Achievement of continuity of (φ,ψ)-derivations without linearity

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Abstract

Suppose that $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space $\mathfrak{H}$, and φ,ψ are mappings from $\mathfrak{A}$ into $B(\mathfrak{H})$ which are not assumed to be necessarily linear or continuous. A (φ,ψ)-derivation is a linear mapping $d:\mathfrak{A}\to B(\mathfrak{H})$ such that

$$d(ab)=\varphi(a)d(b)+d(a)\psi(b)\quad (a,b\in\mathfrak{A}).$$

We prove that if φ is a multiplicative (not necessarily linear) $*$-mapping, then every $*$-(φ,φ)-derivation is automatically continuous. Using this fact, we show that every $*$-(φ,ψ)-derivation d from $\mathfrak{A}$ into $B(\mathfrak{H})$ is continuous if and only if the $*$-mappings φ and ψ are left and right d-continuous, respectively.

1 Introduction

Recently, a number of analysts [2, 4, 12, 13, 14] have studied various generalized notions of derivations in the context of Banach algebras. There are some applications in the other fields of research [7]. Such mappings have been extensively studied in pure algebra; cf. [1, 3, 9]. A generalized concept of derivation is as follows.

Definition 1.1. Suppose that $\mathfrak{B}$ is an algebra, $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, $\mathfrak{X}$ is a $\mathfrak{B}$-bimodule, and $\varphi,\psi:\mathfrak{A}\to\mathfrak{B}$ are mappings. A linear mapping $d:\mathfrak{A}\to\mathfrak{X}$ is a (φ,ψ)-*derivation* if

$$d(ab)=\varphi(a)d(b)+d(a)\psi(b)\quad (a,b\in\mathfrak{A}).$$

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By a *φ-derivation* we mean a (φ, φ)-derivation. Note that we do not have any extra assumptions such as linearity or continuity on the mappings φ and ψ.

The automatic continuity theory is the study of (algebraic) conditions on a category, e.g. C^*-algebras, which guarantee that every mapping belonging to a certain class, e.g. derivations, is continuous. S. Sakai [16] proved that every derivation on a C^*-algebra is automatically continuous. This is an affirmative answer to the conjecture made by I. Kaplansky [11]. J. R. Ringrose [15] extended this result for derivations from a C^*-algebra $\mathfrak{A}$ to a Banach $\mathfrak{A}$-bimodule $\mathfrak{X}$. B. E. Johnson and A. M. Sinclair [10] proved that every derivation on a semisimple Banach algebra is automatically continuous. Automatic continuity of module derivations on JB^*-algebras have been studied in [8]. The reader may find a collection of results concerning these subjects in [6, 17, 18].

M. Brešar and A. R. Villena [4] proved that for an inner automorphism φ, every (id, φ)-derivation on a semisimple Banach algebra is continuous, where id denotes the identity mapping. In [12], it is shown that every (φ, ψ)-derivation from a C^*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathfrak{H}$ into $B(\mathfrak{H})$ is automatically continuous, if φ and ψ are continuous $*$-linear mappings from $\mathfrak{A}$ into $B(\mathfrak{H})$.

This paper consists of five sections. We define the notion of a (φ, ψ)-derivation in the first section and give some examples in the second. In the third section, by using methods of [12], we prove that if φ is a multiplicative (not necessarily linear) $*$-mapping from a C^*-algebra $\mathfrak{A}$ acting on a Hilbert space $\mathfrak{H}$ into $B(\mathfrak{H})$, then every $*$-φ-derivation $d : \mathfrak{A} \to B(\mathfrak{H})$ is automatically continuous. In the fourth section, we show that for not necessarily linear or continuous $*$-mappings $\varphi, \psi : \mathfrak{A} \to B(\mathfrak{H})$ the continuity of a $*$-(φ, ψ)-derivation $d : \mathfrak{A} \to B(\mathfrak{H})$ is equivalent to the left d-continuity of φ and the right d-continuity of ψ. The mapping φ (resp. ψ) is called left (resp. right) d-continuous if $\lim_{x \to 0}\Big(\varphi(x)d(b)\Big) = 0$, for all $b \in \mathfrak{A}$ (resp. $\lim_{x \to 0}\Big(d(b)\psi(x)\Big) = 0$, for all $b \in \mathfrak{A}$). Obviously these conditions happen whenever $\lim_{x \to 0}\varphi(x) = 0 = \lim_{x \to 0}\psi(x)$, in particular whenever φ and ψ are bounded linear mappings. Thus we extend the main results of [12] to a general framework. Furthermore, we prove that if d is a continuous $*$-(φ, ψ)-derivation, then we can replace φ and ψ with mappings with 'at most' zero separating spaces. The last section is devoted to study the continuity of the so-called generalized $*$-(φ, ψ)-derivations from $\mathfrak{A}$ into $B(\mathfrak{H})$.

The reader is referred to [6] for undefined notation and terminology.

2 Examples

In this section, let $\mathfrak{B}$ be an algebra, let $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$, and let $\mathfrak{X}$ be a $\mathfrak{B}$-bimodule. The following are some examples concerning Definition 1.1.

Example 2.1. Every ordinary derivation $d : \mathfrak{A} \to \mathfrak{X}$ is an id-derivation, where $id : \mathfrak{A} \to \mathfrak{A}$ is the identity mapping.

Example 2.2. Every homomorphism $\rho : \mathfrak{A} \to \mathfrak{A}$ is a $(\frac{\rho}{2}, \frac{\rho}{2})$-derivation.

Example 2.3. Let $\varphi, \psi : \mathfrak{A} \to \mathfrak{A}$ be homomorphisms, and let $x \in \mathfrak{X}$ be a fixed element. Then the linear mapping $d_x : \mathfrak{A} \to \mathfrak{X}$ defined by

$$d_x(a) := \varphi(a)x - x\psi(a) \qquad (a \in \mathfrak{A}),$$

is a (φ,ψ)-derivation, which is called the inner (φ,ψ)-derivation corresponding to x.

Example 2.4. Assume that $\gamma,\theta:\mathcal{C}[0,2]\to\mathcal{C}[0,2]$ are arbitrary mappings, where $\mathcal{C}[0,2]$ denotes the C^*-algebra of all continuous complex valued functions on $[0,2]$. Take $\lambda\in\mathbb{C}$ and fixed elements f_1, f_2, and h_0 in $\mathcal{C}[0,2]$ such that

$$f_1h_0 = 0 = f_2h_0.$$

For example, let us take

$$\begin{aligned} h_0(t) &:= (1-t)\chi_{[0,1]}(t),\\ f_1(t) &:= (t-1)\chi_{[1,2]}(t),\\ f_2(t) &:= (t-\frac{3}{2})\chi_{[\frac{3}{2},2]}(t), \end{aligned}$$

where χ_E denotes the characteristic function of E. Define $\varphi,\psi,d:\mathcal{C}[0,2]\to\mathcal{C}[0,2]$ by

$$\begin{aligned} \varphi(f) &:= \lambda f+\gamma(f)f_1,\\ \psi(f) &:= (1-\lambda)f+\theta(f)f_2,\\ d(f) &:= fh_0. \end{aligned}$$

Then d is a (φ,ψ)-derivation, since

$$\begin{aligned} \varphi(f)d(g)+d(f)\psi(g) &= \Big(\lambda f+\gamma(f)f_1\Big)(gh_0)+(fh_0)\Big((1-\lambda)g+\theta(f)f_2\Big)\\ &= \lambda fgh_0+\gamma(f)gf_1h_0+(1-\lambda)fgh_0+\theta(f)fh_0f_2\\ &= fgh_0\\ &= d(fg). \end{aligned}$$

Moreover, we may choose γ and θ such that φ and ψ are neither linear nor continuous.

3 Multiplicative mappings

In this section, we are going to show how a multiplicative property gives us the linearity. We start our work with some elementary properties of (φ,ψ)-derivations.

Lemma 3.1. *Let $\mathfrak{B}$ be an algebra, let $\mathfrak{A}$ be a subalgebra of $\mathfrak{B}$, and let $\mathfrak{X}$ be a $\mathfrak{B}$-bimodule. If $d:\mathfrak{A}\to\mathfrak{X}$ is a (φ,ψ)-derivation, then*
(i) $\Big(\varphi(ab)-\varphi(a)\varphi(b)\Big)d(c)=d(a)\Big(\psi(bc)-\psi(b)\psi(c)\Big)$;
(ii) $\Big(\varphi(a+b)-\varphi(a)-\varphi(b)\Big)d(c)=0=d(a)\Big(\psi(b+c)-\psi(b)-\psi(c)\Big)$;
(iii) $\Big(\varphi(\lambda a)-\lambda\varphi(a)\Big)d(b)=0=d(a)\Big(\psi(\lambda b)-\lambda(\psi(b)\Big)$;
for all $a,b,c\in\mathfrak{A}$, and all $\lambda\in\mathbb{C}$.

Proof. Let $a,b,c\in\mathfrak{A}$ and $\lambda\in\mathbb{C}$. For the first equation we have

$$\begin{aligned} 0 &= d\Big((ab)c\Big)-d\Big(a(bc)\Big)\\ &= \varphi(ab)d(c)+d(ab)\psi(c)-\varphi(a)d(bc)-d(a)\psi(bc)\\ &= \varphi(ab)d(c)+\Big(\varphi(a)d(b)+d(a)\psi(b)\Big)\psi(c)\\ &\quad -\varphi(a)\Big(\varphi(b)d(c)+d(b)\psi(c)\Big)-d(a)\psi(bc)\\ &= \Big(\varphi(ab)-\varphi(a)\varphi(b)\Big)d(c)-d(a)\Big(\psi(bc)-\psi(b)\psi(c)\Big). \end{aligned}$$

Calculating $d\big((a+b)c\big) - d(bc) - d(ac)$ and $d\big(a(\lambda b)\big) - \lambda d(ab)$, we obtain the other equations. ■

We also need the following lemma. Recall that, for a Hilbert space $\mathfrak{H}$, a subset Y of $B(\mathfrak{H})$ is said to be self-adjoint if $u^* \in Y$ for each $u \in Y$.

Lemma 3.2. *Let $\mathfrak{H}$ be a Hilbert space, and let Y be a self-adjoint subset of $B(\mathfrak{H})$. Assume that $\mathfrak{L}_0 = \bigcup_{u\in Y} u(\mathfrak{H})$, and $\mathfrak{L}$ is the closed linear span of $\mathfrak{L}_0$. If $\mathfrak{K} = \mathfrak{L}^\perp$, then $\mathfrak{K} = \bigcap_{u\in Y} \ker(u)$.*

Proof. Let $k \in \mathfrak{K}$. Then $\langle \ell, k\rangle = 0$, for all $\ell \in \mathfrak{L}$. Since Y is self-adjoint,

$$\langle u(k), h\rangle = \langle k, u^*(h)\rangle = 0$$

for all $h \in \mathfrak{H}$ and all $u \in Y$. Therefore $u(k) = 0$, for all $u \in Y$. Hence $\mathfrak{K} \subseteq \bigcap_{u\in Y} \ker(u)$. The inverse inclusion can be proved similarly. ■

Recall that a mapping $f : \mathfrak{A} \to B(\mathfrak{H})$ is called a $*$-mapping, if $f(a^*) = f(a)^*$, for all $a \in \mathfrak{A}$. If we define $f^* : \mathfrak{A} \to B(\mathfrak{H})$, by $f^*(a) := \big(f(a^*)\big)^*$, then f is a $*$-mapping if and only if $f^* = f$.

Theorem 3.3. *Suppose that $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space $\mathfrak{H}$. Let $\varphi : \mathfrak{A} \to B(\mathfrak{H})$ be a $*$-mapping and let $d : \mathfrak{A} \to B(\mathfrak{H})$ be a $*$-φ-derivation. If φ is multiplicative, i.e. $\varphi(ab) = \varphi(a)\varphi(b) \quad (a, b \in \mathfrak{A})$, then d is continuous.*

Proof. Set

$$Y := \{\varphi(\lambda b) - \lambda\varphi(b) \mid \lambda \in \mathbb{C}, b \in \mathfrak{A}\} \cup \{\varphi(b+c) - \varphi(b) - \varphi(c) \mid b, c \in \mathfrak{A}\}.$$

Since φ is a $*$-mapping, Y is a self-adjoint subset of $B(\mathfrak{H})$. Put

$$\mathfrak{L}_0 := \bigcup\{u(h) \mid u \in Y, h \in \mathfrak{H}\}.$$

Let $\mathfrak{L}$ be the closed linear span of $\mathfrak{L}_0$, and let $\mathfrak{K} := \mathfrak{L}^\perp$. Then $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{L}$, and by Lemma 3.2, we get

$$\mathfrak{K} = \bigcap\{\ker(u) \mid u \in Y\}. \tag{3.1}$$

Suppose that $p \in B(\mathfrak{H})$ is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{K}$, then by (3.1) we have

$$\varphi(\lambda b)p \;=\; \lambda\varphi(b)p, \tag{3.2}$$

$$\varphi(b+c)p = \varphi(b)p + \varphi(c)p, \tag{3.3}$$

for all $b, c \in \mathfrak{A}$, and all $\lambda \in \mathbb{C}$. Now we claim that

$$p\varphi(a) = \varphi(a)p\,, \;\; pd(a) = d(a) = d(a)p \qquad (a \in \mathfrak{A}). \tag{3.4}$$

To show this, note that by Lemma 3.1, we have

$$d(a)\Big(\varphi(\lambda b) - \lambda\varphi(b)\Big) = 0,$$
$$d(a)\Big(\varphi(b+c) - \varphi(b) - \varphi(c)\Big) = 0,$$

for all $a, b, c \in \mathfrak{A}$, and all $\lambda \in \mathbb{C}$. Thus $d(a)\mathfrak{L} = 0$, and so $d(a)(1-p) = 0$, for all $a \in \mathfrak{A}$. Since d is a $*$-mapping, it follows that $pd(a) = d(a) = d(a)p$, for all $a \in \mathfrak{A}$. Similarly, since φ is multiplicative, we deduce from (3.2) and (3.3) that

$$\begin{aligned}\Big(\varphi(\lambda b) - \lambda\varphi(b)\Big)\varphi(a)p &= \varphi(\lambda b)\varphi(a)p - \lambda\varphi(b)\varphi(a)p \\ &= \lambda\varphi(ba)p - \lambda\varphi(ba)p \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\Big(\varphi(b+c) - \varphi(b) - \varphi(c)\Big)\varphi(a)p &= \varphi(b+c)\varphi(a)p - \varphi(b)\varphi(a)p - \varphi(c)\varphi(a)p \\ &= \varphi\Big((b+c)a\Big)p - \varphi(ba)p - \varphi(ca)p \\ &= 0,\end{aligned}$$

for all $a, b, c \in \mathfrak{A}$, and all $\lambda \in \mathbb{C}$. Therefore $\varphi(a)(\mathfrak{K}) \subseteq \mathfrak{K}$. Since φ is a $*$-mapping, we conclude that $p\varphi(a) = \varphi(a)p$, for all $a \in \mathfrak{A}$. Now define the mapping $\Phi : \mathfrak{A} \to B(\mathfrak{H})$ by $\Phi(a) = \varphi(a)p$. One can easily see that Φ is a $*$-homomorphism, and so it is automatically continuous. Using (3.4), we obtain

$$\begin{aligned}d(ab) &= d(ab)p \\ &= \varphi(a)d(b)p + d(a)\varphi(b)p \\ &= \varphi(a)pd(b) + d(a)\varphi(b)p \\ &= \Phi(a)d(b) + d(a)\Phi(b) \qquad (a, b \in \mathfrak{A}).\end{aligned}$$

Hence d is a Φ-derivation. We deduce from Theorem 3.7 of [12] that d is continuous.

■

4 d-continuity

We start this section with the following definition.

Definition 4.1. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two normed algebras, and $T, S : \mathfrak{A} \to \mathfrak{B}$ are two mappings.
(i) The mapping T is *left (resp. right) S-continuous* if $\lim_{x \to 0}\Big(T(x)S(b)\Big) = 0$, for all $b \in \mathfrak{A}$ (resp. $\lim_{x \to 0}\Big(S(b)T(x)\Big) = 0$, for all $b \in \mathfrak{A}$). If T is both left and right S-continuous, then it is simply called S-continuous.
(ii) As for linear mappings, the *separating space* of a mapping T is defined to be

$$\mathfrak{S}(T) := \{b \in \mathfrak{B} \mid \exists\{a_n\} \subseteq \mathfrak{A},\ a_n \to 0,\ T(a_n) \to b\}.$$

We notice that for a nonlinear mapping T, this set is not necessarily a linear subspace and it may even be empty. Recall that if $\mathfrak{A}$ and $\mathfrak{B}$ are complete spaces and T is linear, then the closed graph theorem implies that T is continuous if and only if $\mathfrak{S}(T) = \{0\}$.

In Example 2.4, if we consider γ and θ to be $*$-mappings, then the continuity of d together with Corollary 4.5 below imply that φ and ψ are d-continuous mappings.

Lemma 4.2. *Let $\mathfrak{A}$ be a normed algebra, let $\mathfrak{X}$ be a normed $\mathfrak{A}$-bimodule, and let $d : \mathfrak{A} \to \mathfrak{X}$ be a (φ, ψ)-derivation for two mappings $\varphi, \psi : \mathfrak{A} \to \mathfrak{A}$. If φ and ψ are left and right d-continuous mappings, respectively, then*

$$a\Big(\psi(bc) - \psi(b)\psi(c)\Big) = 0 = \Big(\varphi(bc) - \varphi(b)\varphi(c)\Big)a\,,$$

for all $b, c \in \mathfrak{A}$ and all $a \in \mathfrak{S}(d)$.

Proof. Suppose that $a \in \mathfrak{S}(d)$. Then $a = \lim_{n\to\infty} d(a_n)$ for some sequence $\{a_n\}$ converging to zero in $\mathfrak{A}$. By Lemma 3.1 (i), we have

$$\begin{aligned} a\Big(\psi(bc) - \psi(b)\psi(c)\Big) &= \lim_{n\to\infty} d(a_n)\Big(\psi(bc) - \psi(b)\psi(c)\Big) \\ &= \lim_{n\to\infty} \Big(\varphi(a_n b) - \varphi(a_n)\varphi(b)\Big)d(c) \\ &= \lim_{n\to\infty} \varphi(a_n b)d(c) \\ &\quad - \lim_{n\to\infty} \varphi(a_n)d(bc) + \lim_{n\to\infty} \varphi(a_n)d(b)\psi(c) \\ &= 0 \qquad (b, c \in \mathfrak{A}), \end{aligned}$$

since φ is left d-continuous. ∎

In the rest of this section, we assume that $\mathfrak{A}$ is a C^*-subalgebra of $B(\mathfrak{H})$, the C^*-algebra of all bounded linear operators on a Hilbert space $\mathfrak{H}$. Also φ, ψ, and d are mappings from $\mathfrak{A}$ into $B(\mathfrak{H})$. Removing the assumption 'linearity' and weakening the assumption 'continuity' on φ and ψ, we extend the main result of [12] as follows.

Theorem 4.3. *Let φ be a $*$-mapping and let d be a $*$-φ-derivation. If φ is left d-continuous, then d is continuous. Conversely, if d is continuous then φ is left d-continuous.*

Proof. Set

$$Y := \{\varphi(bc) - \varphi(b)\varphi(c) \mid b, c \in \mathfrak{A}\}.$$

Since φ is a $*$-mapping, Y is a self-adjoint subset of $B(\mathfrak{H})$. Put

$$\mathfrak{L}_0 := \bigcup\{u(h) \mid u \in Y, h \in \mathfrak{H}\}.$$

Let $\mathfrak{L}$ be the closed linear span of $\mathfrak{L}_0$, and let $\mathfrak{K} := \mathfrak{L}^\perp$. Then $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{L}$, and by Lemma 3.2, we have

$$\mathfrak{K} = \bigcap\{\ker(u) \mid u \in Y\}. \tag{4.1}$$

Suppose that $p \in B(\mathfrak{H})$ is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{K}$, then by (4.1), we have

$$\varphi(bc)p = \varphi(b)\varphi(c)p, \tag{4.2}$$

for all $b, c \in \mathfrak{A}$. We claim that

$$p\varphi(a) = \varphi(a)p\,, \;\; pd(a) = d(a)p \qquad (a \in \mathfrak{A}). \tag{4.3}$$

To prove this, note that by Lemma 3.1 and (4.2), we have

$$\Big(\varphi(bc) - \varphi(b)\varphi(c)\Big)d(a)p = d(b)\Big(\varphi(ca) - \varphi(c)\varphi(a)\Big)p = 0,$$

for all $a, b, c \in \mathfrak{A}$. Thus $ud(a)p = 0$, for all $a \in \mathfrak{A}$, and all $u \in Y$. So

$$ud(a)(\mathfrak{K}) = ud(a)p(\mathfrak{H}) = \{0\}.$$

This means that $d(a)(\mathfrak{K})$ is contained in the kernel of each u in Y. So by (4.1), $d(a)(\mathfrak{K}) \subseteq \mathfrak{K}$. Since d is a $*$-mapping, it follows that $pd(a) = d(a)p$. Similarly, using (4.2), we get

$$\begin{aligned}\Big(\varphi(bc) - \varphi(b)\varphi(c)\Big)\varphi(a)p &= \varphi(bc)\varphi(a)p - \varphi(b)\varphi(c)\varphi(a)p \\ &= \varphi(bca)p - \varphi(bca)p \\ &= 0 \qquad (a, b, c \in \mathfrak{A}).\end{aligned}$$

Therefore $\varphi(a)(\mathfrak{K}) = \varphi(a)p(\mathfrak{H}) \subseteq \mathfrak{K}$. Since φ is a $*$-mapping we conclude that $p\varphi(a) = \varphi(a)p$, for all $a \in \mathfrak{A}$. Now define the mappings $\Phi, D : \mathfrak{A} \to B(\mathfrak{H})$ by $\Phi(a) := \varphi(a)p$, and $D(a) := d(a)p$. We show that Φ is a multiplicative $*$-mapping and D is a $*$-Φ-derivation. Clearly (4.2) implies that Φ is multiplicative, and by (4.3) we have

$$\Big(\Phi(a)\Big)^* = \Big(\varphi(a)p\Big)^* = p^*\Big(\varphi(a)\Big)^* = p\varphi(a^*) = \Phi(a^*) \qquad (a \in \mathfrak{A}).$$

Thus Φ is a multiplicative $*$-mapping. Now for $a, b \in \mathfrak{A}$,

$$\begin{aligned}D(ab) &= d(ab)p \\ &= \varphi(a)d(b)p + d(a)\varphi(b)p \\ &= \varphi(a)pd(b)p + d(a)p\varphi(b)p \\ &= \Phi(a)D(b) + D(a)\Phi(b).\end{aligned}$$

Thus D is a $*$-Φ-derivation, and it is continuous by Theorem 3.3. Now, we show that $\mathfrak{S}(d) = \{0\}$. Let $a \in \mathfrak{S}(d)$, then there exists a sequence $\{a_n\}$ converging to 0 in $\mathfrak{A}$ such that $d(a_n) \to a$ as $n \to \infty$. Take $h = k + \ell \in \mathfrak{K} \oplus \mathfrak{L} = \mathfrak{H}$. By Lemma 4.2, and by the fact that each $a \in \mathfrak{S}(d)$ is a bounded operator on $\mathfrak{H}$, and that $\ell \in \mathfrak{L}$ is in the closed linear span of elements of the form $\Big(\varphi(bc) - \varphi(b)\varphi(c)\Big)h$, where $b, c \in \mathfrak{A}, h \in \mathfrak{H}$, we have $a(\ell) = 0$. It follows from continuity of D that

$$a(k) = a\Big(p(h)\Big) = \lim_{n\to\infty} d(a_n)\Big(p(h)\Big) = \lim_{n\to\infty} D(a_n)(h) = 0.$$

Thus $a(h) = a(k) + a(\ell) = 0$, and so d is continuous.

Conversely, let d be a continuous φ-derivation. Then

$$\lim_{x\to 0} \varphi(x)d(b) = \lim_{x\to 0} d(xb) - \lim_{x\to 0} d(x)\varphi(b) = 0 \quad (b \in \mathfrak{A}). \qquad \blacksquare$$

Theorem 4.4. *Every φ-derivation d is automatically continuous, provided that φ is d-continuous, and at least one of φ or d is a $*$-mapping.*

Proof. Let $d : \mathfrak{A} \to B(\mathfrak{H})$ be a φ-derivation. Then clearly d^* is a φ^*-derivation. Set

$$\varphi_1 := \frac{1}{2}(\varphi + \varphi^*), \quad \varphi_2 := \frac{1}{2\mathbf{i}}(\varphi - \varphi^*), \quad d_1 := \frac{1}{2}(d + d^*), \quad d_2 := \frac{1}{2\mathbf{i}}(d - d^*).$$

Obviously these are $*$-mappings, $\varphi = \varphi_1 + \mathbf{i}\varphi_2$, $d = d_1 + \mathbf{i}d_2$, and φ_k is d_j-continuous for $1 \le k, j \le 2$. A straightforward calculation shows that if φ is a $*$-mapping then $\varphi_1 = \varphi = \varphi_2$, and d_1, d_2 are φ-derivations. Similarly, if d is a $*$-mapping, then $d_1 = d = d_2$ and d is a φ_j-derivation for $j = 1, 2$. Since φ or d is a $*$-mapping, then φ_k is a $*$-mapping and d_j is a $*$-φ_k-derivation for $1 \le k, j \le 2$. By Theorem 4.3, the d_j's are continuous, and so $d = d_1 + \mathbf{i}d_2$ is also continuous. ∎

Corollary 4.5. *Let φ and ψ be $*$-mappings and let d be a $*$-(φ, ψ)-derivation. Then d is automatically continuous if and only if φ and ψ are left and right d-continuous, respectively.*

Proof. Suppose that φ and ψ are left and right d-continuous, respectively. Since d, φ, and ψ are $*$-mappings, then both φ and ψ are d-continuous. Hence $\frac{\varphi+\psi}{2}$ is also d-continuous. We have

$$\begin{aligned} 2d(ab) &= d(ab) + d^*(ab) \\ &= \varphi(a)d(b) + d(a)\psi(b) + \big(\varphi(b^*)d(a^*) + d(b^*)\psi(a^*)\big)^* \\ &= \varphi(a)d(b) + d(a)\psi(b) + \psi(a)d(b) + d(a)\varphi(b) \\ &= (\varphi + \psi)(a)d(b) + d(a)(\varphi + \psi)(b) \qquad (a, b \in \mathfrak{A}). \end{aligned}$$

Thus d is a $*$-$\frac{\varphi+\psi}{2}$-derivation. It follows from Theorem 4.4 that d is continuous.

Conversely if d is a continuous (φ, ψ)-derivation, then

$$\lim_{x\to 0} \varphi(x)d(b) = \lim_{x\to 0} d(xb) - \lim_{x\to 0} d(x)\psi(b) = 0 \quad (b \in \mathfrak{A}),$$

and

$$\lim_{x\to 0} d(b)\psi(x) = \lim_{x\to 0} d(bx) - \lim_{x\to 0} \varphi(b)d(x) = 0 \quad (b \in \mathfrak{A}).$$

So φ and ψ are left and right d-continuous, respectively. ∎

Lemma 4.6. *Let d be a (φ, ψ)-derivation. Then there are two mappings Φ and Ψ with $\Phi(0) = 0 = \Psi(0)$ such that d is a (Φ, Ψ)-derivation.*

Proof. Define Φ and Ψ by

$$\Phi(a) := \varphi(a) - \varphi(0),$$
$$\Psi(a) := \psi(a) - \psi(0),$$

for all $a \in \mathfrak{A}$. We have

$$0 = d(0) = d(a \cdot 0) = \varphi(a)d(0) + d(a)\psi(0) = d(a)\psi(0) \qquad (a \in \mathfrak{A}),$$
$$0 = d(0) = d(0 \cdot a) = \varphi(0)d(a) + d(0)\psi(a) = \varphi(0)d(a) \qquad (a \in \mathfrak{A}).$$

Thus

$$\Phi(a)d(b) + d(a)\Psi(b) = \varphi(a)d(b) + d(a)\psi(b) = d(ab) \qquad (a, b \in \mathfrak{A}).$$

Hence d is a (Φ, Ψ)-derivation. ∎

Corollary 4.7. *If $*$-mappings φ and ψ are continuous at zero, then every $*$-(φ,ψ)-derivation d is automatically continuous.*

Proof. Apply Lemma 4.6 and Corollary 4.5. ∎

Clearly the assumption of Corollary 4.7 comes true whenever φ and ψ are linear and bounded. The next theorem states that when we deal with a continuous $*$-(φ,ψ)-derivation, we may assume that φ and ψ have 'at most' zero separating spaces.

Theorem 4.8. *Let φ and ψ be $*$-mappings. If d is a continuous $*$-(φ,ψ)-derivation, then there are $*$-mappings $\varphi^{'}$ and $\psi^{'}$ from $\mathfrak{A}$ into $B(\mathfrak{H})$ with 'at most' zero separating spaces such that d is a $*$-$(\varphi^{'},\psi^{'})$-derivation.*

Proof. Set

$$\begin{aligned} Y &:= \{d(a) \mid a \in \mathfrak{A}\}, \\ \mathfrak{L}_0 &:= \cup\{d(a)h \mid a \in \mathfrak{A}, h \in \mathfrak{H}\}. \end{aligned}$$

Let $\mathfrak{L}$ be the closed linear span of $\mathfrak{L}_0$ in $\mathfrak{H}$, and let $\mathfrak{K} := \mathfrak{L}^{\perp}$. Suppose that $p \in B(\mathfrak{H})$ is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{L}$. It follows from continuity of operators $d(a)$ that $d(a)(\mathfrak{L}) \subseteq \mathfrak{L} \quad (a \in \mathfrak{A})$, and so

$$pd(a) = d(a)p \qquad (a \in \mathfrak{A}),$$

and

$$p\varphi(a) = \varphi(a)p, \qquad p\,\psi(a) = \psi(a)p \qquad (a \in \mathfrak{A}).$$

For a typical element $\ell = d(b)h$ of $\mathfrak{L}_0$, we have

$$\begin{aligned} \varphi(a)\ell &= \varphi(a)d(b)h \\ &= d(ab)h - d(a)\psi(b)h \in \mathfrak{L}. \end{aligned}$$

Therefore $\varphi(a)(\mathfrak{L}_0) \subseteq \mathfrak{L}$ and hence $\varphi(a)(\mathfrak{L}) \subseteq \mathfrak{L}$.
The same argument shows that $\psi(a)(\mathfrak{L}) \subseteq \mathfrak{L}$. Now we define $\varphi^{'}$ and $\psi^{'}$ from $\mathfrak{A}$ into $B(\mathfrak{H})$ by $\varphi^{'}(a) := \varphi(a)p \quad (a \in \mathfrak{A})$, and $\psi^{'}(a) := \psi(a)p \quad (a \in \mathfrak{A})$. Clearly $\varphi^{'}$ and $\psi^{'}$ are $*$-mappings. Furthermore, d is a $*$-$(\varphi^{'},\psi^{'})$-derivation. In fact for all $a, b \in \mathfrak{A}$, and $h \in \mathfrak{H}$, we have

$$\begin{aligned} d(ab)h &= \varphi(a)d(b)h + d(a)\psi(b)h \\ &= \varphi(a)pd(b)h + pd(a)\psi(b)h \\ &= \varphi^{'}(a)d(b)h + d(a)\psi^{'}(b)h\,, \end{aligned}$$

since p commutes with $d(a), \varphi(a)$, and $\psi(a)$. Suppose that $\mathfrak{S}(\varphi^{'}) \neq \emptyset$ and $a \in \mathfrak{S}(\varphi^{'})$. If $a = \lim_{n\to\infty} \varphi^{'}(a_n)$ for some sequence $\{a_n\}$ in $\mathfrak{A}$ converging to zero, then

$$\begin{aligned} a(k) &= \lim_{n\to\infty} \varphi^{'}(a_n)k \\ &= \lim_{n\to\infty} \varphi(a_n)pk \\ &= \lim_{n\to\infty} \varphi(a_n)0 \\ &= 0 \qquad (k \in \mathfrak{K}). \end{aligned}$$

Let $\ell_0 = d(b)h \in \mathfrak{L}_0$, where $b \in \mathfrak{A}$ and $h \in \mathfrak{H}$. Then

$$\begin{aligned} a(\ell_0) &= \lim_{n\to\infty} \varphi'(a_n)d(b)h \\ &= \lim_{n\to\infty} d(a_nb)h - \lim_{n\to\infty} d(a_n)\psi(b)h \\ &= 0, \end{aligned}$$

and by continuity of operator a, we get $a(\ell) = 0 \quad (l \in \mathfrak{L})$, and hence $a = 0$. Similarly the separating space of ψ' is empty or $\{0\}$. ∎

5 Generalized (φ, ψ)-derivations

In this section, we study the continuity of generalized (φ, ψ)-derivations.

Definition 5.1. Suppose that $\mathfrak{B}$ is an algebra, $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, $\mathfrak{X}$ is a $\mathfrak{B}$-bimodule, $\varphi, \psi : \mathfrak{B} \to \mathfrak{B}$ are mappings, and $d : \mathfrak{A} \to \mathfrak{X}$ is a (φ, ψ)-derivation. A linear mapping $\delta : \mathfrak{A} \to \mathfrak{X}$ is a *generalized* (φ, ψ)*-derivation* corresponding to d if

$$\delta(ab) = \varphi(a)d(b) + \delta(a)\psi(b) \quad (a, b \in \mathfrak{A}).$$

Proposition 5.2. *Suppose that $\mathfrak{A}$ is a C^*-algebra acting on a Hilbert space $\mathfrak{H}$. A generalized $*$-(φ, ψ)-derivation $\delta : \mathfrak{A} \to B(\mathfrak{H})$ corresponding to the (φ, ψ)-derivation d is automatically continuous provided that $\varphi : \mathfrak{A} \to B(\mathfrak{H})$ is a left d-continuous $*$-mapping, and $\psi : \mathfrak{A} \to B(\mathfrak{H})$ is both a right d-continuous and a δ-continuous $*$-mapping.*

Proof. Suppose that $\{a_n\}$ is a sequence in $\mathfrak{A}$, and $a_n \to 0$ as $n \to \infty$. By the Cohen factorization theorem, there exist a sequence $\{b_n\}$ in $\mathfrak{A}$, and an element $c \in \mathfrak{A}$ such that $a_n = cb_n$, for all $n \in \mathbb{N}$, and $b_n \to 0$ as $n \to \infty$. By Corollary 4.5, d is continuous, so $d(a_n) = d(cb_n) \to 0$ as $n \to \infty$. A straightforward computation shows that

$$(\delta - d)(xy) = (\delta - d)(x)\psi(y) \qquad (x, y \in \mathfrak{A}).$$

Thus

$$\begin{aligned} \delta(a_n) &= (\delta - d)(a_n) + d(a_n) \\ &= (\delta - d)(c)\psi(b_n) + d(a_n), \end{aligned}$$

which converges to zero as $n \to \infty$, since ψ is right$(\delta - d)$-continuous. ∎

Corollary 5.3. *Let $\mathfrak{A}$ be a C^*-algebra acting on a Hilbert space $\mathfrak{H}$. Suppose that $\varphi, \psi : \mathfrak{A} \to B(\mathfrak{H})$ are continuous at zero. Then every generalized $*$-(φ, ψ)-derivation $d : \mathfrak{A} \to B(\mathfrak{H})$ is automatically continuous.*

Proof. Using the same argument as in the proof of Lemma 4.6, we may assume that $\varphi(0) = 0 = \psi(0)$. Thus φ and ψ are S-continuous for each mapping S. Now the result is obtained from Theorem 5.2. ∎

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