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Accurate solution of the Orr–Sommerfeld stability equation

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The Orr–Sommerfeld equation is solved numerically using expansions in Chebyshev polynomials and the QR matrix eigenvalue algorithm. It is shown that results of great accuracy are obtained very economically. The method is applied to the stability of plane Poiseuille flow; it is found that the critical Reynolds number is 5772·22. It is explained why expansions in Chebyshev polynomials are better suited to the solution of hydrodynamic stability problems than expansions in other, seemingly more relevant, sets of orthogonal functions.

1. Introduction

In this paper we reconsider the problem of the stability of plane Poiseuille flow, using expansions in Chebyshev polynomials to approximate the solutions of the Orr–Sommerfeld equation. We obtain results that are considerably more accurate than those obtained previously (and, apparently, at considerably less computational expense). Our methods extend to stability problems for a wide variety of flows including Couette flows and Poiseuille flow in a pipe (Davey & Nguyen 1971).

The present work originated with the author's development of Chebyshev polynomial approximations to time-dependent viscous flows within rigid boundaries (Orszag 1971*a*). It has been shown that Chebyshev approximations require considerably less computer time and storage to achieve reasonably accurate flow simulations than are required by finite-difference approximations. Also, Chebyshev approximations permit simulations of very high accuracy with little extra computation. It is the latter advantage that is particularly significant for the present paper.

The stability problem that we wish to study numerically is that of plane Poiseuille flow in a channel. We measure all lengths in units of the half-width of the channel and velocities in units of the undisturbed stream velocity at the centre of the channel. In the Poiseuille case the undisturbed stream velocity in the x direction is $\overline{u}(y) = 1-y^2$. The side walls are at $y = \pm 1$ and the Reynolds number based on channel half-width and centre-stream velocity is $R = 1/\nu$, where ν is the kinematic viscosity.

We assume a two-dimensional disturbance for which the y component of the perturbation velocity is proportional to the real part of the expression

$$V = v(y)\exp\left[i\alpha(x-\lambda t)\right], \tag{1}$$

with α real. It may be shown (Lin 1955, §1.3) that the velocity perturbation equations obtained by linearization of the Navier–Stokes equations are reducible to the Orr–Sommerfeld equation

$$\frac{d^4v}{dy^4}-2\alpha^2\frac{d^2v}{dy^2}+\alpha^4v-i\alpha R\left[(\bar{u}-\lambda)\left(\frac{d^2v}{dy^2}-\alpha^2v\right)-\frac{d^2\bar{u}}{dy^2}v\right]=0, \tag{2}$$

with boundary conditions

$$v(y)=0,\quad v'(y)=0\quad \text{at}\quad y=\pm 1. \tag{3}$$

According to (1) a solution to (2) and (3) with $\mathrm{Im}(\lambda)>0$ is an unstable linear eigenmode, in the sense that the amplitude of the disturbance grows exponentially with time.

In §2 we explain some properties of the orthogonal polynomial expansions used to solve the eigenvalue problem posed by (2) with (3). In particular, we explain why Chebyshev expansions are superior to expansions in other sets of orthogonal functions that may seem *a priori* to be more relevant to the solution of (2) with (3). In §3 we develop Chebyshev approximations to the solution of (2) with (3) for Poiseuille flow. In §4 some numerical results are discussed. Finally, in §5 we comment on the extension of the present methods to more general stability problems. Some results that are required in §§3 and 5 involving manipulations of Chebyshev polynomial expansions are given in the appendix.

2. Convergence of orthogonal expansions

An important difference between finite-difference approximations to the eigenvalues and eigenfunctions of the Orr–Sommerfeld equation and the Chebyshev approximations advocated here is their order of accuracy. Finite-difference approximations give only a finite order of accuracy in the sense that errors behave asymptotically like $(\Delta x)^p$ for some finite p when the grid scale Δx approaches zero. On the other hand, if the undisturbed velocity profile $\bar{u}(y)$ is infinitely differentiable, the Chebyshev polynomial approximations used here are of infinite order in the sense that errors decrease more rapidly than any power of $1/N$ as $N\to\infty$, where N is the number of Chebyshev polynomials retained in the approximation.

The latter statement is verified as follows. If $\bar{u}(y)$ is infinitely differentiable all the eigenfunctions $v(y)$ of the Orr–Sommerfeld equation (2) are infinitely differentiable for $-1\leqslant y\leqslant 1$ (with one-sided derivatives at the end-points). Let $T_n(x)$ denote the nth-degree Chebyshev polynomial of the first kind, defined by

$$T_n(\cos\theta)=\cos n\theta \tag{4}$$

for all non-negative integers n (see, e.g. Hamming 1962 (chapter 19) or Fox & Parker 1968). Some examples are $T_0(x)=1,\ T_1(x)=x,\ T_2(x)=2x^2-1$. It is possible to expand $v(y)$ in the interval $-1\leqslant y\leqslant 1$ as

$$v(y)=\sum_{n=0}^{\infty}a_nT_n(y), \tag{5}$$

where

$$a_n=\frac{2}{\pi c_n}\int_{-1}^{1}v(y)\,T_n(y)\,(1-y^2)^{-\frac{1}{2}}\,dy, \tag{6}$$

with $c_0 = 2$, $c_n = 1$ $(n > 0)$. The rapidity of convergence of (5) for $|y| \leqslant 1$ is easily demonstrated by observing that

$$f(\theta) = v(\cos\theta)$$

is an infinitely differentiable, even, *periodic* function of θ. Consequently the theory of Fourier series ensures that $f(\theta)$ possesses a Fourier cosine expansion

$$f(\theta) = \sum_{n=0}^{\infty} a_n \cos n\theta \tag{7}$$

with the property that the error after N terms decreases more rapidly than any power of $1/N$ as $N \to \infty$. The expansion (7) is precisely (5) for $y = \cos\theta$. Alternatively, by directly estimating the orders of magnitude of a_n and derivatives of $T_n(x)$ as $n \to \infty$, it also follows that Chebyshev expansions give infinite-order approximations that may be differentiated termwise an arbitrary number of times throughout the interval $-1 \leqslant y \leqslant 1$ (Orszag 1971*a*). It is possible to establish the same result for expansions in Legendre polynomials (Orszag 1971*b*). Legendre expansions are also conveniently applied to the Poiseuille flow stability problem; they give results that are quantitatively quite close to those reported in §4.

The infinite-order accuracy of Chebyshev and Legendre polynomial approximations to infinitely differentiable functions, *no matter what the boundary values of the functions or their derivatives*, should be contrasted with the situation when other sets of orthogonal functions are used. In most cases, although expansions of $v(y)$ are made in terms of orthogonal functions that seem to bear much closer relation to the Orr–Sommerfeld eigenfunctions than do the orthogonal polynomials, only finite-order rates of convergence are obtained. For example, Grosch & Salwen (1968) studied the stability of plane Poiseuille flow using expansions in orthogonal functions defined by

$$\frac{d^4\phi_n}{dy^4} - 2\alpha^2 \frac{d^2\phi_n}{dy^2} + \alpha^4\phi_n = \lambda_n \phi_n, \tag{8}$$

$$\phi_n(\pm 1) = \phi_n'(\pm 1) = 0. \tag{9}$$

These orthogonal functions are generalizations of the Chandrasekhar–Reid functions (Chandrasekhar 1961, appendix V). The latter functions are obtained by setting $\alpha = 0$ in (8). It follows from (8) and (9) that

$$\frac{d^4\phi_n}{dy^4} - 2\alpha^2 \frac{d^2\phi_n}{dy^2} = 0 \tag{10}$$

at $y = \pm 1$. Hence, unless $v(y)$ satisfies (10) at $y = \pm 1$, the expansion of $v(y)$ in terms of $\phi_n(y)$ cannot be differentiated termwise four times at $y = \pm 1$. However, it follows from (2) and (3) that

$$\frac{d^4 v}{dy^4} - 2\alpha^2 \frac{d^2 v}{dy^2} = -i\alpha R\lambda \frac{d^2 v}{dy^2} \tag{11}$$

at $y = \pm 1$. The right-hand side of (11) is generally non-zero at $y = \pm 1$, as it is proportional to the perturbation of viscous wall stress. Upon performing suitable integration by parts in the expression

$$\int_{-1}^{1} v(y)\,\phi_n(y)\,dy$$

for the expansion coefficients of $v(y)$ in terms of normalized ϕ_n and noting (11), it may be shown that the nth expansion coefficient of $v(y)$ is generally of order $1/n^5$ as $n \to \infty$; it may also be shown that the residue after N terms of the expansion is of order $1/N^5$. The origin of this behaviour is the non-uniform convergence of the four-times differentiated series near the end-points. Consequently, the results obtained by expansion of $v(y)$ in a series of $\phi_n(y)$ should not be expected to be significantly better than results obtained by fifth-order finite-difference approximations [with errors of order $(\Delta x)^5$ for a grid interval Δx].

Similarly, expansions of $v(y)$ in terms of Chandrasekhar–Reid functions, as used by Gallagher & Mercer (1962) in Orr–Sommerfeld equation studies, give only fifth-order rates of convergence. Dolph & Lewis (1958) studied the stability of plane Poiseuille flow using expansions in the functions $\phi_n(y)$ defined by

$$\frac{d^4\phi_n}{dy^4} - 2\alpha^2\frac{d^2\phi_n}{dy^2} + \alpha^4\phi_n = \lambda_n\left(\alpha^2\phi_n - \frac{d^2\phi_n}{dy^2}\right) \tag{12}$$

with $\phi_n(\pm 1) = \phi_n'(\pm 1) = 0$. The error after N terms of this expansion is generally of order $1/N^4$ as $N \to \infty$ for functions $v(y)$ satisfying (3); for example, equation (34) of Dolph & Lewis's paper shows that the expansion of $y^2\phi_m(y)$ in terms of $\phi_n(y)$ converges only like $1/N^4$.

The numerical results to be reported in §4 will illustrate the much greater accuracy achieved by Chebyshev expansions than by expansions in the functions defined by (8) or (12). Another advantage of Chebyshev expansions over expansions in the orthogonal functions (8) or (12) is the efficiency with which the coefficients may be determined from the differential equation to be solved. The N coefficients of Chebyshev expansions truncated at $T_{N-1}(x)$ may be determined in roughly the same number of arithmetic operations required to solve the differential equation (2) by finite-difference methods with N grid points (cf. §3), viz. order N operations for both calculations.

3. Chebyshev approximations for Poiseuille flow

We seek an approximate solution of (2) and (3) of the form

$$v(y) = \sum_{n=0}^{N} a_n T_n(y). \tag{13}$$

Equations for the expansion coefficients a_n are found by formally substituting (13) into (2) with $\bar{u}(y) = 1 - y^2$, re-expanding the left-hand side of (2) in terms of Chebyshev polynomials and equating the coefficients of the various $T_n(y)$ to zero.

Some details of this process are given in the appendix. The result is

$$\begin{aligned}\tfrac{1}{24} \sum_{\substack{p=n+4\\ p\equiv n\ (\mathrm{mod}\ 2)}}^{N} & [p^3(p^2-4)^2-3n^2p^5+3n^4p^3-pn^2(n^2-4)^2]\,a_p \\ - \sum_{\substack{p=n+2\\ p\equiv n\ (\mathrm{mod}\ 2)}}^{N} & \{[2\alpha^2+\tfrac{1}{4}i\alpha R(4-4\lambda-c_n-c_{n-1})]\,p(p^2-n^2)-\tfrac{1}{4}i\alpha Rc_np[p^2-(n+2)^2] \\ & -\tfrac{1}{4}i\alpha Rd_{n-2}p[p^2-(n-2)^2]\}\;a_p+i\alpha Rn(n-1)\,a_n+\{\alpha^4+i\alpha R[(1-\lambda)\,\alpha^2-2]\}\;c_n a_n \\ & -\tfrac{1}{4}i\alpha^3R[c_{n-2}a_{n-2}+c_n(c_n+c_{n-1})\,a_n+c_na_{n+2}] = 0 \qquad (14)\end{aligned}$$

for $n \geqslant 0$, where $c_n = 0$ if $n < 0$, $c_0 = 2$, $c_n = 1$ if $n > 0$, and $d_n = 0$ if $n < 0$, $d_n = 1$ if $n \geqslant 0$. The boundary conditions (3) become

$$\sum_{\substack{n=0\\ n\equiv 0\ (\mathrm{mod}\ 2)}}^{N} a_n = 0, \qquad \sum_{\substack{n=0\\ n\equiv 0\ (\mathrm{mod}\ 2)}}^{N} n^2a_n = 0, \qquad (15)$$

$$\sum_{\substack{n=1\\ n\equiv 1\ (\mathrm{mod}\ 2)}}^{N} a_n = 0, \qquad \sum_{\substack{n=1\\ n\equiv 1\ (\mathrm{mod}\ 2)}}^{N} n^2a_n = 0 \qquad (16)$$

upon using the properties $T_n(\pm 1) = (\pm 1)^n$ and $T'_n(\pm 1) = (\pm 1)^{n-1}n^2$.

Before discussing methods of solving the system (14)–(16), a simplification can be made. The equations (14)–(16) separate into two sets with no coupling between coefficients a_n for odd and even n. Therefore there exists a set of solutions with $a_n = 0$ for n odd; the corresponding solution $v(y)$ is symmetric, i.e. $v(y) = v(-y)$. Conversely, the solutions with $a_n = 0$ for n even are antisymmetric, i.e.

$$v(y) = -v(-y).$$

It turns out that the only unstable eigenmode of plane Poiseuille flow is symmetric. For the remainder of §3 we confine discussion to the symmetric modes, so that the relevant equations are (14) and (15) with $a_n = 0$ for odd n. Some results for antisymmetric modes are given in §4.

It is convenient to choose N even so that $N = 2M$. The $M+1$ unknowns are a_{2n} for $n = 0, 1, \ldots, M$. If (14) were applied for $n = 0, 2, 4, \ldots, N$, we would have $M+3$ equations from (14) and (15) in $M+1$ unknowns, with only the trivial solution $a_n = 0$ for all n no matter what the value of λ. The trouble is that (13) is generally not exact so that the formal operations leading to (14) cannot be correct for all n.

There are at least two satisfactory ways to resolve this dilemma. The first is to use Galerkin's method for constructing equations for the coefficients a_n (Orszag 1971*a*). Here the expansion (13) [with $a_n = 0$ for n odd] is specialized to an expansion in terms of the functions

$$q_{2n}(y) = T_{2n}(y) - n^2T_2(y) + (n^2-1)\,T_0(y),$$

for $n = 2, 3, \ldots, M$. The functions $q_n(y)$ are symmetric and satisfy

$$q_n(1) = q'_n(1) = 0,$$

so that the modified expansion (13) automatically satisfies the boundary conditions (3). Galerkin equations are obtained by substituting the new expansion into the left-hand side of (2) and demanding that the resulting formal expression be orthogonal to $q_{2n}(y)$ $(n = 2, \ldots, M)$ with respect to the inner product

$$(f, g) = \int_{-1}^{1} f(y) g(y) (1-y^2)^{-\frac{1}{2}} dy.$$

In this way there result $M-1$ equations for the $M-1$ coefficients of $q_{2n}(y)$ $(n = 2, \ldots, M)$, with a non-trivial solution existing only for certain eigenvalues λ. These equations for the coefficients of $q_n(y)$ are easily reformulated as equations for the $M+1$ coefficients a_{2n} $(n = 0, \ldots, M)$ of (13). These latter equations are precisely (14) for $n = 0, 2, 4, \ldots, N$ with a term $b_0 + n^2 b_2$ added to the left-hand side of the nth equation. The two new unknowns b_0 and b_2 ('boundary' constants) give a total of $M+3$ unknowns to be found from the modified version of (14) for $n = 0, 2, \ldots, N$ and the two boundary conditions (15). The boundary conditions (15) ensure that $v(y)$ expanded as in (13) may be re-expanded in terms of $q_{2n}(y)$ for $n = 2, \ldots, M$. A well-posed eigenvalue problem results.

The second method to determine a_n from (14) and (15) is slightly simpler and often gives more accurate results. It is the method used to obtain the numerical results reported below. We have in mind Lanczos's tau method (Lanczos 1956, chapter 7) as developed and extensively applied to ordinary differential equations by Fox and his school (Fox 1962; Fox & Parker 1968). The idea of the tau method is to apply (14) for $n = 0, 2, 4, \ldots, N-4$ only so that with the two boundary conditions (15) there are $M+1$ equations in $M+1$ unknowns. In other words, the high frequency (i.e. high n) behaviour of the solution is determined not by the dynamical equation (14) but rather by the boundary conditions (15). A more palatable way of explaining the tau method (and the origin of its name) is the observation that (14) for $n = 0, 2, 4, \ldots, N-4$ and (15) follow if a solution of the form (13) is sought for the boundary-value problem in which the right-hand side of (2) is not zero, but rather is

$$\tau_{N-2} T_{N-2}(y) + \tau_N T_N(y) + \tau_{N+2} T_{N+2}(y),$$

and the boundary conditions are still (3). The parameters τ_{N-2}, τ_N, τ_{N+2} are not fixed, but are determined by a_{2n} $(n = 0, 2, 4, \ldots, N)$ from (14) with the right-hand side zero replaced by τ_n for $n = N-2$, N, $N+2$. The series (13) solves this modified problem *exactly*. The magnitude of τ_n gives a measure of the error; frequently, knowledge of τ_n can be used to give actual error bounds (cf. Fox & Parker 1968).

Once the tau method has been selected to obtain equations for the expansion coefficients a_n, there remains the problem of determining the eigenvalues λ. Two principal choices are available. First, it is possible to make an eigenvalue search in the complex-λ plane. Here a guess for λ is made and equations (14) with $n = 0, 2, \ldots, N-4$ and, say, the first of (15) are solved for a_n subject to the condition that, say, $a_N = 1$. For each such guessed λ, determination of a_n $(n = 0, 2, \ldots, N)$ requires only order N arithmetic operations (as commented at the end of § 2) *if* care is taken to *accumulate* sums of the form $\sum_{p=n}^{N} p^5 a_p$.†

† Induced numerical instability due to rapidly growing special solutions of (2) [that cannot satisfy (3)] is not a problem if (14) is solved for a_n by backwards recurrence from $n = N-4$ to $n = 0$.

Then λ is varied according to some prescription, in order to minimize the residue in the second equation of (15). If a good initial guess for λ is available this procedure usually works quite well. It is the method used in most previous numerical studies of hydrodynamic stability.†

The second method for determination of λ, which is the one used here, may be less efficient when a good initial guess for λ is available, but has a better chance of success when such a good estimate is not available. The eigenvalues λ are determined as the eigenvalues of the linear algebraic equations (14) and (15) using a matrix eigenvalue algorithm, as previously done for hydrodynamic stability problems by Dolph & Lewis (1958), Gallagher & Mercer (1962), Grosch & Salwen (1968), and Gary & Helgason (1970), amongst others. We use the QR matrix eigenvalue algorithm (Wilkinson 1965) as adapted for problems of the present type by Gary & Helgason (1970). The advantages of this matrix method are the accuracy of the eigenvalues and the fact that a number of low-stability modes are determined along with the most unstable mode. The disadvantages are that the number of operations to determine the eigenvalues scales as N^3 and computer storage proportional to N^2 must be allocated, where N is the total number of Chebyshev polynomials retained in the approximation. However, the total computer time involved in the present calculations is so nominal that the convenience and accuracy of the matrix method easily outweigh its disadvantages.

On the National Center for Atmospheric Research Control Data 6600 computer used for the calculations reported in § 4, the timings given in table 1 were observed using single precision arithmetic (48 significant bits) and Fortran codes.

4. Numerical results

A critical comparison between the accuracy of various methods for the solution of the Orr–Sommerfeld eigenvalue problem is conveniently made for the most unstable mode of plane Poiseuille flow with $\alpha = 1, R = 10000$. This mode is a symmetric eigenmode. The results obtained by the Chebyshev approximation of § 3 and the QR algorithm are given in table 2. The values of $M+1$ listed in table 2 give the number of even-degree Chebyshev polynomials used to represent the eigenfunction; $v(y)$ is expanded in terms of $T_n(y)$ for $n = 0, 2, 4, \ldots, 2M$. From these results it is plausible to infer that the exact eigenvalue equals

$$0{\cdot}23752649 + 0{\cdot}00373967i$$

to within one part in 10^8. From the results reported in table 2 and similar results obtained for other values of α and R, it has been inferred that, when more than 25 even-degree Chebyshev polynomials ($M > 25$) are used in the method of § 3, the results are accurate to eight decimal places using single-precision arithmetic on the CDC 6600, at least up to a Reynolds number of 50000 with $\alpha = O(1)$. It should also be noticed from table 2 that there is rapid convergence of the eigenvalue with increasing M, as expected on the basis of the results of § 2.

† It is also possible to implement orthogonalization or parallel-shooting methods to avoid strong-instability problems (see Wright (1964) for these Chebyshev techniques, applied using collocation methods).

The round-off error of the computer is a very significant quantity. By artificially increasing the round-off error from about one part in 10^{14} to one part in 10^8 and one part in 10^{12}, the eigenvalues determined by the Chebyshev method of § 3 are changed significantly, as shown by the results given in table 3.

Number of Chebyshev polynomials $(M+1)$	CDC 6600 time (sec)
17	0·42
23	0·94
26	1·31
32	2·35
38	3·75
50	8·24

TABLE 1. CDC 6600 time to find eigenvalues by the Chebyshev approximation and QR method.

$M+1$	λ
14	$0{\cdot}23713751+0{\cdot}005563644i$
15	$0{\cdot}23690887+0{\cdot}00365516i$
17	$0{\cdot}23743315+0{\cdot}00372248i$
20	$0{\cdot}23752676+0{\cdot}00373427i$
23	$0{\cdot}23752670+0{\cdot}00373982i$
26	$0{\cdot}23752648+0{\cdot}00373967i$
29	$0{\cdot}23752649+0{\cdot}00373967i$
32	$0{\cdot}23752649+0{\cdot}00373967i$
38	$0{\cdot}23752649+0{\cdot}00373967i$
50	$0{\cdot}23752649+0{\cdot}00373967i$

TABLE 2. Chebyshev approximation to the most unstable mode of plane Poiseuille flow for $\alpha = 1$, $R = 10000$.

$M+1$	λ (round-off $\simeq 10^{-8}$)	λ (round-off $\simeq 10^{-12}$)
20	$0{\cdot}23752685+0{\cdot}00373451i$	$0{\cdot}23752676+0{\cdot}00373427i$
23	$0{\cdot}23754139+0{\cdot}00383489i$	$0{\cdot}23752670+0{\cdot}00373982i$
26	$0{\cdot}23749300+0{\cdot}00368897i$	$0{\cdot}23752646+0{\cdot}00373965i$
38	$0{\cdot}23714159+0{\cdot}00352930i$	$0{\cdot}23752648+0{\cdot}00373966i$
44	$0{\cdot}23348160+0{\cdot}00534311i$	$0{\cdot}23752648+0{\cdot}00373965i$
50	$0{\cdot}23813295-0{\cdot}00296273i$	$0{\cdot}23752655+0{\cdot}00373979i$

TABLE 3. Effect of round-off error on the most unstable mode of plane Poiseuille flow for $\alpha = 1$, $R = 10000$.

The results reported in table 2 may be compared with results obtained by other investigators. The first accurate numerical calculation of λ for $\alpha = 1$, $R = 10000$ was apparently done by Thomas (1953). Thomas used a five-point Numerov finite-difference method with up to 100 grid points in the interval $0 \leqslant y \leqslant 1$ and a complex-eigenvalue search to find the most unstable symmetric mode. The truncation error of Thomas's scheme is fourth-order in Δx. With 50 grid points Thomas found

$$\lambda = 0{\cdot}2375006+0{\cdot}0035925i \quad \text{and} \quad \lambda = 0{\cdot}2375243+0{\cdot}0037312i$$

with 100 grid points. By extrapolating the error to zero Δx, Thomas found $\lambda = 0{\cdot}2375259+0{\cdot}0037404i$.

Gary & Helgason (1970) reported calculations of λ using finite-difference schemes of various orders of accuracy, along with a 'stretched' co-ordinate to account for the expected detailed structure of the eigenfunctions near the side-walls. Using a sixth-order finite-difference scheme and grid points uniformly spaced in terms of the stretched co-ordinate $z = ye^{y^2-1}$ for $0 \leqslant z \leqslant 1$ (only symmetric modes being sought), Gary & Helgason found $\lambda = 0{\cdot}23752964+0{\cdot}00374248i$ with 43 grid points and $\lambda = 0{\cdot}23752650+0{\cdot}00373969i$ with 100 grid points. Without the stretched co-ordinate, the same finite-difference scheme gives $\lambda = 0{\cdot}23730744+0{\cdot}00375620i$ with 43 grid points.

Grosch & Salwen (1968) found $\lambda = 0{\cdot}237413+0{\cdot}003681i$ for $\alpha = 1$, $R = 10000$ using expansions involving up to 50 symmetric eigenmodes of the problem (8) and (9) and a matrix eigenvalue algorithm. This result seems to be disproportionately in error; it is possible that round-off error is significant. Dolph & Lewis (1958) found $\mathrm{Im}(\lambda) = 0{\cdot}034649$ using eight symmetric modes of (12) and $\mathrm{Im}(\lambda) = 0{\cdot}003772$ using twenty symmetric modes of (12).

These comparisons should make clear the important increase in accuracy achieved by use of Chebyshev approximations. The fact that results of great accuracy are achieved using less than half the number of degrees of freedom required by other methods is significant because, as stated in §3, the computer time required to use matrix eigenvalue routines is proportional to the cube of the number of degrees of freedom and the memory required is proportional to the square.

We have determined the critical Reynolds number for instability of plane Poiseuille flow using the Chebyshev method of §3. The critical Reynolds number R_c is defined as the smallest value of R for which an unstable eigenmode exists. The mode that becomes unstable at R_c is symmetric so that we may again assume that the Chebyshev coefficients a_n with n odd are zero. We find that the critical Reynolds number is

$$R_c = 5772{\cdot}22;$$

the first unstable mode appears with $\alpha_c = 1{\cdot}02056 \pm 0{\cdot}00001$. In table 4 we report the values of λ for the most unstable symmetric mode with $\alpha_c = 1{\cdot}02056$ and $R = 5772{\cdot}22$ and $5772{\cdot}23$ as a function of $M+1$, the number of retained Chebyshev polynomials. The behaviour of $\mathrm{Im}(\lambda)$ for the results reported in table 4 also shows that single-precision arithmetic on the CDC 6600 allows determination of the eigenvalues to about one part in 10^8.

The values $R_c = 5772{\cdot}22$, $\alpha_c = 1{\cdot}02056$ may be compared with those found previously. Using the methods of asymptotic analysis developed by Lin (1955), Shen (1954) found $R_c = 5360$, $\alpha_c = 1{\cdot}05$. Thomas (1953) found $R_c = 5780$, $\alpha_c = 1{\cdot}026$ using finite-difference methods. Nachtsheim (1964), as reported by Betchov & Criminale (1967), found $R_c = 5767$, $\alpha_c = 1{\cdot}02$ using finite-difference methods. Grosch & Salwen (1968) found $R_c = 5750$, $\alpha_c = 1{\cdot}025$ using expansions in the orthogonal functions defined by (8) and (9).

The Chebyshev approximation matrix method of §3 also gives accurate values for a number of the stable eigenvalues of the Orr–Sommerfeld equation. By comparison of the set of eigenvalues obtained for $M+1 = 31, 44, 47$ and 50, a set of 22 of the least stable eigenvalues for symmetric modes with $\alpha = 1$, $R = 10000$ has been found. Similarly, by use of (14) and (16) with $a_n = 0$ for n even, we obtained accurate eigenvalues for the 21 least stable antisymmetric eigenmodes for $\alpha = 1$, $R = 10000$. The results for the 32 least stable of these modes are given in table 5. Only significant figures are given.

$M+1$	λ (R = 5772·22)	λ (R = 5772·23)
20	0·26400095 + 4·3i (−7)	0·26400087 + 4·5i (−7)
26	0·26400174 − 3·1i (−9)	0·26400166 + 1·3i (−8)
38	0·26400174 − 3·2i (−9)	0·26400166 + 1·4i (−8)
44	0·26400174 − 1·7i (−9)	0·26400166 + 1·3i (−8)
50	0·26400174 + 5·9i (−10)	0·26400166 + 1·9i (−8)

TABLE 4. Chebyshev approximation to the most unstable symmetric mode for plane Poiseuille flow at the critical Reynolds number and wavenumber $\alpha_c = 1·02056$.

Mode number	Symmetric (S) Anti-symmetric (A)	Eigenvalue λ
1	S	0·23752649 + 0·00373967i
2	A	0·96463092 − 0·03516728i
3	S	0·96464251 − 0·03518658i
4	A	0·27720434 − 0·05089873i
5	A	0·93631654 − 0·06320150i
6	S	0·93635178 − 0·06325157i
7	A	0·90798305 − 0·09122274i
8	S	0·90805633 − 0·09131286i
9	A	0·87962729 − 0·11923285i
10	S	0·87975570 − 0·11937073i
11	S	0·34910682 − 0·12450198i
12	A	0·41635102 − 0·13822652i
13	A	0·8512458 − 0·1472339i
14	S	0·8514494 − 0·1474256i
15	A	0·8228350 − 0·1752287i
16	S	0·8231370 − 0·1754781i
17	S	0·1900592 − 0·1828219i
18	A	0·794388 − 0·203221i
19	S	0·794818 − 0·203529i
20	A	0·532045 − 0·206465i
21	S	0·474901 − 0·208731i
22	A	0·76588 − 0·23119i
23	S	0·76649 − 0·23159i
24	S	0·36850 − 0·23882i
25	A	0·73741 − 0·25872i
26	S	0·73812 − 0·25969i
27	A	0·63672 − 0·25988i
28	A	0·38399 − 0·26511i
29	S	0·58721 − 0·26716i
30	A	0·71232 − 0·28551i
31	S	0·51292 − 0·28663i
32	S	0·70887 − 0·28765i

TABLE 5. Least stable eigenvalues for $\alpha = 1$, $R = 10000$.

We observe from table 5 that modal pairs 2 and 3, 5 and 6, 7 and 8, 9 and 10, 13 and 14, 15 and 16, 18 and 19, 22 and 23, 25 and 26, and 30 and 32 are nearly degenerate. Notice that all these nearly degenerate pairs are 'fast' modes in the sense that $\mathrm{Re}(\lambda)$ is close to 1 so that the phase speed of the modes is close to the centre-stream velocity. On the other hand, the 'slow' modes with $\mathrm{Re}(\lambda) < \frac{2}{3}$ (which is the average velocity in the channel) are not nearly degenerate. The nearly degenerate modal pairs of symmetric and antisymmetric eigenfunctions are *not* exactly degenerate; the figures given in table 5 are significant, as they do not vary when the order of truncation of the Chebyshev expansion is varied. The behaviour of the higher eigenvalues (not listed in table 5) is consistent with equations (58) and (59) of Grosch & Salwen (1968); in particular $\mathrm{Re}(\lambda) \simeq \frac{2}{3}$.

5. Generalizations

The considerations of § 3 may be generalized to give Chebyshev approximations to the Orr–Sommerfeld equation for arbitrary nearly parallel flows. It is possible to include arbitrary undisturbed velocity profiles $\overline{u}(y)$, e.g. the Blasius boundary-layer profile, and boundary conditions including flexible walls, rigid walls and free streams.

We consider in detail the case of arbitrary nearly parallel flow within rigid walls. Equations (2) and (3) are to be solved with $\overline{u}(y)$ a given function of y and the only change from § 3 concerns the re-expansion in a Chebyshev series of the terms multiplied by $-i\alpha R$ in (2). At this point the choice of Chebyshev polynomials over other sets of orthogonal polynomials as expansion functions is very convenient. Chebyshev polynomials obey the extremely simple multiplication law

$$2T_n(x)\,T_m(x) = T_{n+m}(x) + T_{|n-m|}(x)$$

which ensures that the product terms multiplying $-i\alpha R$ in (2) take on a simple form in the equations for a_n. Some details of the manipulations necessary to derive equations for a_n are given in the appendix. The result, analogous to (14), is

$$\begin{aligned}\tfrac{1}{24} &\sum_{\substack{p=n+4\\ p\equiv n\ (\mathrm{mod}\ 2)}}^{N} [p^3(p^2-4)^2 - 3n^2p^5 + 3n^4p^3 - pn^2(n^2-4)^2]\,a_p \\ &- (2\alpha^2 - i\alpha R\lambda) \sum_{\substack{p=n+2\\ p\equiv n\ (\mathrm{mod}\ 2)}}^{N} p(p^2-n^2)\,a_p + (\alpha^4 - i\alpha^3 R\lambda)\,c_n a_n \\ &- \tfrac{1}{2}i\alpha R\Bigg(\sum_{p=2}^{N} a_p \sum_{\substack{m\equiv p\ (\mathrm{mod}\ 2)\\ |m|\leqslant p-2\\ |n-m|\leqslant N}} p(p^2-m^2)\,\overline{b}_{n-m} - \alpha^2 \sum_{\substack{|p|\leqslant N\\ |n-p|\leqslant N}} \overline{a}_p \overline{b}_{n-p} \\ &- \sum_{\substack{|p|\leqslant N\\ |n-p|\leqslant N}} \overline{a}_p \sum_{\substack{m=|n-p|+2\\ m+n\equiv p\ (\mathrm{mod}\ 2)}}^{N} m[m^2-(n-p)^2]\,b_m\Bigg) = 0, \qquad (17)\end{aligned}$$

where $a_n, b_n (n = 0, \ldots, N)$ are the Chebyshev coefficients of $v(y)$, $\overline{u}(y)$, respectively and $\overline{a}_n = c_{|n|}a_{|n|}$, $\overline{b}_n = c_{|n|}b_{|n|}$ for $-N \leqslant n \leqslant N$, where $c_0 = 2$ and $c_n = 1$ for

$n > 0$.† The boundary conditions are (15) and (16) as before. By carefully accumulating sums such as

$$\sum_{\substack{m=|n-p|+2 \\ m+n\equiv p \ (\mathrm{mod}\ 2)}}^{N} (m, m^3)\, b_m,$$

it is possible to evaluate all the N^2 elements of the matrix of coefficients of the system (15)–(17) using only order N^2 arithmetical operations, assuming the Chebyshev coefficients b_n of $\bar{u}(y)$ are given. If only $\bar{u}(y)$ is given, then approximate $b_n (0 \leqslant n \leqslant N)$ are calculable in order $N \log N$ operations, since the relation

$$\bar{u}\left(\cos\frac{\pi j}{N}\right) = \sum_{m=0}^{N} b_m \cos\frac{\pi m j}{N} \quad (j = 0, \ldots, N), \tag{18}$$

which is a convenient way to define approximate values for b_m $(0 \leqslant m \leqslant N)$, is invertible using properties of discrete Fourier transforms as

$$\bar{c}_m b_m = \frac{2}{N} \sum_{j=0}^{N} \bar{c}_j^{-1} \bar{u}\left(\cos\frac{\pi j}{N}\right) \cos\frac{\pi m j}{N} \quad (0 \leqslant m \leqslant N), \tag{19}$$

where $\bar{c}_0 = \bar{c}_N = 2$, $\bar{c}_j = 1$ for $0 < j < N$. The discrete Fourier transform (19) is efficiently computed in order $N \log N$ operations by means of the fast Fourier transform algorithm (Cooley, Lewis & Welch 1970).

It is also a straightforward matter to use Chebyshev polynomials to study the stability of pipe flows (Davey & Nguyen 1971). In contrast with expansion in series of Bessel functions, used by Davey & Drazin (1969) and others, expansion in series of Chebyshev polynomials gives infinite-order accuracy. Furthermore, the recurrence relation

$$x^{-1}[T_{n+1}(x) + T_{n-1}(x)] = 2T_n(x) \quad (n \geqslant 1)$$

ensures that division by x is readily accomplished within the Chebyshev series. This fact is important for the simple evaluation of terms like $r^{-1}\, dv/dr$ in the stability equation expressed in cylindrical co-ordinates.

Although we have not done so here, it is possible to formulate efficiently implementable Chebyshev approximations using stretched co-ordinates in order to resolve the boundary-layer structure better.

In conclusion, expansions in orthogonal polynomials, especially Chebyshev polynomials, give convenient, accurate, and efficient approximations to the solutions of hydrodynamic stability problems.

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† If $b_n = 0$, except for $b_0 = -b_2 = \frac{1}{2}$, then (17) reduces to (14).

Appendix

In the appendix we present some of the manipulations with Chebyshev series necessary to derive (14) and (17) [see also Fox & Parker 1968]. Let the Chebyshev expansion of an infinitely differentiable function $v(y)$ be (5) and let the Chebyshev expansions of its derivatives d^qv/dy^q be

$$d^qv/dy^q = \sum_{n=0}^{\infty} a_n^{(q)} T_n(y), \tag{A 1}$$

where $a_n^{(0)} = a_n$. It follows easily from (4) that

$$\frac{c_n}{n+1} T'_{n+1}(x) - \frac{d_{n-2}}{n-1} T'_{n-1}(x) = 2T_n(x) \tag{A 2}$$

for $n \geqslant 0$, where $c_n = d_n = 0$ if $n < 0$, $c_0 = 2$, $d_0 = 1$, $c_n = d_n = 1$ if $n > 0$. Consequently,

$$\begin{aligned} \frac{d}{dy} \sum_{n=0}^{\infty} a_n^{(q-1)} T_n(y) = \frac{d^qv}{dy^q} &= \sum_{n=0}^{\infty} a_n^{(q)} T_n(y) \\ &= \frac{d}{dy} \frac{1}{2} \sum_{n=0}^{\infty} a_n^{(q)} \left[\frac{c_n}{n+1} T_{n+1}(y) - \frac{d_{n-2}}{n-1} T_{n-1}(y) \right], \end{aligned}$$

so that equating coefficients of $T_n(y)$ for $n \geqslant 1$ gives

$$c_{n-1} a_{n-1}^{(q)} - a_{n+1}^{(q)} = 2n a_n^{(q-1)} \quad (n \geqslant 1). \tag{A 3}$$

It follows from (A 3) that

$$c_n a_n^{(1)} = 2 \sum_{\substack{p=n+1 \\ p+n \equiv 1 \ (\mathrm{mod}\ 2)}}^{\infty} p a_p \quad (n \geqslant 0), \tag{A 4}$$

where $a \equiv b (\mathrm{mod}\ 2)$ that means $a-b$ is divisible by 2. Further, it follows from (A 3) and (A 4) that

$$\begin{aligned} c_n a_n^{(2)} &= 2 \sum_{\substack{m=n+1 \\ m+n \equiv 1 \ (\mathrm{mod}\ 2)}}^{\infty} m a_m^{(1)} \\ &= 4 \sum_{\substack{p=n+2 \\ p \equiv n \ (\mathrm{mod}\ 2)}}^{\infty} p a_p \sum_{\substack{m=n+1 \\ m+n \equiv 1 \ (\mathrm{mod}\ 2)}}^{p-1} m, \end{aligned}$$

so that

$$c_n a_n^{(2)} = \sum_{\substack{p=n+2 \\ p \equiv n \ (\mathrm{mod}\ 2)}}^{\infty} p(p^2-n^2) a_p \quad (n \geqslant 0). \tag{A 5}$$

Similarly, it may be shown that

$$c_n a_n^{(4)} = \frac{1}{24} \sum_{\substack{p=n+4 \\ p \equiv n \ (\mathrm{mod}\ 2)}}^{\infty} p[p^2(p^2-4)^2 - 3n^2p^4 + 3n^4p^2 - n^2(n^2-4)^2] a_p. \tag{A 6}$$

Using the recurrence relation

$$2xT_n(x) = c_n T_{n+1}(x) + d_{n-1} T_{n-1}(x),$$

it follows that the nth Chebyshev coefficient of $2yv(y)$ is $c_{n-1}a_{n-1}+a_{n+1}$ for $n \geqslant 0$; similarly, the nth Chebyshev coefficient of $4y^2v(y)$ is

$$c_{n-2}a_{n-2}+(c_n+c_{n-1})\,a_n+a_{n+2} \quad \text{for} \quad n \geqslant 0.$$

These latter facts together with (A 5) and (A 6) may be applied to (2) to give (14)

The derivation of (17) is only slightly more complicated. It is sufficient to give a rule for computing the Chebyshev coefficients of the product $v(y)\,w(y)$ given that

$$v(y) = \sum_{n=0}^{\infty} a_n T_n(y), \quad w(y) = \sum_{n=0}^{\infty} b_n T_n(y).$$

To find such a rule, it is simplest to define

$$\overline{T}_n(x) = \exp\,[in\cos^{-1}x]$$

for $|x| \leqslant 1$ and $-\infty < n < \infty$. It follows from (4) that

$$2T_n(x) = \overline{T}_n(x)+\overline{T}_{-n}(x),$$

while it is trivially verified that $\overline{T}_n(x)\,\overline{T}_m(x) = \overline{T}_{n+m}(x)$. It follows that

$$2v(y) = \sum_{n=-\infty}^{\infty} \bar{a}_n\overline{T}_n(y), \quad 2w(y) = \sum_{n=-\infty}^{\infty} \bar{b}_n\overline{T}_n(y),$$

where $\bar{a}_n = c_{|n|}a_{|n|}$ and $\bar{b}_n = c_{|n|}b_{|n|}$ for $-\infty < n < \infty$. Therefore,

$$4v(y)\,w(y) = \sum_{n=-\infty}^{\infty} \bar{e}_n\overline{T}_n(y) = 2\sum_{n=0}^{\infty} e_n T_n(y),$$

where

$$e_n = \frac{1}{c_n}\sum_{m=-\infty}^{\infty} \bar{a}_{n-m}\bar{b}_m, \quad \bar{e}_n = c_{|n|}e_{|n|}. \tag{A 7}$$

Consequently the nth Chebyshev coefficient of $v(y)\,w(y)$ is $\frac{1}{2}e_n$ for $n \geqslant 0$. This latter result together with those used to derive (14) is sufficient to give (17).

REFERENCES

Betchov, R. & Criminale, W. O. 1967 *Stability of Parallel Flows*. Academic.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

Cooley, J. W., Lewis, P. A. W. & Welch, P. D. 1970 The fast Fourier transform algorithm: programming considerations in the calculation of sine, cosine, and Laplace transforms. *J. Sound Vib.* **12**, 315–337.

Davey, A. & Drazin, P. G. 1969 The stability of Poiseuille flow in a pipe. *J. Fluid Mech.* **36**, 209–218.

Davey, A. & Nguyen, H. P. F. 1971 Finite-amplitude stability of pipe flow. *J. Fluid Mech.* **45**, 701–720.

Dolph, C. L. & Lewis, D. C. 1958 On the application of infinite systems of ordinary differential equations to perturbations of plane Poiseuille flow. *Quart. Appl. Math.* **16**, 97–110.

Fox, L. 1962 Chebyshev methods for ordinary differential equations. *Computer J.* **4**, 318–331.

Fox, L. & Parker, I. B. 1968 *Chebyshev Polynomials in Numerical Analysis*. Oxford University Press.

GALLAGHER, A. P. & MERCER, A. McD. 1962 On the behaviour of small disturbances in plane Couette flow. *J. Fluid Mech.* **13**, 91–100.

GARY, J. & HELGASON, R. 1970 A matrix method for ordinary differential eigenvalue problems. *J. Comp. Phys.* **5**, 169–187.

GROSCH, C. E. & SALWEN, H. 1968 The stability of steady and time-dependent plane Poiseuille flow. *J. Fluid Mech.* **34**, 177–205.

HAMMING, R. W. 1962 *Numerical Methods for Scientists and Engineers.* McGraw-Hill.

LANCZOS, C. 1956 *Applied Analysis.* Prentice-Hall.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability.* Cambridge University Press.

NACHTSHEIM, P. R. 1964 An initial value method for the numerical treatment of the Orr–Sommerfeld equation for the case of plane Poiseuille flow. *N.A.S.A. Tech. Note*, D-2414.

ORSZAG, S. A. 1971*a* Galerkin approximations to flows within slabs, spheres, and cylinders. *Phys. Rev. Lett.* **26**, 1100–1103.

ORSZAG, S. A. 1971*b* Numerical simulation of incompressible flows within simple boundaries I. Galerkin (spectral) representations. *Studies in Appl. Math.* **50**, 293–327.

SHEN, S. F. 1954 Calculated amplified oscillations in plane Poiseuille and Blasius flows. *J. Aero. Sci.* **21**, 62–64.

THOMAS, L. H. 1953 The stability of plane Poiseuille flow. *Phys. Rev.* **91**, 780–783.

WILKINSON, J. H. 1965 *The Algebraic Eigenvalue Problem.* Oxford University Press.

WRIGHT, K. 1964 Chebyshev collocation methods for ordinary differential equations. *Computer J.* **6**, 358–363.