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# Achievable Casimirs and its implications on control of port-Hamiltonian systems

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In this paper we extend results on interconnections of port-Hamiltonian systems to infinite-dimensional port-Hamiltonian systems and to mixed finite and infinite dimensional port-Hamiltonian systems. The problem of achievable Dirac structures is now studied for systems with dissipation, in the finite-dimensional, infinite-dimensional and the mixed finite and infinite-dimensional case. We also characterize the set of achievable Casimirs and study its application for the control of port-Hamiltonian systems.

## 1. Introduction

Network modelling of complex physical systems (with components from different physical domains), both lumped and distributed parameter, leads to a class of non-linear systems called port-Hamiltonian systems. Port-Hamiltonian systems are defined with respect to a Dirac structure (which formalizes the power-conserving interconnection structure of the system), an energy function (the Hamiltonian) and a resistive relation. Key property of a Dirac structure is that a power conserving interconnection (composition) of a number Dirac structures again defines a Dirac structure. This implies that any power-conserving interconnection of a port-Hamiltonian system is again a port-Hamiltonian system, with Dirac structure being the composition of Dirac structures of its constituent parts, Hamiltonian being the sum of individual Hamiltonians and total resistive relation determined by the resistive relations of the components taken together. As a result power-conserving interconnections of port-Hamiltonian systems can be studied to a large extent in terms of composition of their Dirac structures.

In this paper we extend results on composition of Dirac structures (both finite and infinite dimensional in nature)

and the theory of achievable Dirac structures to systems with dissipation. The composition of a Dirac structure and a resistive relation are also studied both in the finite and infinite-dimensional case. In the case of infinite dimensional systems we analyse the case of dissipation entering into the system through the spatial domain (distributed resistance) and also the case of terminating the boundary of the infinite-dimensional system with a resistive relation. We study interconnections of finite dimensional systems with infinite dimensional systems, the interconnection being through the boundary of the infinite dimensional systems. We then prove that such an interconnection is again a port-Hamiltonian system, the case of which we call a mixed port-Hamiltonian system.

We also investigate the achievable Casimirs for the closed-loop system and study its implications on control by interconnection of port-Hamiltonian systems. We characterize the set of achievable Casimirs in terms of the plant state and in the finite dimensional case see how without *a priori* knowledge of the controller, whether or not Casimirs exist for the closed-loop system and hence the applicability of the control by interconnection (or the Energy Casimir) method, for stabilizing a system. Also in the finite dimensional case we show in general that for a function to be a Casimir for one non-degenerate resistive relation at the resistive port it actually needs to be a Casimir for all resistive relations.

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**2. Port-Hamiltonian systems and Dirac structures**

It is well known (van der Schaft (2000), van der Schaft and Maschke (2002)) that the notion of power conserving interconnections can be formulated by a geometric structure called a Dirac structure. We briefly discuss these concepts both for finite and infinite-dimensional systems with scalar spatial variable.

**2.1 Finite dimensional systems with dissipation**

To define the notion of Dirac structures for finite dimensional systems, we start with a space of power variables  $\mathcal{F} \times \mathcal{F}^*$ , for some linear space  $\mathcal{F}$ , with power defined by

$$P = \langle e|f \rangle, \quad (f, e) \in \mathcal{F} \times \mathcal{F}^*,$$

where  $\langle e|f \rangle$  denotes the duality product, that is, the linear functional  $e \in \mathcal{F}^*$  acting on  $f \in \mathcal{F}$ .  $\mathcal{F}$  is called the space of flows and  $\mathcal{F}^*$  the space of efforts, with the power of a signal  $(f, e) \in \mathcal{F} \times \mathcal{F}^*$  denoted as  $\langle e|f \rangle$ .

There exists on  $\mathcal{F} \times \mathcal{F}^*$  a canonically defined bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle$ , defined as

$$\langle\langle (f^a, e^a), (f^b, e^b) \rangle\rangle := \langle e^a|f^b \rangle + \langle e^b|f^a \rangle, \quad (f^a, e^a)(f^b, e^b) \in \mathcal{F} \times \mathcal{F}^*. \quad (1)$$

**Definition 1** (van der Schaft 2000): A constant Dirac structure on  $\mathcal{F} \times \mathcal{F}^*$  is a subspace  $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$  such that  $\mathcal{D} = \mathcal{D}^\perp$  with respect to the bilinear form (1).

As an immediate corollary of the definition we see that for all  $(f, e) \in \mathcal{D}$  we have that  $\langle e|f \rangle = 0$ . Hence a Dirac structure defines a power conserving relation.

Consider a lumped-parameter physical system given by power-conserving interconnection defined by a constant Dirac structure  $\mathcal{D}$  and energy storing elements with energy variables  $x$ . For simplicity we assume that the energy variables are living in a linear space  $\mathcal{X}$  although everything can be generalized to the case of manifolds. The constitutive relations of the energy storing elements are specified by their stored energy functions  $H(x)$ .

The space of flows is naturally partitioned as  $\mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}$  with  $f_S \in \mathcal{F}_S$ , the flows corresponding to the energy storing elements, and  $f_R \in \mathcal{F}_R$  denoting the flows corresponding to the dissipative elements and  $f \in \mathcal{F}$  denoting the remaining flows (corresponding to ports/sources). Correspondingly, the space of effort variables is split as  $\mathcal{F}_S^* \times \mathcal{F}_R^* \times \mathcal{F}^*$ , with  $e_S \in \mathcal{F}_S^*$  the efforts corresponding to the energy-storing,  $e_R \in \mathcal{F}_R^*$  the efforts corresponding to the dissipative elements and  $e \in \mathcal{F}^*$  the remaining efforts. The Dirac structure  $\mathcal{D}$  can then be given in matrix kernel representation as

$$\begin{aligned} \mathcal{D} = & \{ (f_S, e_S, f_R, e_R, f, e) \in \mathcal{F}_S \times \mathcal{F}_S^* \times \mathcal{F}_R \times \mathcal{F}_R^* \times \mathcal{F} \times \mathcal{F}^* | \\ & F_S f_S + E_S e_S + F_R f_R + E_R e_R + F f + E e = 0 \} \\ & E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + E F^T + F E^T = 0 \end{aligned}$$

with  $\text{rank} [F_S \dot{ : } E_S \dot{ : } F_R \dot{ : } E_R \dot{ : } F \dot{ : } E] = \dim(\mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F})$ . (2)

Now the flows of the energy storing elements are given by  $\dot{x}$ , and equated with  $-f_S$  (the negative sign is included to have a consistent energy flow direction). The efforts,  $e_S$ , corresponding to the energy storing elements are given as  $(\partial H / \partial x) = e_S$ . Similarly, restricting to linear resistive elements, the flow and effort variables connected to the resistive elements are related as  $f_R = -R e_R$ . Substituting these into (2) leads to the description of the physical system by the set of DAEs

$$\begin{aligned} -F_S \dot{x}(t) + E_S \frac{\partial H}{\partial x}(x(t)) - F_R R e_R \\ + E_R e_R + F f(t) + E e(t) = 0 \end{aligned} \quad (3)$$

with  $f, e$  the port power variables. The system of equation (3) is called a port-Hamiltonian system with dissipation.

By the power conserving property of a Dirac structure it follows that any port-Hamiltonian system with dissipation satisfies the energy balance

$$\frac{dH}{dt}(x(t)) = \left\langle \frac{\partial H}{\partial x}(x(t)) | \dot{x}(t) \right\rangle = -e_R^T(t) R e_R(t) + e^T(t) f(t)$$

which means that the increase in internal energy of the port-Hamiltonian system is equal to the externally supplied power minus the power dissipated in the energy-dissipating elements.

**2.2 Infinite dimensional systems**

The key concept in order to define an infinite-dimensional port-Hamiltonian system on a bounded spatial domain, with non-zero energy flow through the boundary, is the introduction of a special type of Dirac structure on suitable spaces of differential forms on the spatial domain and its boundary, making use of Stokes' theorem; see van der Schaft and Maschke (2002). Let  $Z$  be an  $n$ -dimensional manifold with a smooth  $(n - 1)$  dimensional boundary  $\partial Z$ , representing the space of spatial variables. Define now the linear space

$$\mathcal{F}_{p,q} := \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z)$$

for any pair  $p, q$  of positive integers satisfying  $p + q = n + 1$ , and correspondingly define

$$\mathcal{F}_{p,q}^* := \Omega^{n-p} \times \Omega^{n-q} \times \Omega^{n-q}(\partial Z).$$

Here  $\Omega^k(Z)$ ,  $k=0, 1, \dots, n$ , is the space of exterior  $k$ -forms on  $Z$ , and  $\Omega^k(\partial Z)$ ,  $k=0, 1, \dots, n-1$ , the space of  $k$ -forms on  $\partial Z$ .

There is a natural pairing between  $\Omega^k(Z)$  and  $\Omega^{n-k}(Z)$  (similarly between  $\Omega^k(\partial Z)$  and  $\Omega^{n-k}(\partial Z)$ ) given by

$$\langle \beta | \alpha \rangle := \int_Z \beta \wedge \alpha, \quad (\in \mathbb{R}) \tag{4}$$

with  $\alpha \in \Omega^k(Z)$ ,  $\beta \in \Omega^{n-k}(Z)$ , with  $\wedge$  the usual wedge product of differential forms yielding the  $n$ -form  $\beta \wedge \alpha$ .

Then the pairing (4) yields a pairing between  $\mathcal{F}_{p,q}$  and  $\mathcal{F}_{p,q}^*$ , and symmetrization of this pairing leads to the following bilinear form on  $\mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^*$  with values in  $\mathbb{R}$ :

$$\begin{aligned} & \left\langle \left( f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1 \right), \left( f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2 \right) \right\rangle \\ & := \int_Z \left[ e_p^1 \wedge f_p^2 + e_q^1 \wedge f_q^2 + e_p^2 \wedge f_p^1 + e_q^2 \wedge f_q^1 \right] \\ & \quad + \int_{\partial Z} \left[ e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1 \right], \end{aligned} \tag{5}$$

where for  $i = 1, 2$

$$\begin{aligned} f_p^i & \in \Omega^p(Z), & f_q^i & \in \Omega^q(Z) \\ e_p^i & \in \Omega^{n-p}(Z), & e_q^i & \in \Omega^{n-q}(Z) \\ f_b^i & \in \Omega^{n-p}(\partial Z), & e_b^i & \in \Omega^{n-q}(\partial Z) \end{aligned}$$

The spaces of differential forms  $\Omega^p(Z)$  and  $\Omega^q(Z)$  represent the energy variables of two different physical energy domains interacting with each other, while  $\Omega^{n-p}(\partial Z)$  and  $\Omega^{n-q}(\partial Z)$  denote the boundary variables whose (wedge) product represents the boundary energy flow. It has thus been shown in van der Schaft and Maschke (2002) that the following system defines a port-Hamiltonian system

$$\begin{aligned} \begin{bmatrix} f_p \\ f_q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}; & \begin{bmatrix} f_b \\ e_b \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_p|_{\partial Z} \\ e_q|_{\partial Z} \end{bmatrix} \end{aligned} \tag{6}$$

with  $|_{\partial Z}$  denoting the restriction to the boundary  $\partial Z$  and  $r := pq + 1$ . The space of all admissible flows and efforts satisfying (6) represents a Dirac structure called Stokes–Dirac structure.

### 3. Achievable Casimirs for systems with dissipation: finite dimensions

Casimirs are functions that are conserved quantities of the system for every Hamiltonian

(see van der Schaft (2000)), and they are completely characterized by the Dirac structure of the port-Hamiltonian systems. The existence of such functions has immediate consequences on stability analysis of systems. Suppose we want to stabilize a plant port-Hamiltonian system around a desired equilibrium  $x^*$ , and we would like to design a controller port-Hamiltonian system such that the closed-loop system is asymptotically stable around  $x^*$ . The closed-loop system necessarily satisfies

$$\frac{d}{dt}(H_P + H_C) \leq 0.$$

In case  $x^*$  is not a minimum for  $H_P$ , then a possible strategy is that we generate Casimir functions  $\mathcal{C}(x, \xi)$  for the closed-loop system by appropriately choosing the controller port-Hamiltonian system. The candidate Lyapunov function is then given by the sum of the plant and controller Hamiltonians and the corresponding Casimir function,

$$V(x, \xi) := H_P(x) + H_C(\xi) + \mathcal{C}(x, \xi).$$

The objective is to generate Casimir function  $\mathcal{C}(x, \xi)$  in such way that  $V$  has a minimum at  $(x^*, \xi^*)$ , with  $\xi^*$  still to be chosen. This strategy is based on characterizing all the achievable Casimirs of the closed-loop systems. Since the closed-loop Casimirs are based on the closed-loop Dirac structures, the problem reduces to finding all the closed-loop Dirac structures.

#### 3.1 Recall of systems without dissipation

A Casimir function  $\mathcal{C}: \mathcal{X} \rightarrow \mathbb{R}$  for a port-Hamiltonian system is a function which is constant along all the trajectories of the port-Hamiltonian system irrespective of the Hamiltonian. Consider the following subspace

$$G_1 := \{f \in \mathcal{F} \mid \exists e \in \mathcal{F}^* \text{ s.t. } (f, e) \in \mathcal{D}\}.$$

A function  $\mathcal{C}: \mathcal{X} \rightarrow \mathbb{R}$  is a Casimir function if  $(d\mathcal{C}/dt)(x(t)) = (\partial^T \mathcal{C} / \partial x)(x(t)) \dot{x}(t) = 0$  for all  $\dot{x}(t) \in G_1$ . Hence  $\mathcal{C}: \mathcal{X} \rightarrow \mathbb{R}$  is a Casimir function for the port-Hamiltonian system if and only if

$$\frac{\partial^T \mathcal{C}}{\partial x}(x) \in G_1^\perp.$$

Geometrically this can be formulated by defining the following subspace of the dual space of efforts

$$P_0 = \{e \in \mathcal{F}^* \mid (0, e) \in \mathcal{D}\}.$$

It can easily be seen that  $G_1^\perp = P_0$  where  $\perp$  denotes the orthogonal complement with respect to the duality product  $\langle \cdot | \cdot \rangle$ . Hence  $\mathcal{C}$  is a Casimir function if and only if  $(\partial^T \mathcal{C} / \partial x)(x) \in P_0$ . In short we can say that a

Casimir function for a port-Hamiltonian system is any function  $\mathcal{C}: \mathcal{X} \rightarrow \mathcal{R}$  such that its gradient  $e = (\partial\mathcal{C}/\partial x)$  satisfy

$$(0, e) \in \mathcal{D}. \quad (7)$$

In case of a non-autonomous system, where now the elements of the Dirac structure are  $(f, e, f', e') \in \mathcal{D}$ , with  $(f', e')$  connected to the control ports, a Casimir is a function  $\mathcal{C}: \mathcal{X} \rightarrow \mathcal{R}$ , such that its gradient now satisfies

$$(0, e, f_c, e_c) \in \mathcal{D} \quad (8)$$

for some  $f_c, e_c$ . This will imply that  $(d\mathcal{C}/dt)$  no longer equals zero, but will depend on the variables at the control ports. Indeed, from (8) we have that

$$\left(0, \frac{\partial\mathcal{C}}{\partial x}, f_c, e_c\right) \in \mathcal{D} = \mathcal{D}^\perp.$$

This implies that

$$-\frac{\partial^T \mathcal{C}}{\partial x} \dot{x} = 0 \cdot e_S + f_c e' + e_c f' = 0$$

for all  $(-\dot{x}, e_S, f', e') \in \mathcal{D}$ , which means that

$$\frac{d\mathcal{C}}{dt} = f_c e' + e_c f'.$$

Thus  $d\mathcal{C}/dt$  is a linear function of  $f'$  and  $e'$ .

### 3.2 Composition of Dirac structure and a resistive relation

**Proposition 1:** *Let  $\mathcal{D}$  be a Dirac structure defined with respect to  $\mathcal{F}_S \times \mathcal{F}_S^* \times \mathcal{F}_R \times \mathcal{F}_R^*$ . Furthermore, let  $\mathcal{R}$  be a resistive relation defined with respect to  $\mathcal{F}_R \times \mathcal{F}_R^*$  given by*

$$R_f f_R + R_e e_R = 0,$$

where the square matrices  $R_f$  and  $R_e$  satisfy the symmetry and semi-positive definiteness condition

$$R_f R_e^T = R_e R_f^T \geq 0.$$

Define the composition  $\mathcal{D} \parallel \mathcal{R}$  of the Dirac structure and the resistive relation in the same way as the composition of two Dirac structures. Then

$$(\mathcal{D} \parallel \mathcal{R})^\perp = \mathcal{D} \parallel -\mathcal{R}, \quad (9)$$

where  $-\mathcal{R}$  denotes the pseudo-resistive relation given by

$$R_f f_R - R_e e_R = 0$$

( $-\mathcal{R}$  is called a pseudo-resistive relation since it corresponds to negative instead of positive resistance).

**Proof:** We follow the same steps as in the proof (see van der Schaft and Cervera (2002); Cervera et al. (2006))

that the composition of two Dirac structures is again a Dirac structure. Because of the sign difference in the definition of a resistive relation as compared with the definition of a Dirac structure we immediately obtain the stated proposition.  $\square$

**Remark 1:** Similarly we can also view interconnections of two resistive relations with partially shared variables. If we consider a resistive relation  $\mathcal{R}_1$  defined with respect to  $\mathcal{V}_1 \times \mathcal{V}_1^* \times \mathcal{V}_2 \times \mathcal{V}_2^*$  and  $\mathcal{R}_2$  defined on  $\mathcal{V}_2 \times \mathcal{V}_2^* \times \mathcal{V}_3 \times \mathcal{V}_3^*$ , then it can be proved that the interconnection  $\mathcal{R}_1 \parallel \mathcal{R}_2$  is a resistive relation defined on  $\mathcal{V}_1 \times \mathcal{V}_1^* \times \mathcal{V}_3 \times \mathcal{V}_3^*$ , with the property that

$$(\mathcal{R}_1 \parallel \mathcal{R}_2)^\perp = -\mathcal{R}_1 \parallel -\mathcal{R}_2$$

with  $-\mathcal{R}_1, -\mathcal{R}_2$  denoting the pseudo resistive relations corresponding to negative resistances.

### 3.3 Achievable Dirac structures

The problem of control by interconnection of a plant port-Hamiltonian system  $P$  is to find a controller port-Hamiltonian system  $C$  such that the closed-loop system has the desired properties. The closed-loop system is again a port-Hamiltonian system with Hamiltonian equal to the sum of the Hamiltonians of the plant and the controller system, a total resistive relation depending on the resistive relations of the plant and controller systems and the Dirac structure being the composition of the Dirac structure of the plant and controller port-Hamiltonian systems. Desired properties of the closed-loop may include for example internal stability of the system and behavior at the interaction port.

Within the framework of control by interconnection of port-Hamiltonian systems, discussed in this paper, which relies on the existence of Casimirs for the closed-loop system, the problem is restricted to finding achievable Dirac structures of the closed-loop system, that is given a  $\mathcal{D}_P$  with a  $\mathcal{R}_P$  (i.e, a plant system with dissipation) and a (to be designed)  $\mathcal{D}_C$  with  $\mathcal{R}_C$  (a controller system with dissipation), what are the achievable  $(\mathcal{D}_P \parallel \mathcal{R}_P) \parallel (\mathcal{D}_C \parallel \mathcal{R}_C)$ . For ease of notation we henceforth use  $\mathcal{DR}_P$  for  $(\mathcal{D}_P \parallel \mathcal{R}_P)$  and  $\mathcal{DR}_C$  for  $(\mathcal{D}_C \parallel \mathcal{R}_C)$ . Consider here the case where  $\mathcal{DR}_P$  is given a Dirac structure with dissipation (finite-dimensional), and  $\mathcal{DR}_C$  a to be designed controller Dirac structure with dissipation. We investigate what are the achievable  $\mathcal{DR}_P \parallel \mathcal{DR}_C$ , the closed-loop structures.

**Theorem 1:** *Consider a (given) plant Dirac structure with dissipation  $\mathcal{DR}_P$  with port variables  $f_1, e_1, f_{R1}, e_{R1}, f, e$  and a desired Dirac structure with dissipation  $\mathcal{DR}$  with port-variables  $f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, f_{R2}, e_{R2}$ . Here  $(f_1, e_1), (f_{R1}, e_{R1})$  respectively denote the flow and effort variables corresponding to the energy storing*

elements and the energy dissipating elements of the plant system and similarly for the controller system. Then there exists a controller system  $\mathcal{DR}_C$  such that  $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_C$  if and only if the following two conditions are satisfied

$$\mathcal{DR}_P^0 \subset \mathcal{DR}^0 \tag{10}$$

$$\mathcal{DR}^\pi \subset \mathcal{DR}_P^\pi \tag{11}$$

where

$$\left. \begin{aligned} \mathcal{DR}_P^0 &:= \{(f_1, e_1, f_{R1}, e_{R1}) \mid (f_1, e_1, f_{R1}, e_{R1}, 0, 0) \in \mathcal{DR}_P\} \\ \mathcal{DR}_P^\pi &:= \{(f_1, e_1, f_{R1}, e_{R1}) \mid \exists (f, e) \text{ s.t. } (f_1, e_1, f_{R1}, e_{R1}, f, e) \in \mathcal{DR}_P\} \\ \mathcal{DR}^0 &:= \{(f_1, e_1, f_{R1}, e_{R1}) \mid (f_1, e_1, f_{R1}, e_{R1}, 0, 0, 0, 0) \in \mathcal{DR}\} \\ \mathcal{DR}^\pi &:= \{(f_1, e_1, f_{R1}, e_{R1}) \mid \exists (f_2, e_2, f_{R2}, e_{R2}) \text{ s.t. } ((f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, f_{R2}, e_{R2}) \in \mathcal{DR})\}. \end{aligned} \right\} \tag{12}$$

**Proof:** The proof is again based on the copy  $\mathcal{DR}_P^*$  of the plant system defined as

$$\mathcal{DR}_P^* := \left\{ (f_1, e_1, f_{R1}, e_{R1}, f, e) \right. \\ \left. \times (-f_1, e_1, -f_{R1}, e_{R1}, -f, e) \in \mathcal{DR}_P \right\} \tag{13}$$

and defining a controller system

$$\mathcal{DR}_C := \mathcal{DR}_P^* \parallel \mathcal{DR}.$$

We follow the same procedure for the proof as in the case of achievable Dirac structures van der Schaft and Cervera (2002) and Cervera *et al.* (2006).

Necessity of conditions (10) and (11) is obvious. Sufficiency is shown by using the controller Dirac structure with dissipation

$$\mathcal{DR}_C := \mathcal{DR}_P^* \parallel \mathcal{DR}.$$

To check that  $\mathcal{DR} \subset \mathcal{DR}_P \parallel \mathcal{DR}_C$ , consider  $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, f_{R2}, e_{R2}) \in \mathcal{DR}$ . Because  $(f_1, e_1, f_{R1}, e_{R1}) \in \mathcal{DR}^\pi$ , applying (11) yields that  $\exists (f, e)$  such that  $(f_1, e_1, f_{R1}, e_{R1}, f, e) \in \mathcal{DR}_P$ . This implies that  $(-f_1, e_1, -f_{R1}, e_{R1}, -f, e) \in \mathcal{DR}_P^*$ , with the following interconnection constraints (see figure 1):

$$f = -f^*, \quad e = e^*, \quad f_1^* = -f_1, \quad e_1^* = e_1.$$

By taking  $(f'_1, e'_1, f'_{R1}, e'_{R1}) = (f_1, e_1, f_{R1}, e_{R1})$  in figure 1 it follows that  $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, f_{R2}, e_{R2}) \in \mathcal{DR}_P \parallel \mathcal{DR}_C$  and hence  $\mathcal{DR} \subset \mathcal{DR}_P \parallel \mathcal{DR}_C$

To check that  $\mathcal{DR}_P \parallel \mathcal{DR}_C \subset \mathcal{DR}$ , consider  $(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, f_{R2}, e_{R2}) \in \mathcal{DR}_P \parallel \mathcal{DR}_C$ . Then there exists  $f = -f^*, e = e^*, f_1^* = -f_1, e_1^* = e_1, f_{R1}^* = -f_{R1}, e_{R1}^* = e_{R1}$  such that

$$(f_1, e_1, f_{R1}, e_{R1}, f, e) \in \mathcal{DR}_P \tag{14}$$

$$\left. \begin{aligned} (f_1^*, e_1^*, f_{R1}^*, e_{R1}^*, f^*, e^*) &\in \mathcal{DR}_P^* \\ \iff (f'_1, e'_1, f'_{R1}, e'_{R1}, f, e) &\in \mathcal{DR}_P \end{aligned} \right\} \tag{15}$$

$$(f'_1, e'_1, f'_{R1}, e'_{R1}, f_2, e_2, f_{R2}, e_{R2}) \in \mathcal{DR} \tag{16}$$

subtracting (15) from (14) and also by making use of the linearity on  $\mathcal{DR}_P$  we get

$$\left. \begin{aligned} (f_1 - f'_1, e_1 - e'_1, f_{R1} - f'_{R1}, e_{R1} - e'_{R1}, 0, 0) &\in \mathcal{DR}_P \\ \iff (f_1 - f'_1, e_1 - e'_1, f_{R1} - f'_{R1}, e_{R1} - e'_{R1}) &\in \mathcal{DR}_P^0. \end{aligned} \right\} \tag{17}$$

Using (17) and (10) we get

$$(f_1 - f'_1, e_1 - e'_1, f_{R1} - f'_{R1}, e_{R1} - e'_{R1}, 0, 0, 0, 0) \in \mathcal{DR}. \tag{18}$$

Finally, adding (18) and (16) we get

$$(f_1, e_1, f_{R1}, e_{R1}, f_2, e_2, f_{R2}, e_{R2}) \in \mathcal{DR}$$

hence  $\mathcal{DR}_P \parallel \mathcal{DR}_C \subset \mathcal{DR}$ . □

**Remark 2:** It can easily be checked that the conditions (10) and (11) are no more equivalent as in the case of systems without dissipation, see Cervera *et al.* (2006). This is primarily due to the compositional property of a Dirac structure with a resistive relation given by (9).

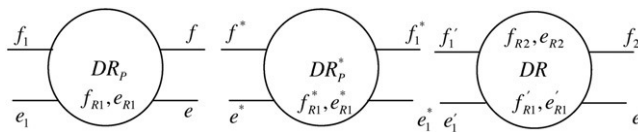


Figure 1.  $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_P^* \parallel \mathcal{DR}$ .

**3.3.1 Properties of  $\mathcal{DR}_p^*$ .** Consider the following input-state-output port-Hamiltonian plant system with inputs  $f$  and outputs  $e$

$$\left. \begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H_P}{\partial x}(x) + g(x)f, & x \in \mathcal{X}, f \in R^m \\ e &= g^T(x) \frac{\partial H_P}{\partial x}(x), & e \in R^m, \end{aligned} \right\} \quad (19)$$

where  $J(x)$  is the interconnection matrix and  $R(x)$  corresponds to the dissipation. The corresponding Dirac structure is given by the graph of the map

$$\begin{bmatrix} f_p \\ e \end{bmatrix} = \begin{bmatrix} -[J(x) - R(x)] & -g(x) \\ g^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_p \\ f \end{bmatrix}. \quad (20)$$

Now, going by the definition of  $\mathcal{DR}_p^*$  (see equation (13)), we can write it as

$$\begin{bmatrix} f_p^* \\ e^* \end{bmatrix} = \begin{bmatrix} -[J^*(x) - R^*(x)] & -g^*(x) \\ g^{*T}(x) & 0 \end{bmatrix} \begin{bmatrix} e_p^* \\ f^* \end{bmatrix}. \quad (21)$$

This implies that the interconnection matrix  $J^*(x)$ , the dissipation matrix  $R^*(x)$  and the input vector field  $g^*(x)$  of  $\mathcal{DR}_p^*$  would relate to the interconnection matrix  $J(x)$ , the dissipation matrix  $R(x)$  and the input vector field  $g(x)$  of  $\mathcal{DR}_p$  as follows

$$\left. \begin{aligned} J^*(x) &= -J(x), & R^*(x) &= -R(x) \\ g^*(x) &= g(x). \end{aligned} \right\} \quad (22)$$

A standard plant-controller interconnection would result in a closed-loop Dirac structure of the form, which we call as the desired closed-loop system

$$\left. \begin{aligned} \begin{bmatrix} f_p \\ f_c \end{bmatrix} &= \begin{bmatrix} -J(x) & g(x)g_c^T(\xi) \\ -g_c(\xi)g^T(x) & -J_c(\xi) \end{bmatrix} + \begin{bmatrix} R(x) & 0 \\ 0 & R_c(\xi) \end{bmatrix} \begin{bmatrix} e_p \\ e_c \end{bmatrix} \\ \begin{bmatrix} e \\ \tilde{e} \end{bmatrix} &= \begin{bmatrix} g^T(x) & 0 \\ 0 & g_c^T(\xi) \end{bmatrix} \begin{bmatrix} e_p \\ e_c \end{bmatrix}. \end{aligned} \right\} \quad (23)$$

It can easily be checked that such a Dirac structure would satisfy the conditions (10, 11) and hence we can construct a controller Dirac structure as in Theorem 1. The controller Dirac structure is defined as  $\mathcal{DR}_C = \mathcal{DR}_p^* \parallel \mathcal{DR}$ . Interconnecting  $\mathcal{DR}_p^*$  and  $\mathcal{DR}$  with the following interconnection constraints

$$\begin{aligned} f_p^* &= -f_p \\ e_p^* &= e_p \end{aligned}$$

would result in the following

$$\left. \begin{aligned} f_c &= -[J_c(\xi) - R_c(\xi)]e_c - g_c(\xi)g^T(x)e_p \\ g(x)g_c^T(\xi)e_c &= -g(x)f \end{aligned} \right\} \quad (24)$$

but we know from (23) that

$$e = g^T(x)e_p = \tilde{f}$$

and we also have, due to the left invertibility of  $g(x)$ , the following:

$$g_c^T(\xi)e_c = f = \tilde{e}$$

and hence we can rewrite (24) as

$$\left. \begin{aligned} f_c &= -[J_c(\xi) - R_c(\xi)]e_c - g_c(\xi)\tilde{f} \\ \tilde{e} &= -f \end{aligned} \right\} \quad (25)$$

which gives the controller Dirac structure, with the input of the controller given by the output of the plant system and the output of the controller given by negative of the plant input, the case of such interconnection is called the gyrative interconnection. It then directly follows that  $\mathcal{DR} = \mathcal{DR}_p \parallel \mathcal{DR}_C$ .

### 3.4 Casimirs for a system with dissipation

We define a Casimir for a port-Hamiltonian system with dissipation to be any function  $\mathcal{C}: \mathcal{X} \rightarrow \mathcal{R}$  such that its gradient satisfies

$$(0, e, 0, 0) \in \mathcal{DR}$$

which implies that

$$\frac{d\mathcal{C}}{dt} = e^T f_p = 0. \quad (26)$$

At this point one may think that the definition of Casimir function may be relaxed by requiring that the above expression holds only for a specific resistive relation

$$R_f f_R + R_e e_R = 0, \quad (27)$$

where the square matrices  $R_f$  and  $R_e$  satisfy the symmetry and positive semi definiteness condition

$$R_f R_e^T = R_e R_f^T \geq 0$$

together with the dimensionality condition

$$\text{rank}[R_f | R_e] = \dim f_R.$$

In this case, the condition for a function to be a conserved quantity for one resistive relation will actually imply that it is a conserved quantity for all resistive relations.

Indeed, let  $\mathcal{C}: \mathcal{X} \rightarrow R$  be a function satisfying (26) for a specific resistive port  $\mathcal{R}$  specified by matrices  $R_f$  and  $R_e$  as above. This means that  $e = (\partial\mathcal{C}/\partial x)(x)$  satisfies

$$\begin{aligned} e^T f_p &= 0, \forall f_p \quad \text{for which } \exists e_p, f_R, e_R \text{ s.t.} \\ &\times (f_p, e_p, f_R, e_R) \in \mathcal{DR} \quad \text{and} \\ &\times R_f f_R + R_e e_R = 0. \end{aligned}$$

However, this implies that  $(0, e, 0, 0) \in (\mathcal{D}\|\mathcal{R})^\perp$ . We also know that  $(\mathcal{D}\|\mathcal{R})^\perp = \mathcal{D}\|(-\mathcal{R})$ , and thus there exists  $\tilde{f}_R, \tilde{e}_R$  such that  $R_f \tilde{f}_R - R_e \tilde{e}_R = 0$  and

$$(0, e, \tilde{f}_R, \tilde{e}_R) \in \mathcal{DR}$$

Hence,

$$0 = e^T \cdot 0 + \tilde{e}_R^T \tilde{f}_R = \tilde{e}_R^T \tilde{f}_R$$

By writing the pseudo resistive relation  $-\mathcal{R}$  in image representation van der Schaft (2000),  $\tilde{f}_R = R_e^T \lambda$ ,  $\tilde{e}_R = R_f^T \lambda$ , it follows that

$$\lambda^T R_f R_e^T \lambda = 0$$

and by the positive definiteness condition  $R_f R_e^T = R_e R_f^T > 0$  this implies that  $\lambda = 0$ , whence  $\tilde{f}_R = \tilde{e}_R = 0$ . Hence not only  $(0, e, \tilde{f}_R, \tilde{e}_R) \in \mathcal{D}$ , but actually  $(0, e, 0, 0) \in \mathcal{D}$ , implying that  $e$  is the gradient of the Casimir function.

Of course, the above argument does not fully carry through if the resistive relations are only positive semi-definite. In particular this is the case if  $R_f R_e^T = 0$  (implying zero dissipation), corresponding to the presence of ideal power conserving constraints.

### 3.5 Achievable Casimirs for any resistive relation

We now consider the question of characterizing the set of achievable Casimirs for the closed-loop system  $\mathcal{DR}_P \|\mathcal{DR}_C$  for all resistive relations and every port behavior. Here  $\mathcal{DR}_P$  is the Dirac structure of the plant port-Hamiltonian system with dissipation with Hamiltonian  $H_P$ , and  $\mathcal{DR}_C$  is the controller

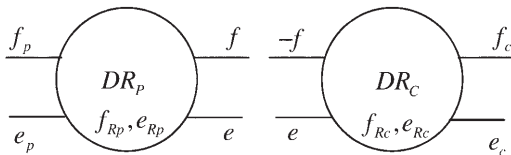


Figure 2.  $\mathcal{DR}_P \|\mathcal{DR}_C$ .

Dirac structure. Then the Casimirs depend on the plant state  $x$ , and also on the controller state  $\xi$ , with the controller Hamiltonian  $H_C(\xi)$  at our own disposal.

Consider the notation as in figure 2 and assume that the ports in  $(f_p, e_p), (f_{Rp}, e_{Rp})$  are respectively connected to the (given) energy storing elements and the energy dissipating elements of the plant port-Hamiltonian system. Similarly  $(f_c, e_c)$  are connected to the (to be designed) energy storing elements of the controller port-Hamiltonian system with dissipation; that is  $(f_c = -\dot{\xi}, e_c = (\partial^T H_C / \partial \xi))$  and  $(f_{Rc}, e_{Rc})$  are connected to the energy dissipation elements of the controller system. In this situation the achievable Casimirs are functions  $\mathcal{C}(x, \xi)$  such that  $(\partial^T \mathcal{C} / \partial x)(x, \xi)$  belongs to the space

$$\begin{aligned} P_{Cas} &= \{e_p \mid \exists \mathcal{DR}_C \text{ s.t. } \exists e_c : \\ &(0, e_p, 0, 0, 0, e_c, 0, 0) \in \mathcal{DR}_P \|\mathcal{DR}_C\} \end{aligned}$$

The following theorem then addresses the question of characterizing the achievable Casimirs of the closed-loop system, regarded as functions of the plant state  $x$  by characterization of the space  $P_{Cas}$ .

**Proposition 2:** *The space  $P_{Cas}$  defined above is equal to the space*

$$\tilde{P} = \{e_1 \mid \exists (f, e) \text{ s.t. } (0, e_1, 0, 0, f, e) \in \mathcal{DR}_P\}.$$

**Proof:** We see that  $P_{Cas} \subset \tilde{P}$  trivially and by using the controller Dirac structure  $\mathcal{DR}_C = \mathcal{DR}_P^*$  we obtain  $P \subset P_{Cas}$ .  $\square$

### 3.6 Achievable Casimirs for a given resistive relation

If  $\mathcal{C}: \mathcal{X} \rightarrow R$  is a Casimir function for a specific resistive relation  $\mathcal{R}$  given by (27), then this means that  $e = (\partial\mathcal{C}/\partial x)(x)$  satisfies

$$\begin{aligned} \frac{\partial^T \mathcal{C}}{\partial x}(x) f_p &= 0 \quad \text{for all } f_p \text{ s.t. } \exists e_p, f_R, e_R \text{ s.t.} \\ &\times (f_p, e_p, f_R, e_R) \in \mathcal{DR} \quad \text{and} \\ &\times R_f f_R + R_e e_R \end{aligned}$$

which means that  $(0, e, 0, 0) \in (\mathcal{D}\|\mathcal{R})^\perp$ . Since we know by proposition 1 that  $(\mathcal{D}\|\mathcal{R})^\perp = \mathcal{D}\|-\mathcal{R}$ , and thus  $\mathcal{C}$  is a Casimir function if there exist  $(f_R, e_R)$  such that

$$\left(0, \frac{\partial \mathcal{C}}{\partial x}, -f_R, e_R\right) \in \mathcal{DR}.$$

We now consider the question of finding all the achievable Casimirs for closed-loop system  $\mathcal{DR}_P \|\mathcal{DR}_C$ , with  $\mathcal{DR}_P$  the Dirac structure of the plant port-Hamiltonian system with dissipation with Hamiltonian  $H_P$ , and  $\mathcal{DR}_C$  is the controller Dirac structure; for a



given resistive relations and every port behavior. Consider  $\mathcal{DR}_P$  and  $\mathcal{DR}_C$  as above, and in this case the achievable Casimirs are functions  $\mathcal{C}(x, \xi)$  such that  $(\partial^T \mathcal{C} / \partial x)(x, \xi)$  belongs to the space

$$P_{Cas} = \{e_p \mid \exists \mathcal{DR}_C \text{ s.t. } \exists e_c, f_{Rp}, e_{Rp}, f_{Rc}, e_{Rc} : \\ (0, e_p, -f_{Rp}, e_{Rp}, 0, e_c, -f_{Rc}, e_{Rc}) \in \mathcal{DR}_P \parallel \mathcal{DR}_C\}.$$

**Proposition 3:** *The space  $P_{Cas}$  defined above is equal to the linear space*

$$\tilde{P} = \{e_p \mid \exists (f_{Rp}, e_{Rp}, f, e) \text{ s.t. } (0, e_p, -f_{Rp}, e_{Rp}, f, e) \in \mathcal{DR}_P\}.$$

**Proof:** The proof follows the same procedure as in Proposition 2.  $\square$

**Remark 3:** The characterization in terms of plant state is useful in the sense that given a plant Dirac structure we can, without defining a controller, determine whether or not there exist Casimir functions for the closed-loop system as will be shown in examples below. This is in addition to the fact the we can also know the Casimir functions for all  $R$  and  $R_c$ , with  $R_c \geq 0$  and  $R_c \geq 0$ .

**Example 1:** Consider the port-Hamiltonian system with  $(f_p, e_p)$  respectively the flows and efforts corresponding to the energy storage elements,  $(f_R, e_R)$  the flows and efforts corresponding to the energy dissipating elements and inputs  $f$  and outputs  $e$ . The corresponding Dirac structure is given by

$$f_p = -J(x)e_p - g_R(x)f_R - g(x)f \\ \begin{bmatrix} e_R \\ e \end{bmatrix} = \begin{bmatrix} g_R^T(x) \\ g^T(x) \end{bmatrix} e_p.$$

The characterization of the space  $P_{Cas}$  is given by

$$P_{Cas} = \left\{ e_p \mid \exists f_p \text{ s.t. } \begin{cases} 0 = -J(x)e_p - g(x)f \\ \text{and } 0 = g_R^T(x)e_p \end{cases} \right\} \quad (28)$$

The above expression implies that the achievable Casimirs do not depend on the coordinate where dissipation enters into the system (follows from the second line, and well known in literature as the ‘‘dissipation obstacle’’), in addition to that they are also the Hamiltonian functions corresponding to the input vector fields given by the columns of  $g(x)$ .

**Example 2** (The series RLC circuit): The dynamics of the circuit are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & R \end{bmatrix} \begin{bmatrix} x_1/C \\ x_2/L \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

and the corresponding Dirac structure given by

$$\begin{bmatrix} -\dot{x}_1 \\ -\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1/C \\ x_2/L \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ Rx_2/L \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ e_R = \begin{bmatrix} 0 \\ x_2/L \end{bmatrix} \\ e_p = \frac{x_2}{L}$$

comparing with the previous example we have

$$(f_p, e_p, f_R, e_R, f, e) \\ = \left( -[\dot{x}_1, \dot{x}_2]^T, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T, \begin{bmatrix} 0 & Rx_2/L \end{bmatrix}^T, \begin{bmatrix} 0 & x_2/L \end{bmatrix}^T, u, e \right), \\ g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad g_R(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In this case the achievable Casimirs (in terms of the plant state  $x$ ) should satisfy the following set of equations

$$\frac{\partial \mathcal{C}}{\partial x_2}(x, \xi) = 0.$$

The above expression implies that any Casimir function for this system does not depend on  $x_2$  term, which is precisely where dissipation enters into the system. There however do exist Casimirs depending on the  $x_1$  term.

**Example 3** (The parallel RLC circuit): We next consider the case of a parallel RLC circuit whose dynamics are given by the following set of equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1/R & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1/C \\ x_2/L \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

and the corresponding Dirac structure given by

$$\begin{bmatrix} -\dot{x}_1 \\ -\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1/C \\ x_2/L \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1/RC \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

where

$$(f_p, e_p, f_R, e_R, f, e) = \left( -[\dot{x}_1, \dot{x}_2]^T, \begin{bmatrix} x_1/C, x_2/L \end{bmatrix}^T, \right. \\ \left. \times \begin{bmatrix} x_1/RC & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & x_2/L \end{bmatrix}^T, u, e \right), \\ g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad g_R(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

As above the achievable Casimirs in terms of the plant state  $x$  should be such that

$$\frac{\partial \mathcal{C}}{\partial x_1}(x, \xi) = \frac{\partial \mathcal{C}}{\partial x_2}(x, \xi) = 0$$

which means that we cannot find any Casimir functions for the closed-loop system which depend on the plant state  $x$  (the only possible Casimirs are the ‘‘trivial

Casimirs” which are constant), hence we cannot directly apply the control by interconnection method for such systems.

**4. Achievable Casimirs for systems with dissipation: Infinite dimensions**

In this section we discuss interconnection properties of infinite-dimensional systems defined by a Stokes–Dirac structures, in particular systems with dissipation. Dissipation can enter into a infinite dimensional port-Hamiltonian system in two ways: either by terminating its boundary or boundaries by a resistive relation or through the spatial domain where we terminate some or all of the distributed ports by a resistive relation. In this section we focus on the latter case where we have dissipation entering into the system through the spatial domain. The case of terminating the boundary of the system by a resistive relation can be considered as a special case of interconnection of mixed finite and infinite dimensional port-Hamiltonian systems as will be discussed on the next section.

**4.1 Composition of Dirac structure and a resistive relation**

We discuss here the composition of a Stokes–Dirac structure and a resistive relation, where the dissipation enters into the system through the spatial domain (part or whole of it).

**Proposition 4:** Let  $\mathcal{D}$  be a Stokes–Dirac structure defined with respect to  $\mathcal{F}_{p,q} \times \mathcal{E}_{p,q} \times \mathcal{F}_{R_{p,q}} \times \mathcal{E}_{R_{p,q}} \times \mathcal{F}_b \times \mathcal{E}_b$  as follows:

$$\left. \begin{aligned} \begin{bmatrix} f_p \\ f_q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} - g_R \begin{bmatrix} f_{Rp} \\ f_{Rq} \end{bmatrix}; \\ \begin{bmatrix} e_{Rp} \\ e_{Rq} \end{bmatrix} &= g_R^T \begin{bmatrix} e_p \\ e_q \end{bmatrix}; \\ \begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_p & |_{\partial Z} \\ e_q & |_{\partial Z} \end{bmatrix}. \end{aligned} \right\} \quad (29)$$

Furthermore let  $\mathcal{R}$  be a resistive relation defined with respect to  $\mathcal{F}_{R_{p,q}} \times \mathcal{E}_{R_{p,q}}$ . Let  $S^*: \Omega^{n-k}(Z) \rightarrow \Omega^k(Z)$  be a map satisfying

$$\int_Z e_R \wedge (S * e_R) = \int_Z (S * e_R) \wedge e_R \geq 0, \quad \forall e_R \in \Omega^{n-k}(Z), \quad R \in \mathcal{R}. \quad (30)$$

We consider here a typical case where the flows and the efforts of the energy dissipating elements are related as

$$f_R = -S * e_R.$$

Here  $f_R$  and  $e_R$  correspond to the flows and effort variables of the resistive elements in both the  $p$  and  $q$  energy domains, i.e.

$$\begin{aligned} f_R &= [f_{Rp} \quad f_{Rq}]^T \\ e_R &= [e_{Rp} \quad e_{Rq}]^T. \end{aligned}$$

Similarly  $S$  also incorporates the dissipation in both the energy domains. Typically  $S$  is a block of the form

$$S = \begin{bmatrix} G & 0 \\ 0 & R \end{bmatrix}. \quad (31)$$

Defining interconnections of  $\mathcal{D}$  and  $\mathcal{R}$  in the standard way, we have the composed structure as follows:

$$\left. \begin{aligned} \begin{bmatrix} f_p \\ f_q \end{bmatrix} &= \begin{bmatrix} 0 & (-1)^r d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} - g_R \begin{bmatrix} f_{Rp} \\ f_{Rq} \end{bmatrix}; \\ \begin{bmatrix} f_{Rp} \\ f_{Rq} \end{bmatrix} &= - \begin{bmatrix} G^* & 0 \\ 0 & R^* \end{bmatrix} \begin{bmatrix} e_{Rp} \\ e_{Rq} \end{bmatrix} \\ \begin{bmatrix} e_{Rp} \\ e_{Rq} \end{bmatrix} &= g_R^T \begin{bmatrix} e_p \\ e_q \end{bmatrix} \end{aligned} \right\} \quad (32)$$

which has the property that

$$(\mathcal{D} \parallel \mathcal{R})^\perp = \mathcal{D} \parallel -\mathcal{R},$$

where  $\mathcal{R}$  again is a pseudo resistive relation (corresponding the negative resistance).

**Proof:** For simplicity of the proof we take a system with a 1D spatial domain and assume zero boundary conditions (meaning that all the boundary variables are set to zero). Then the bilinear form on  $\mathcal{D} \parallel \mathcal{R}$  is given by

$$\begin{aligned} &\langle \langle (f_p^1, f_q^1, e_p^1, e_q^1), (f_p^2, e_p^2, f_q^2, e_q^2) \rangle \rangle \\ &= \int_Z [e_p^2 \wedge f_p^1 + e_p^1 \wedge f_p^2 + e_q^2 \wedge f_q^1 + e_q^1 \wedge f_q^2] \quad (33) \end{aligned}$$

Part I: Take a  $(f_p^1, f_q^1, e_p^1, e_q^1) \in \mathcal{D} \parallel \mathcal{R}$  and take any other  $(f_p^2, f_q^2, e_p^2, e_q^2) \in \mathcal{D} \parallel (-\mathcal{R})$ . By substituting (29) into (33), the right hand side of (33) becomes

$$\begin{aligned} &\int_Z [e_p^2 \wedge (de_q^1 + G * e_p^1) + e_p^1 \wedge (de_q^2 - G * e_p^2) \\ &\quad + e_q^2 \wedge (de_p^1 + R * e_q^1) + e_q^1 \wedge (de_p^2 - R * e_q^2)] \\ &= \int_Z [e_p^2 \wedge de_q^1 + e_p^2 \wedge G * e_p^1 + e_p^1 \wedge de_q^2 - e_p^1 \wedge G * e_p^2 + e_q^2 \\ &\quad \wedge de_p^1 + e_q^2 \wedge R * e_q^1 + e_q^1 \wedge de_p^2 - e_q^1 \wedge R * e_q^2]. \quad (34) \end{aligned}$$

We now use the following properties of the exterior derivative and the Hodge star operator

$$\begin{aligned} d(\alpha \wedge \beta) &= d\alpha \wedge \beta + \alpha \wedge d\beta \\ \alpha \wedge \beta &= \beta \wedge \alpha \\ \alpha \wedge (\beta * \gamma) &= (\beta * \alpha) \wedge \gamma \end{aligned}$$

Before we proceed, we would like to mention here that since we are dealing here with an infinite-dimensional system with a 1D spatial domain, we deal only with zero-forms and one-forms. The above properties hold for the case where  $\alpha$  is a one form,  $\beta$  and  $\gamma$  are zero forms. For general case of  $n$ -forms refer to Abraham *et al.* (1988).

Using the above properties and the Stokes' theorem, equation (34) can be written as

$$\begin{aligned} \int_Z & \left[ d(e_p^2 \wedge e_q^1) - de_p^2 \wedge e_q^1 + G * e_p^2 \wedge e_p^1 \right. \\ & + d(e_p^1 \wedge e_q^2) - de_p^1 \wedge e_q^2 - G * e_p^2 \wedge e_p^1 + de_p^1 \wedge e_q^2 \\ & \left. + R * e_q^2 \wedge e_q^1 + de_p^2 \wedge e_q^1 - R * e_q^2 \wedge e_q^1 \right] = 0 \end{aligned}$$

Hence  $(\mathcal{D}\|-\mathcal{R}) \subset (\mathcal{D}\|\mathcal{R})^\perp$ .

Part II: Let  $(f_p^1, f_q^1, e_p^1, e_q^1) \in \mathcal{D}\|\mathcal{R}$  and let  $(f_p^2, f_q^2, e_p^2, e_q^2) \in (\mathcal{D}\|-\mathcal{R})^\perp$ , hence the right hand side of (33) is zero for these elements and hence by substitution, we have

$$\begin{aligned} \int_Z & \left[ e_p^2 \wedge (de_q^1 + G * e_p^1) + e_p^1 \wedge f_p^2 \right. \\ & \left. + e_q^2 \wedge (de_p^1 + R * e_q^1) + e_q^1 \wedge f_q^2 \right] = 0 \\ \Rightarrow \int_Z & \left[ e_p^2 \wedge de_q^1 + e_p^2 \wedge G * e_p^1 + e_p^1 \wedge f_p^2 + e_q^2 \wedge de_p^1 \right. \\ & \left. + e_q^2 \wedge R * e_q^1 + e_q^1 \wedge f_q^2 \right] = 0. \end{aligned}$$

Now, again using the above mentioned properties of the exterior derivative and the Hodge star operator, we get

$$\begin{aligned} \int_Z & \left[ d(e_p^2 \wedge e_q^1) - de_p^2 \wedge e_q^1 + G * e_p^2 \wedge e_p^1 + e_p^1 \wedge f_p^2 + d(e_q^2 \wedge e_p^1) \right. \\ & \left. - de_q^2 \wedge e_p^1 + R * e_q^2 \wedge e_q^1 + e_q^1 \wedge f_q^2 = 0 \right] \end{aligned}$$

Since, we assume zero boundary conditions and applying the Stokes theorem the above expression can be written as

$$\begin{aligned} \int_Z & - \left( de_p^2 - R * e_q^2 \right) \wedge e_q^1 - \left( de_q^2 - G * e_p^2 \right) \wedge e_p^1 + f_p^2 \wedge e_p^1 \\ & + f_q^2 \wedge e_q^1 = 0 \end{aligned}$$

which implies that

$$\begin{aligned} f_p^2 &= de_q^2 - G * e_p^2 \\ f_q^2 &= de_p^2 - R * e_q^2 \end{aligned}$$

showing that  $(f_p^2, f_q^2, e_p^2, e_q^2) \in \mathcal{D}\|(-\mathcal{R})$ , which means that  $(\mathcal{D}\|\mathcal{R})^\perp \subset \mathcal{D}\|(-\mathcal{R})$ , completing the proof.  $\square$

**Remark 4:** Equation (32), defines an infinite-dimensional port-Hamiltonian system with dissipation. The port-Hamiltonian system with dissipation now satisfies the energy balance inequality; also see van der Schaft and Maschke (2002)

$$\begin{aligned} \frac{dH}{dt} &= \int_Z f_b \wedge e_b - \int_Z e_R \wedge S(e_R) \\ &\leq \int_{\partial Z} f_b \wedge e_b. \end{aligned}$$

### 4.2 Achievable Dirac structures

Similar to the finite dimensional case we investigate what are the achievable closed-loop Dirac structures interconnecting a given plant Stokes–Dirac structure with dissipation  $\mathcal{DR}_P$  to a to be designed controller Stokes–Dirac structure with dissipation  $\mathcal{DR}_C$ .

**Theorem 2:** Given a plant Stokes–Dirac structure with dissipation  $\mathcal{DR}_P$ , a certain interconnected  $\mathcal{DR} = \mathcal{DR}_P \|\mathcal{DR}_C$  can be achieved by a proper choice of the controller Stokes–Dirac structure with dissipation if and only if the following two conditions are satisfied

$$\mathcal{DR}_P^0 \subset \mathcal{DR}^0 \tag{35}$$

$$\mathcal{DR}^\pi \subset \mathcal{DR}_P^\pi \tag{36}$$

where

$$\begin{aligned} \mathcal{DR}_P^0 &:= \{ (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, 0, 0) \in \mathcal{DR}_P \}^1 \\ \mathcal{DR}_P^\pi &:= \{ (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid \exists (f_b, e_b) : (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, f_b, e_b) \in \mathcal{DR}_P \} \\ \mathcal{DR}^0 &:= \{ (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, 0, 0, 0, 0) \in \mathcal{DR} \} \\ \mathcal{DR}^\pi &:= \{ (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}) \mid \exists (f_{pq}^c, e_{pq}^c, f_{R_{pq}}^c, e_{R_{pq}}^c) : (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, f_{pq}^c, e_{pq}^c, f_{R_{pq}}^c, e_{R_{pq}}^c) \in \mathcal{DR} \}. \end{aligned}$$

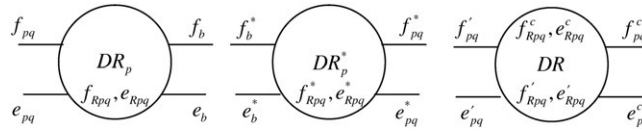


Figure 3.  $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_P^* \parallel \mathcal{DR}$ .

**Proof:** The proof follows the same lines as in the finite dimensional case, which again is based on a ‘‘copy’’ of  $\mathcal{DR}_p$  (also see figure 3) which in this case is defined as

$$\mathcal{DR}_P^* := \left\{ (f_{pq}, e_{pq}, f_{R_{pq}}, e_{R_{pq}}, f_b, e_b) \mid \right. \\ \left. \times (-f_{pq}, e_{pq}, -f_{R_{pq}}, e_{R_{pq}}, -f_b, e_b) \in \mathcal{DR}_P \right\}. \quad (37)$$

□

**4.2.1 Properties of  $\mathcal{DR}_P^*$ .** Consider a 1D infinite-dimensional system with a distributed dissipation defined with respect to a Stokes–Dirac structure  $\mathcal{DR}_P$  given by

$$\left. \begin{aligned} \begin{bmatrix} f_p \\ f_q \end{bmatrix} &= \begin{bmatrix} G^* & d \\ d & R^* \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix} \\ \begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_p \mid \partial z \\ e_q \mid \partial z \end{bmatrix} \end{aligned} \right\} \quad (38)$$

Now, consider the following closed-loop (achievable) Dirac structure  $\mathcal{DR}$ . This is obtained by interconnecting this system to another infinite-dimensional port-Hamiltonian system.

$$\begin{bmatrix} f_p \\ f_q \\ f_p^c \\ f_q^c \end{bmatrix} = \begin{bmatrix} G^* & d & 0 & 0 \\ d & R^* & 0 & 0 \\ 0 & 0 & G^{c*} & d \\ 0 & 0 & d & R^{c*} \end{bmatrix} \begin{bmatrix} e_p \\ e_q \\ e_p^c \\ e_q^c \end{bmatrix}. \quad (39)$$

It can easily be checked that this Dirac structure satisfies the conditions (35, 36). By the definition of  $\mathcal{DR}_P^*$  from equation (37), we can write it as

$$\left. \begin{aligned} \begin{bmatrix} f_p^* \\ f_q^* \end{bmatrix} &= \begin{bmatrix} G^* & d \\ d & R^* \end{bmatrix} \begin{bmatrix} e_p^* \\ e_q^* \end{bmatrix} \\ \begin{bmatrix} f_b^* \\ e_b^* \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_p^* \mid \partial z \\ e_q^* \mid \partial z \end{bmatrix} \end{aligned} \right\} \quad (40)$$

Theorem 2 defines the controller Dirac structure with dissipation as  $\mathcal{DR}_C = \mathcal{DR}_P^* \parallel \mathcal{DR}$ . Now, interconnecting  $\mathcal{DR}_P$  with  $\mathcal{DR}$  with the following interconnection constraints

$$\begin{aligned} f_p^* &= -f_p & f_q^* &= -f_q & f_b^* &= -f_b \\ e_p^* &= e_p & e_q^* &= e_q & e_b^* &= e_b \end{aligned} \quad (41)$$

would, with the help of a few computations, result in the following controller Stokes–Dirac structure with dissipation

$$\left. \begin{aligned} \begin{bmatrix} f_p^c \\ f_q^c \end{bmatrix} &= \begin{bmatrix} G^{c*} & d \\ d & R^{c*} \end{bmatrix} \begin{bmatrix} e_p^c \\ e_q^c \end{bmatrix} \\ \begin{bmatrix} f_b^c \\ e_b^c \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_p^c \mid \partial z \\ e_q^c \mid \partial z \end{bmatrix} \end{aligned} \right\} \quad (42)$$

It then immediately follows that  $\mathcal{DR} = \mathcal{DR}_P \parallel \mathcal{DR}_C$

**Remark 5:** If we consider infinite-dimensional Dirac structures defined on Hilbert spaces, then the compositional property is not immediate, as shown in Golo (2002). Necessary and sufficient conditions have been derived in Kurula *et al.* (2006) for the composition of two or more Dirac structures on Hilbert spaces to again define a Dirac structure. The infinite-dimensional Dirac structures we focus on here are of a particular kind, which we call the Stokes–Dirac structure, which are defined on spaces of differential forms. We have shown that a power-conserving interconnection of a number of Stokes–Dirac structures is again a Stokes–Dirac structure. Now, relating this to this Hilbert space setting, it follows that the composition of Stokes–Dirac structure satisfies the necessary and sufficient conditions as derived in Kurula *et al.* (2006) for the composition to again define a Stokes–Dirac structure.

**4.3 Achievable Casimirs**

A Casimir function for an infinite dimensional port-Hamiltonian system with dissipation is any functional  $\mathcal{C}: \mathcal{X} \rightarrow \mathcal{R}$  such that the Casimir gradients satisfy

$$(0, e_{pq}, -f_{R_{pq}}, e_{R_{pq}}) \in \mathcal{D}$$

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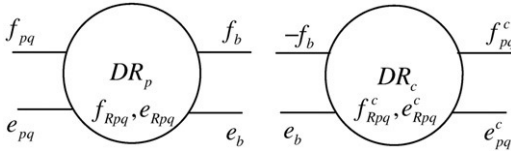


Figure 4.  $\mathcal{DR}_P \parallel \mathcal{DR}_C$ .

implying that

$$\frac{dC}{dt} = \int_Z e_p \wedge f_p + e_q \wedge f_q = 0.$$

To address the question of finding all the achievable Casimirs for the closed loop system  $\mathcal{DR}_P \parallel \mathcal{DR}_C$ , we consider the case where both  $\mathcal{DR}_P$  and  $\mathcal{DR}_C$  are infinite dimensional port-Hamiltonian systems with dissipation, and are defined with respect to a Stokes–Dirac structure. The interconnection between  $\mathcal{DR}_P$  and  $\mathcal{DR}_C$  takes place through the boundary of the system (see figure 4). It can easily be shown that such an interconnection (through the boundary) leads to another port-Hamiltonian system with dissipation. In this case the achievable Casimirs are functional  $C$  such that  $e_{pq} = \delta_{pq}C$  belongs to the space

$$P_{Cas} = \left\{ e_{pq} \mid \exists \mathcal{DR}_C \text{ s.t. } \exists e_{pq}^c, f_{Rpq}, e_{Rpq}, f_{Rpq}^c, e_{Rpq}^c : \right. \\ \left. \begin{aligned} &(0, e_{pq}, -f_{Rpq}, e_{Rpq}, 0, e_{pq}^c \\ &-f_{Rpq}^c, e_{Rpq}^c \in \mathcal{DR}_P \parallel \mathcal{DR}_C \end{aligned} \right\}.$$

Here  $\delta_{pq}C = [\delta_p C \ \delta_q C]$  are variational derivatives of  $C$  with respect to  $p$  and  $q$  respectively. See van der Schaft and Maschke (2002) and the references therein.

The characterization of the set of achievable Casimirs of the closed-loop system in terms of the plant state, by finding characterization of the space  $P_{Cas}$ , is addressed by the following Proposition.

**Proposition 4:** *The space  $P_{Cas}$  defined above is equal to the linear space*

$$\tilde{P} = \left\{ e_{pq} \mid \exists (f_{Rpq}, e_{Rpq}, f_b, e_b) : \right. \\ \left. (0, e_{pq}, -f_{Rpq}, e_{Rpq}, f_b, e_b) \in \mathcal{DR}_P \right\}$$

**Proof:** The proof follows the same steps as before by taking  $\mathcal{DR}_C = \mathcal{DR}_P^*$ , where  $\mathcal{DR}_P^*$  is as defined above.  $\square$

**Example 4:** Consider a transmission line with distributed dissipation over the entire spatial domain with  $Z = [0, l] \in R$ . The flow variables are the charge density one-form  $Q = Q(z, t)dz \in \Omega^1([0, l])$ , and the flux density one-form  $\phi = \phi(z, t)dz \in \Omega^1([0, l])$ , hence  $p = q = n = 1$ .

The total energy stored at time  $t$  in the transmission line is given by

$$H(Q, \phi) = \int_0^l \frac{1}{2} \left( \frac{Q^2(z, t)}{C(z)} + \frac{\phi^2(z, t)}{L(z)} \right) dz$$

with effort variables

$$\delta_Q H = \frac{Q(z, t)}{C(z)} = V(z, t), \quad \text{the voltage}$$

$$\delta_\phi H = \frac{\phi(z, t)}{L(z)} = I(z, t), \quad \text{the current}$$

with  $C(z)$ ,  $L(z)$  are, respectively, the distributed capacitance and distributed inductance of the line. The resulting dynamics of the transmission line with dissipation are given by

$$\begin{aligned} \begin{bmatrix} -\partial_t Q(z, t) \\ -\partial_t \phi(z, t) \end{bmatrix} &= \left( \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} + \begin{bmatrix} G^* & 0 \\ 0 & R^* \end{bmatrix} \right) \begin{bmatrix} \delta_Q H \\ \delta_\phi H \end{bmatrix} \\ \begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_Q H & \mid_{\partial Z} \\ \delta_\phi H & \mid_{\partial Z} \end{bmatrix}, \end{aligned}$$

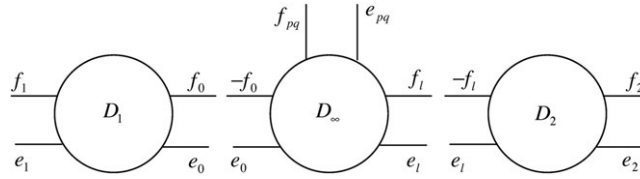
where  $d: \Omega^0(Z) \rightarrow \Omega^1(Z)$ , denotes the exterior derivative and  $*$ :  $\Omega^0(Z) \rightarrow \Omega^1(Z)$ , the Hodge star operator and  $G$  and  $R$  respectively being the distributed conductance and distributed resistance in the transmission line. Now, by applying proposition (4) and after some simple computations, we see that the achievable Casimirs are all functional  $C(Q(z, t), \phi(z, t))$  which satisfy (see also Macchelli and Melchiorri (2005))

$$\begin{aligned} d\delta_\phi C - G * \delta_Q C &= 0 \\ d\delta_Q C - R * \delta_\phi C &= 0. \end{aligned}$$

**Remark 6:** Contrast to the case of a transmission line without dissipation Rodriguez *et al.* (2001), Pasumarthy and van der Schaft (2004), the clear distinction here is that we do not have Casimirs which are constant with respect to the spatial variable  $z$ . This is clearly due to the presence of dissipation in the transmission line.

**5. Achievable Casimirs for systems with dissipation: Mixed finite and infinite dimensions**

Mixed port-Hamiltonian systems arise by interconnections of finite dimensional port-Hamiltonian systems with infinite dimensional port-Hamiltonian systems; see Rodriguez *et al.* (2001) and Macchelli and Melchiorri (2005), for example. We here study interconnections of such systems and show that the interconnection is again a Dirac structure, or in turn a port-Hamiltonian system. We also study what are the closed-loop Dirac structures


 Figure 5.  $\mathcal{D}_1 \parallel \mathcal{D}_\infty \parallel \mathcal{D}_2$ .

that can be achieved by interconnecting a given plant port-Hamiltonian system with a to-be-designed controller port-Hamiltonian system in the mixed case and then finally study its implications on control of port-Hamiltonian systems.

### 5.1 Interconnection of mixed finite and infinite dimensional systems

We consider here composition of two Dirac structures, without dissipation, (denoted  $\mathcal{D}_1$  and  $\mathcal{D}_2$  respectively) interconnected to each other via a Stokes–Dirac structure, also without dissipation, (denoted  $\mathcal{D}_\infty$ ). We consider here the simple case  $p = q = n = 1$  throughout, for the Stokes–Dirac structure (though it can be extended, if not easily, to the higher dimensional case). An immediate example of the case  $p = q = n = 1$  is that of a transmission line.

First we consider the composition of the two Dirac structures  $\mathcal{D}_1$  and  $\mathcal{D}_\infty$ . Consider  $\mathcal{D}_1$  on the product space  $\mathcal{F}_1 \times \mathcal{F}_0$  of two linear spaces  $\mathcal{F}_1$  and  $\mathcal{F}_0$ , and the Stokes–Dirac structure  $\mathcal{D}_\infty$  on the product space

We define the two interconnections as follows. The interconnection of the two Dirac structures  $\mathcal{D}_1$  and  $\mathcal{D}_\infty$  is defined as

$$\begin{aligned} \mathcal{D}_1 \parallel \mathcal{D}_\infty := & \{f_1, e_1, f_p, f_q, e_p, e_q, f_l, e_l\} \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \\ & \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^* | \exists (f_0, e_0) \in \mathcal{F}_0 \times \mathcal{F}_0^* \text{ s.t.} \\ & (f_1, e_1, f_0, e_0) \in \mathcal{D}_1 \quad \text{and} \\ & (-f_0, e_0, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty \}. \end{aligned}$$

Similarly, the interconnection of  $\mathcal{D}_\infty$  and  $\mathcal{D}_2$  is defined as

$$\begin{aligned} \mathcal{D}_\infty \parallel \mathcal{D}_2 := & \{-f_0, e_0, f_p, f_q, e_p, e_q, f_2, e_2\} \in \mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \\ & \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^* | \exists (f_l, e_l) \in \mathcal{F}_l \times \mathcal{F}_l^* \text{ s.t.} \\ & (-f_0, e_0, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty \\ & \text{and } (-f_l, e_l, f_2, e_2) \in \mathcal{D}_2 \}. \end{aligned}$$

Hence we can define the total interconnection of  $\mathcal{D}_1$ ,  $\mathcal{D}_\infty$  and  $\mathcal{D}_2$  as (also see figure 5).

$$\left. \begin{aligned} \mathcal{D}_1 \parallel \mathcal{D}_\infty \parallel \mathcal{D}_2 := & \left\{ (f_1, e_1, f_p, f_q, e_p, e_q, f_2, e_2) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^* \right. \\ & \exists (f_0, e_0) \in \mathcal{F}_0 \times \mathcal{F}_0^* \text{ s.t. } (f_1, e_1, f_0, e_0) \in \mathcal{D}_1 \\ & \text{and } (-f_0, e_0, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty \\ & \exists (f_l, e_l) \in \mathcal{F}_l \times \mathcal{F}_l^* \text{ s.t. } (-f_0, e_0, f_p, f_q, e_p, e_q, f_l, e_l) \in \mathcal{D}_\infty \\ & \left. \text{and } (-f_l, e_l, f_2, e_2) \in \mathcal{D}_2 \right\} \end{aligned} \right\} \quad (43)$$

$\mathcal{F}_0 \times \mathcal{F}_{p,q} \times \mathcal{F}_l$ , with  $\mathcal{F}_0$  and  $\mathcal{F}_l$  being linear spaces (representing the space of boundary variables of the Stokes–Dirac structure) and  $\mathcal{F}_{p,q}$  an infinite dimensional function space with  $p, q$  representing the two different physical energy domains interacting with each other. The linear space  $\mathcal{F}_0$  is the space of shared flow variables and its dual  $\mathcal{F}_0^*$ , the space of shared effort variables between  $\mathcal{D}_1$  and  $\mathcal{D}_\infty$ . Next consider the composition of  $\mathcal{D}_\infty$  and  $\mathcal{D}_2$ . Considering  $\mathcal{D}_2$  as defined on the product space  $\mathcal{F}_l \times \mathcal{F}_2$  of two linear spaces, we have the linear space  $\mathcal{F}_l$  the space of shared flow variables and its dual  $\mathcal{F}_l^*$ , the space of shared effort variables between  $\mathcal{D}_2$  and  $\mathcal{D}_\infty$ .

This yields the following bilinear form on  $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$ :

$$\begin{aligned} & \left\langle \left\langle (f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a), (f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \right\rangle \right\rangle \\ & := \langle e_1^b | f_1^a \rangle + \langle e_1^a | f_1^b \rangle + \langle e_2^b | f_2^a \rangle + \langle e_2^a | f_2^b \rangle \\ & + \int_Z \left[ e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^a \wedge f_q^b + e_q^b \wedge f_q^a \right]. \end{aligned} \quad (44)$$

**Theorem 3:** Let  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and  $\mathcal{D}_\infty$  be Dirac structures as said above (defined respectively with respect to  $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_0 \times \mathcal{F}_0^*$ ,  $\mathcal{F}_l \times \mathcal{F}_l^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$  and

$\mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^*$ ). Then  $\mathcal{D} = \mathcal{D}_1 \parallel \mathcal{D}_\infty \parallel \mathcal{D}_2$  is a Dirac structure defined with respect to the bilinear form on  $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$  given by (44).

We use the following facts for the proof (as we know that  $\mathcal{D}_1$ ,  $\mathcal{D}_2$  and  $\mathcal{D}_\infty$  individually are Dirac structures). On  $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_0 \times \mathcal{F}_0^*$  the bilinear form is defined as

$$\begin{aligned} & \langle (f_1^a, f_0^a, e_1^a, e_0^a), (f_1^b, f_0^b, e_1^b, e_0^b) \rangle \\ & := \langle e_1^b | f_1^a \rangle + \langle e_1^a | f_1^b \rangle + \langle e_0^b | f_0^a \rangle + \langle e_0^a | f_0^b \rangle \end{aligned} \quad (45)$$

and  $\mathcal{D}_1 = \mathcal{D}_1^\perp$  with respect to the bilinear form as in (45).

Similarly on  $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_l \times \mathcal{F}_l^*$  the bilinear form is defined as

$$\begin{aligned} & \langle (-f_l^a, e_l^a, f_2^a, e_2^a), (-f_l^b, e_l^b, f_2^b, e_2^b) \rangle \\ & := \langle e_2^b | f_2^a \rangle + \langle e_2^a | f_2^b \rangle - \langle e_l^b | f_l^a \rangle - \langle e_l^a | f_l^b \rangle \end{aligned} \quad (46)$$

and  $\mathcal{D}_2 = \mathcal{D}_2^\perp$  with respect to the bilinear form as in (46).

On  $\mathcal{F}_0 \times \mathcal{F}_0^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_l \times \mathcal{F}_l^*$  the bilinear form takes the following form

$$\begin{aligned} & \langle (f_p^a, f_q^a, f_b^a, e_p^a, e_q^a, e_b^a), (f_p^b, f_q^b, f_b^b, e_p^b, e_q^b, e_b^b) \rangle \\ & := \int_Z [e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^a \wedge f_q^b + e_q^b \wedge f_q^a \\ & \quad + [\langle e_l^a | f_l^b \rangle + \langle e_l^b | f_l^a \rangle - \langle e_0^a | f_0^b \rangle - \langle e_0^b | f_0^a \rangle] \end{aligned} \quad (47)$$

and  $\mathcal{D}_\infty = \mathcal{D}_\infty^\perp$  with respect to the bilinear form as in (47).

**Proof:**

(i)  $\mathcal{D} \subset \mathcal{D}^\perp$ : Let  $(f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a) \in \mathcal{D}$  and consider any other  $(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathcal{D}$  and the bilinear form on  $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_{p,q} \times \mathcal{F}_{p,q}^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$  as in (44). Then  $\exists (f_0^a, e_0^a), (f_l^a, e_l^a)$  s.t.  $(f_1^a, e_1^a, f_0^a, e_0^a) \in \mathcal{D}_1$ ,  $(-f_0^a, e_0^a, f_p^a, e_p^a, f_q^a, e_q^a, f_l^a, e_l^a) \in \mathcal{D}_\infty$  and  $(-f_l^a, e_l^a, f_2^a, e_2^a) \in \mathcal{D}_2$  and  $\exists (f_0^b, e_0^b), (f_l^b, e_l^b)$  s.t.  $(f_1^b, e_1^b, f_0^b, e_0^b) \in \mathcal{D}_1$ ,  $(-f_0^b, e_0^b, f_p^b, e_p^b, f_q^b, e_q^b, f_l^b, e_l^b) \in \mathcal{D}_\infty$  and  $(-f_l^b, e_l^b, f_2^b, e_2^b) \in \mathcal{D}_2$ . Since  $\mathcal{D}_\infty$  is a Dirac structure with respect to (47)

$$\begin{aligned} & \int_Z [e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^a \wedge f_q^b + e_q^b \wedge f_q^a] \\ & = -\langle e_l^a | f_l^b \rangle - \langle e_l^b | f_l^a \rangle + \langle e_0^a | f_0^b \rangle + \langle e_0^b | f_0^a \rangle \end{aligned} \quad (48)$$

Substituting (48) in (44) and using the fact that the bilinear form (45) is zero on  $\mathcal{D}_1$  and (46) is zero on  $\mathcal{D}_2$ , we get

$$\begin{aligned} & \langle e_1^b | f_1^a \rangle + \langle e_1^a | f_1^b \rangle + \langle e_2^b | f_2^a \rangle + \langle e_2^a | f_2^b \rangle \\ & + \int_Z [e_p^a \wedge f_p^b + e_p^b \wedge f_p^a + e_q^a \wedge f_q^b + e_q^b \wedge f_q^a] = 0 \end{aligned}$$

and hence  $\mathcal{D} \subset \mathcal{D}^\perp$

(ii)  $\mathcal{D}^\perp \subset \mathcal{D}$ : We know that the flow and effort variables of  $\mathcal{D}_\infty$  are related as

$$\begin{aligned} \mathcal{D}_\infty & \triangleq \left\{ (f, e) \in \mathcal{F} \times \mathcal{F}^* \mid \begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} \right. \\ & \left. = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e_p |_{\partial Z} \\ e_q |_{\partial Z} \end{bmatrix} \right\}. \end{aligned} \quad (49)$$

Let  $(f_1^a, f_p^a, f_q^a, f_2^a, e_1^a, e_p^a, e_q^a, e_2^a) \in \mathcal{D}^\perp$ , then for all  $(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathcal{D}$  the right side of equation (44) is zero. Now consider the vectors  $(f_1^b, f_p^b, f_q^b, f_2^b, e_1^b, e_p^b, e_q^b, e_2^b) \in \mathcal{D}$  with  $f_1^b = f_2^b = e_1^b = e_2^b = 0$  and also  $f_0^b = e_0^b = f_l^b = e_l^b = 0$ . Then from (49) and (44) we have

$$\int_Z [e_p^a \wedge de_q^b + e_p^b \wedge f_p^a + e_q^b \wedge f_q^a + e_q^a \wedge de_p^b] = 0. \quad (50)$$

This implies (see the proof of Theorem 2.1 in van der Schaft and Maschke (2002))

$$f_p^a = de_q^a \quad \text{and} \quad f_q^a = de_p^a. \quad (51)$$

Substituting (51) in (44) we have

$$\begin{aligned} & \langle e_1^b | f_1^a \rangle + \langle e_1^a | f_1^b \rangle + \langle e_2^b | f_2^a \rangle + \langle e_2^a | f_2^b \rangle \\ & + \int_Z [e_p^a \wedge de_q^b + e_p^b \wedge de_q^a + e_q^b \wedge de_p^a + e_q^a \wedge de_p^b] = 0. \end{aligned}$$

This yields by Stokes' theorem

$$\langle e_1^b | f_1^a \rangle + \langle e_1^a | f_1^b \rangle + \langle e_2^b | f_2^a \rangle + \langle e_2^a | f_2^b \rangle + \left[ \langle e_p^a | e_q^b \rangle + \langle e_p^b | e_q^a \rangle \right]_0^l = 0$$

for all  $e_p, e_q$ . Expanding the above and substituting for the boundary conditions

$$\begin{aligned} & \langle e_1^b | f_1^a \rangle + \langle e_1^a | f_1^b \rangle + \langle e_0^a | f_0^b \rangle + \langle e_0^b | f_0^a \rangle \\ & + \langle e_2^b | f_2^a \rangle + \langle e_2^a | f_2^b \rangle - \langle e_l^a | f_l^b \rangle - \langle e_l^b | f_l^a \rangle = 0 \end{aligned} \quad (52)$$

since  $(f_0^b, e_0^b, f_l^b, e_l^b)$  are arbitrary and with  $f_l^b = e_l^b = f_2^b = e_2^b = 0$  the above equation reduces to

$$\langle e_1^b | f_1^a \rangle + \langle e_1^a | f_1^b \rangle + \langle e_0^a | f_0^b \rangle + \langle e_0^b | f_0^a \rangle = 0$$

which implies that  $(f_1^a, e_1^a, f_0^a, e_0^a) \in \mathcal{D}_1$ .

With similar arguments (with  $f_l^b = e_l^b = f_1^b = e_1^b = 0$ ) equation (52) reduces to

$$\langle e_2^b | f_2^a \rangle + \langle e_2^a | f_2^b \rangle - \langle e_l^a | f_l^b \rangle - \langle e_l^b | f_l^a \rangle = 0$$

implying  $(-f_l^a, e_l^a, f_2^a, e_2^a) \in \mathcal{D}_2$  and hence  $\mathcal{D}^\perp \subset \mathcal{D}$ , completing the proof.  $\square$

Similarly we can also study interconnections of mixed finite and infinite dimensional systems where we also have dissipation in the respective subsystems, which would again result in a port-Hamiltonian system with dissipation, as stated in the following corollary.

**Corollary 1:** *Let  $\mathcal{D}_1\|\mathcal{R}_1, \mathcal{D}_2\|\mathcal{R}_2$  and  $\mathcal{D}_\infty\|\mathcal{R}_\infty$  be Dirac structures as defined above interconnected to their respective resistive relations (representing their dissipation), then the composed system will again have a structure of the form  $\mathcal{D}\|\mathcal{R}$  with the property that  $(\mathcal{D}\|\mathcal{R})^\perp = \mathcal{D}\|-\mathcal{R}$  where  $-\mathcal{R}$  is a pseudo resistive relation (corresponding to negative resistance).  $\mathcal{D}$  is the composition of the individual Dirac structures and  $\mathcal{R}$  is the composition of the individual resistances of the subsystems.*

**Remark 7:** If we replace the two finite dimensional Dirac structures in Theorem 3 with resistive relations it would amount to terminating the boundary ports with resistive relations, which is one of the cases of dissipation entering into an infinite dimensional system. The composed structure would then have the following property

$$(\mathcal{R}_1\|\mathcal{D}_\infty\|\mathcal{R}_2)^\perp = (-\mathcal{R}_1)\|\mathcal{D}_\infty\|(-\mathcal{R}_2)$$

with the  $-\mathcal{R}$ 's again denoting the pseudo resistive relations corresponding to negative resistances.

### 5.2 Interconnections in a higher dimensional case

In the previous subsection on interconnections of infinite-dimensional systems with finite-dimensional system through the boundary of the infinite-dimensional system, we considered the simple case where  $p=q=n=1$ , for the infinite-dimensional system, given by a Stokes–Dirac structure. This corresponds to the case of a system with a 1D spatial domain. In this subsection we highlight briefly on how this could be extended to the case where we have a higher dimensional spatial domain, i.e  $n > 1$  and how these systems could be interconnected through the boundary to finite-dimensional systems.

The dynamics of 2D shallow water equations are given by (Pedlosky (1986))

$$\begin{aligned} \partial_t h + \partial_{z_1}(hu) + \partial_{z_2}(hv) &= 0 \\ \partial_t u + \partial_{z_1}\left(\frac{1}{2}u^2 + gh\right) + v\partial_{z_2}u &= 0 \\ \partial_t v + u\partial_{z_1}v + \partial_{z_2}\left(\frac{1}{2}v^2 + gh\right) &= 0. \end{aligned} \tag{53}$$

In vector notation the 2D shallow-water equations can be written as

$$\left. \begin{aligned} \partial_t h + \nabla(h\vec{v}) &= 0 \\ \partial_t \vec{v} + \vec{v}\nabla\vec{v} + \frac{1}{2}(\vec{v}\vec{v} + gh) &= 0, \end{aligned} \right\} \tag{54}$$

where  $\vec{v}$  is the velocity vector with components  $(u, v)$ . The formulation of above equation as a port-Hamiltonian system is given as follows. Let  $W \subset \mathbb{R}^2$  be a given domain through which the water flows. We assume the existence of a Riemannian metric  $\langle \cdot | \cdot \rangle$  on  $W$ , usually the standard Euclidian metric on  $\mathbb{R}^2$ . Let  $Z \subset W$  be any two-dimensional manifold with boundary  $\partial Z$ . We identify the height  $h$  (which represents the mass density) with a two-form on  $Z$ , that is with elements in  $\Omega^2(Z)$ . Furthermore we identify the Eulerian vector field  $V$  with a one-form on  $Z$ , that is, with an element in  $\Omega^1(Z)$ . The spaces  $\mathcal{F}_{pq}$  and  $\mathcal{E}_{pq}$  are given by

$$\begin{aligned} \mathcal{F}_{pq} &= \Omega^2(Z) \times \Omega^1(Z) \times \Omega^0(\partial Z) \\ \mathcal{E}_{pq} &= \Omega^0(Z) \times \Omega^1(Z) \times \Omega^1(\partial Z) \end{aligned}$$

we can now define the corresponding Stokes–Dirac structure  $\mathcal{D}$  on  $\mathcal{F}_{pq} \times \mathcal{E}_{pq}$ . We now have the following modified Stokes–Dirac structure

$$\begin{aligned} D &:= \left\{ (f_h, f_V, f_b, e_h, e_V, e_b) \in \mathcal{F}_{pq} \times \mathcal{E}_{pq} \right\}. \\ \begin{bmatrix} f_h \\ f_V \end{bmatrix} &= \begin{bmatrix} de_V \\ de_h + \frac{1}{*h}((*dV) \wedge (*e_V)) \end{bmatrix}; \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} e_V|_{\partial Z} \\ -e_h|_{\partial Z} \end{bmatrix} \end{aligned} \tag{55}$$

In terms of the 2D shallow water equations this would correspond to

$$\begin{aligned} f_h &= -\frac{\partial}{\partial t}h(z, t), \quad e_h = \delta_h H = \frac{1}{2}\langle V^\#, V^\# \rangle + f(*h) \\ f_V &= -\frac{\partial}{\partial t}V(z, t), \quad e_V = \delta_V H = (*h)(*V) \end{aligned} \tag{56}$$

together with the boundary variables  $f_b = \delta_h H|_{\partial Z}$  called the Bernoulli function and  $e_b = \delta_V H|_{\partial Z}$  denoting the boundary mass flow. Here  $V^\#$  denotes the vector field corresponding to the one-form  $V$  (“index lowering”), also see van der Schaft *et al.* (2002b).

Consider interconnection of such an infinite-dimensional system, with finite-dimensional systems through its boundaries, see figure 6. Note that a major difference with the 1D case considered before is that in the 2D case the boundary  $\partial Z$  is a one-dimensional manifold and thus one of the boundary variables  $(f_b, e_b)$  is a distributed



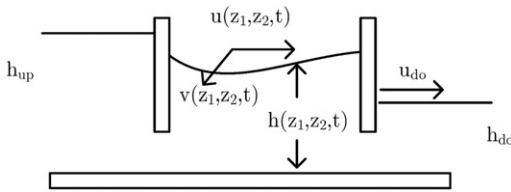


Figure 6. The 2-D water flow.

quantity, which cannot be directly interconnected to a finite-dimensional system. The finite-dimensional systems can be thought of water reservoirs given in port-Hamiltonian form as

$$\begin{aligned} \dot{x}_i &= u_i \\ y_i &= \frac{\partial H_i(x_i)}{\partial x_i}; \quad i = 1, 2. \end{aligned}$$

The indices (1,2) correspond to the left and right reservoirs respectively, with  $(u_i, y_i)$  denoting the respective inputs and outputs. Note that this notation should not be confused with  $(h_{up}, u_{up})$  and  $(h_{do}, u_{do})$  which represent the water heights and velocities respectively of the left and the right reservoirs.

The interconnection constraints at the gates would then be as follows

$$\begin{aligned} u_1 &= \int f_{b0} = \int (*h * V)|_0; \\ u_2 &= \int f_{b1} = \int (*h * V)|_1 \\ y_1 &= -e_{b0} = \frac{1}{2}(\langle V^\#, V^\# \rangle + g(*h))|_0; \\ y_2 &= -e_{b1} = \frac{1}{2}(\langle V^\#, V^\# \rangle + g(*h))|_1 \end{aligned}$$

In the above equation  $u_1$  equals the total mass flow through the one-dimensional boundary  $\partial Z$  (left gate), while the second equation involving  $y_1$  implies that the Bernoulli function at the boundary should be constant and equal to  $-e_{b0}$ . Similar explanations also hold for the variables  $(u_2, y_2)$  (at the right gate). It can easily be seen that such interconnection constraints are indeed power conserving and the total interconnection is again a Dirac structure. This is simply because the above equation ensures that the total power flow through the boundary of the infinite-dimensional system is equal to the total power flow going into the water reservoirs. This is thus an example of an infinite-dimensional system with  $n > 1$ , interconnected through its boundaries to finite-dimensional systems.

Similarly we can also extend this to the case of infinite-dimensional systems with an  $n$ -dimensional spatial domain, interconnected to the boundary to finite-dimensional systems, in which case the

interconnection constraints at the boundary take the following form

$$\left. \begin{aligned} f_0 &= -\int_{\partial Z} f_{b0}, & f_l &= -\int_{\partial Z} f_{bl} \\ e_0 &= e_{b0}, & e_l &= e_{bl}, \end{aligned} \right\} \quad (57)$$

where  $(f_0, e_0), (f_l, e_l)$  correspond to the port variables of the two finite-dimensional systems which are to be interconnected to the boundaries of the infinite-dimensional systems, and  $(f_{b0}, e_{b0}), (f_{bl}, e_{bl})$  correspond to the boundary variables of the infinite-dimensional system.

### 5.3 Achievable Dirac structures

The mixed finite and infinite-dimensional case we will consider here (and the rest of the section) is the case where the plant Dirac structure  $\mathcal{DR}_P$  is the interconnection of a Stokes-Dirac structure with a finite-dimensional Dirac structure connected to one of its boundary, the controller Dirac structure  $\mathcal{DR}_C$  being a finite-dimensional Dirac structure connected to the other end of the Stokes-Dirac structure. This would correspond to a case where  $\mathcal{DR}_P = \mathcal{DR}_1 \parallel \mathcal{DR}_\infty$  and  $\mathcal{DR}_C = \mathcal{DR}_2$ , compare with figure 5. This typically is a case where we wish to control a plant which is interconnected to a controller through an infinite-dimensional system.

Finding all the achievable Dirac structures of the closed-loop system in this case follows, by a simple extension, the same procedure as in the previous sections by defining the respective subspaces  $\mathcal{DR}_P^0, \mathcal{DR}_P^\pi, \mathcal{DR}^0, \mathcal{DR}^\pi$  and also  $\mathcal{DR}_P^*$ . Hence, we omit the details here.

### 5.4 Achievable Casimirs

In this case the achievable Casimirs are functionals  $C(x, \bar{q}(z, t))$  such that  $\delta C(x, \bar{q}(z, t))$  belongs to the space

$$\begin{aligned} P_{Cas} &= \left\{ e_1, e_{pq} \mid \exists \mathcal{DR}_C \text{ s.t. } \exists e_2 : \right. \\ &\quad \left( 0, e_1, -f_{R1}, e_{R1}, 0, e_{pq}, \right. \\ &\quad \left. \left. -f_{Rpq}, e_{Rpq}, 0, e_2, -f_{R2}, e_{R2} \right) \in \mathcal{DR}_P \parallel \mathcal{DR}_C \right\} \end{aligned}$$

with  $(f_1, e_1), (f_{R1}, e_{R1})$  respectively denoting the flows and efforts variables of the storage and the dissipation terms in the finite-dimensional part of the plant Dirac structure, and  $(f_2, e_2), (f_{R2}, e_{R2})$  the flow and effort variables associated with the storage and dissipation terms in the controller Dirac structure (finite-dimensional).

Similar to the finite-dimensional case, the following theorem addresses the question of characterizing the achievable Casimirs of the closed-loop system, regarded

as functions of the plant state by characterization of the space  $P_{Cas}$ .

**Proposition 5:** *The space  $P_{Cas}$  defined above is equal to the space*

$$\tilde{P} = \left\{ e_1, e_{pq} \mid \exists (f_b, e_b) \text{ s.t.} \right. \\ \left. \times (0, e_1, f_{R1}, 0, e_{R1}, 0, e_{pq}, -f_{Rpq}, e_{Rpq}, f_b, e_b) \in \mathcal{DR}_P \right\}$$

where  $(f_b, e_b)$  are the boundary variables.

**Proof:** The inclusion  $\tilde{P} \subset P_{Cas}$  is again obtained by taking  $\mathcal{DR}_2 = \mathcal{DR}_1^*$ .  $\square$

**Example 5:** A simple example in this case would be to consider a plant system where we interconnect the transmission line at one end to a finite-dimensional port-Hamiltonian system, the Dirac structure of which would be given as

$$\begin{bmatrix} -\dot{x}_1 \\ -\partial_t Q \\ -\partial_t \phi \end{bmatrix} = \left( \begin{bmatrix} -J(x) & 0 & 0 \\ 0 & 0 & d \\ 0 & d & 0 \end{bmatrix} + \begin{bmatrix} R(x) & 0 & 0 \\ 0 & G^* & 0 \\ 0 & 0 & R^* \end{bmatrix} \right) \\ \times \begin{bmatrix} \partial_x H \\ \delta_Q H \\ \delta_\phi H \end{bmatrix} + \begin{bmatrix} -g(x) \\ 0 \\ 0 \end{bmatrix} \delta_Q H|_0; \\ \begin{bmatrix} f_l \\ e_l \end{bmatrix} = \begin{bmatrix} -\delta_\phi H|_l \\ \delta_Q H|_l \end{bmatrix}; \delta_\phi H|_0 = g^T(x)e_1.$$

The achievable Casimirs in this case are all functional  $\mathcal{C}$  such that

$$\begin{aligned} J(x)\partial_x \mathcal{C} + g(x)\delta_Q \mathcal{C}|_0 &= 0 \\ g^T(x)\partial_x \mathcal{C} &= 0 \\ d\delta_\phi \mathcal{C} - G^* \delta_Q \mathcal{C} &= 0 \\ d\delta_Q \mathcal{C} - R^* \delta_\phi \mathcal{C} &= 0. \end{aligned}$$

We see that the first two conditions are the same as that obtained for the finite-dimensional case (28) and the last two conditions are those corresponding to the transmission line.

## 6. Conclusions

In this paper we have extended the results on composition of Dirac structures to the case of infinite-dimensional systems and also mixed finite and infinite dimensional systems and have shown that the composition is again a Dirac structure. We have also discussed the case of interconnections where there is dissipation entering into the system. Next, the characterization of the set of achievable Dirac structures in the composition

of a given plant Dirac structure with a to-be-designed controller Dirac structure has been extended to the case of systems with dissipation and a canonical construction for the controller system with dissipation has been provided.

We also see how this leads to the characterization of the set of achievable Casimirs for the closed-loop system. In particular, for the case of finite-dimensional systems with dissipation, we see how under certain conditions if a function is a conserved quantity that is a Casimir for a given resistive relation it is also a Casimir for all resistive relations. Moreover, in the finite-dimensional case we also see how without the knowledge of the controller system, the characterization of the set of achievable Casimirs in terms of the plant state enable us to see whether or not there exist Casimirs for the closed-loop system.

Future work could focus on making use of these results for stability problems of infinite-dimensional systems, from the control by interconnection point of view. Furthermore, the interconnection constraints (57) hold for classes of systems where one of the boundary variables is a zero-form (or a function which holds value at points). Examples of such a case are the 3D fluid flow, the  $nD$  wave equation etc. However, things change when none of the boundary variables is a function, as in the case of Maxwell's equations where the boundary variables are the electric field intensity and the magnetic field intensity both being one-forms. To interconnect systems of this sort through the boundary with finite-dimensional systems remains an open issue.

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