# Achieving the Capacity of the N-Relay Gaussian Diamond Network Within log N Bits

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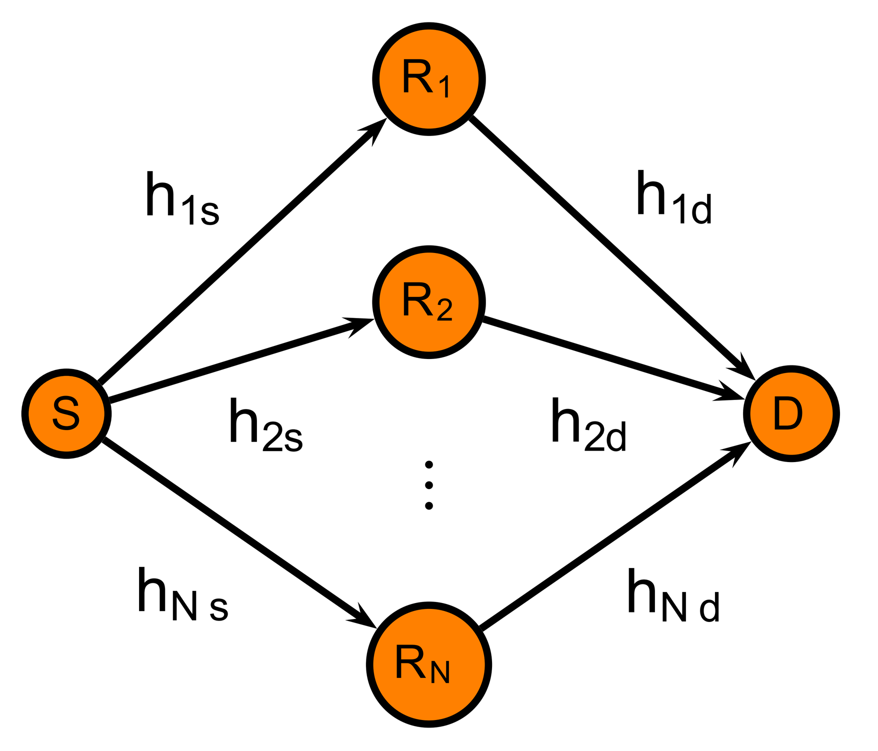
## Abstract

**Can we achieve the information theoretic cutset upper bound on the capacity of the Gaussian $N$-relay diamond network within $O(\log N)$ bits?**

Best capacity approximations currently available for this network are within $O(N)$ bits to the cutset upper bound. We show that several strategies can achieve the capacity of this network within O($\log N$) bits, independent of the channel configurations and the operating SNR.

## Gaussian Diamond Network




- Broadcast from source to relays:

$$Y_i[t] = h_{is}X_s[t] + Z_i[t]$$

- Superposition of relay signals at destination:

$$Y_d[t] = \Sigma_{i=1}^{N} h_{id}X_i[t] + Z[t]$$

- Arbitrary channel gains and SNR's.

## Cutset Bound

The information theoretic cutset upper bound on the capacity of this network is given by

$$\overline{C} = \sup_{X,X_1,\dots,X_N} \min_{\Lambda\subseteq\mathcal{N}} I(X, X_\Lambda; Y, Y_{\bar{\Lambda}} \mid X_{\bar{\Lambda}}),$$

and it can be upper bounded as

$$\overline{C} \le \min_{\Lambda\subseteq\mathcal{N}} \left( \log\left(1 + \mathsf{SNR}\Sigma_{i\in\bar{\Lambda}}|h_{is}|^2\right) + \log\left(1 + \mathsf{SNR}\left(\Sigma_{i\in\Lambda}|h_{id}|\right)^2\right)\right).$$

## Previous Results

- Quantize-map-and-forward [1], noisy network coding [2] and compress and forward [3] achieve

$$R \ge \overline{C} - O(N).$$

- Amplify-and-forward achieves [5]

$$R \ge \frac{1}{O(\log^4(N))}\overline{C}.$$

- Using a subset $k$ of the relays achieves [4]

$$R \ge \frac{k}{k+1}\overline{C} - O(k) - O(\log N).$$

## Main Result

**Theorem 1.** Let $\overline{C}$ be the information-theoretic cutset upper bound on the capacity of the $N$-relay diamond network. Then a partial decode-and-forward strategy at the relays achieves a rate

$$R_{PDF} \ge \overline{C} - G_1,$$

where $G_1 = 2\log N$.

**Theorem 2.** Noisy network coding at the relays can achieve a rate

$$R_{NNC} \ge \overline{C} - G_2,$$

where $G_2 = \log(N+1) + \log N + 1$. The same performance can be achieved by quantize-map-and-forward or compress-and-forward.

## Partial Decode-and-Forward

Treat the first stage of communication as a broadcast channel and the second stage as a multiple access channel.

- Source sends independent messages to the relays at rates $R_i$ that lie in the intersection of the broadcast and multiple access capacity regions.
- Relays decode these messages and re-encode and forward them to the destination over the MAC channel.
- The rate achieved by this strategy is given by

$$R_{PDF} = \Sigma R_i \quad \text{if} \quad \{R_1,\dots,R_N\} \in \mathcal{C}_{BC} \cap \mathcal{C}_{MAC}$$

- $\mathcal{C}_{MAC}$ is polymatroidal and $\mathcal{C}_{BC}$ contains the polymatroidal capacity region of the dual MAC.
- By Edmond's polymatroid intersection theorem

$$\max\{\Sigma_i R_i : (R_1,\dots,R_N) \in \mathcal{C}_{BC} \cap \mathcal{C}_{MAC}\} = \min_{\Lambda\subseteq\mathcal{N}} \quad f(\bar{\Lambda}) + g(\Lambda),$$

where $f$ and $g$ are the submodular functions associated with the polymatroids

$$f(S) = \log\left(1 + \Sigma_{i\in S}|h_{id}|^2\,\mathsf{SNR}\right),$$
$$g(S) = \log\left(1 + \Sigma_{i\in S}|h_{is}|^2\,\mathsf{SNR}/N\right).$$

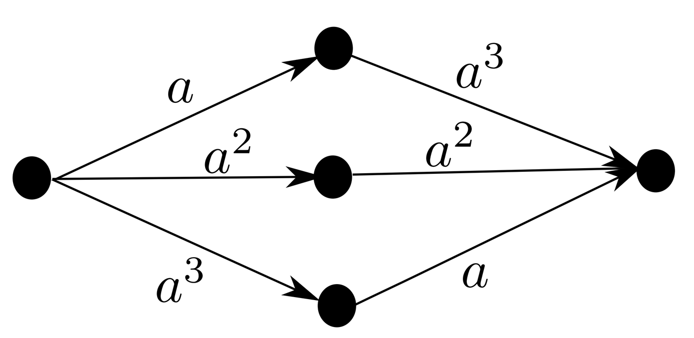
- Therefore, partial decode and forward can achieve a rate $R_{PDF}$

$$\min_{\Lambda\subseteq\mathcal{N}}\left(\log\left(1 + \Sigma_{i\in\bar{\Lambda}}|h_{is}|^2\,\mathsf{SNR}/N\right) + \log\left(1 + \Sigma_{i\in\Lambda}|h_{id}|^2\,\mathsf{SNR}\right)\right).$$

- The gap to the cutset upper bound is bounded as

$$\overline{C} - R_{PDF} \le 2\log N.$$

## Discussion




The deterministic model suggests each relay should carry information at rate approximately $\log a$. These correspond to the rates of the superposed codebooks with partial decode-and-forward.

- A natural choice for the powers of superposed codebooks is $P_1 = 1 - 1/a - 1/a^2$, $P_2 = 1/a$, and $P_3 = 1/a^2$. At large $a$, this corresponds to rates

$$R_1 \approx \log a - 1, \quad R_2 \approx \log a - 1, \quad \dots \quad R_N \approx \log a,$$

yielding a gap of $O(N)$.

- Instead, if we choose powers as $P_1 = 1 - \Sigma_{i=2}P_i$ and $P_i = \frac{N-i+1}{a^{i-1}}$ for $i = 2,\dots,N$, we obtain the rates

$$R_1 \approx \log a - \log N, \quad R_2 \approx \log a, \quad \dots \quad R_N \approx \log a.$$

Then the rate is only $O(\log N)$ bits away from the SIMO capacity.

## Noisy Network Coding

Performance achieved by noisy network coding is:

$$R_{NNC} = \min_{\Lambda\subseteq\mathcal{N}} I(X_s, X_\Lambda; Y_d, \hat{Y}_{\bar{\Lambda}}|X_{\bar{\Lambda}}) - I(Y_\Lambda; \hat{Y}_\Lambda|X, X_\mathcal{N}, \hat{Y}_{\bar{\Lambda}}, Y_d)$$

for some joint probability distribution $\Pi p(x_i)p(\hat{y}_i|y_i, x_i)$.

Choosing $X_i$ to be i.i.d. circularly symmetric Gaussian of variance $P$ and

$$\hat{Y}_i = Y_i + \hat{Z}_i, \qquad i \in \mathcal{N},$$

where $\hat{Z}_i$ are i.i.d. circularly symmetric and complex Gaussian random variables of variance $N\sigma^2$, the second term can be upper bounded as

$$I(Y_\Lambda; \hat{Y}_\Lambda|X, X_\mathcal{N}, \hat{Y}_{\bar{\Lambda}}, Y_d) = |\Lambda|\log(1 + \frac{1}{N}) \le 1.$$

The first mutual information becomes

$$I(X_s, X_\Lambda; Y_d, \hat{Y}_{\bar{\Lambda}}|X_{\bar{\Lambda}}) = \log\left(1 + \Sigma_{i\in\bar{\Lambda}}|h_{is}|^2\,\mathsf{SNR}/(N+1)\right) + \log\left(1 + \Sigma_{i\in\Lambda}|h_{id}|^2\,\mathsf{SNR}\right),$$

since the quantized observations are corrupted by quantization and thermal noise with total variance $(N+1)\sigma^2$.

The total gap to the cutset upper bound is bounded by

$$\overline{C} - R_{NNC} \le \log(N+1) + \log(N) + 1.$$

The same performance can be achieved by quantize-map-and-forward and compress-and-forward.

## References

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