick's solution (11), (12) satisfies Eq. (13) and if these parameters are non-zero but small, we can expect to obtain a reasonable approximation for λ if we replace ϕ by ϕ_0 in Eq. (13). In order to reduce Eq. (13) to an algebraic equation, we multiply both sides by a function of q^2 and integrate over d^4q. This function has to ensure convergence and for convenience we have chosen ϕ_0 (11), but the final result is independent of the specific form of this function. We observe that in Eq. (13), μ and $m_B = 2p_0$

occur quadratically, while m_1 and m_2 enter linearly (and, for symmetry reasons, only as their sum). We therefore seek a perturbative formula of the form

$$\lambda(\mu^2, m_B^2, m_1+m_2) = \lambda(0,0,0) + \left(\frac{m_B}{m}\right)^2 \lambda_1 + \left(\frac{m_1+m_2}{m}\right) \lambda_2 + \frac{\mu^2}{m^2}\lambda_3 + O(m_B^4, \mu^4, m_1^2, m_2^2) \tag{14}$$

$\lambda(0,0,0)$ is of course 2, which is Wick's value. By carrying out the integrals arising from the left-hand side of Eq. (13) as Feynman integrals, we obtain for the second and third terms

$$\lambda_1 = -\frac{2}{5} \quad , \quad \lambda_2 = 2 \tag{15}$$

The fourth term is somewhat tricky to evaluate since the inner integrand in

$$\lambda(\mu) = \frac{\pi^4}{6m^4}\left[\int d^4q \frac{1}{(q^2-m^2)^3}\int d^4k \frac{1}{(k^2-m^2)^3} \frac{1}{(q-k)^2-\mu^2}\right]^{-1} \tag{16}$$

has no analytic expansion around $\mu = 0$. This is understandable since for small μ Eq. (16) is expected to behave like $\mu^2 \log \mu^2$ due to the singularity in the exchange propagator. We shall therefore calculate $\lambda(\mu) - \lambda(0)$, to be identified with the fourth term of Eq. (14). By going through the standard techniques, we obtain from (16)

$$\lambda(\mu) = \frac{1}{3m^4}\left[\int_0^1\int_0^1 dx\, dy \frac{x^3y^2(1-x)(1-y)}{\left[m^2x(1-xy)+\mu^2(1-x)(1-y)\right]^3}\right]^{-1} \tag{17}$$

which is evaluated numerically and fitted to a linear dependence on $\frac{\mu^2}{m^2}\log\frac{\mu^2}{m^2}$ which is valid over a wide range of values of $\frac{\mu}{m}$. We arrive at the relationship

$$\lambda = 2.0 - \frac{2}{5}\left(\frac{m_B}{m}\right)^2 + 2\,\frac{(m_1+m_2)}{m} + \frac{\mu^2}{m^2}\left[a \log \frac{\mu^2}{m^2} + b\right] \tag{18}$$

where

$$a = -0.85\,, \quad b = 1.69 \tag{19}$$

We see that Eq. (18) predicts a larger coupling for a more tightly bound system and also a larger coupling for a more massive exchange particle. This is in accordance with what we expect since, for a massive exchange particle, the potential becomes less singular. In Table 1 we have compared some values of λ obtained on the computer as a solution to Eq. (6) against those following from relations (18) and (19) with $m_1 = m_2 = 0$ and $m_B = \mu$. We may comment that for large values of m/μ the perturbative formula is more reliable and more accurate than the numerical solution of Eq. (6), while for $m \sim \mu$ it should not be used. In the special case of $\mu = 0$ a similar formula was apparently used by Gürsey, Lee and Nauenberg [16] in deriving a quadratic mass formula for mesons [17].

3. SPIN $\frac{1}{2}$ PARTICLES

We consider now the B-S equation describing the fermion-antifermion system (spin 1/2, mass m) bound by the exchange of a pseudoscalar meson of mass μ

$$\Psi(p,q) = \frac{ig^2}{(2\pi)^4}\int d^4k\, \frac{(\not p + \not q + m)}{(p+q)^2 - m^2}\,\gamma_5\, \frac{\Psi(p,k)}{(k-q)^2-\mu^2}\,\gamma_5\, \frac{(\not q - \not p + m)}{(p-q)^2 - m^2} \tag{20}$$

Following the standard procedure (4)-(7), we introduce scalar and vector functions as invariant amplitudes and choose the ansatz:

$$\Psi = \begin{pmatrix} \frac{1}{2}(iS+T)+\frac{1}{2}\vec{\sigma}\cdot(\vec{V}+i\vec{U}) & \frac{1}{2}(B+C)+\frac{i}{2}\vec{\sigma}\cdot(\vec{F}+\vec{G}) \\ \frac{1}{2}(B-C)+\frac{i}{2}\vec{\sigma}\cdot(\vec{F}-\vec{G}) & \frac{1}{2}(iS-T)+\frac{1}{2}\vec{\sigma}\cdot(\vec{V}-i\vec{U}) \end{pmatrix} \tag{21}$$

where $\vec{\sigma}$ is the Pauli spinor [18].

On examination of the behaviour of Ψ under space reflection

$$\Psi(q_0,\vec{q}) \rightarrow \mathcal{P}\,\Psi(q_0,\vec{q}) = \pm\,\gamma_0\,\Psi(q_0,-\vec{q})\,\gamma_0 \tag{22}$$

we associate even intrinsic parity with the amplitudes S, T, $\vec{U}$, $\vec{V}$ and odd parity with the rest. Eq. (20) is invariant under the combined operation of charge conjugation $\times$ parity: CP which is equivalent to spin exchange invariance: $\mathcal{E}$ [19]. Under CP, we have

$$\Psi(q_0,\vec{q}) \rightarrow \Psi^{\mathcal{E}}(q_0,\vec{q}) = -\gamma_2\,\Psi^T(-q_0,\vec{q})\,\gamma_2 \; . \tag{23}$$

In terms of the components of Ψ as contained in (21) it can be seen that the CP invariance separates the sixteen amplitudes of Eq. (21) into two disconnected groups:

$$\begin{aligned} &S,\ V_2,\ U_1,\ U_3,\ F_1,\ F_3,\ G_1,\ G_3 : \ A^{\mathcal{E}}(q_0,\vec{q}) = A(-q_0,\vec{q})\ ; \\ &T,\ V_1,\ V_3,\ U_2,\ B,\ C,\ F_2,\ G_2 : \ A^{\mathcal{E}}(q_0,\vec{q}) = -A(-q_0,\vec{q}) \end{aligned} \tag{24}$$

The subscripts 1, 2, 3 denote the Cartesian components of the three-vector functions.

Inserting the ansatz (21) into Eq. (20) and using the explicit representation [20] for the γ matrices leads to two sets of eight coupled integral equations for the scalar functions $S(p,q)\dots T(p,q)$. For $p^2 < m^2$ the integral in Eq. (20) defines these functions as analytic functions in a cut q_0 plane. The familiar two cuts and four poles in the k_0 plane of integration are avoided by an infinitesimal deformation of the contour according to the Feynman prescription. In this way, the k_0 path of integration would still pass close to these singularities which is not good for numerical integration. We therefore carry out a Wick rotation of the contour of integration and replace $q_0 \to iq_4$ and $k_0 \to ik_4$. This rotation is allowed here in the spin $\frac{1}{2}$ case because the motion of the contour does not encounter any singularities in the first and third quadrant. The infinite contour which is picked up does not contribute if the integral exists in the first place. We will assure this in the following by assuming the presence of an analytic cut-off function. We show in the next Section that such a cut-off is actually necessary to obtain physically meaningful solutions. This rotation of k_0 cannot affect the Fredholm type denominator because the latter is a function of p_0 only.

In order to be able to expand our functions on the basis of the Euclidean hyperspherical harmonics, we also rotate p_0 to ip_4. This secondary rotation is for convenience only and has nothing to do with the solubility of the problem at hand. At the end of the calculation, we undo the p_0 rotation by analytic continuation of our functions to the physical region. We could entirely avoid this last rotation by expanding our functions on the basis of $O(3,1)$ instead.

The coupled equations take the form

$$A^i(p,q) = \frac{g^2}{(2\pi)^4}\left[(p+q)^2+m^2\right]^{-1}\left[(q-p)^2+m^2\right]^{-1}\int d^4k \; K^{ij}(p,q)\,\frac{A^j(p,k)}{(k-q)^2+\mu^2} \qquad (25)$$

where the 8x8 matrix K^{ij} is given in Table 2, $A^i = (S, T, \vec{V}, \vec{U}, B, C, \vec{F}, \vec{G})$. Our next task is to project, from Eq. (6), the bound states of well-defined quantum numbers n, J, m. We expand the scalar amplitudes in hyperspherical harmonics and the vector amplitudes in vector-hyperspherical harmonics :

$$
\begin{aligned}
S(p,q) &= \sum_{nJm} S_{nJm}(pq)\, Y_{nJm}(\Omega_{pq}) \\
\vec{V}(p,q) &= \sum_{nJLm} \vec{V}_{nJLm}(pq)\, \vec{Y}_{nJLm}(\Omega_{pq})
\end{aligned}
\tag{26}
$$

The harmonic $\vec{Y}_{nJLm}$ is a weighted sum of Y_{nLm} :

$$
\vec{Y}_{nJLm}(\Omega) = \sum_{\mu=-1}^{+1} C(J\,m \,\|\, L\; m-\mu\,;\, 1\,\mu)\, Y_{nL\,m-\mu}(\Omega)\, \vec{\varepsilon}_\mu \tag{27}
$$

where $\vec{\varepsilon}_\mu$ is a unit vector in the spherical basis. In addition to the orthonormality of the harmonics, we also need the following well-known projection formulae [21)]:

$$
\begin{aligned}
\vec{q}\, Y_{nJm} &= q \sin\theta \sqrt{\frac{J+1}{2J+1}}\, \vec{Y}_{nJ\ell+1\,m} + q \sin\theta \sqrt{\frac{J}{2J+1}}\, \vec{Y}_{nJ\ell-1\,m} \\
\vec{q}\cdot\vec{Y}_{nJJ+1\,m} &= q \sin\theta \sqrt{\frac{J+1}{2J+1}}\, Y_{nJm} \\
\vec{q}\cdot\vec{Y}_{nJJ-1\,m} &= q \sin\theta \sqrt{\frac{J}{2J+1}}\, Y_{nJm} \\
\vec{q}\cdot\vec{Y}_{nJJm} &= 0
\end{aligned}
\tag{28}
$$

and similar formulae for $\vec{q}\times\vec{Y}$. Here q denotes the magnitude of the Euclidean four-vector q. On inserting (26) into Eq. (25), and making use of Eqs. (5) and (28), we obtain 16 coupled integral equations (i, j = 1 to 16)

$$
\begin{aligned}
A^i_{nJLm}(q) = \frac{g^2}{2\pi^2} \int dk\, d\theta\; k^3 \sin^2\theta \sum_{n'=0}^{\infty} \Delta_{n'}(kq)\, M^{ij}(p,q,J,\theta) \\
\times \frac{\mathcal{C}^L_n(\theta)\, \mathcal{C}^{L'}_{n'}(\theta)}{(p^2+q^2+m^2)^2 - 4p^2q^2\cos^2\theta}\; A^j_{n'JL'm}
\end{aligned}
\tag{29}
$$

Here, A^i $(i=1$ to $16)$ denotes the column matrix :

$$A^i = (S_J, T_J, V_{J+1}, V_{J-1}, V_J, U_{J+1}, ----) \tag{30}$$

where V_{J+1}, V_{J-1} and V_J are the components in the spherical basis of $\vec{V}(p,q)$. The values of L and L' in Eq. (29) are specified by $J, J, J+1, ..$ for $i,j = 1,2,3,...$ as indicated by the subscripts in Eq. (30). The parts of the matrix M^{ij} which are of interest to us will be discussed below.

It is possible to separate the singlet and triplet states. In view of Eq. (24), we define odd and even combinations :

$$A^{e,o}(q_0,\vec{q}) = \frac{1}{2}\left[A(q_0,\vec{q}) \pm A(-q_0,\vec{q})\right] \tag{31}$$

On rewriting Eq. (25) using the spherical basis, we can immediately separate the triplet and singlet amplitudes according to the evenness and oddness under spin exchange invariance. Next, as a consequence of Eqs. (22) and (26), we separate the amplitudes of parity $(-1)^J$ from those of parity $(-1)^{J+1}$. The 16 amplitudes, together with their doubling as in Eq. (31), thus fall into four classes as shown in Table 3. In general, therefore, a state (triplet or singlet) of given parity is described by a set of 8×8 integral equations. For the special case of $J^P = 0^-$, the corresponding kernel reduces trivially to a 4×4 matrix. The 1^- is still described by an 8×8 kernel.

The coupled equations can be cast in the form

$$A^i_{nJLm}(pq) = \frac{g^2}{2\pi^2}\int dk\, k^3 \sum_{n'} \Delta_{n'}(kq)\, K^{ij}(pqk)\, A^j_{n'JL'm}(pk) \tag{32}$$

if the angular integrations of Eq. (29) are performed. For this purpose, we utilize Eq. (4) together with the well-known properties of the Gegenbauer functions to express the $\mathcal{C}^L_N(\theta)$ in terms of simple trigonometric functions [22]. For a few low values of L and n, the expressions for $\mathcal{C}^L_N(\theta)$ are shown in Table 4. For each element (i,j), we pick the lowest occurring value of n (and the same for n' which occurs in the sum over n') by the criterion that $n-L$ is even (odd) for A^e (A^o). The resulting kernels are shown in the tables as follows.

For the case of 0^-, we express $K^{ij}.\Delta_{n'}$ as

$$\Delta_{n'} K^{ij} = \frac{2}{\pi}\int_0^{\pi} d\theta \sin^2\theta \, \frac{R^{ij}_{(0)}(p,q,\theta)}{(p^2-q^2-m^2)^2+4p^2q^2\cos^2\theta} \tag{32.a}$$

and for 1^-

$$\Delta_{n'} K^{ij} = \int d\theta \sin^2\theta \, R^{ij}_{(1)}(p,q,\theta) \frac{\mathcal{C}^{L}_{n}(\theta)\mathcal{C}^{L'}_{n'}(\theta)}{(p^2-q^2-m^2)^2+4p^2q^2\cos^2\theta} \tag{32.b}$$

The $R^{ij}_{(0)}$ and $R^{ij}_{(1)}$ are shown in Tables 5 and 6, respectively.

Both these systems of one-dimensional integral equations do not exceed the memory of the computer even if we allow as much as 12 points for each variable. This is sufficient by use of Legendre Gaussian integration. The integrations over the angle θ in the kernel were also performed numerically and the results tabulated for faster use in the actual matrix calculation. We have made all integrals well defined by introducing a cut-off at $k^2 = m^2$ and have evaluated the Fredholm denominators $D(S)$ for the two states 0^- and 1^-. Figure 2 shows the behaviour of D for 0^- as a function of the coupling λ. We see that there exists an anomalous solution for negative coupling $\lambda = -2.8$. The smallest positive coupling $\lambda = 2.31$ $(m_B = 0,\ m/\mu = 10)$ is the one of physical interest, while the figure shows that one can also get solutions for higher λ. A similar situation holds for 1^-. In Figs. 3 and 4, we have plotted the couplings for 0^- and 1^- against the bound state mass squared. We observe that both are monotonic decreasing functions, a result one might expect intuitively, although it has not been proved analytically for any one of the B-S equations. In Fig. 5, we have drawn both the couplings for 0^- and 1^- for the value $\alpha = m/\mu = 10$ on the same plot. The interesting observation is that for weak binding and for a given coupling constant, the 1^- (triplet state) is bound more tightly than the 0^- (singlet). For strong binding this situation is reversed in that again for a given coupling the 1^- becomes a heavier particle than the 0^-. For larger values of the cut-off, the numerical values of λ decrease but we found that the qualitative situation as given in Figs. 2 to 5 is maintained.

4. INHERENT CUT-OFF DEPENDENCE OF THE SPINOR B-S PROBLEM

An investigation of the special case $\mu = 0$, $m_B = 0$ reveals a fundamental feature of the spinor case (in contrast to the spinless problem), namely, the need for introducing a cut-off in order to obtain physically meaningful solutions. In this Section we propose to study the cut-off dependence of the solutions by discussing the limiting case of the B-S equation describing the 0^- state. Before taking up Eq. (32), it is instructive to discuss briefly the solutions obtained by Goldstein[5] for the homogeneous equation when the exchanged particle as well as the bound state has vanishing mass.

Retaining only the S waves, the wave function for a + parity bound state is decomposable into scalar invariant amplitudes, using a basis of the sixteen γ matrices, as[23]

$$\psi(q) = \left[\psi_1(q) + \not{q}\, \psi_2(q) + \not{p}\, \psi_3(q) + (\not{p}\not{q} - \not{q}\not{p})\, \psi_4(q) \right] \tag{33}$$

since p,q are the only momenta available in the problem. The wave function here transforms according to

$$\mathcal{P}\, \psi(q_0, \vec{q}) = \gamma_0\, \psi(q_0, -\vec{q})\, \gamma_0 = \psi(q_0, \vec{q}) \tag{34}$$

under spatial reflections (in the c.m. system) whereas for odd intrinsic parity, a factor γ_5 will be inserted in the right-hand side of Eq. (33). When the total energy is zero, corresponding to a massless bound state, the ansatz (33) degenerates into

$$\psi(q) = \psi_1(q) + \not{q}\,\psi_2(q) \tag{35}$$

Now the B-S equation for the fermion-fermion bound state problem is $(p=0,\ \mu=0)$:

$$\psi(q) = \frac{i\lambda}{\pi^2}\int d^4k\, \frac{(\not{q}+m)\,\Gamma\,\psi(k)\,\Gamma\,(m-\not{q})}{(q^2-m^2)^2\ \ (q-k)^2} \tag{36}$$

where $\Gamma = 1$, γ_5 or γ_μ depending on the nature of the interaction. If the form (35) is inserted above, one derives, for ψ_1 :

$$\psi_1(q) = -\frac{i\lambda}{\pi^2}\int d^4k\,(\Gamma)^2\,\frac{\psi_1(k)}{(q^2-m^2)(q-k)^2} \tag{37}$$

It should be noted that the equations do not diagonalize in such a way if one considers negative parity, or the fermion-antifermion bound state with positive intrinsic parity. It is easily verified, however, that for the case of our interest, viz., fermion-antifermion bound state of negative parity that causes the singlet 0^- ground state, one again obtains Eq. (37).

This equation can be solved as a differential hypergeometric equation and is known [5)] to possess solutions for all positive values of λ. This situation is related to the highly singular nature of the potential at the origin (in x space). As can be verified by power counting, Eq. (37) is seen to have a solution only if $\psi(q)$ falls off asymptotically as fast as q^{-3}. The exact solution, on the other hand, has an asymptotic behaviour which is governed by the coupling constant and hence makes the integrals diverge, as was shown by Goldstein. A well-defined procedure would then be the introduction of a cut-off at some large momentum Ω . Goldstein used this procedure to select one specific solution for λ out of the continuum by the observation that for $\lambda = 1/4$ his solution apparently becomes independent of Ω. So it would appear that the only solution for ψ that is insensitive to the cut-off is the one for $\lambda = 1/4$. We point out, however, that if one sets $\lambda = 1/4$ in Goldstein's solution [his equation (17.c)] :

$$\Psi(s) = \frac{1}{\alpha}\Big[C_1(\alpha)\, F(1+\alpha, \alpha, 2\alpha; s^{-1})(-s)^{-\alpha-1} + C_2(\alpha)\, F(2-\alpha, 1-\alpha, 2-2\alpha; s^{-1})(-s)^{\alpha-2}\Big] \tag{38}$$

where $s = q^2$ and $\lambda = \alpha(1-\alpha)$, one finds that ψ vanishes identically for all q^2. This comes about because at this point the two hypergeometric functions F become identical and $C_1 = -C_2$. Equation (38) therefore is no longer the non-trivial solution and we conclude that it is necessary to accept the introduction of a cut-off as a meaningful procedure and to look for solutions which are cut-off dependent. One cannot imagine that momenta which are much higher than the highest mass in the problem should play an important role in a well-defined bound state problem. Also in an even more realistic situation our vertices which are simple coupling constants should be replaced by form factors which then provide the cut-off and guarantee convergence of the integrals.

Goldstein only discusses the full equation (37) and therefore the principal quantum number n does not appear in his analysis. We can show that the ^1S state with $n = 0$ indeed satisfies Goldstein's equation. A look at our Table 5 tells us that for $p_0 = 0$ the four coupled equations reduce to block diagonal form 3 + 1. One solution is provided by solving only the equation satisfied by the amplitude A_1 (in the notation of Section 3) :

$$A_1(q) = 8\lambda \int k^3 dk \; \Delta_0(k,q) \frac{1}{(q^2+m^2)} A_1(k) \tag{39}$$

Setting $q^2 = s$, $k^2 = t$ we obtain for $\mu = 0$

$$s(s+m^2)A_1(s) = \lambda \int_0^s t\,dt\,A_1(t) + \lambda s \int_s^\infty A_1(t)\,dt \tag{40}$$

which is equivalent to Eq. (32) of Ref. 5). We have investigated the solutions of this equation by introducing a cut-off Ω on the computer. We find that for finite Ω the eigenvalue λ behaves roughly like Ω^{-1}.

Considering this situation, we must accept the fact that the solutions of the spinor equation are cut-off dependent. In the entire calculation of the present note we have kept the cut-off equal to the highest mass appearing in the problem : $\Omega = m$.

5. DISCUSSION

We have presented mainly in Section 3 an accurate solution of the B-S equation for a fermion-antifermion pair forming a 0^- or a 1^- bound state via the exchange of a pseudoscalar meson. The equations were found to have meaningful solutions only after the introduction of a cut-off. For the present analysis this was kept fixed ($\Omega = m$) at the highest mass which is present in the problem. The main results are contained in the Figs. 1-6. Therefore we will keep this discussion short and refer to these figures for the specific numbers. The main qualitative results are as follows. We find that the anomalous solutions which are known as odd solutions in the spinless equal mass case are still present in that our systems have solutions for negative coupling constants. Restricting our discussion to the smallest positive coupling constant we found that this one is a monotonic function of the bound state mass even in the spin $\frac{1}{2}$ problem. Figure 4 shows that for a given ratio m/μ the two curves representing $\lambda(m_B)$ cross each other. This means that for a strongly bound system the 1^- particle is heavier than the 0^-. This situation is observed in the elementary particle spectrum where the vector mesons are generally heavier than the pseudoscalar particles. For weak binding we found that for given λ the 0^- is less tightly bound than the 1^- a situation which is familiar from the deuteron where only the triplet state is bound [24).

We may recall that in an earlier note we have investigated the influence of orbital angular momentum excitation (Regge recurrences) in a scalar B-S equation quark model [25) on the masses of $q\text{-}\bar{q}$ bound states. It was found that as long as we admit only the single Yukawa type potential in the kernel of the equation the states with orbital angular momentum $\ell \geq 1$ become much too massive to be interpreted as meson states. This result persists in the spin case where we have verified that for $J = 2$ we would again get a mass which is out of range. The result of the present calculation shows that for strong coupling one can get quite close to the observed mass ratio 4:1 between vector mesons and pseudoscalar mesons. Figure 6 shows that for a quark mass of 10μ we find a ratio 4.2:1. To our knowledge this is the first accurate calculation which gives anything so close to the realistic ratio in the bound state region of the quark model. Unless one introduces an effectively non-relativistic square well type potential and a hard core which in momentum space corresponds to a cut-off much lower than the one we have used, one may consider this result as a hint to look for the ρ meson actually as a 1^- triplet state of a $q\text{-}\bar{q}$ system and that ℓ excitation is negligible. The obvious difficulty with this model is that the $J = 2$ mesons now must be constructed out of four quarks. A more detailed investigation of this question, as well as an analysis of the influence of vector meson exchange using the tools presented in this note, is under investigation. We also hope to generalize the perturbation formula derived for the scalar system in Section 2 to the spin $\frac{1}{2}$ B-S equation with cut-off.

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12) See, e.g., Tables of Integral Transforms, Vol.II, Bateman Manuscript Project, McGraw Hill Co. (1954), Editor A. Erdélyi, p.283.

13) It should be remarked that Gourdin's final equations [Ref. 4] are the result of an approximate evaluation of the integral in Eq. (8) obtained by putting the two particles on their mass shells. This approximation introduces an error of at least 30% in the coupling constant and is not good at all in the lower part of the bound state region where we are interested in.

14) We show below (Fig. 2) that the spin equations have solutions for both positive and negative discrete λ ; therefore the kernel has no definiteness properties.

15) We have used the CERN CDC 6600. The evaluation of the determinant of one 100×100 matrix takes about seven seconds.

16) F. Gürsey, T.D. Lee and M. Nauenberg, Phys.Rev. 135, B647 (1964).

17) D. Holdsworth, in Oxford, has derived a similar perturbation formula (private communication).

18) Factors 2 and i have been inserted in Eq. (21) to make later formulae look simpler.

19) J.M. Jauch and F. Rohrlich, The Theory of Photons and Electrons, Addison Wesley Publ.Co. (1959), p.275.

20) Our convention for the γ matrices is

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} , \quad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} , \quad \gamma_5 = -i \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

We use the metric so that $a \cdot b = a_0 b_0 - \vec{a} \cdot \vec{b}$.

21) A.R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press (1957), p.81.

A complete list of the formulae is obtained by generalizing to Euclidean four-space the ones given in the Appendix of Ref. 7).

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23) In contrast to the more general ansatz (21), this form restricts J to be equal to L. It suffices for our purpose in this Section since we are interested in the S waves only.

24) In the case of positronium, the singlet state is lower than the triplet state but this is a consequence of the fact that there the major correction to the Coulomb potential comes from a vector exchange.

25) We have not included here any references on the quark models. See Ref. 3) for a list of references on this subject.

TABLE CAPTIONS

Table 1 Comparison of the eigenvalues of the scalar B-S equation as obtained by numerical solution and from relations (18) and (19) for different values of m/μ and for zero and unit mass of the bound state.

Table 2 The kernel matrix of the full spin $\frac{1}{2}$ equation corresponding to the K_{ij} of Eq. (25). $q^2 = \vec{q}^{\,2}$ here and $\alpha, \beta, \gamma = 1,2,3$ refer to the space components.

Table 3 Classification of the amplitudes into even and odd parity triplet and singlet states. Superscripts (o,e) refer to the (odd, even) character of the functions under $q_o \to -q_o$.

Table 4 Explicit expressions for $\mathcal{C}_n^L(\theta)$ for a few low and integer values of n and L

Table 5 The kernel $R^{ij}_{(0)}$ of Eq. (32.a) for the 0^- state. $q = \sqrt{q_o^2+q^2}$.

Table 6 The kernel $R^{ij}_{(1)}$ of Eq. (32.b) for the 1^- state. $q = \sqrt{q_o^2+q^2}$.

Table 1

m/μ	λ (B.S. Eqn. $m_B = 0$)	λ (Formula, $m_B = 0$)	λ (B.S. Eqn. $m_B = \mu$)	λ (Formula, $m_B = 0$)
1	3.4182	3.693	2.9395	3.293
10	2.047	2.06298	2.058	2.05898
20	2.020	2.01921	2.021	2.01821
50	2.004	2.00381	2.004	2.00365
70	2.002	2.00208	2.002	2.00199
100	2.000	2.00109	2.000	2.00105

Table 2

	S	T	V_β	U_β	B	C	F_β	G_β
S	$p_o^2-q_o^2-q^2+m^2$	$-2mq_o$	0	0	0	0	$2ip_oq_\beta$	$-2mq_\beta$
T	$-2mq_o$	$q_o^2-q^2-p_o^2-m^2$	0	0	0	0	0	$2q_oq_\beta$
V_α	0	0	$m^2-q_o^2+q^2+p_o^2$ $-2q_\alpha q_\beta$	$2mq_o$	$2ip_oq_\alpha$	$-2mq_\alpha$	$-2iq_o\epsilon_{\alpha\beta\gamma}q_\gamma$	0
U_α	0	0	$2mq_o$	$q_o^2+q^2-m^2-p_o^2$ $-2q_\alpha q_\beta$	0	$-2q_oq_\alpha$	$2im\epsilon_{\alpha\beta\gamma}q_\gamma$	$2p_o\epsilon_{\alpha\beta\gamma}q_\gamma$
B	0	0	$-2ip_oq_\beta$	0	$m^2-p_o^2+q_o^2+q^2$	$-2mip_o$	0	0
C	0	0	$-2mq_\beta$	$-2q_oq_\beta$	$2mip_o$	$p_o^2-m^2-q_o^2+q^2$	0	0
F_α	$-2ip_oq_\alpha$	0	$-2iq_o\epsilon_{\alpha\beta\gamma}q_\gamma$	$2im\epsilon_{\alpha\beta\gamma}q_\gamma$	0	0	$m^2-p_o^2-q^2+q_o^2$ $+2q_\alpha q_\beta$	$-2mip_o$
G_α	$-2mq_\alpha$	$2q_oq_\alpha$	0	$-2p_o\epsilon_{\alpha\beta\gamma}q_\gamma$	0	0	$2mip_o$	$p_o^2-m^2-q_o^2-q^2$ $+2q_\alpha q_\beta$

Table 3

Parity $(-1)^J$		Parity $(-1)^{J+1}$	
Singlet	Triplet	Singlet	Triplet
S^o_J T^e_J U^o_J V^e_J	S^e_J T^o_J U^e_J V^o_J	B^e C^e F^o_J G^o_J	B^o_J C^o_J F^e_J G^e_J
F^o_{J+1} F^o_{J-1} G^o_{J+1} G^o_{J-1}	F^e_{J+1} F^e_{J-1} G^e_{J+1} G^e_{J-1}	V^e_{J+1} V^e_{J-1} U^o_{J+1} U^o_{J-1}	V^o_{J+1} V^o_{J-1} U^e_{J+1} U^e_{J-1}

Table 4

n \ L	0	1	2
0	$\sqrt{\frac{2}{\pi}}$	0	0
1	$\sqrt{\frac{2}{\pi}}\ 2\cos\Theta$	$\sqrt{\frac{8}{3\pi}}\ \sin\Theta$	0
2	$\sqrt{\frac{2}{\pi}}\ \frac{\sin 3\Theta}{\sin\Theta}$	$\sqrt{\frac{1}{\pi}}\ 4\cos\Theta\sin\Theta$	$\frac{4}{\sqrt{5\pi}}\sin^2\Theta$
3	$\sqrt{\frac{2}{\pi}}\ \frac{\sin 4\Theta}{\sin\Theta}$	$\sqrt{\frac{8}{15\pi}}\ (12\cos^2\Theta - 2)\sin\Theta$	$\frac{16}{\sqrt{10\pi}}\cos\Theta\ \ \sin^2\Theta$

Table 5

$(m^2+p_o^2+q^2)\Delta_o$	$-2mp_o\Delta_o$	$-\frac{4}{\sqrt{3}}\, p_o q\Delta_1 \sin^2\Theta$	0
$2mp_o\Delta_o$	$-\Delta_o[(p_o^2+m^2)+q^2 \cos 2\Theta]$	$-\frac{4}{\sqrt{3}}\, mq\Delta_1 \sin^2\Theta$	$-4\sqrt{2}\, q^2\Delta_2 \sin^2\Theta \cos^2\Theta$
$\frac{4}{\sqrt{3}}\, p_o q\Delta_o \sin^2\Theta$	$-\frac{4}{\sqrt{3}}\, mq\Delta_o \sin^2\Theta$	$\frac{4}{3}(m^2-p_o^2-q^2)\Delta_1 \sin^2\Theta$	$8\sqrt{\frac{2}{3}}\, mq\Delta_2 \sin^2\Theta \cos^2\Theta$
0	$-4\sqrt{2}\, q^2\Delta_o \sin^2\Theta \cos^2\Theta$	$8\sqrt{\frac{2}{3}}\, mq\Delta_1 \sin^2\Theta \cos^2\Theta$	$8\Delta_2 \sin^2\Theta \cos^2\Theta$ $\times[(p_o^2-m^2)+q^2 \cos 2\Theta]$

Table 6

$(q^2-m^2+p^2)$	$2mq\cos\Theta$	$\frac{2}{\sqrt{3}}pq\sin\Theta$	$\sqrt{\frac{8}{3}}pq\sin\Theta$	$-\frac{2}{\sqrt{3}}mq\sin\Theta$	$-\sqrt{\frac{8}{3}}mq\sin\Theta$	0	0
$2mq\cos\Theta$	$(m^2-p^2-q^2\cos 2\Theta)$	0	0	$\frac{1}{\sqrt{3}}q^2\sin 2\Theta$	$\sqrt{\frac{2}{3}}q^2\sin 2\Theta$	0	0
$\sqrt{\frac{8}{3}}pq\sin\Theta$	0	$-p^2-m^2-q^2\cos^2\Theta$ $+\frac{1}{3}q^2\sin\Theta$	$2\frac{\sqrt{2}}{3}q^2\sin^2\Theta$	$-2mp$	0	$-2\sqrt{\frac{2}{3}}mq\sin\Theta$	$\sqrt{\frac{2}{3}}q^2\sin 2\Theta$
$\frac{2}{\sqrt{3}}pq\sin\Theta$	0	$\frac{2\sqrt{2}}{3}q^2\sin^2\Theta$	$-p^2-m^2-q^2\cos^2\Theta$ $-\frac{1}{3}q^2\sin^2\Theta$	0	$-2mp$	$-\frac{2}{\sqrt{3}}mq\sin\Theta$	$\frac{1}{\sqrt{3}}q^2\sin 2\Theta$
$-\sqrt{\frac{8}{3}}mq\sin\Theta$	$\sqrt{\frac{2}{3}}q^2\sin 2\Theta$	$2mp_o$	0	$m^2+p^2+q^2\cos^2\Theta$ $-\frac{1}{3}q^2\sin^2\Theta$	$2\sqrt{\frac{2}{3}}q^2\sin^2\Theta$	$\sqrt{\frac{8}{3}}pq\sin\Theta$	0
$-\frac{4}{\sqrt{3}}mq\sin\Theta$	$\sqrt{\frac{1}{3}}q^2\sin 2\Theta$	0	$2mp_o$	$2\sqrt{\frac{2}{3}}q^2\sin^2\Theta$	$m^2+p^2+q^2\cos^2\Theta$ $-\frac{1}{3}q^2\sin^2\Theta$	$\frac{2}{\sqrt{3}}pq\sin\Theta$	0
0	0	$-2\sqrt{\frac{2}{3}}mq\sin\Theta$	$-\frac{2}{\sqrt{3}}mq\sin\Theta$	$-2\sqrt{\frac{2}{3}}pq\sin\Theta$	$-\frac{2}{\sqrt{3}}pq\sin\Theta$	$m^2-p^2-q^2$	$-2mq\cos\Theta$
0	0	$\sqrt{\frac{2}{3}}q^2\sin 2\Theta$	$\frac{1}{\sqrt{3}}q^2\sin 2\Theta$	0	0	$-2mq\cos\Theta$	$p^2-m^2+q^2\cos 2\Theta$

FIGURE CAPTIONS

Figure 1 Shows the coupling constant λ as function of $S = m_B^2$ for different ratios $\alpha = m/\mu$ for S-waves. The values on the lower right indicate the critical couplings for bound states of zero binding energy. For $\mu = 0$ we get the Coulomb case and know that $\lambda(0) = 2$ and $\lambda(4m^2) = 0$. The curves agree well with the values given in Refs. 2) and 3).

Figure 2 Shows the Fredholm denominator $D(\lambda)$ in arbitrary units as function of the coupling constant λ. By definition $D(0) = 1$. We see that there are zeros at least for $\lambda = -2.8$, 2.31, 5.7 and 12.4. Here we have taken $S = 0$ and $\alpha = 10$.

Figure 3 Shows the coupling constant $\lambda (= g^2(4\pi)^{-2})$ as function of $S = m_B^2$ for a 0^- bound state and for different ratios α. For $\alpha = 1$ the value of λ that gives a zero mass bound state is $\lambda = 6.59$.

Figure 4 Shows the coupling constant λ as function of $S = m_B^2$ for a 1^- bound state and different ratios α. The couplings which give a zero mass bound state for $\alpha = 1,2$ are $\lambda_1 = 8.66$ and $\lambda_2 = 4.38$.

Figure 5 Shows $\lambda(S)$ for both a 0^- and a 1^- bound state and for $\alpha = 5$. It shows that for $S \sim 0.6\ m^2$ the two curves cross over.

Figure 6 Shows $\lambda(m_B)$, linear, for both a 0^- and a 1^- bound state in the strong binding region $0 \leq m_B \leq m$. We see that for $\alpha = 10$ the same coupling constant $\lambda = 2.29$ which gives a 0^- with $m_B = \mu$ produces a 1^- state with $m_B = 4{:}2\mu$.

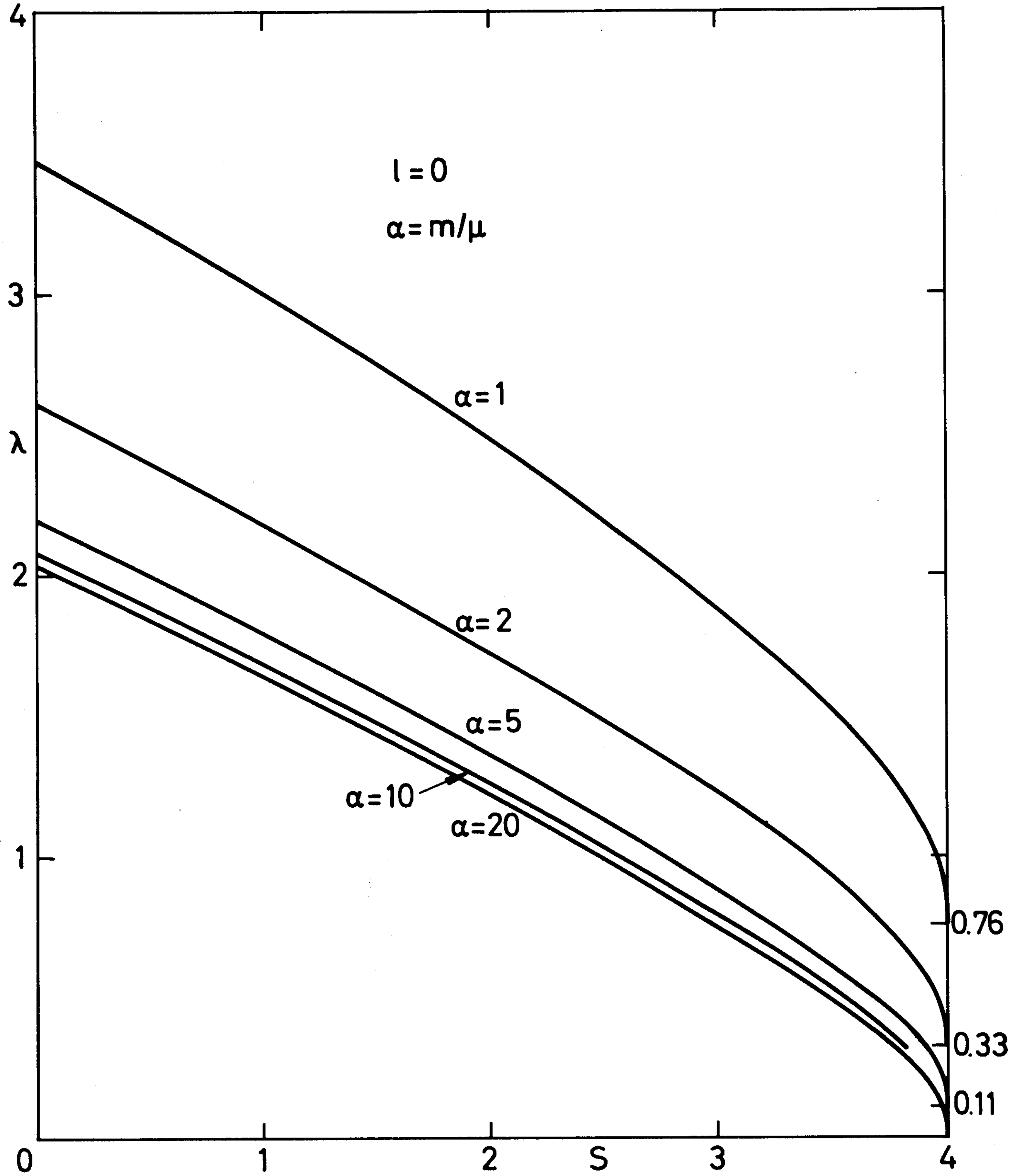

FIG. 1

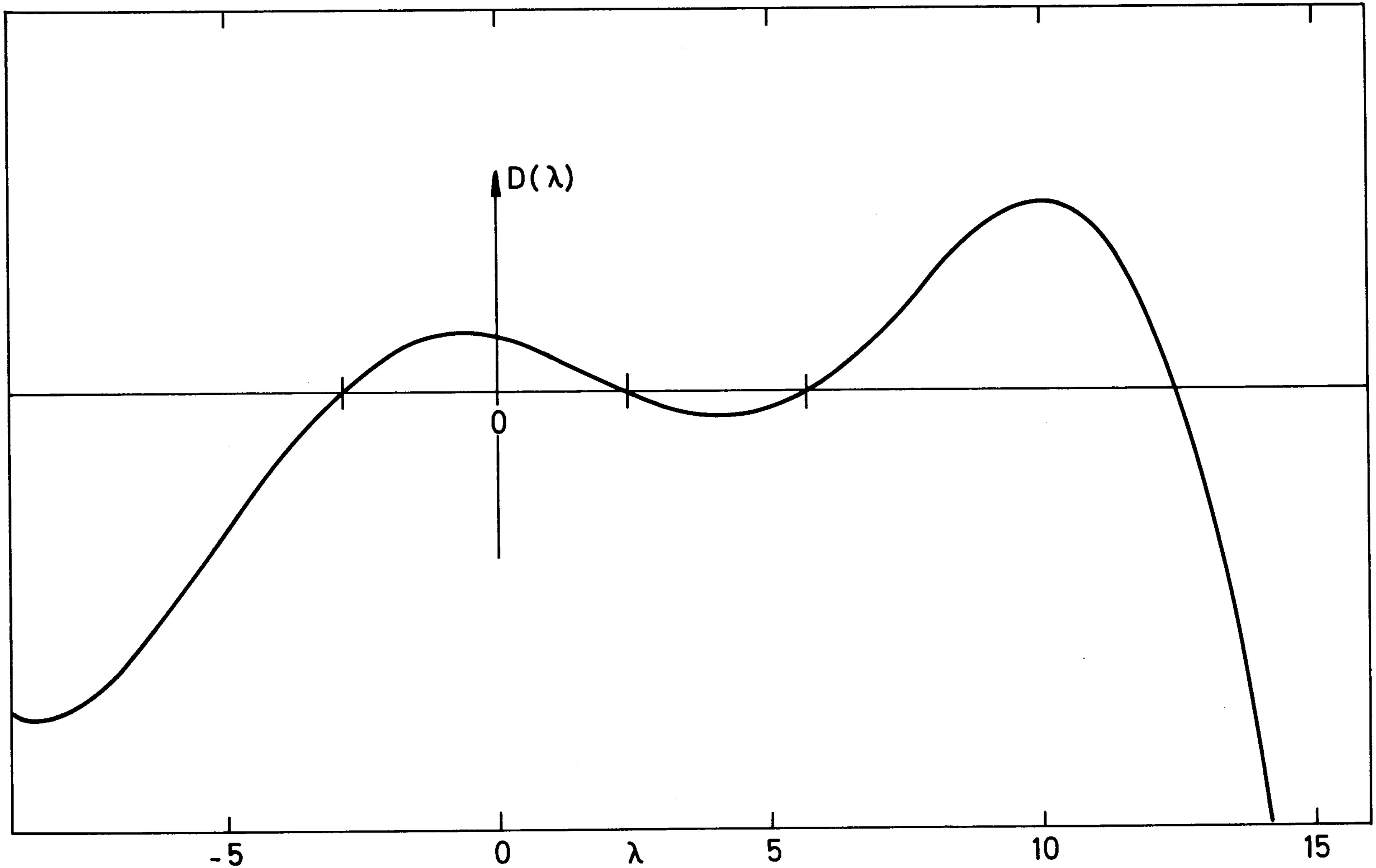

FIG. 2

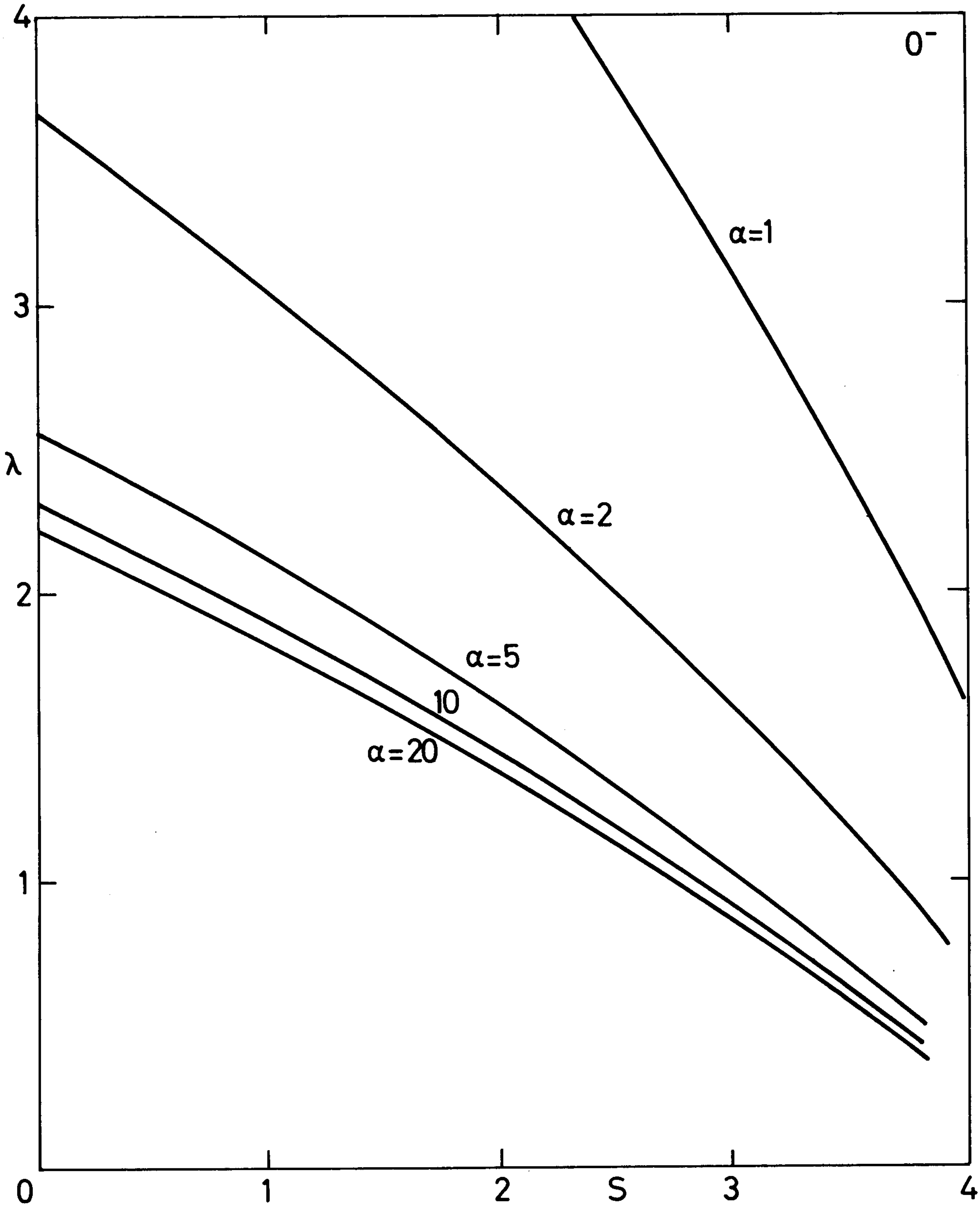

FIG. 3

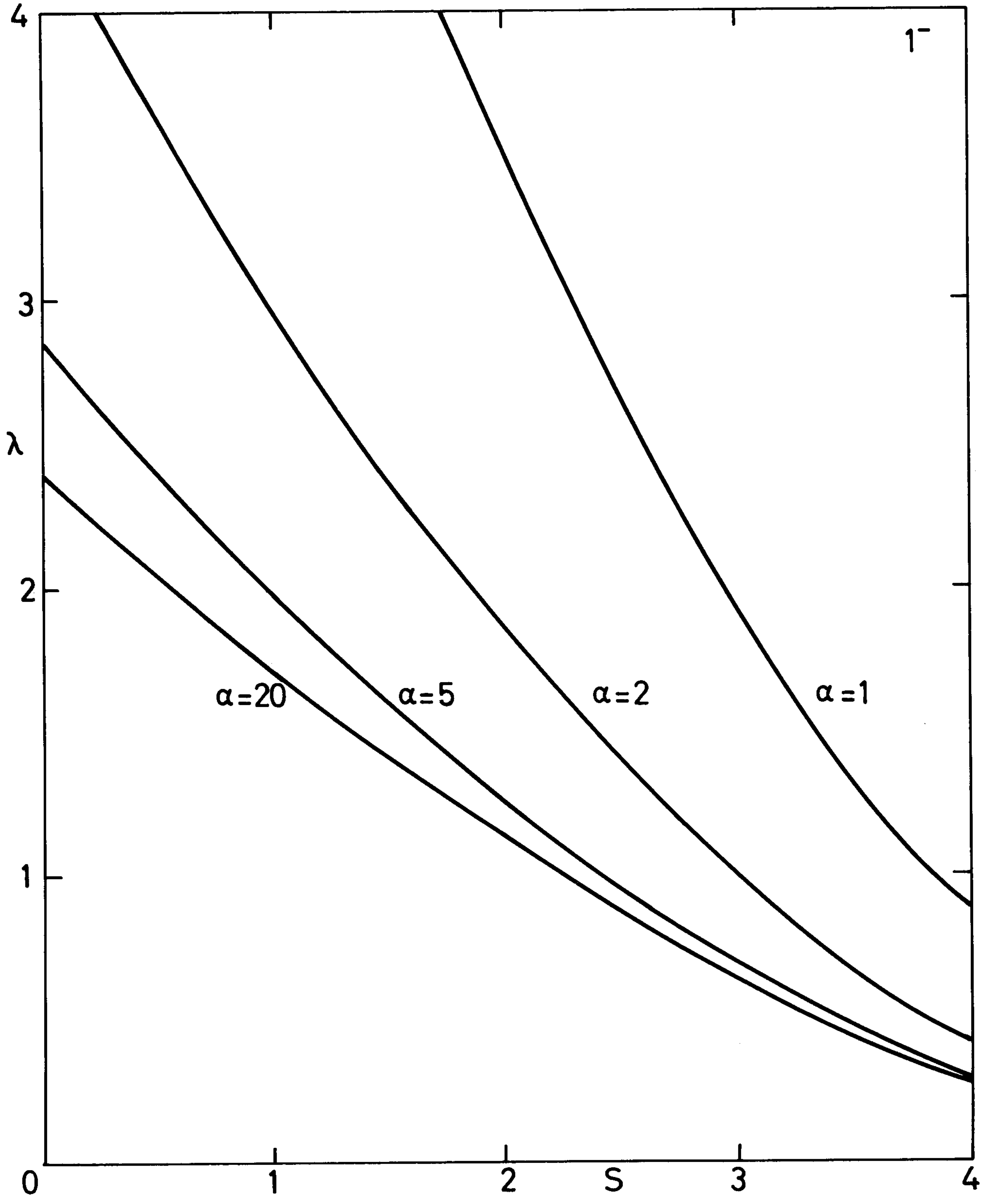

FIG. 4

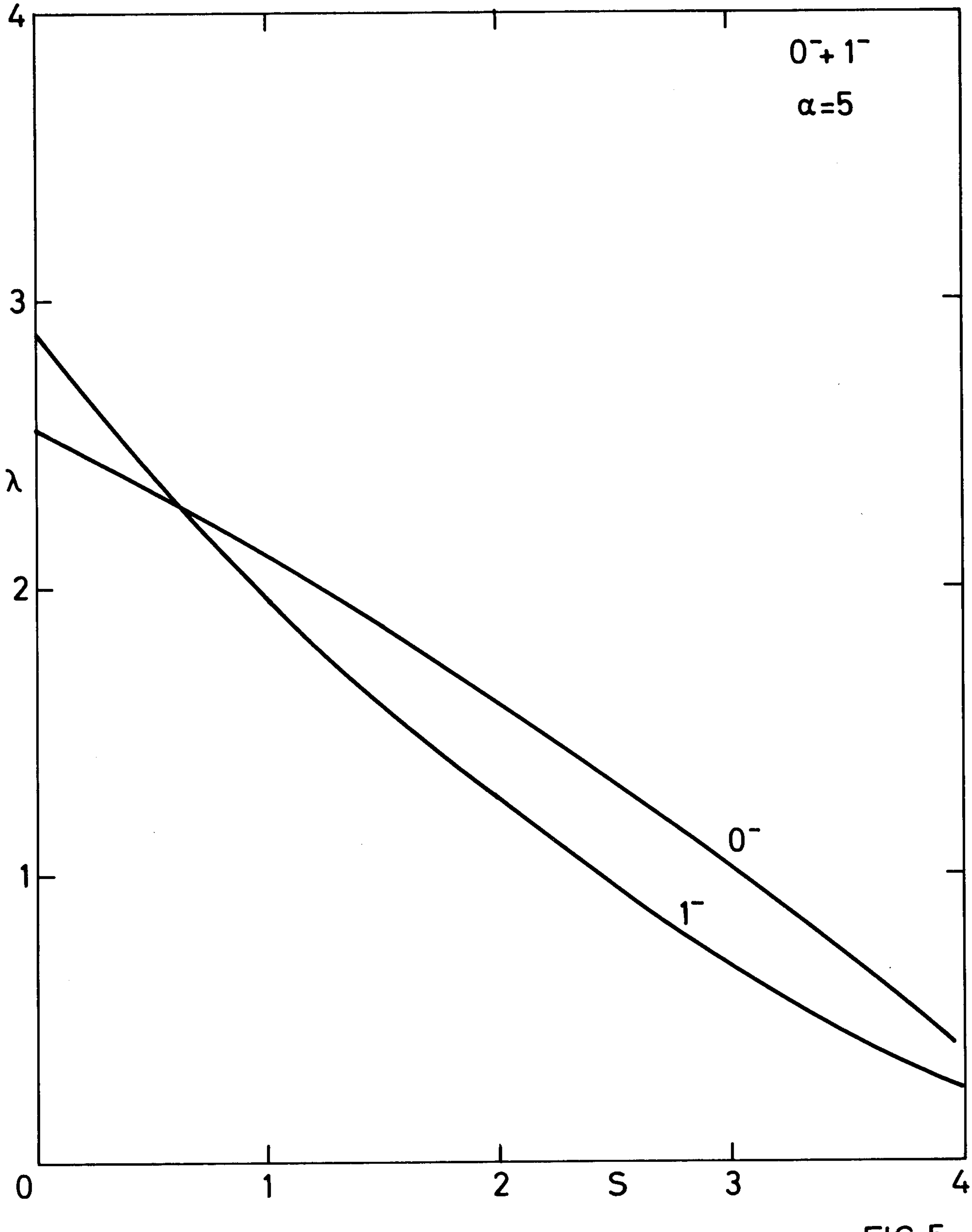

FIG. 5

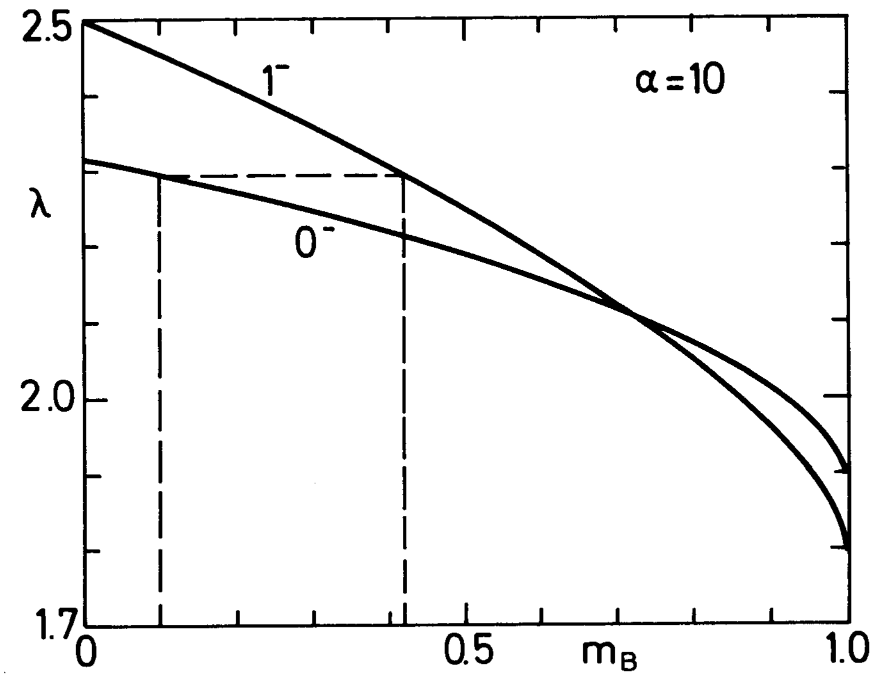

FIG.6